Toric Topology [Buchstaber–Panov]

(moment-angle manifolds) "Toric Space"
new: 1990–2000
↳ $\boxed{\text{polyhedral products}}$

moment-angle complex & polyhedral products: their homotopy theory.

[Equivariant topology (torus action), combinatorics, discret geometry (polytopes) (fans), commutative algebra, homotopy theory, ...]

$\boxed{1 \text{ Polytopes}}$ 55p → $\boxed{2 \text{ Combinatorial structures}}$ 36p → $\boxed{3 \text{ Face Rings}}$ 38p → $\boxed{4 \text{ Moment-angle complexes}}$ 50p → $\boxed{8 \text{ Homotopy Theory}}$ 34p

1 Polytopes
- Convex polytopes
- Combinatorics of faces
- g-theorem
- (Gale duality)
- → nestohedra, graph associahedra,
  - ↳ permutohedra, associahedra
  - truncated cubes

2 Combinatorial structures
- combinatorial structures on the orbit spaces of toric spaces.
  - ↳ convex polytopes - - - → fans
  - simplicial / cubical cxs.

3 Face Rings
- = Stanley-Reisner rings: alg theory
- simplicial cx / posets
- → algebraic & homological (methods)
- → regular sequences, Cohen-Mac aulay / Gorenstein rings
- Tor-alg. local rings.

4 Moment-angle complexes
- $Z_K \in$ Top
- ↑ simplicial cx
- (torus actions & equivariant maps)
- combinatorial invariants of K
- ↕
- topo characteristics of Z_K
- face numbers, Betti of $\mathbb{Z}[K] \leftrightarrow H^*_{\varphi}(Z_K)$
- $\boxed{H^*(Z_K) \cong \mathrm{Tor}_{\mathbb{Z}[v_1,\dots,v_m]}(\mathbb{Z}[K],\mathbb{Z})}$
- (Prod ring model for cellular cochains of Z_K Δ ξ functorial) → Koszul complex
- $Z_K \sim U(K)$: complement of arrangement (↔ conf spaces)

8 Homotopy Theory
- Polyhedral product,
- model category,
- homotopy colimits,
- higher Whitehead \ Samelson products
- Koszulity formality.

Polytopes $\xrightarrow{\text{boundary of simplicial polytopes}}$ **Simplicial cxs**

P simple $\longmapsto K_P := \partial P^*$
" nerve of covering of ∂P by the facets

($\updownarrow$ variétés toriques)

- $K_{P_1} * K_{P_2} = K_{P_1 \times P_2}$ (join)
- triangulation

$\mathbb{Z}[-]$: Simplicial cxs $\to$ **face ring**

$\mathcal{Z}_\cdot, \mathcal{R}_\cdot, (\mathbb{C}P^\infty)^\cdot$: Simplicial cxs $\to$ **Top**

$\mathcal{Z}_K, \mathcal{R}_K, (\mathbb{C}P^\infty)^K, \ldots$

$\overset{H^\bullet}{\longleftrightarrow}$

[homological alg]

Stanley-Reisner alg $k[K] := k[v_1, \ldots, v_m] / (v_I, I \notin K)$

Cohen-Macaulay/Gorenstein

- Tor
- local cohomology
- regular sequences; Koszul complexes

$k[K_1 * K_2] \cong k[K_1] \otimes k[K_2]$

(quadratic $\Longleftrightarrow$ K flag)

$$\mathrm{Tor}^{-i,2j}_{k[v_1,\ldots,v_m]}(k[K], k) \cong \underbrace{\bigoplus_{\substack{J \subset [m] \\ |J| = j}} \tilde{H}^{j-i}(K_J)}_{\text{structure d'algèbre}}$$

$\boxed{\text{Thm}}$ [Reisner] $k[K]$ Cohen-Macaulay $\to$ Gorestein if $\mathrm{Tor}^{-(m-n)}_{k[m]}(k[K], k) \cong k$

$\updownarrow$

$H^\bullet_{simp}(K)$

$\updownarrow$ [Munkres-Stanley]

$|K|$

<u>Ex:</u> triangulated sphere

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Definition [Moment-angle complex]

$\mathbb{D}^m := \{(z_1,\dots,z_m)\in\mathbb{C}^m ;\ |z_i|^2\leq 1\ \forall i\}$ 2m-dim polydisc.

