Characteristic points for the semilinear wave equation

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The equation

$$\begin{cases} \partial_t^2 u = \partial_x^2 u + |u|^{p-1}u, \\ \\ u(0) = u_0 \text{ and } u_t(0) = u_1, \end{cases}$$

where $p > 1$,
$u(t) : x \in \mathbb{R} \to u(x,t) \in \mathbb{R}$,
$u_0 \in \mathrm{H}^1_{\mathrm{loc,u}}(\mathbb{R})$ and $u_1 \in \mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R})$
and

$$\|v\|_{\mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R})} = \sup_{a\in\mathbb{R}} \left(\int_{a-1}^{a+1} |v(x)|^2 dx \right)^{1/2}.$$

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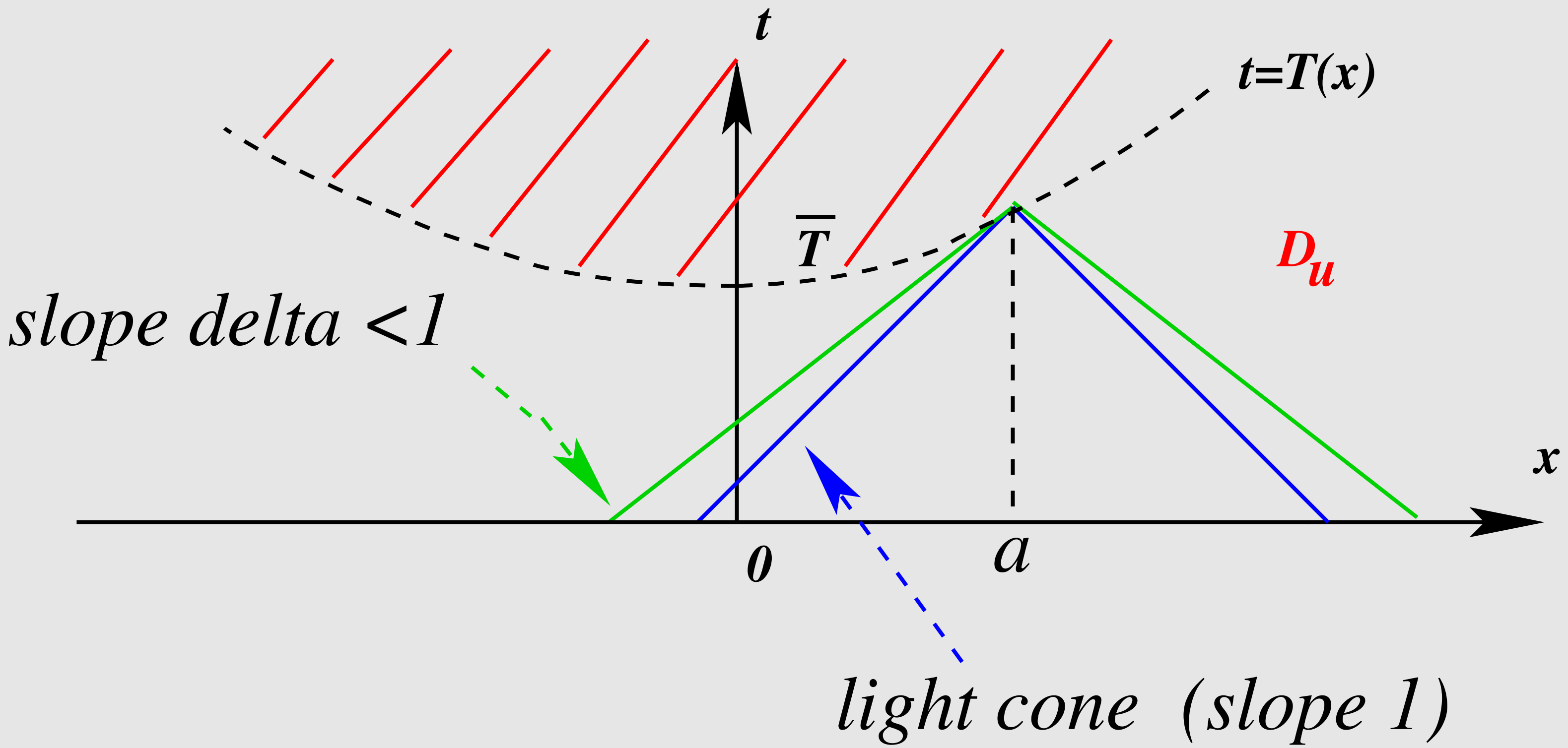
Definition: Non characteristic points and characteristic points

A point a is said *non characteristic* if the domain contains a cone with vertex $(a, T(a))$ and slope $\delta < 1$.



