A pyramid-shaped blow-up surface for the 2d semilinear wave equation

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Introduction: The equation

$$\begin{cases} \partial_t^2 u = \Delta u + |u|^{p-1}u, \\ u(0) = u_0 \text{ and } u_t(0) = u_1, \end{cases}$$

where $1 < p < p_c = 1 + \frac{4}{N-1}$, $u(t) : x \in \mathbb{R}^N \to u(x,t) \in \mathbb{R}$, $u_0 \in H^1(\mathbb{R}^N)$ and $u_1 \in L^2(\mathbb{R}^N)$.
Rk.: The conformal exponent $p_c \equiv 1 + \frac{4}{N-1} < 1 + \frac{4}{N-2}$, the Sobolev exponent.

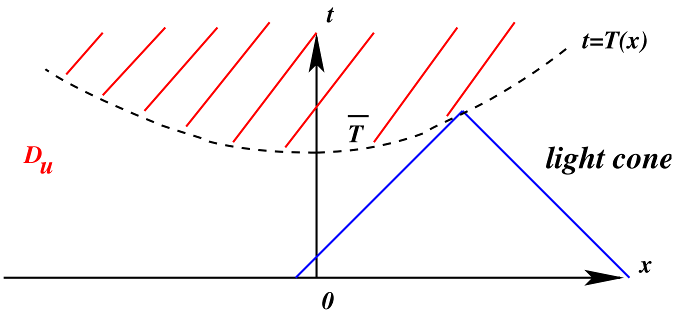
Earlier work: Levine 1974, Caffarelli and Friedman 1985, Ginibre, Soffer and Velo 1992, Kichenassamy and Littman 1993, Alinhac 1995, Lindblad and Sogge 1995, Shatah and Struwe 1998, Killip, Stroval and Vişan 2012, Donninger and Shorkhüber 2012, Schlag, Krieger, Nakanishi, Bizón, etc...

Singular solutions: the maximal influence domain

We consider an arbitrary blow-up solution $u(x,t)$.
From the finite speed of propagation, its domain of definition is

$$D_u = \{(x,t) \mid 0 \le t < T(x)\}$$

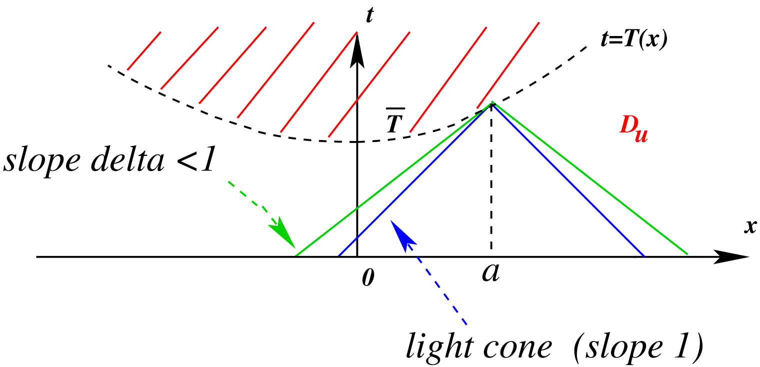
where $x \mapsto T(x)$ is 1-Lipschitz.



Remark: For all $x \in \mathbb{R}^N$, there exists a “local” blow-up time $T(x)$.

Definition: Non characteristic points and characteristic points

A point a is said *non characteristic* if the domain contains a cone with vertex $(a, T(a))$ and slope $\delta < 1$.



The point is said *characteristic* if not.

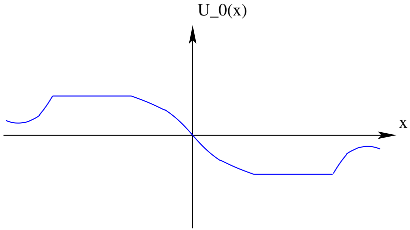
- Notation: $\mathcal{R} \subset \mathbb{R}^N$ is the set of all *non* characteristic points.
- Notation: $\mathcal{S} \subset \mathbb{R}^N$ is the set of all characteristic points ($\mathcal{S} \cup \mathcal{R} = \mathbb{R}^N$).

Case $N = 1$ (and $p > 1$): Existence results

Rk. All blow-up solutions have non-characteristic points ($x_0 = \arg\min T(x)$);

Th (Merle, Z.): There *exist* solutions with characteristic points.

Example: We take odd initial data, with two large plateaus of different signs. Then, the solution blows up, and the origin is a characteristic point with $\forall t < T(0)$, $u(0,t) = 0$.



