Two problems based on infinite uniform planar matchings

Laurent TOURNIER

+ Nicolas CURIEN, Gady KOZMA and Vladas SIDORAVICIUS.

Séminaire de Probas du LAGA

18.1.2017

Planar matchings - A versatile object

↳ i.i.d. sequence $(\omega_x)_{x\in\mathbb{Z}}$ uniform on $\{-1,1\}$

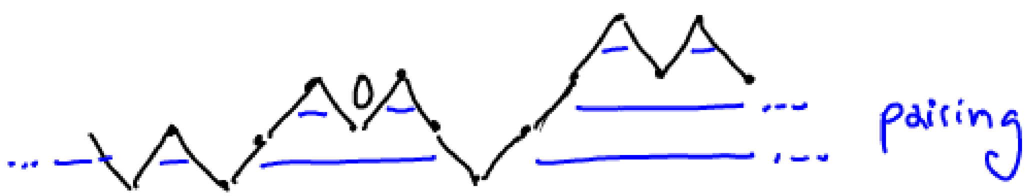
$$\begin{array}{c} \qquad\qquad\qquad\quad 0 \\ \cdots\ -\ +\ -\ +\ +\ -\ +\ -\ -\ +\ +\ +\ -\ +\ -\ \cdots \end{array}$$

↳ viewed as parentheses $\quad \cdots\ )\ (\ )\ (\ (\ )\ (\ )\ )\ (\ (\ (\ )\ (\ )\ \cdots$

⇒ planar matching on $\mathbb{Z}$

(non crossing)

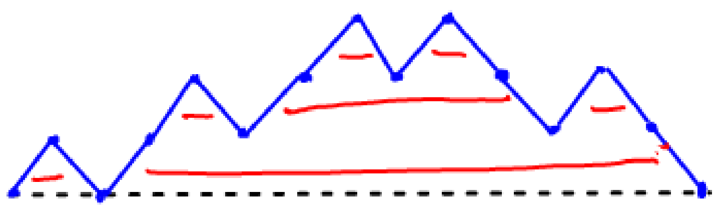
↳ viewed as RW



N.B. Distance to pair = length of excursion of SRW ⇒ non-integrable $\left(P(L>n) \sim \frac{c}{\sqrt{n}}\right)$