The point is said *characteristic* if not.

- Notation: $\mathcal{R} \subset \mathbb{R}$ is the set of all *non* characteristic points.
- Notation: $\mathcal{S} \subset \mathbb{R}$ is the set of all characteristic points ($\mathcal{S} \cup \mathcal{R} = \mathbb{R}$).

Similarity variables

Selfsimilar transformation for all $x_0 \in \mathbb{R}$

$$w_{x_0}(y,s) = (T(x_0)-t)^{\frac{2}{p-1}} u(x,t),\ y = \frac{x-x_0}{T(x_0)-t},\ s = -\log(T(x_0)-t).$$

(x,t) in the light cone of vertex $(x_0, T(x_0)) \Longleftrightarrow (y,s) \in (-1,1) \times [-\log T(x_0), \infty)$.

Equation on $w = w_{x_0}$: For all $(y,s) \in (-1,1) \times [-\log T(x_0), \infty)$:

$$\partial^2_{ss} w - \frac{1}{\rho}\partial_y(\rho(1-y^2)\partial_y w) + \frac{2(p+1)}{(p-1)^2} w - |w|^{p-1} w$$

$$= -\frac{p+3}{p-1}\partial_s w - 2y\partial^2_{sy} w$$

where $\rho(y) = (1-|y|^2)^{\frac{2}{p-1}}$

A Lyapunov functional (Antonini-Merle)

$$E(w) = \int_{-1}^{1}\left(\frac{1}{2}(\partial_s w)^2 + \frac{1}{2}(\partial_y w)^2(1-y^2) + \frac{(p+1)}{(p-1)^2}w^2 - \frac{1}{p+1}|w|^{p+1}\right)\rho dy,$$

Thanks to a Hardy-Sobolev inequality, $E = E(w, \partial_s w)$ is well defined in the energy space

$$\mathcal{H} = \left\{q \in H^1_{loc} \times L^2_{loc}(B) \ \mid \ \|q\|^2_{\mathcal{H}} \equiv \int_{-1}^{1}\left(q_1^2 + (\partial_y q_1)^2(1-y^2) + q_2^2\right)\rho dy < +\infty\right\}.$$

Properties of the Lyapunov functional E

Lemma 1 (Monotonicity (Antonini-Merle)) *For all s_1 and s_2:*

$$E(w(s_2)) - E(w(s_1)) = -\frac{4}{p-1}\int_{s_1}^{s_2}\int_{-1}^{1}(\partial_s w)^2(1-|y|^2)^{\frac{2}{p-1}-1}dyds.$$

Lemma 2 (A blow-up criterion) *Consider a solution W such that $E(W(s_0)) < 0$ for some $s_0 \in \mathbb{R}$, then W blows up in finite time $S > s_0$.*

Regularity of the blow-up set at a *characteristic* point

Th. *The set of characteristic points $\mathcal{S}$ is made of* **isolated points**.
If $a \in \mathcal{S}$, then $T_l'(a) = 1$ and $T_r'(a) = -1$.
Rk. An important step of the proof is to prove first that $\mathcal{S}$ has an empty interior.