Th. (Merle-Z.) If we perturb the constructed initial data, then the new solution blows up and has a characteristic point.

Case $N = 1$ (and $p > 1$): Asymptotic behavior

Introducing similarity variables

$$w_{x_0}(y,s) = (T(x_0) - t)^{\frac{2}{p-1}} u(x,t) \text{ with } y = \frac{x - x_0}{T(x_0) - t} \text{ and } s = -\log(T(x_0) - t),$$

and the soliton

$$\kappa(d,y) = \kappa_0(p)\frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dy)^{\frac{2}{p-1}}},$$

we have as $s \to \infty$:
- if $x_0 \in \mathcal{R}$, then $w_{x_0}(y,s) \to \pm\kappa(d(x_0), y)$;
- if $x_0 \in \mathcal{S}$, then $w_{x_0}(y,s) \sim \pm \sum_{i=1}^{k} (-1)^i \kappa(d_i(s), y)$ **(multi-solitons)**
with
$k \geq 2$ and $d_i(s) = \tanh(C_0(i - \frac{k+1}{2}) \log s + C_1)$.
Th. (Côte, Z.) : Every multi-soliton modality does occur.

Illustration with hyperbolic coordinates when $x_0 \in \mathcal{S}$

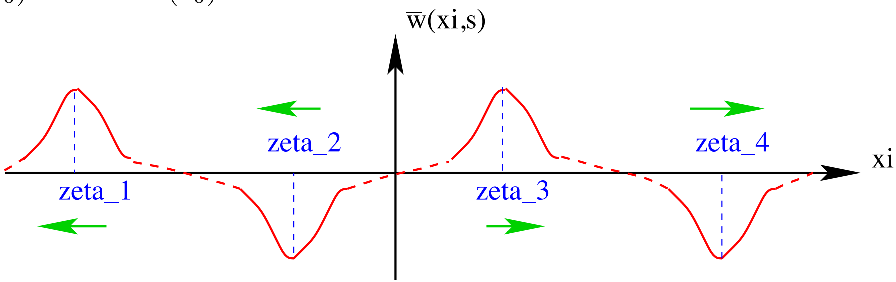
Introducing for $\xi \in \mathbb{R}$,

$$\bar{w}_{x_0}(\xi, s) = (1 - y^2)^{\frac{1}{p-1}} w_{x_0}(y, s) \text{ with } y = \tanh \xi \text{ and } \zeta_i(s) = C_0(i - \frac{k+1}{2}) \log s + C_1),$$

we get

$$\|\bar{w}_{x_0}(\xi, s) - \epsilon(x_0)\kappa_0 \sum_{i=1}^{k(x_0)} (-1)^i \cosh^{-\frac{2}{p-1}}(\xi - \zeta_i(s))\|_{H^1 \cap L^\infty(\mathbb{R})} \to 0 \text{ as } s \to \infty,$$

and with $k(x_0) = 4$ and $\epsilon(x_0) = -1$:



Behavior of the solitons' centers

$(\zeta_i)_{i=1,\dots,k}$ is a solution to the system

$$\dot{\zeta}_i = e^{-\frac{2}{p-1}(\zeta_i-\zeta_{i-1})} - e^{-\frac{2}{p-1}(\zeta_{i+1}-\zeta_i)}, \quad i = 1, \dots, k,$$

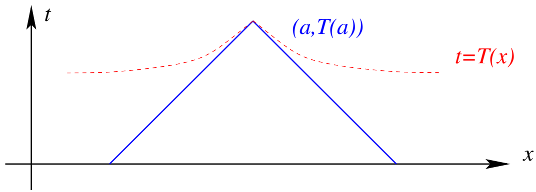
with the convention $\zeta_0(s) \equiv -\infty$, $\zeta_{k+1}(s) \equiv +\infty$. Note that the barycenter is *conserved* $\frac{1}{k}(\zeta_1(s) + \cdots + \zeta_k(s)) \equiv \bar{\zeta}(x_0)$. One can compute explicitely:

$$\zeta_i(s) = \left(i - \frac{k+1}{2}\right)\frac{(p-1)}{2}\ln s + \alpha_i + \bar{\zeta}(x_0),$$

where $\alpha_i = \alpha_i(p,k)$ are chosen adequately.