In the finite case: Correct expression with n pairs of parentheses

- $(\,) \, (\, (\,) \, (\, (\,) \, (\,) \,) \, (\,) \,)$

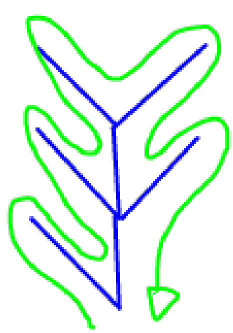
- 

non negative path of length $2n$

NB. Counted by Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$

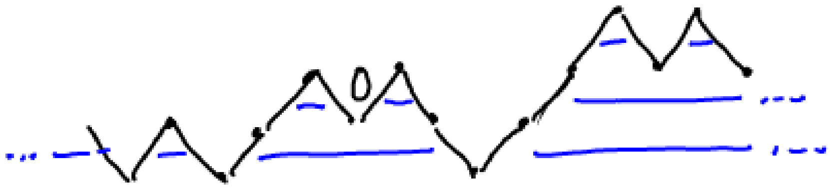
identify matched edges

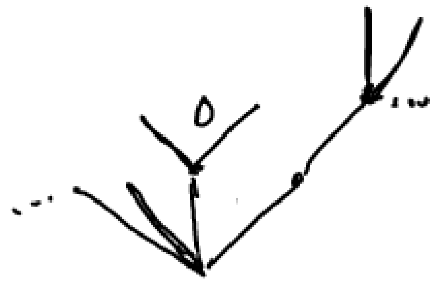
height along contour

- 

ordered tree on n vertices

The tree in the infinite case

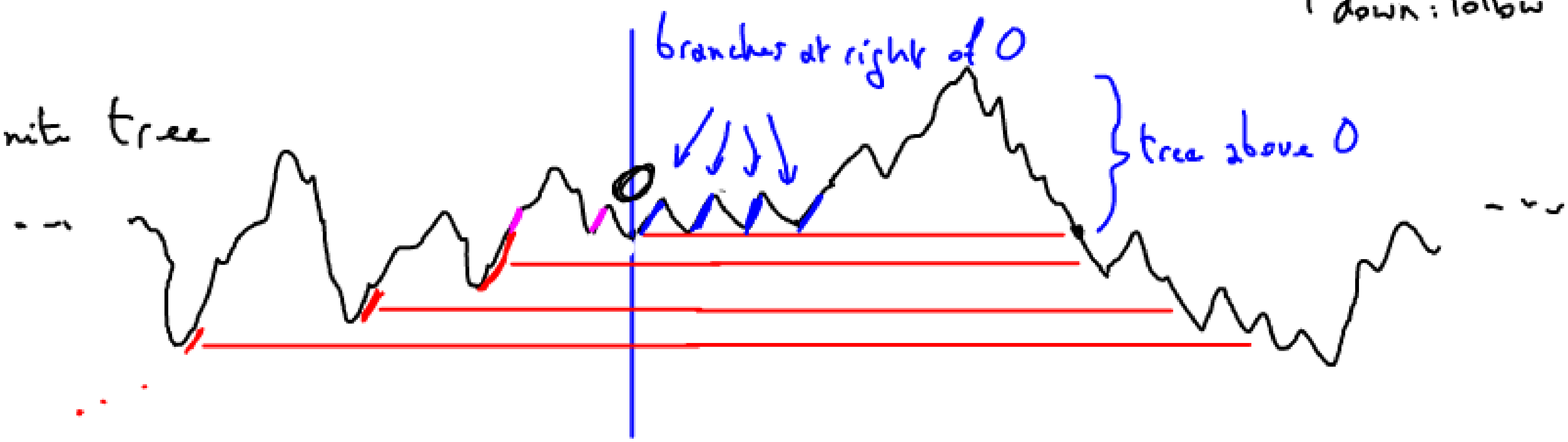




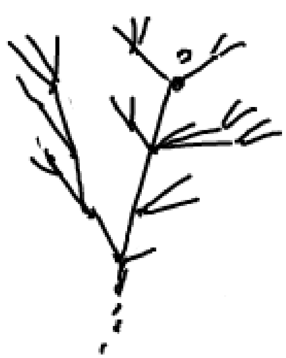
Build along the RW:

- up: new edge
- down: follow edge

→ infinite tree

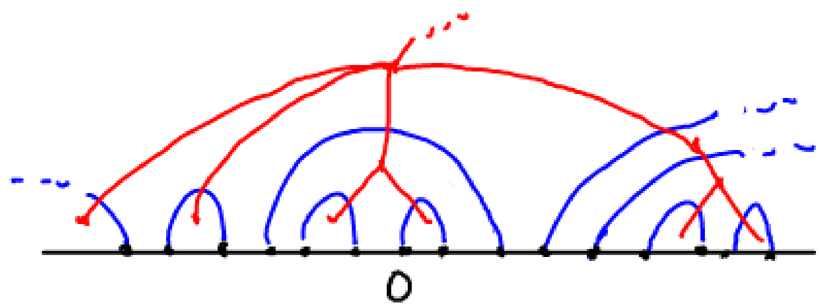


= critical GW tree with geometric(1/2), conditioned on survival (limit)



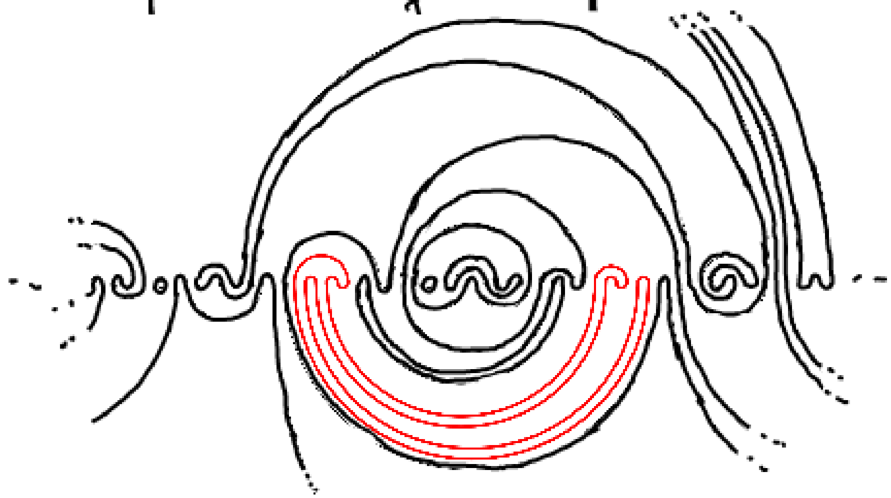
infinite spine, + indep. critical GW trees grafted on left and right

Link with matching = dual graph



Problem #1 : The "elusive infinite noodle"

Superpose 2 independent infinite planar matchings :



Each vertex has degree 2 $\rightarrow$ connected components are cycles or paths

Q: Is there an infinite path ?