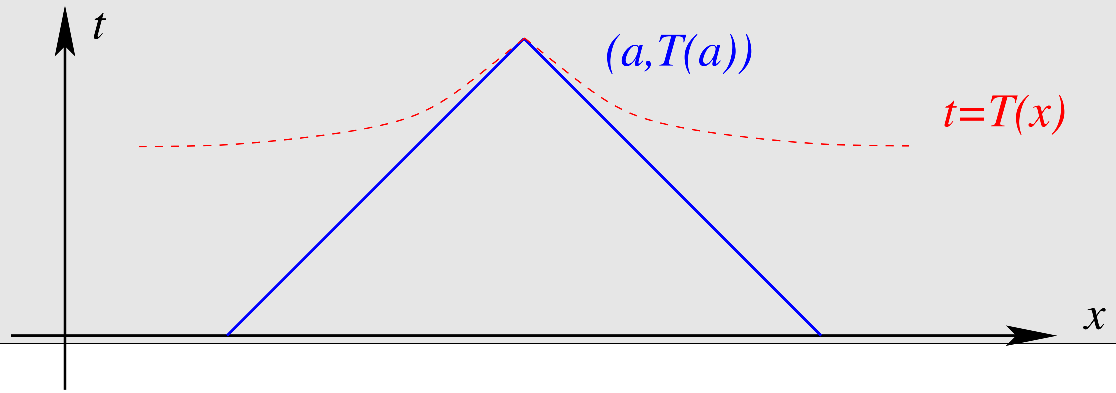
Th. (the corner property) *If $a \in \mathcal{S}$, then for all x near a,*

$$\frac{1}{C}|x-a||\log|x-a||^{-\gamma(a)} \le T(x) - T(a) + |x-a| \le C|x-a||\log|x-a||^{-\gamma(a)} \tag{1}$$

where

$$\gamma(a) = \frac{(k(a)-1)(p-1)}{2} \text{ with } k(a) \in \mathbb{N}, k(a) \ge 2.$$

Rk. Estimate (1) remains valid after differentiation.



Asymptotic behavior at a *characteristic* point

Th. If $x_0 \in \mathbb{R}$ is **characteristic**, then, there exist $k(x_0) \geq 2$, $e(x_0) = \pm 1$ and continuous $d_i(s) = -\tanh \zeta_i(s)$ for $i = 1, ..., k$ such that:
(i)

$$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - e(x_0) \sum_{i=1}^{k(x_0)} (-1)^i \begin{pmatrix} \kappa(d_i(s), \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \to 0 \text{ as } s \to \infty,$$

(ii) Introducing

$$\bar{w}_{x_0}(\xi, s) = (1 - y^2)^{\frac{1}{p-1}} w_{x_0}(y, s) \text{ with } y = \tanh \xi \text{ and } \zeta_i(x_0) = -\operatorname{argth} d_i(s),$$

we get

$$\|\bar{w}_{x_0}(\xi, s) - e(x_0) \sum_{i=1}^{k(x_0)} (-1)^i \cosh^{-\frac{2}{p-1}}(\xi - \zeta_i(s))\|_{H^1 \cap L^\infty(\mathbb{R})} \to 0 \text{ as } s \to \infty,$$

Asymptotic behavior at a *characteristic* point (cont.)

(iii) For all $i = 1, ..., k(x_0)$ and s large enough,

$$\left(i - \frac{(k(x_0)+1)}{2}\right)\frac{(p-1)}{2}\log s - C_0 \le \zeta_i(s) \le \left(i - \frac{(k(x_0)+1)}{2}\right)\frac{(p-1)}{2}\log s + C_0.$$

(iv) $E(w_{x_0}(s)) \to k(x_0)E(\kappa_0)$ as $s \to \infty$.

Rk.
- As $s \to \infty$, w_{x_0} becomes like a **decoupled** sum of *equidistant* stationary solutions (“solitons”), with *alternate* signs.
- In the ξ variable, half of the solitons go to $-\infty$, and the other half to $+\infty$.
- The main difficulty in the proof is to prove that $k(x_0) \ge 2$ (the case $k(x_0) = 0$ is harder to eliminate).
- The $\zeta_i(s)$ satisfy a Toda system:

$$\frac{1}{c_1}\zeta_i'(s) = e^{-\frac{2}{p-1}(\zeta_i - \zeta_{i-1})} - e^{-\frac{2}{p-1}(\zeta_{i+1} - \zeta_i)} + R_i \text{ with } R_i = o\left(\sum_{j=1}^{k-1} e^{-\frac{2}{p-1}(\zeta_{j+1} - \zeta_j)}\right) \text{ as } s \to \infty.$$

Idea of the proof of the results in the *characteristic* case

The results are: the decomposition into solitons, the corner property and the fact that the interior of $\mathcal{S}$ is empty.