Regularity of the blow-up curve

- $\mathcal{R}$ is open and $T_{|\mathcal{R}}$ is C^1; more precisley, if $d(x_0)$ is such that $w_{x_0}(y,s) \sim \pm\kappa(d(x_0),y)$, then, $T'(x_0) = d(x_0)$;
- $\mathcal{S}$ is finite on compact sets, and T is corner shaped near $a \in \mathcal{S}$.



Furthermore, for some $\gamma = \gamma(p) > 0$,

$$T(x) - T(x_0) + |x - x_0| \sim \frac{\gamma e^{2\zeta_0 \mathrm{sgn}(x_0 - x)} |x - x_0|}{|\ln|x - x_0||^{\frac{(k(x_0)-1)(p-1)}{2}}} \text{ as } x \to x_0,$$

where $k(x_0)$ is the solitons' number and $\zeta_0(x_0)$ is their barycenter. Note that

- The number of solitons $k(x_0)$ can be "seen" on the blow-up curve.
- The blow-up curve is never symmetric with respect to x_0, unless maybe if the barycenter of the solitons $\zeta_0(x_0) = 0$.

Generalizations

- **When $N = 1$ with lower order perturbations (M.A. Hamza and Z. 2013)**:

$$\partial_t^2 u = \partial_x^2 u + |u|^{p-1}u + f(u) + g(\partial_t u, \partial_x u, x, t)$$

with $|f(u)| \leq C(1 + |u|^q), \quad |g(\partial_t u, \partial_x u, x, t)| \leq M(1 + |\partial_t u| + |\partial_x u|)$ and $q < p$.

- **When $N \geq 2$, $p < p_c$, with radial symmetry, outside the origin**:

$$\partial_t^2 u = \partial_r^2 u + (N-1)\frac{\partial_r u}{r} + |u|^{p-1}u$$

(this is because the term $\frac{\partial_r u}{r}$ appears as a lower order perturbation).

- **A mixture of both cases (radial + perturbations)**,
- **When $u \in \mathbb{C}$ (by A. Azaiez 2013)**.
- **With “strong” perturbations (by M.A. Hamza and O. Saidi)**, i.e. $|f(u)| \leq C|u|^p \log^{-a}|u|$.

And what about $N \geq 2$ with u not necessarily radial?

We know the blow-up rate (Merle, Z. 2003 and 2005):

- If $x_0 \in \mathcal{R}$, then

$$0 < \epsilon_0(N,p) \leq \|(w_{x_0}(s), \partial_s w_{x_0}(s))\|_{H^1 \times L^2(|y|<1)} \leq K(u_0,u_1);$$

- If $x_0 \in \mathcal{S}$, then

$$\|(w_{x_0}(s), \partial_s w_{x_0}(s))\|_{H^1 \times L^2(|y|<\frac{1}{2})} \leq K(u_0,u_1).$$

Facts about characteristic points when $N \geq 2$:
- No classification (except for radial solutions outside the origin);
- The only known examples are rigorously radial or 1d in some neighborhood of the characteristic point.

Question: Can we find "new" blow-up solutions with characteristic points, i.e. with a "non 1d"-behavior?

General question: Geometry of the set of singular points

Our dream: Find a solution where $\mathscr{S}$ is cross-shaped (open question) ?

More generally, the geometry of the singular set is largely open in PDEs.

For the semilinear heat equation (Sobolev subcritical): We have solutions where the singular set is a single points, a finite number of points, a sphere, a finite number of concentric spheres. **That's it**.

An ellipse in 2d ? open problem.
A cross in 2d? open problem.

For the semilinear wave equation: The good notion for singular points concerns *characteristic* points, since *all points are blow-up points, though at different times*.

A new blow-up solution when $N = 2$

Th (Merle, Z. 2016) There exists a solution $u(x,t)$ which blows up on a 1-Lipschitz graph $x \mapsto T(x)$ such that

$$T(x) - T(0) \sim -\max(|x_1|, |x_2|) \text{ as } x \to 0 \quad \text{(pyramid shape)}$$

with the origin being an isolated characteristic point.

Rk.
- $u(x,t)$ is of course not radial.
- $u(x,t)$ is symmetric with respect to the axis and antisymmetric with respect to bisectrices. In particular $u(x_1, \pm x_1, t) = 0$.
- We could prove that the origin is the *only* characteristic point.