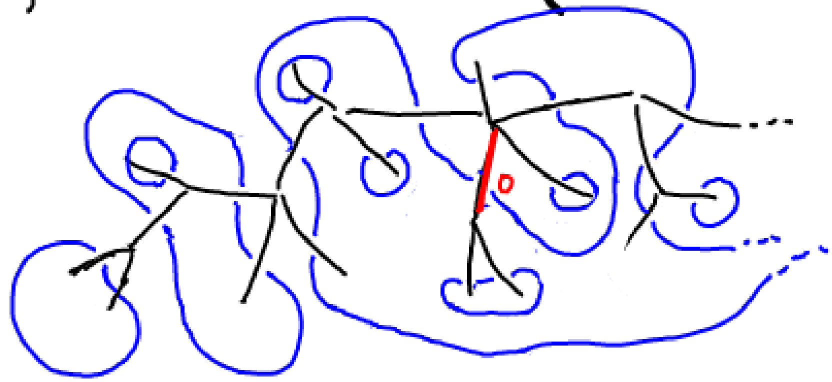
NB. Number N_{paths} is constant a.s. (translation invariant + ergodicity)

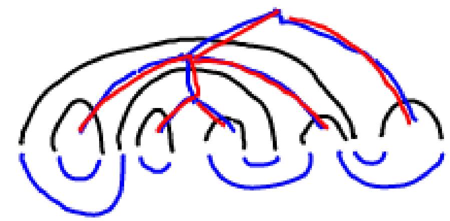
Thm (Curien, Kozma, Sidoravicius, T.): $N_{paths} \leq 1$ a.s.

Open : $N_{paths} = 0$?

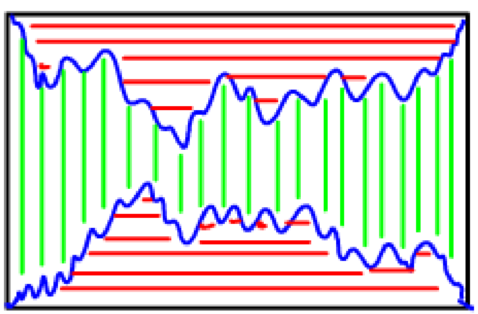
Origin of problem, context (inspiration)

View one matching as a tree; the other as identification between edges:





cf. "mating of trees" by Duplantier-Miller-Sheffield



two Brownian excursions (correlated)

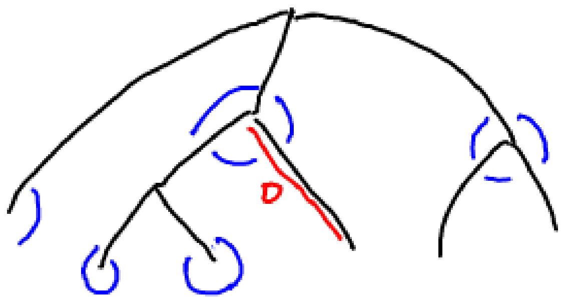
identify (→ Continuous tree)

identify ⇒ topological sphere (+ structure ⇝ brownian map, SLE, imaginary geometry ...)

~

For us: identify edges, cut out if . Edge of 0 eventually cut ⟺ $C(0)$ finite

Ex. with simple arches:

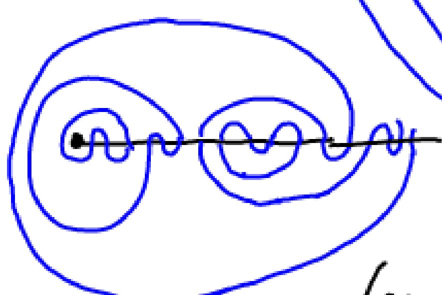


$\Rightarrow c(0)$ finite

NB. More interesting with

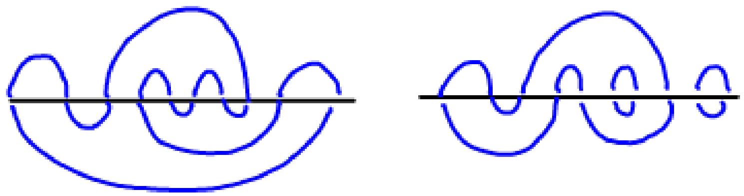
but difficult.

In fact, $\Leftrightarrow$



(same problem on $\frac{1}{2}$-line $\to$ not even translation invariant...)

Context in the finite case: superpose two matchings of $1, \dots, 2n$

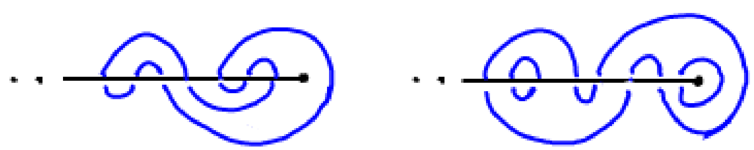


"meanders" cf. Di Francesco, Guitter

→ enumeration of connected meanders on $2n$ points? Asymptotics?