6 main steps are needed:

- ▷ Step 1: Decomposition into a decoupled sum of $k(x_0) \geq 0$ solitons, with no information on the signs or the distance between the solitons' centers (in the ξ variable).
- ▷ Step 2: Characterization of the case $k(x_0) \geq 2$. Proof of *the upper bound* in the corner property if $k(x_0) \geq 2$.
- ▷ Step 3: Excluding the case $k(x_0) = 0$ if $x_0 \in \partial\mathcal{S}$ (note that $\partial\mathcal{S} \subset \mathcal{S}$ since $\mathcal{R} = \mathbb{R} \backslash \mathcal{S}$ is open).
- ▷ Step 4: Characterization of the case where $x_0 \in \partial\mathcal{S}$ and $k(x_0) = 1$.
- ▷ Step 5: We prove that the interior of $\mathcal{S}$ is empty, then that $k(x_0) \geq 2$ for all $x_0 \in \mathcal{S}$ (which gives *the upper bound* in the corner property by Step 2).
- ▷ Step 6: We prove that $\mathcal{S}$ is made of isolated points.

Comments

Rk. 1: A good understading of the *non-characteristic* case is *crucial*.

Rk. 2: Excluding the case $k(x_0) = 0$ is more difficult than excluding the case $k(x_0) = 1$.
In particular, we can't exclude directly the case $k(x_0) = 0$ for all $x_0 \in \mathcal{S}$. We do it first when $x_0 \in \partial\mathcal{S}$, then prove that the interior of $\mathcal{S}$ is empty, hence $\partial\mathcal{S} = \mathcal{S}$.

Step 1: Decomposition into a decoupled sum of $k(x_0) \geq 0$ solitons

Take $x_0 \in \mathbb{R}$ a **characteristic** points. We have two estimates:

- $\|(w_{x_0}(s), \partial_s w_{x_0}(s))\|_{\mathcal{H}} \leq C_0$;
- $\int_{-\log T(x_0)}^{\infty} \int_{-1}^{1} (\partial_s w_{x_0}(y,s))^2 \frac{\rho}{1-y^2} dy \leq C_0$.

Rk. Unlike the non characteristic case, we can't have a covering argument, so we can't obtain the $H^1 \times L^2$ norm bounded (in fact, we will show that it is unbounded).

Step 1: Decomposition into a decoupled sum of $k(x_0) \geq 0$ solitons (cont.)

In the $\bar{w}_{x_0}(\xi, s)$ variable, we have

$$\|\bar{w}_{x_0}(\xi, s)\|_{H^1(\mathbb{R})} \leq C_0.$$

For any sequence ξ_n in $\mathbb{R}$, we find a "local" limit in the sense that for some $s_n \to \infty$, we have

$$\bar{w}_{x_0}(\xi + \xi_n, s + s_n) \to \bar{w}^* \text{ as } n \to \infty,$$

uniformly on compact sets for (ξ, s), with w^* a stationary solution, due to the fact that

$$\int_{-\log T(x_0)}^{\infty} \int_{-1}^{1} (\partial_s w_{x_0}(y, s))^2 \frac{\rho}{1 - y^2} dy \leq C_0.$$

Since the energy is bounded, the number of non zero "local limits" is finite, and we end-up with the following result:

Step 1: Decomposition into a decoupled sum of $k(x_0) \geq 0$ solitons (cont.)

Prop. *There exist $k(x_0) \geq 0$ and continuous $d_i(s) \in (-1,1)$ such that*

$$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - \sum_{i=1}^{k(x_0)} e_i(x_0) \begin{pmatrix} \kappa(d_i(s),\cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \to 0 \text{ as } s \to \infty,$$

with

$$\zeta_{i+1}(s) - \zeta_i(s) \to \infty \text{ as } s \to \infty \text{ and } d_i(s) = -\tanh \zeta_i(s).$$

Rk.

