Construction of a multi-soliton blow-up solution to the semilinear wave equation in one space dimension

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Semilinear wave equations

$u : \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{R}$ solution to

$$\begin{cases} \partial_{tt}u - \partial_{xx}u - |u|^{p-1}u = 0, \\ (u, \partial_t u)|_{t=0} = (u_0, u_1). \end{cases} \qquad \text{(NLW)}$$

where $p > 1$.

Local well-posedness: $(u_0, u_1) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ (Ginibre, Soffer & Velo, Lindblad & Sogge).

Blow-up curve

Blow-up criterion (Levine 74):

$$\text{If } \int \left(\frac{1}{2}|u_1|^2 + \frac{1}{2}|\partial_x u_0|^2 - \frac{1}{p+1}|u_0|^{p+1} \right) dx < 0,$$

then the solution can not be global in time.

If u is a blow-up solution:

- let $D \subset \mathbb{R}^2$ be the maximal domain of influence of u (in space-time), write $D = \{(x,t) | \, 0 \le t < T(x)\}$.
- Blow-up curve $\Gamma = \{(x, T(x))\}$.
- T is 1-Lipschitz (finite speed of propagation).
- $\bar{T} = \inf_{x \in \mathbb{R}} T(x)$ is the blow-up time.

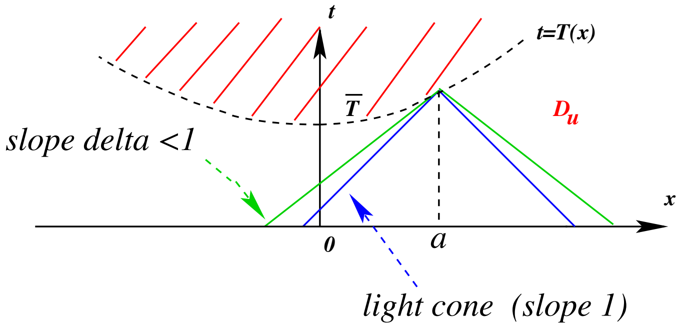
Goal:

- Description of *any* arbitrary blow-up solution;
- Construction of *examples* for each of the blow-up modalities.

Charateristic points

A point $a \in \mathbb{R}$ is *non-characteristic* if D contains a splaying cone

$$\mathscr{C}_\delta(a, T(a)) := \{(x,t) \in \mathbb{R} \times \mathbb{R}^+ | |x-a| \le \frac{T(a)-t}{\delta}\} \subset D \text{ for some } \delta < 1.$$



A point is *characteristic* if it is not the case.

Notation:

- $\mathscr{R}$ is the set of non-characteristic points.
- $\mathscr{S}$ is the set of characteristic points.

Known results:

- $\mathscr{R}$ is never empty ($\bar{x}$ such that $T(\bar{x}) = \bar{T}$).
- $\mathscr{S}$ can be empty (Caffarelli and Friedman 85, 86).

Introduce the similarity variables: for any point $(x_0, T_0) \in \overline{D}$

$$w_{x_0,T_0}(y,s) = (T_0 - t)^{\frac{2}{p-1}} u(x,t), \quad y = \frac{x - x_0}{T_0 - t}, \quad s = -\ln(T_0 - t).$$

Functional space:

$$\mathscr{H} = \left\{ (q,p) \middle| \int_{-1}^{1} \left(|p(y)|^2 + |\partial_y q|^2 (1 - y^2) + |q(y)|^2 \right) (1 - y^2)^{\frac{2}{p-1}} dy < +\infty \right\}.$$

Stationary solutions (for the "w" equation):

$$\kappa(d,y) = \pm \kappa_0 \frac{(1 - d^2)^{\frac{1}{p-1}}}{(1 + dy)^{\frac{2}{d-1}}}, \qquad \kappa_0 = \left(\frac{2(p+1)}{(p-1)^2} \right)^{\frac{1}{p-1}}.$$

Regularity of $\mathscr{R}$

Theorem (Merle & Z. 2007, 2008)

- *$\mathscr{R}$ is open and $T(x)$ is $\mathscr{C}^1$ on $\mathscr{R}$.*
- *There exists $\mu_0 > 0$, C_0, such that for all $x_0 \in \mathscr{R}$, there exist $\varepsilon(x_0) = \pm 1$, $s(x_0) \geq -\ln T(x_0)$ such that $\forall s \geq s(x_0)$,*

$$\left\| \begin{pmatrix} w_{x_0,T(x_0)}(s) \\ \partial_s w_{x_0,T(x_0)}(s) \end{pmatrix} - \varepsilon(x_0) \begin{pmatrix} \kappa(T'(x_0)) \\ 0 \end{pmatrix} \right\|_{\mathscr{H}} \leq C_0 e^{-\mu_0(s-s_0)}.$$

(Nouaili improved the regularity of T to $\mathscr{C}^{1,\mu_0}$).

