Isolatedness of characteristic points for blow-up solutions of a semilinear wave equation (Part 1)

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The equation

$$\begin{cases} \partial_t^2 u = \partial_x^2 u + |u|^{p-1}u, \\ \\ u(0) = u_0 \text{ and } u_t(0) = u_1, \end{cases}$$

where $p > 1$,
$u(t) : x \in \mathbb{R} \to u(x,t) \in \mathbb{R}$,
$u_0 \in \mathrm{H}^1_{\mathrm{loc,u}}(\mathbb{R})$ and $u_1 \in \mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R})$
and

$$\|v\|_{\mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R})} = \sup_{a\in\mathbb{R}} \left(\int_{a-1}^{a+1} |v(x)|^2 dx \right)^{1/2}.$$

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THE CAUCHY PROBLEM IN $\mathrm{H}^1_{\mathrm{loc,u}}(\mathbb{R}) \times \mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R})$

It is a consequence of:

- the Cauchy problem in $H^1 \times L^2(\mathbb{R})$,
- the finite speed of propagation.

Maximal solution in $\mathrm{H}^1_{\mathrm{loc,u}}(\mathbb{R}) \times \mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R})$

- either it exists for all $t \in [0, \infty)$ (**global solution**),
- or it exists for all $t \in [0, \bar{T})$ (**singular solution**).

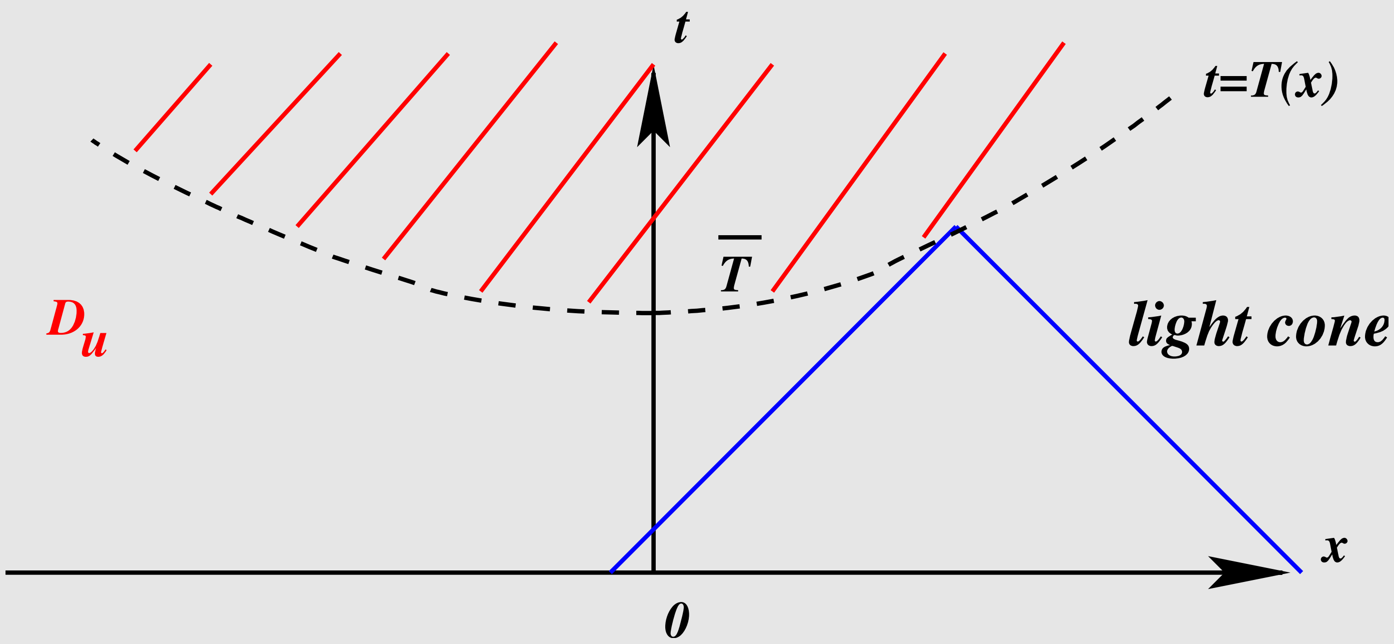
Existence of singular solutions

It's a consequence of ODE techniques and the finite speed of propagation; see also the energy argument by Levine 1974:

if $(u_0, u_1) \in H^1 \times L^2(\mathbb{R})$ *and* $\int_{\mathbb{R}} \left(\frac{1}{2}(u_1)^2 + \frac{1}{2}(\partial_x u_0)^2 - \frac{1}{p+1}|u_0|^{p+1}\right) dx < 0$,

then u is not global.