Regularity of the blow-up graph

Locally near 0, the blow-up graph

$$x \mapsto T(x)$$

has the the regularity of the asymptotic pyramid

$$x \mapsto T(0) - \max(|x_1|, |x_2|).$$

In particular:
- It is C^1 outside the bisectrices with (when $0 \le x_2 < x_1$)

$$\partial_{x_1} T(x) = -1 + c_0(p)|\log x_1|^{-\frac{p-1}{2}} + \ldots \text{ and } \partial_{x_2} T(x) = O(|\log x_1|^{-\frac{p-1}{4}}) \text{ as } x \to 0;$$

- On the bisectrices outside the origin, $x \mapsto T(x)$ has directional derivatives except in the direction of the bisectrix;
- At the origin, we have directional derivatives, except along the bisectrices.
Rk. Unlike the 1d case, we have on the bisectrices *the first example of non characteristic points where T is non differentiable*.

The blow-up behavior of $u(x,t)$ at the origin

If $x_0 = 0$, then (decoupled multi-solitons localized along the axes)

$$\left\| w_0(y,s) - \left(\kappa(\bar d(s)e_1,y) + \kappa(-\bar d(s)e_1,y) - \kappa(\bar d(s)e_2,y) - \kappa(-\bar d(s)e_2,y)\right)\right\|_{\mathcal{H}} \to 0$$

where

$$\kappa(d,y) = \kappa_0(p)\frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dy)^{\frac{2}{p-1}}},$$

$$\bar d(s) = -\tanh \bar\zeta(s) \text{ and } \bar\zeta(s) = \left(\frac{p-1}{4}\right)\log s - \frac{(p-1)}{4}\log\left(\frac{p-1}{4c_4(p)}\right)$$

which is an explicit solution to the ODE

$$\frac{1}{c_4(p)}\bar\zeta'(s) = e^{-\frac{4}{p-1}\bar\zeta(s)}.$$

Rk. Note that $\bar d(s) = -1 + c_5(p)s^{-\frac{p-1}{2}} + \dots$ as $s \to \infty$.

The blow-up behavior of $u(x,t)$ outside the origin

If $x_0 \neq 0$, then $w_{x_0}(s) \to w^*_{x_0}$, a stationary solution in similarity variables with:
- $w^*_{x_0} = \kappa(d(x_0))$ if $0 \le x_{0,2} < x_{0,1}$, and

$$d(x_0) = -1 + c_0 |\log x_{0,1}|^{-\frac{p-1}{2}} + \dots \text{ as } x_0 \to 0,$$

for some $c_0(p) > 0$, with similar behavior whenever $|x_{0,1}| \neq |x_{0,2}|$, from symmetry;

-$w^*_{x_0}$ is a genuinely two-dimensional stationary solution if $|x_{0,1}| = |x_{0,2}|$.

Rk.
- $w^*_{x_0}$ is non radial; it is a *new* stationary solution.

The proof

Two major steps:

- Step 1: The construction in the light cone with vertex $(0,T(0))$ (i.e. the construction of w_0).
- Step 2: Derivation of the behavior of $w_x(s)$ as $s \to \infty$ *and* the behavior of $x \mapsto T(x)$, for x small.

Rk.
- Between the two steps, using the finite speed of propagation, we extend the solution to the region outside the cone, in order to get a solution to the Cauchy problem at $t=0$.
- As usual with blow-up problemes (heat, wave), the asymptotic behavior of the solution at blow-up and the regularity of the blow-up set are linked and advanced side by side in the proof.

Step 1: The construction in the central light cone

Goal: To construct a solution in similarity variables $w_0(y,s)$ for $|y|<1$ and $s \geq s_0$ showing 4 solitons:

$$w_0(y,s) \sim \kappa(\bar{d}(s)e_1, y) + \kappa(-\bar{d}(s)e_1, y) - \kappa(\bar{d}(s)e_2, y) - \kappa(-\bar{d}(s)e_2, y) \text{ as } s \to \infty,$$

where

$$\kappa(d,y) = \kappa_0(p)\frac{(1-|d|^2)^{\frac{1}{p-1}}}{(1+d\cdot y)^{\frac{2}{p-1}}},$$

(here, $d \in \mathbb{R}^2$ and $y \in \mathbb{R}^2$),

$$\bar{d}(s) = -\tanh \bar{\zeta}(s) \text{ and } \bar{\zeta}(s) = \left(\frac{p-1}{4}\right)\log s - \frac{(p-1)}{4}\log\left(\frac{p-1}{4c_4(p)}\right).$$

Step 1: The construction in the central light cone

Framework: Construction of a solution with *prescribed behavior*.
Method: We linearize the equation around the intended behavior, and find three regions in the spectrum:
- Negative spectrum: controlled thanks to a linearized version of the Lyapunov functional ;
- $\lambda = 0$: controlled thanks to modulation in the parameter d in $\kappa(d, y)$;
- $\lambda = 1$: controlled thanks to modulation in the parameter ν in the *generalized solitons*:

$$\kappa^*(d, \nu, y) = \kappa_0 \frac{(1 - |d|^2)^{\frac{1}{p-1}}}{(1 + \nu + d \cdot y)^{\frac{2}{p-1}}}.$$

Rk. We were inspired by the construction of multi-solitons in 1d in Côte-Z. (2013).