- Only one $\kappa(d)$.
- Its parameter it the slope of the blow-up curve.
- Exponential convergence to the profile.

Description of $\mathscr{S}$ and refined asymptotics of characteristic blow-up points

Theorem (Merle & Z. 2012, improved in Côte & Z. 2012)

- *$\mathscr{S}$ is a discrete set.*
- *If $x_0 \in \mathscr{S}$, there exist $k = k(x_0) \geq 2$, $\epsilon(x_0) = \pm 1$ and $\zeta_0 = \zeta_0(x_0)$ s.t. $\forall s \geq s_0$*

$$\left\| \begin{pmatrix} w_{x_0,T(x_0)}(s) \\ \partial_s w_{x_0,T(x_0)}(s) \end{pmatrix} - \varepsilon(x_0) \sum_{i=1}^{k} (-1)^i \begin{pmatrix} \kappa(d_i(s)) \\ 0 \end{pmatrix} \right\|_{\mathscr{H}} \leq C_0 \left(\frac{s_0}{s}\right)^{\eta} \quad \text{for some } \eta > 0.$$

where $d_i(s) = -\tanh \zeta_i(s)$ and $\zeta_i(s)$ is defined on the next slide.

- *Furthermore, the blow-up curve is corner shaped at x_0: for some $\gamma = \gamma(p) > 0$,*

$$T(x) = T(x_0) - |x - x_0| + \frac{\gamma e^{2\zeta_0 \operatorname{sgn}(x_0 - x)} |x - x_0| (1 + o(1))}{|\ln |x - x_0||^{\frac{(k-1)(p-1)}{2}}}.$$

Illustration with hyperbolic coordinates

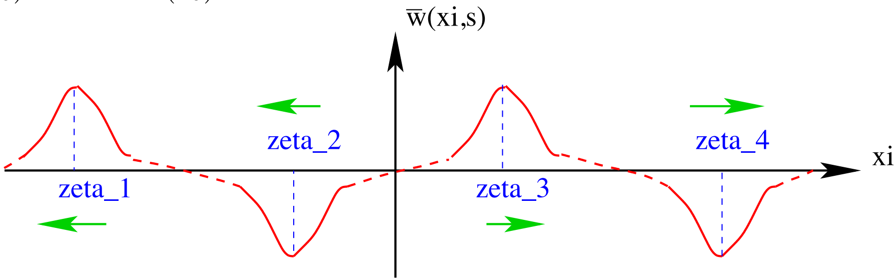
Introducing

$$\bar{w}_{x_0}(\xi, s) = (1 - y^2)^{\frac{1}{p-1}} w_{x_0}(y, s) \text{ with } y = \tanh \xi \text{ and } \zeta_i(s) = -\tanh^{-1} d_i(s),$$

we get

$$\|\bar{w}_{x_0}(\xi, s) - \epsilon(x_0)\kappa_0 \sum_{i=1}^{k(x_0)} (-1)^i \cosh^{-\frac{2}{p-1}}(\xi - \zeta_i(s))\|_{H^1 \cap L^\infty(\mathbb{R})} \to 0 \text{ as } s \to \infty,$$

and with $k(x_0) = 4$ and $\epsilon(x_0) = -1$:



$(\zeta_i)_{i=1,\dots,k}$ is a solution to the system

$$\dot{\zeta}_i = e^{-\frac{2}{p-1}(\zeta_i - \zeta_{i-1})} - e^{-\frac{2}{p-1}(\zeta_{i+1} - \zeta_i)}, \quad i = 1, \dots, k,$$

with the convention $\zeta_0(s) \equiv -\infty$, $\zeta_{k+1}(s) \equiv +\infty$, and barycenter $\frac{1}{k}(\zeta_1(s) + \cdots + \zeta_k(s)) = \zeta_0$. One can compute explicitely for

$$\zeta_i(s) = \left(i - \frac{k+1}{2}\right) \frac{(p-1)}{2} \ln s + \alpha_i + \zeta_0(x_0),$$

where $\alpha_i = \alpha_i(p,k)$ are chosen adequately.