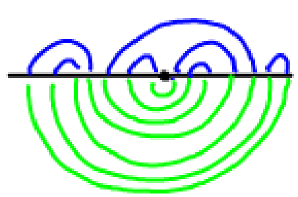
→ $\det\left(q^{\#\text{ c.c. of }(C,C')}\right)_{C,C' \text{ matchings on } 1,\dots,2n}$ can be computed explicitly

Also on $1/2$-line



(cf. stamps problem)

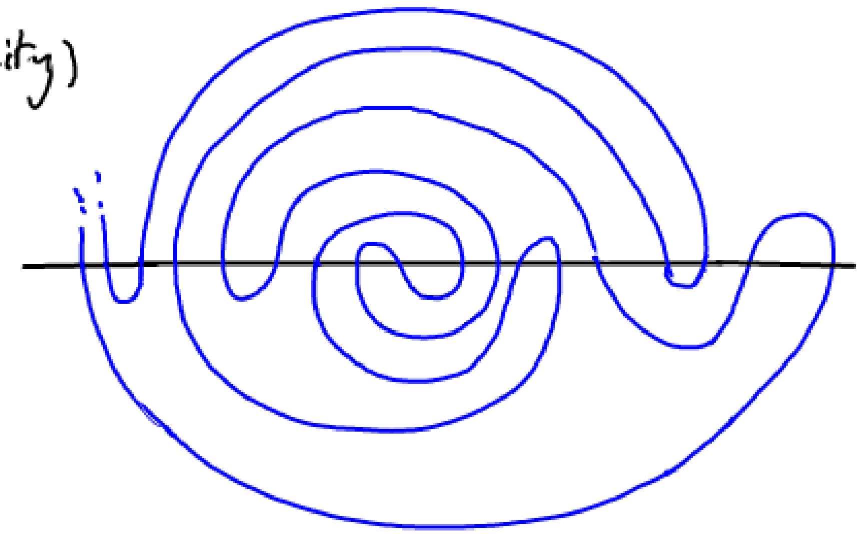
(⇔



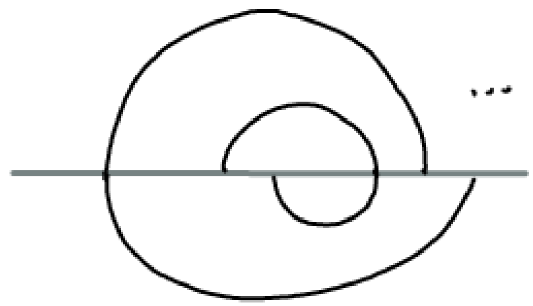
)

What would the infinite path look like?

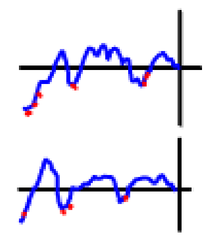
* it has positive density (by ergodicity)
* it crosses 0 infinitely many times



(reason: infinitely many shells:



(common time at minimum for 2 SRW)



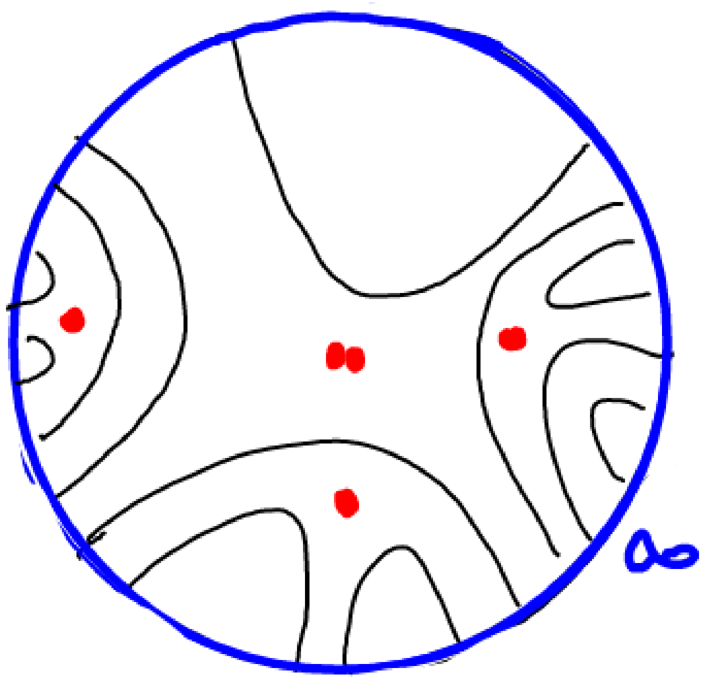
Proof. (Adapts arguments introduced for other models of stat. physics)

* $\boxed{N_{paths} \leq 2}$.