History of the construction with prescribed behavior

More generally, we are in the framework of constructing a solution to some PDE with some *prescribed behavior*:

- NLS: Merle (1990), Martel and Merle (2006); Côte, Martel and Merle (2011)
- KdV (and gKdV): Martel (2005), Côte (2006, 2007), Côte, Martel and Merle,
- water waves: Ming-Rousset-Tzvetkov (2013),
- Schrödinger maps: Merle-Raphaël-Rodniansky (2013),
- semilinear wave equation: Côte and Z. (2013),
- semilinear heat equation: Bressan, Merle (1992), Bricmont and Kupiainen (1993), Merle and Z. (1997), Shweyer,
- nonlinear heat equation with gradient terms : Ebde and Z. (2011), Tayachi and Z. (2015),
- Ginzburg-Landau: Z. (1998), Masmoudi and Z. (2008).
- Keller-Segel: Raphaël-Schweyer, Ghoul-Masmoudi,
- etc.

Step 2: Behavior of w_{x_0} and $T(x_0)$ for $x_0 \neq 0$

Take $x_0 \neq 0$.
We need to know the behavior of w_{x_0} for $|y| < 1$ and $s \geq -\log T(x_0)$.
This is equivalent to knowing the behavior of $u(x,t)$ in the backward light cone $\mathscr{C}_{x_0}$ with vertex $(x_0, T(x_0))$.
But, if x_0 is small, and $t < \min(T(0), T(x_0))$, the sections of $\mathscr{C}_{x_0}$ and $\mathscr{C}_0$ are almost the same.
Moreover, we have the following relation between w_{x_0} and w_0:

$$w_{x_0}(y,s) = (1 - T(x_0)e^s)^{-\frac{2}{p-1}} w_0(Y,S), \quad Y = \frac{y + xe^s}{1 - T(x_0)e^s} \quad S = s - \log(1 - T(x_0)e^s).$$

Since w_0 shows 4 solitons:

$$w_0(y,s) \sim \kappa(\bar{d}(s)e_1, y) + \kappa(-\bar{d}(s)e_1, y) - \kappa(\bar{d}(s)e_2, y) - \kappa(-\bar{d}(s)e_2, y) \text{ as } s \to \infty,$$

the function w_{x_0} also shows 4 (generalized) solitons, though with a *deformation*.

Step 2: Behavior of w_{x_0} and $T(x_0)$ outside the bisectrices

Two cases then arise:

Case 1: If x_0 is not on the bisectrices (say, $0 \le x_{0,2} < x_{0,1}$), only one soliton remains at some time $t^* = T(x_0) - e^{-s^*} = T(0) - e^{-S^*}$:

$$w_{x_0}(y, s^*) \sim \kappa(\bar{d}(S^*)e_1) \text{ with } S^* \sim -\log x_1.$$

Applying our trapping result near solitons, we see that if x_0 is non characteristic, then

$$w_{x_0}(y, s) \to \kappa(\nabla T(x_0), y) \text{ as } s \to \infty \quad (1)$$

with

$$\nabla T(x_0) \sim \bar{d}(S^*)e_1 = (-1 + c_0 S^{*-\frac{p-1}{2}} + ...)e_1 = (-1 + c_0 |\log x_1|^{-\frac{p-1}{2}} + ...)e_1.$$

Rk.

- If x_0 is characteristic, we have no information; later, we will have to show that all points outside the bisectrices are non characteristic.
- Note the link between the asymptotic behavior of w_{x_0} and the regularity of $T(x_0)$ in (1).