Notice that

- The solitons are alternating.
- The number of solitons can be seen on the blow-up curve.
- The blow-up curve is never symetric with respect to x_0, unless maybe if the barycenter of the solitons $\zeta_0(x_0) = 0$.

Characteristic blow-up points with prescribed asymptotics

Theorem (Côte & Z. 2012)

For any integer $k \geq 2$ *and* $\zeta_0 \in \mathbb{R}$, *there exists a blow-up solution* $u(x,t)$ *in* $H^1 \times L^2(\mathbb{R})$ *with* $0 \in \mathscr{S}$ *such that* $T(0) = 1$ *and*

$$\left\| \begin{pmatrix} w_{0,1}(s) \\ \partial_s w_{0,1}(s) \end{pmatrix} - \sum_{i=1}^{k} (-1)^{i+1} \begin{pmatrix} \kappa(d_i(s)) \\ 0 \end{pmatrix} \right\|_{\mathscr{H}} \to 0 \quad as\ s \to \infty,$$

with

$$d_i(s) = -\tanh \zeta_i(s), \quad \zeta_i(s) = \left(i - \frac{k+1}{2}\right) \frac{(p-1)}{2} \ln s + \alpha_i + \zeta_0.$$

Some ideas of proofs : DESCRIPTION, then CONSTRUCTION

Equation on w: let $\rho = (1-y^2)^{\frac{2}{p-1}}$ and $\mathscr{L}w = \frac{1}{\rho}\partial_y(\rho(1-y^2)\partial_y w)$.

$$\partial_{ss}w = \mathscr{L}w - \frac{2(p+1)}{(p-1)^2}w + |w|^{p-1}w - \frac{p+3}{p-1}\partial_s w - 2y\partial^2_{ys}w, \qquad (eqw)$$

Starting point: Monotonicity property in the w variable.

$$E(w) = \int_{-1}^{1}\left(\frac{1}{2}|\partial_s w|^2 + \frac{1}{2}|\partial_y w^2|(1-y^2) + \frac{p+1}{(p-1)^2}w^2 - \frac{1}{p+1}|w|^{p+1}\right)\rho dy.$$

Theorem (Lyapunov functional, Antonini & Merle 02)

$$E(w(s_2)) - E(w(s_1)) = -\frac{4}{p-1}\int_{s_1}^{s_2}\int_{-1}^{1}|\partial_s w|^2(1-y^2)^{\frac{2}{p-1}-1}dyds \le 0.$$

If $E(w((s_0)) < 0$ for some $s_0 \in \mathbb{R}$, w blows-up in finite time.

DESCRIPTION: Decomposition into a sum of solitons

- We have two bounds

$$\|w_{x_0,T(x_0)}, \partial_s w_{x_0,T(x_0)}\|_{\mathscr{H}} \le C,$$
$$\int_{-\ln s_0}^{+\infty} \int_{-1}^{1} |\partial_s w|^2 (1-y^2)^{\frac{2}{p-1}-1} dy ds \le C.$$

- We find local limits: for some sequences $s_n \to +\infty$, in the $\xi = \arg\tanh(y)$ variable,

$$w_{x_0,T(x_0)}(\xi + \xi_n, s + s_n) \to w^* \textit{ stationary} \text{ solution, in } H^1_{\mathrm{loc}}.$$

- Nonzero stationary solutions are exactly the $\pm\kappa(d,\cdot)$. In the $(\zeta,\xi) = (\arg\tanh d, \arg\tanh y)$ variables,

$$\kappa(\zeta,\xi) = \frac{\kappa_0(p)}{\cosh(\xi-\zeta)^{\frac{2}{p-1}}} \quad \text{is a soliton.}$$

Proposition

There exist an integer $k(x_0)$ and $\varepsilon_i \in \{\pm 1\}$ and continuous functions $d_i(s)$ such that

$$\left\| \begin{pmatrix} w_{x_0,T(x_0)}(s) \\ \partial_s w_{x_0,T(x_0)}(s) \end{pmatrix} - \sum_{i=1}^{k(x_0)} \varepsilon_i \begin{pmatrix} \kappa(d_i(s),\cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \to 0 \quad as \quad s \to +\infty.$$

and, with $\zeta_i = \arg\tanh(d_i)$, $\zeta_{i+1}(s) - \zeta_i(s) \to +\infty$.

At this point:

- We may have $k(x_0) = 0$ or 1 or $k(x_0) \geq 2$.
- No control on the signs ε_i.
- If $k(x_0) \geq 2$, we have no control on the size $\zeta_{i+1}(s) - \zeta_i(s)$.