Based on Burton & Keane's trifurcation argument from percolation

Assume $N_{paths} \geq 3$. Consider only infinite paths. "Multifurcation = connected component touching ≥ 3 paths"

Topologically,



• : multifurcations

↪ these are vertices of degree ≥ 3 of the dual graph (= tree)

leaves = # trifurcations + 2

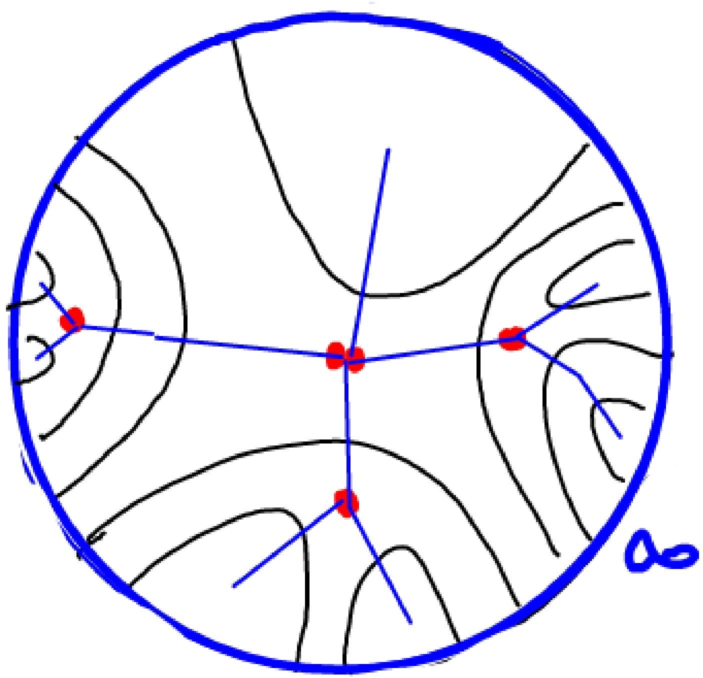
Proof. (Adapts arguments used for other stat. mech. models)

$\ast\ \boxed{N_{paths} \leq 2}$.

Based on Burton & Keane's trifurcation argument (from percolation)

Assume $N_{paths} \geq 3$. Consider only infinite paths. "Multifurcation = connected component touching ≥ 3 paths"

Topologically,

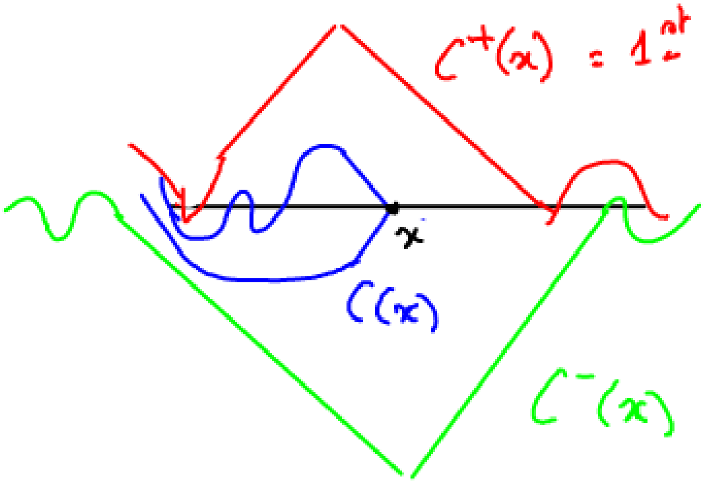


$\bullet$: multifurcations

↪ these are vertices of degree ≥ 3 of the dual graph (= tree)

leaves = # trifurcations + 2

More specifically,



$C^+(x)$ = 1st infinite path above x

$C^-(x)$ = same below

x is a trifurcation if $\begin{cases} \cdot\ C(x),\ C^+(x),\ C^-(x) \text{ distinct} \\ \cdot\ d(x, C^+(x)) \leq M,\ d(x, C^-(x)) \leq M \end{cases}$

(restrictive, but enough)

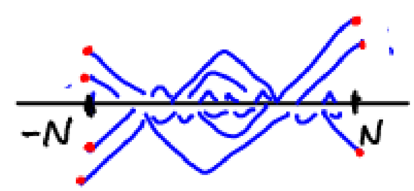
Lemma: There is at most 1 such trif. for 3 clusters

⊤ ——— $n-2$ ——— n — $\forall n \geq 2$

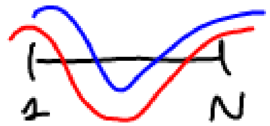
Lemma: If $N_{paths} \geq 3$, $P(0 \text{ is a trif.}) > 0$. (local modification)

$$\Rightarrow \underbrace{\#(\text{trif. in } [-N-M, N+M])}_{\leq \#(\text{clusters in } [N,N]) - 2 \ \leq\ \#(\text{paths exiting } [-N,N]) \ \overset{CLT}{\simeq}\ \sqrt{N}} \simeq 2N \cdot P(0 \text{ is a trif.})$$