Step 2: Behavior of w_{x_0} and $T(x_0)$ on the bisectrices

Case 2: If x_0 is on the bisectrices (say, $x_{0,2} = x_{0,1}$), then w_{x_0} is anti-symmetric with respect to the bisectrix;
therefore, 2 solitons remain at some time $\tilde{t} = T(x_0) - e^{-\tilde{s}} = T(0) - e^{-\tilde{S}}$:

$$w_{x_0}(y, \tilde{s}) \sim \kappa(\bar{d}(\tilde{S})e_1) - \kappa(-\bar{d}(\tilde{S})e_2)$$

with

$$\tilde{S} \sim -\log x_1.$$

From the behavior of the neighbors outside the bisectrix, we derive that x_0 is non-characteristic.
Therefore, from the existence of a Lyapunov functional in similarity variables, we see that

$$\text{as } s \to \infty, w_{x_0}(y, s) \to w^*_{x_0}(y),$$

a stationary solution in similarity variables, with

$$w^*_{x_0}(y) \sim \kappa(\bar{d}(\tilde{S})e_1) - \kappa(-\bar{d}(\tilde{S})e_2).$$

Rk. This is a *new* kind of stationary solutions, which are neither radial, nor 1d.

Step 2: Outside the bisectrices, all the points are non characteristic (The Umbrella Technique)

Goal: Take x outside the bisectrices, for example with $0 \le x_2 < x_1$, and show that x is non characteristic.

Proof: Take $\gamma \in (0, x_1^2]$ and consider a family of cones with vertex (x,t) with $t \le T(x)$ and slope $1 - \gamma < 1$.
Consider $\bar{t}$ the largest value such of t such that the cone touches the graph of $x \mapsto T(x)$ at some point $(\bar{x}, T(\bar{x}))$ (imagine an umbrella under the graph).

Since the slope of the cone is $1 - \gamma < 1$, by definition, $\bar{x}$ is non characteristic.

If $\bar{x} = x$ (the graph touches the umbrella at its vertex), then x is non characteristic; we are done.

The Umbrella Technique (cont.)

If $\bar{x} \neq x$, we will reach a contradiction.

If $\bar{x}$ is on the bisectrices, this is a bit complicated to explain: omitted.

If $\bar{x}$ is not on the bisectrices, say, $0 \leq \bar{x}_2 < \bar{x}_1$, then both the cone and the graph are differentiable, and their slopes have to agree:
- Slope of the cone: $-1 + \gamma \leq -1 + x_1^2$,
- Slope of the graph: $-1 + c_0 |\log \bar{x}_1|^{-\frac{p-1}{2}} + \dots$.
Therefore,

$$c_0 |\log \bar{x}_1|^{-\frac{p-1}{2}} + \dots = \gamma \leq x_1^2, \text{ hence } \bar{x}_1 \ll x_1,$$

on the one hand.

The Umbrella Technique (cont.)

On the other hand, since $(x,T(x))$ is the vertex of the umbrella and $(\bar{x},T(\bar{x}))$ is on the umbrella, it follows that

$$T(x) \geq T(\bar{x}). \qquad (2)$$

Since $x \mapsto T(x)$ is 1-Lipschitz and $\bar{x}_1 \geq \bar{x}_2$, it follows that

$$T(\bar{x}) \geq T(0) - |\bar{x}| \geq T(0) - \bar{x}_1\sqrt{2}. \qquad (3)$$

Since "w_x is bounded" by our work in 2003-2005, we do have the following (non sharp) upper bound:

$$T(0) - \frac{x_1}{2} \geq T(x). \qquad (4)$$

Combining (2), (3) and (4), we see that

$$T(0) - \frac{x_1}{2} \geq T(x) \geq T(\bar{x}) \geq T(0) - \bar{x}_1\sqrt{2}, \text{ hence } x_1 \leq 2\bar{x}_1\sqrt{2}.$$

Recalling from the previous slide that

$$c_0|\log \bar{x}_1|^{-\frac{p-1}{2}} + \ldots = \gamma \leq x_1^2, \text{ hence } \bar{x}_1 \ll x_1,$$

we get a contradiction.

The Umbrella Technique (cont.)

Thus, *all points x outside the bisectrices are non characteristic* and

$$\nabla T(x) = e_1(-1 + c_0|\log x_1|^{-\frac{p-1}{2}}) + \ldots \text{ if } 0 \le x_2 < x_1.$$

Integrating this estimate between 0 and x gives

$$T(x) - T(0) \sim -x_1 \text{ if } 0 \le x_2 < x_1.$$

Extending this by symmetry, we obtain the pyramid shape:

$$T(x) - T(0) \sim -\max(|x_1|, |x_2|).$$

Thank you for your attention