DESCRIPTION at non-characteristics points

If x_0 is a non-characteristic point

- Control in a splaying cone.
- A covering argument gets rid of the weight.

$$\|w_{x_0,T(x_0)},\partial_s w_{x_0,T(x_0)}\|_{H^1\times L^2}\le C.$$

- We get one single limit: $k(x_0)=1$ (otherwise we quit $H^1(-1,1)$).
- Modulation + linear version of the Lyapunov yields

$$\forall s\ge s(x_0),\quad \left\|\begin{pmatrix} w_{x_0,T(x_0)}(s)\\ \partial_s w_{x_0,T(x_0)}(s)\end{pmatrix}-\varepsilon(x_0)\begin{pmatrix}\kappa(d(x_0))\\ 0\end{pmatrix}\right\|_{\mathscr{H}}\le C_0e^{-\mu_0(s-s_0)}.$$

- Stability property $\to d(x_0)=T'(x_0)$ and $\mathscr{R}$ is open.

DESCRIPTION at characteristics points

- The covering argument does not hold:
- We may have $k(x_0) \geq 2$.

Equation on the solitons centers $\zeta_i = \arg\tanh d_i$:

$$\frac{1}{c_1(p)}\dot{\zeta}_i = \left(\varepsilon_i\varepsilon_{i-1}e^{-\frac{2}{p-1}(\zeta_i-\zeta_{i-1})} - \varepsilon_i\varepsilon_{i+1}e^{-\frac{2}{p-1}(\zeta_{i+1}-\zeta_i)}\right) + o(1).$$

- By construction $\zeta_{i+1} - \zeta_i \to +\infty$.
- This implies $\varepsilon_i = (-1)^{i-1}\varepsilon_1$.
- To study further the dynamics, we need an adequate modulated decomposition.
- Introduction of the $\kappa^*(d,\nu,y)$.

Modulation

Define for $\nu > -1+|d|$, $\kappa^*(d,\nu,y) = (\kappa_1^*(d,\nu,y), \kappa_2^*(d,\nu,y))$, where

$$\kappa_1^*(d,\nu,y) = \kappa_0 \frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dy+\nu)^{\frac{2}{p-1}}}, \qquad \kappa_2^*(d,\nu,y) = \nu\partial_\nu \kappa_1^*(d,\nu,y).$$

- For $\mu \in \mathbb{R}$, $\kappa_1^*(d,\mu e^s,y)$ is solution of (eqw).
- If $\mu = 0$, it is $\kappa(d,y)$.
- If $\mu > 0$, it converges to 0 as $s \to +\infty$.
- If $\mu < 0$, it blows up at time $s = \ln\left(\frac{|d|-1}{\mu}\right)$.
- We can write a decomposition

$$\begin{pmatrix} w_{x_0,T(x_0)}(s) \\ \partial_s w_{x_0,T(x_0)}(s) \end{pmatrix} = q(s) + \sum_{i=1}^{k_0} (-1)^j \kappa^*(\hat{d}_i, \hat{\nu}_i), \qquad \|q(s)\|_{\mathscr{H}} \to 0$$

with the projection $\Pi_{\lambda,i}(q(s)) = 0$, for all $i = 1,\dots,k(x_0)$ and eigenvalues $\lambda = 0, 1$ (we have $2k$ nonnegative eigenvalues).

Define

$$J=\sum_{i=2}^{k}e^{-\frac{2}{p-1}(\zeta_i-\zeta_{i-1})},\quad \bar J=\sum_{i=1}^{k}\frac{|\nu_i|}{1-d_i^2},\quad \hat J=\sum_{i=2}^{k}e^{-\frac{\bar p}{p-1}(\zeta_i-\zeta_{i-1})},$$

where $\bar p=\min(p,2-1/100)$.

Proposition (Dynamics of the parameters)

$$\frac{|\dot\nu_i-\nu_i|}{1-d_i^2}\le C(\|q\|_{\mathcal H}^2+J+\|q\|_{\mathcal H}\bar J)$$

$$\left|\frac{1}{c_1(p)}\dot\zeta_i-\left(e^{-\frac{2}{p-1}(\zeta_i-\zeta_{i-1})}-e^{-\frac{2}{p-1}(\zeta_{i+1}-\zeta_i)}\right)\right|\le C(\|q\|_{\mathcal H}^2+(J+\|q\|_{\mathcal H})\bar J+J^{1+\eta})$$

$$\|q(s)\|_{\mathcal H}^2\le Ce^{-\eta(s-s_0)}\|q(s_0)\|_{\mathcal H}^2+C\hat J(s)^2)$$

with $\zeta_i(s)=-\arg\tanh(d_i(s))$.