- If $k(x_0) = 0$, then the above sum is 0.
- At this level, we don't know that $k(x_0) = 0$ and $k(x_0) = 1$ don't occur.
- We have no information on the signs $e_i(x_0)$.
- We have no equivalent for $\zeta_i(s)$ as $s \to \infty$.

Step 2: Case $k(x_0) \geq 2$; A differential equation on the solitons' centers

Here, we assume that $k(x_0) \geq 2$ (we don't prove that fact here).
Linearizing the equation in the $w(y,s)$ setting around the sum of the solitons, we get the following Toda system on the solitons' centers in the ξ variable: for all $i = 1, ..., k$ and s large enough, we have

$$\frac{1}{c_1}\zeta_i' = -e_{i-1}e_i e^{-\frac{2}{p-1}(\zeta_i - \zeta_{i-1})} + e_i e_{i+1} e^{-\frac{2}{p-1}(\zeta_{i+1}-\zeta_i)} + R_i$$

where

$$|R_i| \leq CJ^{1+\delta_0}, \quad J(s) = \sum_{j=1}^{k-1} e^{-\frac{2}{p-1}(\zeta_{j+1}(s) - \zeta_j(s))},$$

$e_0 = e_{k+1} = 0$, for some $c_1 > 0$ and $\delta_0 > 0$.

Step 2: Case $k(x_0) \geq 2$ (cont.)

Since for all $i = 1, ..., k(x_0) - 1$, we have

$$\zeta_{i+1}(s) - \zeta_i(s) \to \infty \text{ as } s \to \infty,$$

using ODE techniques, we find that

$$e_i e_{i+1} = -1 \text{ and } \zeta_i(s) \sim \left(i - \frac{k(x_0)+1}{2}\right) \frac{(p-1)}{2} \log s.$$

The upper bound on the blow-up rate gives the *upper bound* in the corner property.

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Step 3: Excluding the case where $x_0 \in \partial\mathcal{S}$ and $k(x_0) = 0$

By contradiction, if $x_0 \in \partial\mathcal{S}$ and $k(x_0) = 0$, then

$$\|w_{x_0}(s)\|_{\mathcal{H}} \to 0 \text{ and } E(w_{x_0}(s)) \to 0 \text{ as } s \to \infty.$$

Fixing s_0 large enough such that $E(w_{x_0}(s_0)) \leq \frac{1}{4}E(\kappa_0)$, we find x_1 near x_0 such that

$$x_1 \in \mathcal{R} \text{ and } E(w_{x_1}(s_0)) \leq \frac{1}{2}E(\kappa_0).$$

Since $E(w_{x_1}(s)) \to E(\kappa_0)$ as $s \to \infty$ and $E(w_{x_1}(s))$ is decreasing, it follows that

$$E(w_{x_1}(s_0)) \geq E(\kappa_0).$$

Contradiction.

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Step 4: Characterization of the case where $x_0 \in \partial\mathcal{S}$ and $k(x_0) = 1$

In this case,

$$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - e_1 \begin{pmatrix} \kappa(d_1(s), y) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \to 0 \text{ as } s \to \infty \text{ and } E(w_{x_0}(s)) \geq E(\kappa_0).$$

Our "trapping" result implies that for some $d(x_0) \in (-1, 1)$,

$$w_{x_0}(s) \to \kappa(d(x_0)) \text{ as } s \to \infty.$$

Some elementary geometry and the precise knowledge of the case of non characteristic points gives that x_0 is either left-non-characteristic or right-non-characteristic.

Step 5: Conclusion without Isolatedness

Using the previous steps, we prove in the same time that $k(x_0) \geq 2$ and the interior of $\mathcal{S}$ is empty, together with precise estimate on the location of the solitons' centers.
We also get *the upper bound* in the corner property.