$\mathbb{T}^m := \{ \underline{\hspace{6em}} = 1 \}$ m-dim torus.

$\mathbb{D}^m/\mathbb{T}^m \cong \mathbb{I}^m \subset \mathbb{R}^m$, $\mathbb{I}^m = [0,1]^m$, $\mathbb{T}^m$ par rotation

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Foncteur monoïdal fort : $(\text{Simpl cx}, *) \xrightarrow[(\mathcal{X},\mathcal{A})]{\mathcal{Z}} (\text{Top}_{\mathbb{T}}, \times)$ T-action

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$(\mathcal{X},\mathcal{A}) = ((X_1,A_1),\dots,(X_m,A_m))$

$BS^1 \cong \mathbb{C}P^\infty$ universal principle S^1-bundle : ∞ Hopf bundle

$BT^m \cong (\mathbb{C}P^\infty)^m$ $S^\infty \to \mathbb{C}P^\infty \Rightarrow ET^m \cong (S^\infty)^m$

$H^*(BT^m;\mathbb{Z}) \cong \mathbb{Z}[v_1,\dots,v_m]$ $|v_i| = 2$

$(\mathbb{C}P^\infty)^K$ cellular sub-cx of $(\mathbb{C}P^\infty)^m$

$\mathcal{Z}_K \cong \bigcup_{I\in K} \left(\prod_{i\in I} D^2 \times \prod_{i\notin I} S^1\right) := \text{Colim } F_K$ (et ?)

$F_K : \text{Cat}(K) \longrightarrow \text{Top}$

$\text{Cat}(K) = \text{Cat}(\Pi_K)$ ← face poset

I face $\longmapsto B_I = \{(z_1,\dots,z_m)\in\mathbb{C}^m \mid |z_i|^2\leq 1$ et $|z_j|^2 = 1\ j\notin I\}$ $=: (D^2,S^1)^I$

Exemples

$[\mathbb{R}]$

① $K = \Delta^{m-1} \Rightarrow \mathcal{Z}_{\Delta^{m-1}} = \mathbb{D}^m$ ① $\mathcal{R}_{\Delta^{m-1}} \cong \mathbb{I}^m$

② $\mathcal{Z}_{\phi_{[m]}} = \mathbb{T}^m$ ② $\mathcal{R}_{\phi_{[m]}} \cong \{-1,1\}^m$

③ $\mathcal{Z}_{\bullet\ \bullet} = D^2\times S^1 \cup S^1\times D^2 \cong \partial(D^2\times D^2) \cong S^3$

④ $\mathcal{Z}_{K\sqcup\{m+1\}} = \mathcal{Z}_K \times S^1$ ④ $\mathcal{R}_{K\sqcup\{m+1\}} \cong \mathcal{R}_K \sqcup \mathcal{R}_K$

⑤ $\mathcal{Z}_{\partial\Delta^{m-1}} = \mathbb{D}^{m-1}\times S^1 \cup \mathbb{D}^{m-2}\times S^1\times \mathbb{D}^1 \cup \dots \cup S^1\times\mathbb{D}^{m-1}$
$= \partial(\underbrace{\mathbb{D}^m}_{(D^2)^m}) \cong S^{2m-1}$ ⑤ $\mathcal{R}_{\partial\Delta^{m-1}} \cong \partial\mathbb{I}^m \cong S^{m-1}$

Proposition: $\mathcal{Z}_{K_1 * K_2} \cong \mathcal{Z}_{K_1}\times\mathcal{Z}_{K_2}$