- The decomposition in generalized solitons κ^* is *stable* in some sense:

$$\zeta_i\sim\left(i-\frac{k+1}{2}\right)\frac{(p-1)}{2}\ln s,\quad J\sim s^{-2},\quad \hat J\sim s^{-\bar p},\quad \|q\|_{\mathcal H}\le s^{-\bar p},\quad \frac{|\nu_i|}{1-d_i^2}\le s^{-\bar p}.$$

CONSTRUCTION: Prescribed non-characteristic blow-up

Parameters are given: number of solitons $k \geq 2$ integer and their barycenter $\zeta_0 \in \mathbb{R}$.
Define $\bar{\zeta}_i$, $\bar{d}_i$ be the "perfect" centers of mass:

$$\bar{d}_i(s_0) = \tanh \bar{\zeta}_i(s), \qquad \bar{\zeta}_i(s) = \left(i - \frac{k+1}{2}\right)\frac{(p-1)}{2}\ln s + \alpha_i.$$

Step 1: Construction of w decomposing into k solitons as $s \to +\infty$, without the condition on the barycenter.

Recall there is a 1 to 1 correspondence between w and $(q, (d_i)_i, \nu_i)$ around a sum of k decoupled soliton.

Goal: Find initial conditions $(q(s_0), (d_i(s_0))_i, (\nu_i(s_0))_i)$ such that w is defined on $[s_0, +\infty)$ and

$$q(s) \to 0, \quad d_i(s) \sim \bar{d}_i(s) \quad \text{and} \quad \nu_i(s) \to 0 \quad \text{as} \quad s \to +\infty.$$

Equations describing the dynamics up to leading order:

$$\dot{\nu}_i \sim \nu_i,$$

$$\frac{1}{c_1(p)}\dot{\zeta}_i \sim \left(e^{-\frac{2}{p-1}(\zeta_i-\zeta_{i-1})} - e^{-\frac{2}{p-1}(\zeta_{i+1}-\zeta_i)}\right),$$

$$\|q(s)\|_{\mathscr{H}}^2 \le Ce^{-\eta(s-s_0)}\|q(s_0)\|_{\mathscr{H}}^2 + C\hat{J}(s)^2.$$

- q has some (spectral) stability property: Negative part of the spectrum $\lambda \le -\eta$,
- The ν_i correspond to $\lambda = 1$. They are *transversally* unstable.
- The ζ_i correspond to $\lambda = 1$. Fortunately, we have almost a Lyapunov Theorem for the ODE system of ζ_i: stability property except for the barycenter.
  More precisely, consider the linearized system of ζ_i around $\bar{\zeta}_i$.

  Let $\boldsymbol{\xi} = (\zeta_i - \bar{\zeta}_i)_i$, up to a linear change of variable $\boldsymbol{\phi} = P\boldsymbol{\xi}$, it writes

$$\dot{\boldsymbol{\phi}} \sim \frac{1}{s}M\boldsymbol{\phi}, \quad \text{where} \quad (M\boldsymbol{\phi}, \boldsymbol{\phi}) \le -\sum_{i=2}^{k}\boldsymbol{\phi}_i^2, \quad \text{and} \quad M\boldsymbol{\phi}_1 = 0.$$

  Hence, the ϕ_i for $i \ge 2$ are controlled, and ϕ_1 "doesn't change much".

CONCLUSION: we only need to control ν_i for $i = 1, \dots, k$.

Define the rescaling $\Gamma_s : (\nu_1, \ldots, \nu_k) \mapsto (s^{-1/2-|\gamma_1|}\nu_1, \ldots, s^{-1/2-|\gamma_k|}\nu_k)$ where

$$\gamma_i = (i - \frac{k+1}{2})\frac{(p-1)}{2}.$$

Consider initial data of the type

$$(0, (\bar{d}_i(s_0))_i, (\nu_i(s_0)_i) \quad \text{that is} \quad w(s_0) = 0 + \sum_{i=1}^{k} \kappa^*(\bar{d}_i(s_0), \nu_i(s_0))$$

where $\nu_i(s_0)$ belongs to $\mathbb{B} := [-1, 1]^k$ after rescaling

$$(\nu_i(s_0))_i = \Gamma_{s_0}((\nu_{i,0})_i), \quad \nu_{i,0} \in [-1, 1].$$

For any $\boldsymbol{\nu} = (\nu_{i,0})_i$, let $(q(s), ((d_i(s))_i, (\nu_i(s))_i)$ the evolution with such initial conditions at $s = s_0$.