Step 6: Characteristic points are isolated

Consider $x_0 \in \mathcal{S}$. From translation invariance of the equation in $u(x,t)$, we can assume that $x_0 = T(x_0) = 0$, hence,

$$0 \in \mathcal{S} \text{ and } T(0) = 0.$$

We have just proved that for some integer $k = k(0) \geq 2$, for some continuous functions $d_i(s)$, $C_0 > 0$ and $s_0 \in \mathbb{R}$, we have

$$\left\| \begin{pmatrix} w_0(s) \\ \partial_s w_0(s) \end{pmatrix} - \begin{pmatrix} \sum_{i=1}^{k} (-1)^i \kappa(d_i(s)) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \to 0 \text{ as } s \to \infty,$$

$$\forall s \geq s_0, \qquad \left| \operatorname{argth} d_i(s) - \frac{\gamma_i}{2} \log s \right| \leq C_0 \text{ where } \gamma_i = (p-1)\left(\frac{k+1}{2} - i\right).$$

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Step 6: Characteristic points are isolated (cont.)

Introducing for $x \neq 0$, $B = B(x)$ by

$$-\frac{T(x)}{|x|} = 1 - B(x),$$

we translate *the upper bound* in the corner property as follows

$$0 < B \leq \frac{C_0}{|\log|x||^{\gamma_1}}.$$

We proceed in two parts:

- In Part 1, we use the algebraic relation between w_0 and w_x and a dynamical study to derive the expansion of w_x where x is near $0 \in \mathcal{S}$.

- In Part 2, we show that x is non characteristic and measure the distance of $T'(x)$ to 1 when $x < 0$ (and to -1 when $x > 0$).

Part 1: Expansion for w_x.

Algebraic transformation

Recalling the selfsimilar change of variables for w_0 and w_x:

$$\begin{aligned} w_0(Y,S) &= (-\tau)^{\frac{2}{p-1}} u(\xi,\tau),\ Y = \frac{\xi}{-\tau},\ S = -\log(-\tau), \\ w_x(y,s) &= (T(x)-\tau)^{\frac{2}{p-1}} u(\xi,\tau),\ y = \frac{\xi - x}{T(x)-\tau},\ s = -\log(T(x)-\tau), \end{aligned}$$

we get the following algebraic relation between w_x and w_0

$$w_x(y,s) = (1-(1-B)xe^s)^{-\frac{2}{p-1}} w_0(Y,S),\ \ Y = \frac{y+xe^s}{1-(1-B)xe^s}\ \ S = s - \log(1-(1-B)xe^s).$$

This means that the expansion for w_0 translates into an expansion for w_x:

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Part 1: Expansion for w_x (cont.)

Prop. We have

$$\lim_{L\to\infty}\left(\lim_{x\to 0^-}\sup_{L\le s\le L+|\log|x||}\left\|\begin{pmatrix} w_x(s)\\ \partial_s w_x(s)\end{pmatrix}-\sum_{i=1}^{k}(-1)^i\kappa^*\left(\hat{d}_i(s),\hat{\nu}_i(s)\right)\right\|_{\mathcal{H}}\right)=0$$

where

$$\hat{\nu}_i(x,s)=[B-(1-\hat{d}_i(x,s))]xe^s,\ \ \hat{d}_i(x,s)=d_i(S) \text{ and } -e^{-S(x,s)}=x(1-B)-e^{-s}.$$

Moreover, for any $d\in(-1,1)$ and $\mu\in\mathbb{R}$, $\kappa^*(d,\mu e^s,y)$ **is a particular solution** of the equation in selfsimilar variables, given by...

Part 1: Definition of $\kappa^*(d,\nu,y)$

.... $\kappa^*(d,\nu,y) = (\kappa_1^*,\kappa_2^*)(d,\nu,y)$ where

$$\kappa_1^*(d,\nu,y) = \kappa_0 \frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dy+\nu)^{\frac{2}{p-1}}} \text{ and } \kappa_2^*(d,\nu,y) = \nu\partial_\nu \kappa_1^*(d,\nu,y) = -\frac{2\kappa_0\nu}{p-1}\frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dy+\nu)^{\frac{p+1}{p-1}}}$$

where $d \in (-1,1)$ and $\nu > -1+|d|$.