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§4. Moment-Angle complexes

4.1 Basic definitions

• Moment-Angle complexes

Thm K: triangulated manifold $\Rightarrow$

$\mathcal{Z}_K \setminus \mu^{-1}(1,\dots,1)$ non-compact manifold

Proof: (simple polytope, $|\mathrm{Cone}(K)|$, nerve, barycentric subdivision) $\square$

Def: • Moment-Angle manifold: $\mathcal{Z}_K \leftarrow$ triangulated sphere

• Polytopal " : $\mathcal{Z}_K \leftarrow$ nerve cx of simple polytopes $\Big\}$ smooth

• Real moment-angle complex

$(D^2, S^1) \longrightarrow (\overset{D^1}{I}, \overset{S^0}{\partial I})$, $\quad I = [-1,1]$, $\partial I = \{-1,1\}$

Prop $\mathcal{R}_{\partial(m\text{-gon})} \cong$ oriented surface of genus $g = 1 + (m-4)2^{m-3}$

4.2 Polyhedral products

Def: polyhedral products

Ex: $(X, pt)^K$

4.3 Homotopical properties

Prop $H^*((\mathbb{C}P^\infty)^K) \cong \mathbb{Z}[K]$.

Thm $h: (\mathbb{C}P^\infty)^K \xrightarrow{\sim} ET^m \times_{T^m} \mathcal{Z}_K =: DJ(K)$

$$\begin{tikzcd} (\mathbb{C}P^\infty)^K \arrow[r, "\sim"] \arrow[dr, bend right] & ET^m \times_{T^m} \mathcal{Z}_K \arrow[d, "\text{fibre } \mathcal{Z}_K"] \\ & BT^m \simeq (\mathbb{C}P^\infty)^m \end{tikzcd}$$

↳ $\mathcal{Z}_K$ homotopy fibre of the inclusion $(\mathbb{C}P^\infty)^K \hookrightarrow (\mathbb{C}P^\infty)^m$

Cor $H^*_{T^m}(\mathcal{Z}_K) \cong \mathbb{Z}[K]$

Prop • 2-connexe

• $\pi_i(\mathcal{Z}_K) \cong \pi_i((\mathbb{C}P^\infty)^K)$ $i \geq 3$

• K q-neighbourly $\Rightarrow \pi_i(\mathcal{Z}_K) = 0$ for $i < 2q+1$

4.4 Cell decomposition

$\mathbb{D} =$ D • 1 ⇝ bigraded $\begin{cases}\text{cochain cx} \\ \text{Betti numbers}\end{cases}$

4.5 Cohomology ring

Algebraic model for cellular cochains:

Lemma $R^*(K) := \Lambda[u_1,\dots,u_m] \otimes \mathbb{Z}[K] \,/\, (v_i^2;\ u_i v_i) \simeq C^*(\mathcal{Z}_K)$

Cellular diagonal approximation

Davis-Januszkiewicz space

canonical functorial $\tilde{\Delta}: \mathcal{Z}_K \to \mathcal{Z}_K \times \mathcal{Z}_K$

↳ product on $C^*(\mathcal{Z}_K)$

$\Downarrow$

Lemma iso rings $H(R^*(K)) \cong H^*(\mathcal{Z}_K)$

Thm Functorial iso of bigraded com algebras

$$H^{*,*}(\mathcal{Z}_K) \cong \mathrm{Tor}_{\mathbb{Z}[v_1,\dots,v_m]}(\mathbb{Z}[K], \mathbb{Z}) \cong H(\Lambda[u_1,\dots,u_m] \otimes \mathbb{Z}[K], d)$$

Proof: A.2.6 : Koszul resolution $\square$

Thm Functorial iso $H^*_T(\mathcal{Z}_K) \cong \mathbb{Z}[K]$.

Proof: $\square$

(Case of Cohen-Macaulay cxs)

↳ description of the product in terms of full subcomplexes

Cas $H^*((X,A)^K)$ et $H^*(\mathcal{R}_K)$

product .... higher Massey ....