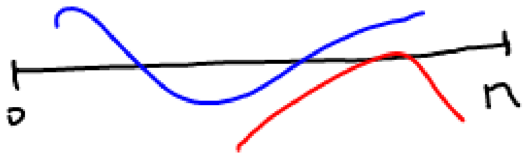
→ contradiction



$*$ $\boxed{N_{paths} \neq 2}$. Assume $N_{paths} = 2$: C_1 C_2

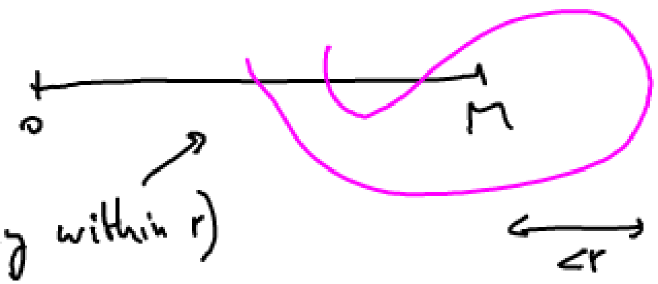
Let N s.t. $P\Big($ 

$\Big) \geq \frac{3}{4}$

M s.t. $P\Big($ 

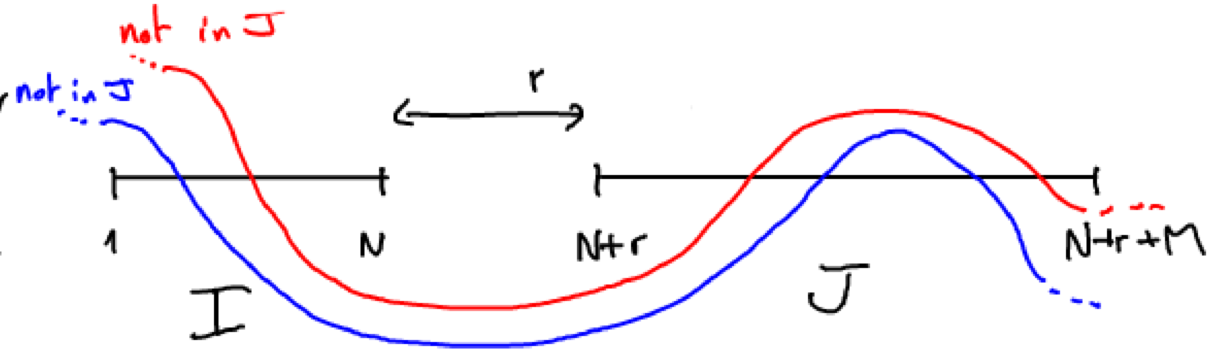
$\Big) \geq 1 - \frac{1}{4} 4^{-N} \left(> \frac{3}{4}\right)$ $\qquad (*)$

r s.t.

$P\Big($ 

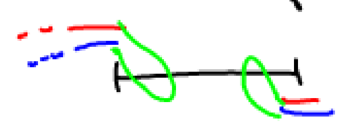
$\Big) \geq \frac{3}{4}$

(all loops stay within r)

Then $P\Big($ 

$\Big) > 1 - \frac{1}{4} - \frac{1}{4} - \frac{1}{4} = \frac{1}{4}$

But, by modifying configuration in I (cost 4^N), we can contradict $(*)$: connect the "left" ends.



Problem #2 : Annihilating particles

Back to planar matching:

(Belitsky-Ferrari) . density of particles at time t $\sim C t^{-1/2}$

. formation of aggregates (...)

More generally, consider particles . starting from a Poisson point process on $\mathbb{R}$

. travelling at iid speeds with law μ (bounded support)

. annihilating at collision

↪ Gives a matching ? i.e. does every particle collide ?

NB. All surviving particles ought to have same speed. If μ is symmetric, only $v=0$ may survive

<u>Example</u>: μ on $\{-1, 0, 1\}$, symmetric: $\mu(0) = p$, $\mu(\pm 1) = \frac{1-p}{2}$

Is there survival ?

Conjecture (?) : survival $\Leftrightarrow p > \frac{1}{4}$.