Note that for any $\mu \in \mathbb{R}$, $(y,s) \mapsto \kappa^*(d,\mu e^s,y)$ is an explicit solution to the equation in similarity variables. Moreover,

- when $\mu = 0$, we recover the stationary solutions $\kappa(d,y)$;
- when $\mu > 0$, the solution exists for all $(y,s) \in (-1,1)\times\mathbb{R}$ and converges to 0 in $\mathcal{H}$ as $s \to \infty$;
- when $\mu < 0$, the solution exists for all $(y,s) \in (-1,1)\times\left(-\infty,\log\left(\frac{|d|-1}{\mu}\right)\right)$ and blows up at time $s = \log\left(\frac{|d|-1}{\mu}\right)$.

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Part 1: Proof: First, algebraic technique

Rk. The algebraic technique gives *explicit* parameters but *not on the whole interval* $(-1,1)$.

Starting from the expansion of w_0 and the algebraic relation between w_x and w_0, we get the result with the norm restricted to

$$y > y_1(x,s)$$

for some $y_1(x,s) > -1$:

$$\lim_{L\to\infty}\left(\lim_{x\to 0^-}\sup_{L\le s\le L+|\log|x||}\left\|\begin{pmatrix} w_x(s) \\ \partial_s w_x(s)\end{pmatrix} - \sum_{i=1}^{k}(-1)^i\kappa^*\left(\hat{d}_i(s),\hat{\nu}_i(s)\right)\right\|_{\mathcal{H}(y>y_1(x,s))}\right) = 0$$

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Part 1, Proof: Second, analytic technique

Rk. The analytic technique gives *non explicit* parameters, but *on the whole interval* $(-1,1)$.

Since the result holds for w_0 on the square $(y,s) \in (-1,1) \times [L, L+1]$, by continuity, it holds also for w_x when $|x|$ small on the same square. Performing a modulation technique around the sum of $\kappa^*(d_i, v_i)$, we propagate the estimate with non explicit parameters up to

$$s = L + |\log|x||,$$

in the sense that

$$\lim_{L\to\infty}\left(\lim_{x\to 0^-}\sup_{L\le s\le L+|\log|x||}\left\|\begin{pmatrix} w_x(s) \\ \partial_s w_x(s)\end{pmatrix} - \sum_{i=1}^{k}(-1)^i\kappa^*\left(\bar d_i(s), \bar\nu_i(s)\right)\right\|_{\mathcal{H}}\right) = 0.$$

Part 1: Conlusion of the proof of the expansion for w_x

Since $y_1(x,s)$ is “close” to the center of the first soliton, comparing the two expansions for $y \in (y_1,(x,s),1)$ gives that **the parameters $(\hat{d}_i(s),\hat{v}_i(s))$ and $(\bar{d}_i(s),\bar{v}_i(s))$ are close**, and we get to the conclusion of the proposition:

$$\lim_{L\to\infty}\left(\lim_{x\to 0^-}\sup_{L\le s\le L+|\log|x||}\left\|\begin{pmatrix} w_x(s)\\ \partial_s w_x(s)\end{pmatrix}-\sum_{i=1}^{k}(-1)^i\kappa^*\left(\hat{d}_i(s),\hat{v}_i(s)\right)\right\|_{\mathcal{H}}\right)=0.$$

Part 2: Conclusion of the fact that x is isolated

It happens that when $s = L + |\log|x||$, all the solitons for $i \geq 2$ vanish, in the sense that

$$\forall i \geq 2, \quad \lim_{L\to\infty} \left\| \kappa^* \left(\hat{d}_i(|\log|x|| + L), \hat{\nu}_i(|\log|x|| + L) \right) \right\|_{\mathcal{H}} = 0.$$

Therefore, given $\epsilon > 0$, for L large enough and $|x|$ small enough, we have

$$\left\| \begin{pmatrix} w_x(|\log|x|| + L) \\ \partial_s w_x(|\log|x|| + L) \end{pmatrix} + \kappa^* \left(\hat{d}_1(|\log|x|| + L), \hat{\nu}_i(|\log|x|| + L) \right) \right\|_{\mathcal{H}} \leq \epsilon.$$

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Part 2: Using the energy behavior

Since we have the following

Prop. *(Energy minimum)*

$$\forall x \in \mathbb{R}, \ \ \forall s \geq -\log T(x), \ \ E(w_x(s)) \geq E(\kappa_0),$$

it follows that

$$E\left(\kappa^*\left(\hat{d}_1(|\log|x||+L), \hat{\nu}_i(|\log|x||+L)\right)\right) \geq E(\kappa_0) - C\epsilon \qquad (2)$$

on the one hand.