Singular solutions: the maximal influence domain



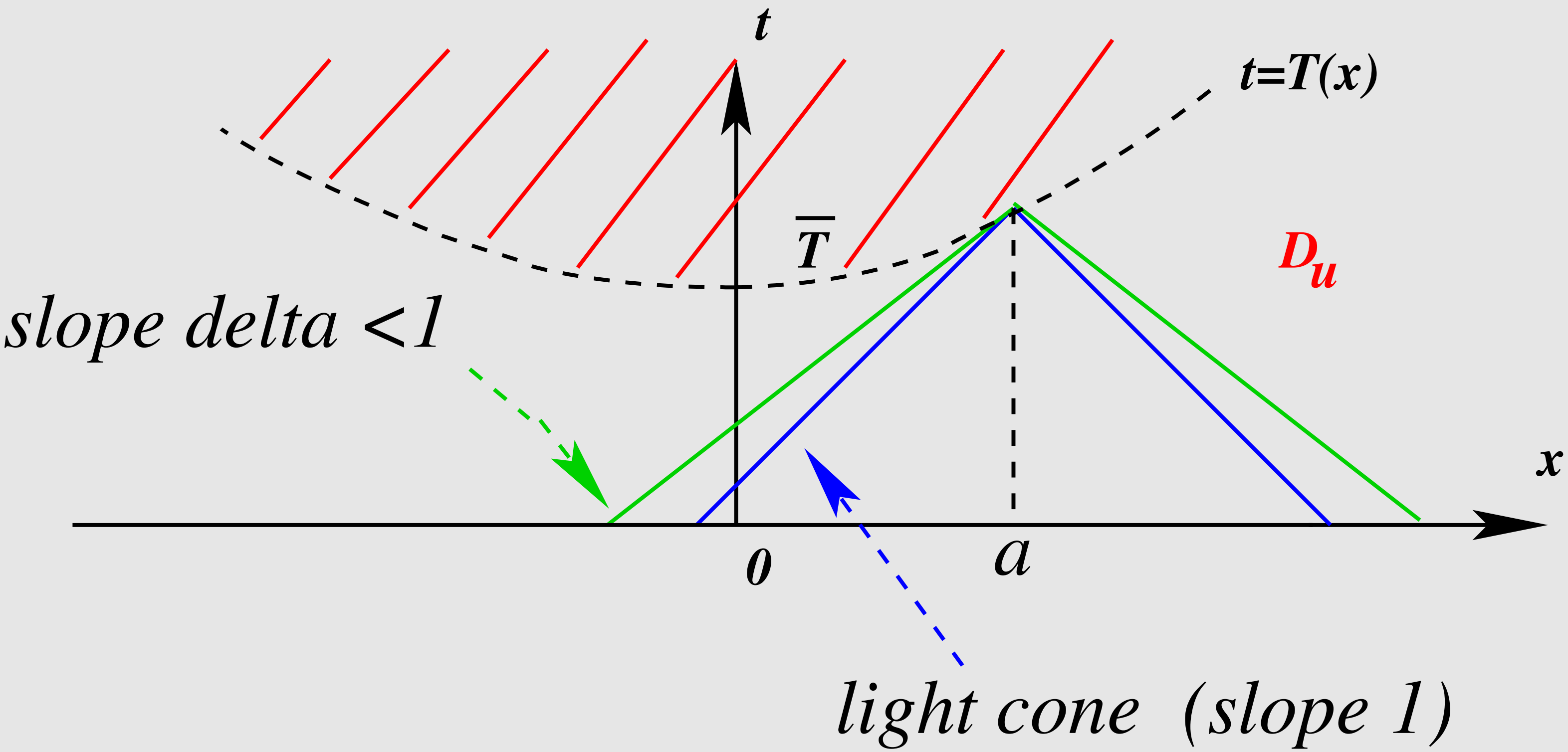
The blow-up set $t \to T(x)$ is 1-Lipschitz (**finite speed of propagation**).

Remark: $\bar{T} = \inf T(x)$ is the **blow-up time**. For all $x \in \mathbb{R}^N$, there exists a "local" blow-up time $T(x)$.

The aim of this talk: To describe precisely the blow-up set, and the solution near the blow-up set, *for an arbitrary blow-up solution*.

Definition: Non characteristic points and characteristic points

A point a is said *non characteristic* if the domain contains a cone with vertex $(a, T(a))$ and slope $\delta < 1$.



The point is said *characteristic* if not.

- Notation: $\mathcal{R} \subset \mathbb{R}$ is the set of all *non* characteristic points.
- Notation: $\mathcal{S} \subset \mathbb{R}$ is the set of all characteristic points ($\mathcal{S} \cup \mathcal{R} = \mathbb{R}$).

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Known results, for an arbitrary solution

- The blow-up set $\Gamma = \{(x, T(x))\} \subset \mathbb{R}^2$.