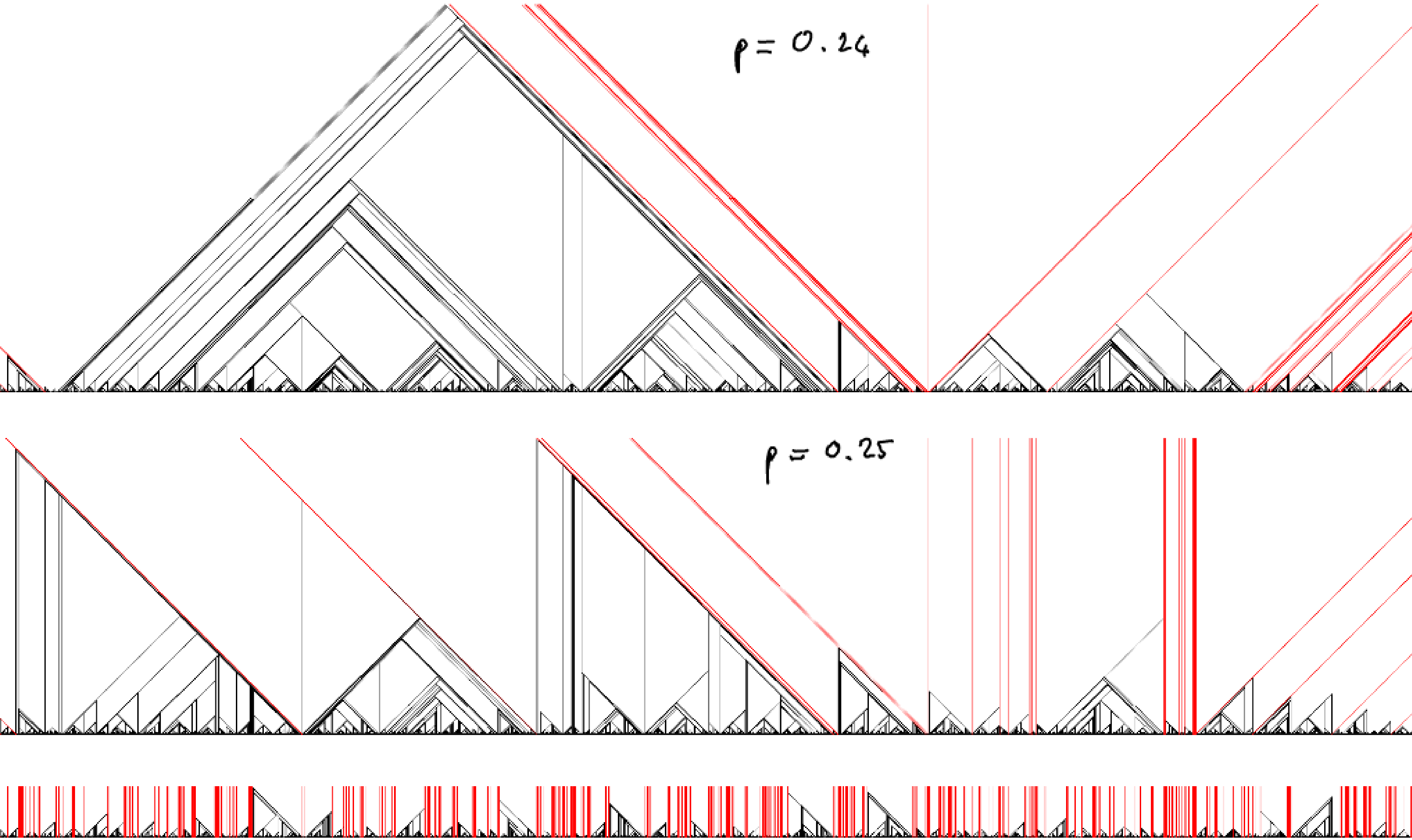
- survival at $p > \frac{1}{3}$, simple
- extinction at $p \simeq 0$, surprisingly hard??

Piasecki (Phys. Rev. Letters 1995) : $p_c = \frac{1}{4}$ (and asymptotics of densities)

(based on "exact" derivation and resolution of diff. equation for p.d.f. of interdistances ; partly formal)

↪ rigorous ?

↪ very rigid, no intuition



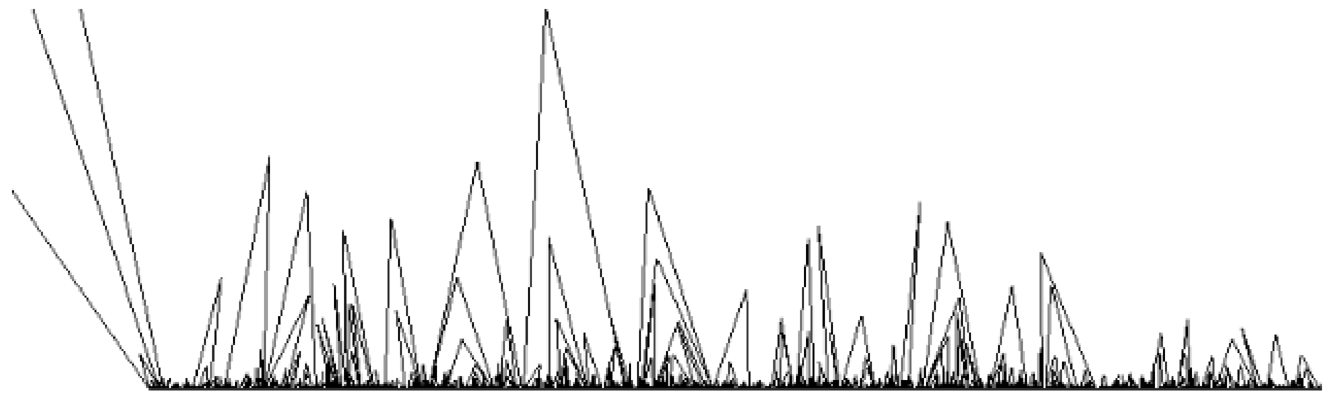
p = 0.26

Variant on half-line

Assume particles start on $\mathbb{R}_+$ only.