On the other hand, we have by direct computation

$$E(\kappa_0) \leq E(\kappa^*(d,\nu)) \leq E(\kappa_0)\left(3\lambda^2 - (2-\epsilon)\lambda^3\right) \text{ where } \lambda = \frac{(1-d^2)}{(1+\nu)^2 - d^2}. \qquad (3)$$

From (2) and (3), we see that

$$3\lambda^2 - (2-\epsilon)\lambda^3 \geq 1 - C\epsilon, \text{ hence } |\lambda - 1| \leq C\epsilon.$$

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Part 2: Using the energy behavior (cont.)

Since we have in this regime

$$\left\|\kappa^*(d,\nu) - \begin{pmatrix} \kappa\left(\frac{d}{1+\nu},0\right) \\ 0 \end{pmatrix}\right\|_{\mathcal{H}} \leq C|\lambda - 1|,$$

it follows that

$$\left\|\begin{pmatrix} w_x(|\log|x|| + L) \\ \partial_s w_x(|\log|x|| + L) \end{pmatrix} + \kappa\left(\frac{\hat{d}_1(|\log|x|| + L)}{1 + \hat{\nu}_i(|\log|x|| + L)}, 0\right)\right\|_{\mathcal{H}} \leq C\epsilon.$$

Part 2: A trapping argument

Now, we recall the following result (from the non-characteristic case):

Prop. *(Trapping result) There exists* $\epsilon^* > 0$ *such that if for some* $x^* \in \mathbb{R}$*,* $s^* \geq -\log T(x^*)$ *and* $d^* \in (-1,1)$ *we have*

$$\|w_{x^*}(s^*) + \kappa(d^*, y)\|_{\mathcal{H}} \leq \epsilon^*,$$

then, $w_{x^*}(s) \to \kappa(\bar{d})$ *as* $s \to \infty$ *for some* $\bar{d}$ *such that*

$$\left|\operatorname{argth} \bar{d} - \operatorname{argth} d^*\right| \leq C\epsilon^*.$$

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Part 2: Application to our case

Therefore, in our case, for some $\bar{d}(x)$ such that

$$\left| \operatorname{argth} \bar{d}(x) - \operatorname{argth} \frac{\hat{d}_1(|\log|x|| + L)}{1 + \hat{v}_i(|\log|x|| + L)} \right| \leq C\epsilon^*,$$

we have $w_x(s) \to -\kappa(\bar{d}(x))$ as $s \to \infty$.

From the knowledge of the non-characteristic case and the characteristic case,
x is not characteristic !!!!!

Moreover, $T'(x) = \bar{d}(x)$.

$$\left| \operatorname{argth} T'(x) - \operatorname{argth} \frac{\hat{d}_1(|\log|x|| + L)}{1 + \hat{v}_i(|\log|x|| + L)} \right| \leq C\epsilon^*.$$

Part 2: Final conclusion

This gives a bound on $T'(x)$:

$$\frac{1}{C|\log|x||^{\frac{(k(x_0)-1)(p-1)}{2}}} \le T'(x) - 1 \le \frac{C}{|\log|x||^{\frac{(k(x_0)-1)(p-1)}{2}}}$$

which gives by integration

$$\frac{|x|}{C|\log|x||^{\frac{(k(x_0)-1)(p-1)}{2}}} \le T(x) - x \le \frac{C|x|}{|\log|x||^{\frac{(k(x_0)-1)(p-1)}{2}}}$$

(remember that

$$x_0 = T(x_0) = 0).$$

— LaTeX prosper —