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4.6 Bigraded Betti numbers

avec $\mathcal{R}^\circ(K)$

caractéristique d'Euler

h-vecteur

$H^*(\mathcal{Z}_K, k)$ PD algebra : $A \xrightarrow{\cong} \mathrm{Hom}(A, A)$, $a \mapsto ax-$

Thm $\mathcal{Z}_K$: Poincaré duality space over k iff K : Gorenstein complex over k

Cohen-Macaulay si $\mathbb{Z}[K]$ Cohen-Macaulay alg ($\leftrightarrow$ regular sequences)

4.7 Coordinate subspace arrangements

$L_I := \{(z_1, \dots, z_m) \in \mathbb{C}^m \mid z_{i_1} = \dots = z_{i_k} = 0\}$

$\mathcal{A}(K) := \{L_I \mid I \notin K\}$ Ex: $\mathcal{A}(\Delta^1) =$ 

$\mathcal{A}(\cdot^0 \cdot^2 \cdot) =$ 

$U(K) := \mathbb{C}^m \setminus \bigcup_{I \notin K} L_I$: contravariant functor simpl cx $\longrightarrow$ Top

$\|$

$(\mathbb{C}, \mathbb{C}^\times)^K$

Thm $\mathbb{T}^m$-equivariant deformation retract

$\mathcal{Z}_K \hookrightarrow U(K) \xrightarrow{\simeq} \mathcal{Z}_K$

Proof: cubical cx □

Thm $K = sd^i \Delta^{m-1} \Rightarrow U(K) \simeq \bigvee_{k=i+2}^{m} (S^{i+k+1})^{\vee \binom{m}{k}\binom{k-1}{i+1}}$

Prop $\widetilde{H}^q(U(K)) \cong \bigoplus_{\hat{I} \in K} \widetilde{H}_{2m-2|\hat{I}|-q-2}(\mathrm{lk}_K \hat{I})$

Proof : Alexander duality □

4.9 Massey products in the cohomology of $\mathcal{Z}_K$

$\hookrightarrow$ sur $(\Lambda[u_1, \dots, u_m] \otimes \mathbb{Z}[K], d)$ ($\uparrow$ Koszul complex)

$\mu_3 \neq 0 \Rightarrow$ not formal

$\hookrightarrow$ Golod complexes ($:= \forall$ higher Massey products $\equiv 0$)

Thm [Berglund – Jöllenbeck] K Golod iff $\mu_2 \equiv 0$

$\mathrm{Tor}_{k[v_1, \dots, v_m]}(k[K], k) \equiv 0$

4.10 Moment-angle complexes from simplicial posets

§8. Homotopy theory of polyhedral products

8.1 Rational homotopy theory of polyhedral products

Thm $\mathcal{X} = (X_1, \dots, X_m)$ formal $\Rightarrow \mathcal{X}^K$ formal $\forall K$

Proof: (co)limit Sullivan models □ formal

Ex: $\mathbb{C}P^\infty$ $\Rightarrow$ $DJ(K)$ spaces formal

formality over $\mathbb{Z}$ (Exercise 8.1.11)

(Doesn't work for pairs $(\mathcal{X}, \mathcal{A})$)

Prop $w \in \mathcal{Z}_K \simeq \mathrm{hocolim}_I\, T^m / T_I$

8.2 Wedges of spheres & connected sums of sphere products

(~~[illegible]~~)

[diff top & surgery]

Thm K : discrete, tree, shifted

$\Rightarrow \mathcal{Z}_K \simeq \bigvee$ spheres

8.3 Stable decomposition of polyhedral products

polyhedral smash product $(\mathcal{X}, \mathcal{A})^{\wedge K}$

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8.4 Loop spaces, Whitehead and Samelson products