Define

- The rescaled flow $\Phi : \ (s, \boldsymbol{\nu}) \mapsto \Gamma_s^{-1}((\nu_1(s), \ldots, \nu_k(s))$, and
- The exit time $s^*(\boldsymbol{\nu}) = \sup\{s \geq s_0 | \ \forall \tau \in [s_0, s], \ \Phi(\boldsymbol{\nu}) \in \mathbb{B}\}$.

Goal: find $\boldsymbol{\nu} \in \mathbb{B}$ such that $s^*(\boldsymbol{\nu}) = +\infty$.

We argue by contradiction: assume that for all $\boldsymbol{\nu} \in \mathbb{B}$, $s^*(\nu) < +\infty$. Define

$$\Psi : \mathbb{B} \to \mathbb{B}, \quad \boldsymbol{\nu} \mapsto \Phi(s^*(\boldsymbol{\nu}), \boldsymbol{\nu}).$$

Then, denoting $\mathbb{S} = \partial\mathbb{B}$ the boundary

1. $\forall \nu \in \mathbb{B}, \quad \Psi(\nu) \in \mathbb{S}$.
2. $\forall \nu \in \mathbb{B}, s \in [s_0, s^*(\nu)], \quad s^{1/2+\eta}\|q(s)\|_{\mathscr{H}} + \sum_{i=2}^{k} s^\eta |\phi_i|(s) + s_0^\eta |\phi_1(s)| \leq 1.$
3. $\forall \nu \in \mathbb{S}, \quad s^*(\nu) = s_0$ and $\Psi(\nu) = \nu$.
4. Ψ is continuous.

We used:

- Stability properties for 2).
- Transversality of the flow on the boundary $\mathbb{S}$ for 3) and 4).

$\Psi : \mathbb{B} \to \mathbb{S}$ continuous such that $\Psi|_{\mathbb{S}} = \mathrm{Id}$. This contradicts Brouwer's Theorem.

Hence there exists $\boldsymbol{\nu}^\sharp \in \mathbb{B} = [-1, 1]^k$ such that $s^*(\boldsymbol{\nu}^\sharp) = +\infty$.

- Conclusion: existence of $w^\sharp \in \mathscr{C}([s_0,+\infty),\mathscr{H})$ satisfying the alternating k-soliton decomposition, with barycenter $|\zeta_0| \le s_0^\eta$.
- Define $u^\sharp \in H^1_{\text{uloc}} \times L^2_{\text{uloc}}$ solution to (NLW), such that the trace on $(-1,1)$ is $w^\sharp$:

$$u^\sharp(0)|_{(-1,1)} = w(x,s), \quad \partial_t u^\sharp(x,0)|_{(-1,1)} = \partial_s w^\sharp(x,s_0) + \frac{2}{p-1} w^\sharp(x,s_0) + x\partial_y w^\sharp(x,s_0).$$

- Check that for $t \in [0,1)$ and $|x| < 1-t$,

$$u^\sharp(x,t) = (1-t)^{-\frac{2}{p-1}} w^\sharp\left(\frac{x}{1-t}, s_0 - \ln(1-t)\right).$$

  $u^\sharp$ has characteristic point 0 as desired (without the barycenter condition).
- Fix barycenter using a Lorentz transform on $u^\sharp$.
- Corollary (prescribing multiple characteristic points) follows from finite speed of propagation.

Conclusion

Summary

- Complete description of the blow-up.
- Results are specific to semilinear equation:
  e.g. $\partial_{tt}u - \partial_{xx}u = \partial_x u \partial_t u$, explicit solution with blow-up curve $\{(x, |x| + 1\}$, $\mathscr{S} = \mathbb{R}^*$.
- Results are very sensitive to the nonlinearity, especially to sign change, but the method is robust:
  e.g. $\partial_{tt}u - \partial_{xx}u = |u|^p$: one always has $\mathscr{S} = \varnothing$.

Some open questions

- Extension to $\mathbb{R}^d$? Ok in the radial case.
  Problem in the general case: classification of stationnary solutions.
- Can one construct a blow-up solution with prescribed characteristic set $\mathscr{S}$?
- Given a 1-Lipshitz (smooth) curve Γ, can one give a solution u with blow-up curve Γ? (See Killip & Visan).