Prop: If μ symmetric,
- positive-speed particles die a.s.
- 0 is crossed i.o. a.s.

(with particles on $\mathbb{R}_+$ initially)



Open: What of negative speeds?

- For any μ, is there a critical v_c?

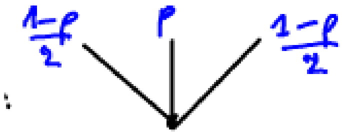
Lemma: On full-line. If there is no survival, then 0 is still crossed i.o.

[Aizenman-Nachtergaele's argument] Assume #(hulls over 0) $< \infty$

Then parity of #(hulls above 0^+)
- alternates with shift $\Rightarrow$ = even/odd w.p. $1/2$
- is invariant by even shift $\Rightarrow$ = constant a.s. (ergodicity)

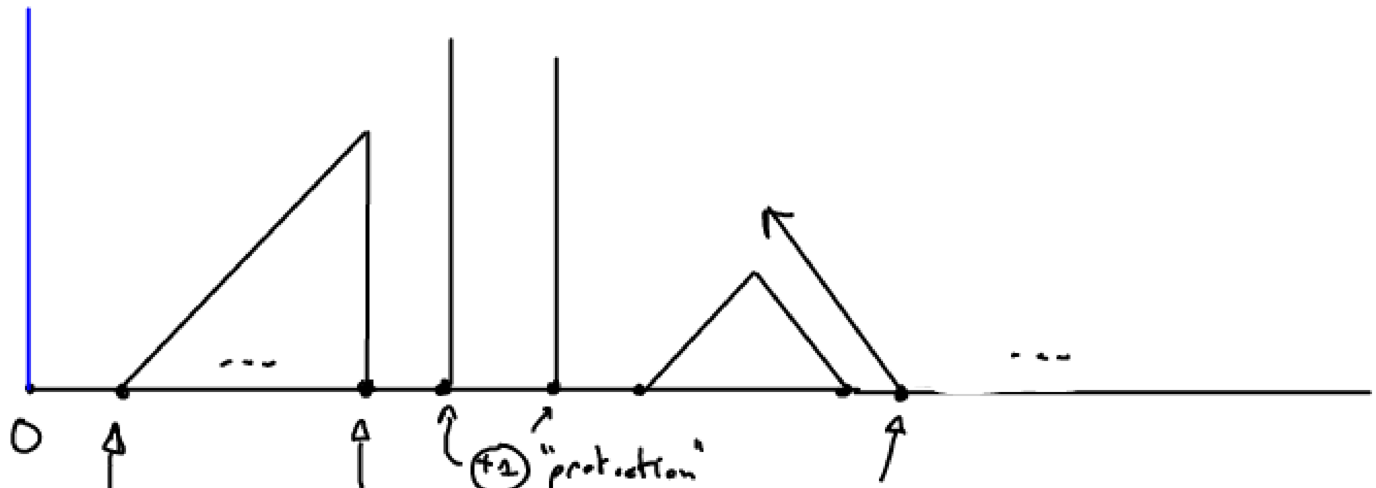
Contradiction!

— : even nb of hulls

<u>Case -1,0,1</u>, symmetric: 

survival at $p > 1/3$.

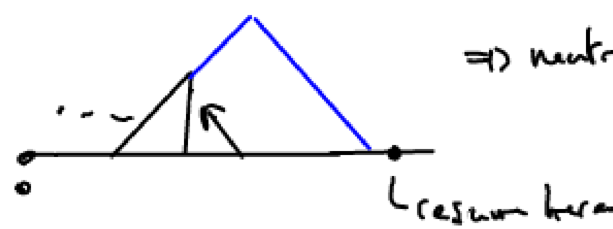
Idea: Assume $v_0 = 0$. <u>Exploration</u> to the right:



speed 1
$\Rightarrow$ eventually dies

1^{st} collision
<u>in exploration</u>

either, breaks protection (-1)
. hits 0 (if protection = 0)
. hits vertical wall that liberates a (+1)-speed particle
$\hookrightarrow$ dies eventually; explore until 1^{st} collision



$\Rightarrow$ neutral effect $0 \geq -1$

$\hookrightarrow$ if $P(\underset{-1}{\nwarrow}) < P(\underset{0}{\uparrow})$, by LLN, protections accumulate faster than (potential) attacks

hence <u>survival of 0 w.p.p.</u>