Prop exact sequence of homotopy Lie algebras

$$0 \to \pi_*(\Omega Z_K) \otimes \mathbb{Q} \to \pi_*(\Omega (\mathbb{C}P^\infty)^K) \otimes \mathbb{Q} \to Lie^{ab}(u_1, \dots, u_m) \to 0$$

exact sequence of Pontryagin algebras

$$0 \to H_*(\Omega Z_K, k) \to H_*(\Omega (\mathbb{C}P^\infty)^K; k) \to \Lambda[u_1, \dots, u_m] \to 0$$

$$T(u_1, \dots, u_m) / (u_i^2, u_i u_j + u_j u_i \text{ if } \{i,j\} \in K)$$

Topological models for Ω-

Cor $\Omega((\mathbb{C}P^\infty)^K) \simeq \operatorname{hocolim}^{T\text{top}}_{I \in K} T^I$

Algebraic models

face coalgebra $k\langle K \rangle$

Prop iso of graded alg: $H_*(\Omega (\mathbb{C}P^\infty)^K, k) \cong H(\Omega_* k\langle K \rangle)$

$\to$ Lie models

Examples.

8.5 The case of flag complexes K

Prop K flag $\Longrightarrow$ $k[K]$. Koszul alg

every missing face has 2 vertices

($\Leftrightarrow$ pairwise connected vertices $\Rightarrow$ simplex)

K flag cx $\Rightarrow$ $H_*(\Omega(\mathbb{C}P^\infty)^K, k)$

π_*

(right-angled Coxeter / Artin groups)

classifying space = polyhedral power

Thm $(\mathbb{C}P^\infty)^K$ coformal iff K flag (Koszul)

Thm K flag cx TFAE

ⓐ $k[K]$ Golod ring

ⓑ $H^*(Z_K, k)$ trivial algebra

ⓒ $\Gamma = K^1$ chordal graph

ⓓ $Z_K \simeq \vee S^*$

§1. Geometry and Combinatorics of Polytopes

Polytopes (↔ linear programming & optimisation).

§2. Combinatorial structures

simplicial complexes (↔ discrete and computational geometry)

both

topological ⊥ combinatorial

convex polyhedral cone, normal fan

simplicial cx, barycentric subdivision, Alexander duality.

triangulated manifolds, stellar subdivision

simplicial posets, cubical cxs.

§3. Combinatorial algebra of face rings

Face/Stanley-Reisner ring, Tor, Betti #,

Cohen-Macaulay complexes

Gorenstein complexes and Dehn-Sommerville relations.

face ring of simplicial poset

§4. Moment-Angle complexes

$(D^2,S^1)^K$ $\quad (X,A)^K$: a polyhedral product

$\mathcal{Z}$: simp Cx $\not\longrightarrow$ Toric spaces : functor

$\mathrm{colim}_{I \in cat(K)} (X,A)$

§8. Homotopy Theory of Polyhedral Products

$\boxed{\text{Thm}}$ $K \Rightarrow \mathcal{Z}_K \sim$ wedge of spheres

$\boxed{\text{Thm}}$ Davis-Januszkiewicz space $DJ(K) = (\mathbb{C}P^\infty)^K$ formal [Notbohm-Ray]

$(X,A)^K \simeq \mathrm{hocolim}_{I \in Cat(K)}$ $\quad T^m/T^I$ $\quad$ pow(D^2,S^1).

↳ Quillen model category

coformal iff K flag

$\boxed{\text{Thm}}$ Toric manifolds formal

$\boxed{\text{Thm}}$ $\Sigma (X,A)^K \simeq \bigwedge_{I \in [m]} \Sigma \ldots$

$\Omega \mathcal{Z}_K$, $\Omega (X,A)^K$: higher Whitehead/Samelson products.

$\underbrace{H_*(\Omega (\mathbb{C}P^\infty)^K) \quad H_*(\Omega \mathcal{Z}_K)}_{\text{gen+rel.}}$

Plus loin: Toric manifolds

$\mathbb{C}$ cobordism

[Matthias FRANZ] Gerstenhaber formality for DJ spaces $\Rightarrow H^\circ(\wedge_{\Omega DJ} DJ)$

$\underbrace{\hat{N}^\circ}_{\text{normalized singular cochains.}}$

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