- By definition, Γ is 1-Lipschitz.

- $\mathcal{R} \neq \emptyset$ (Indeed, $\bar{x}$ such that $T(\bar{x}) = \min_{x \in \mathbb{R}} T(x)$ is non characteristic).

- Caffarelli and Friedman (1985 and 1986) had two criteria to have $\mathcal{R} = \mathbb{R}$ and $x \mapsto T(x)$ of class C^1 (using the **positivity of the fundamental solution**):

  - ▷ either when $p \geq 3$, with $u_0 \geq 0$, $u_1 \geq 0$ and $(u_0, u_1) \in C^4 \times C^3(\mathbb{R})$,
  - ▷ or under conditions on initial data that ensure that

$$u \geq 0 \text{ and } \partial_t u \geq (1 + \delta_0)|\partial_x u|$$

  for some $\delta_0 > 0$.

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Questions and new results

- **Existence**
  - Are there characteristic points? *yes,* $\mathcal{S} \neq \emptyset$.
- **Regularity**
  - Is $\mathcal{R}$ open? *yes*
  - Is Γ (or $\Gamma_{\mathcal{R}}$) of class C^1 ? *yes*
  - "How is" $\mathcal{S}$? *isolated points*
  - How does Γ look like near $\mathcal{S}$? *corner shaped*
- **Asymptotic behavior (profile)**
  - How does the solution behave near a non characteristic point? *we have the profile*
  - and near a characteristic point? *we have a precise decomposition into solitons*

Rk. Regularity and asymptotic behavior are linked.

The plan

- ▷ Part 1: Existence of characteristic points.
- ▷ Part 2: A Liouville theorem and regularity of the blow-up set.
- ▷ Part 3: A Lyapunov functional and the blow-up rate.
- ▷ Part 4: Asymptotic behavior near *non characteristic* points (the blow-up profile).
- ▷ Part 5: Asymptotic behavior near *characteristic* points (decomposition into solitons).

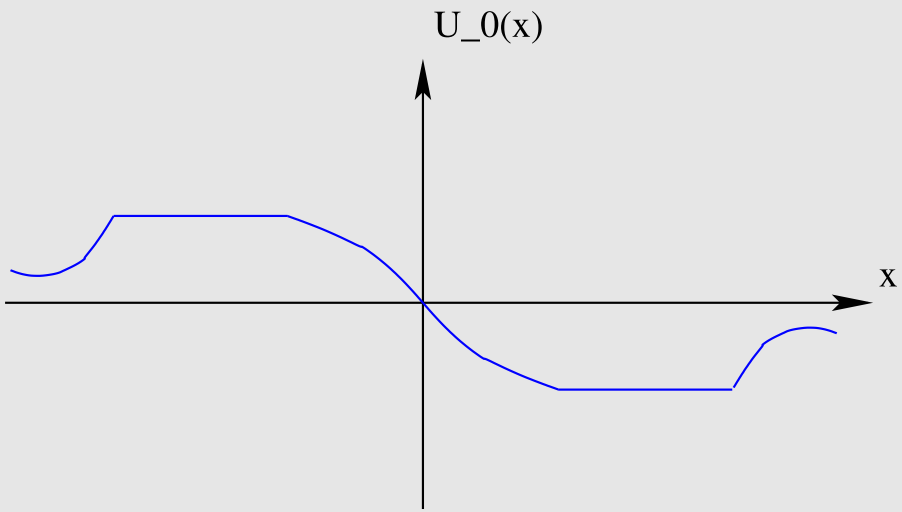
Rk. The order of this presentation goes from the easiest (to state) to the most complicated. The chronological order is actually 3, 4, 1, 2, 5.

Part 1: Existence of characteristic points

We recall: Any *solution to the Cauchy problem has (at least) a non characteristic point (the minimum of the blow-up set).*

Th. There exist *initial data which give solutions with a characteristic point.*

Example: We take odd initial data, with two large plateaus of different signs. Then, the solution blows up, and the origin is a characteristic point with $\forall t < T(0),\ u(0,t) = 0$.



Th. If we perturb initial data, then then new solution blows up and has a characteristic point.

Part 2: Regularity of the blow-up set

- Near a non characteristic point:

  Th. *The set of non characteristic points* $\mathcal{R}$ **is open** *and* $T(x)$ **is of class** C^1 *on this set* ($C^{1,\alpha}$ *by N. Nouaili CPDE 2008*).

- Near a characteristic point:

  Th. *The set of characteristic points* $\mathcal{S}$ *is made of* **isolated points.**

  If $a \in \mathcal{S}$, *then* $T_l'(a) = 1$ *and* $T_r'(a) = -1$.

  Cor. *There is no solution with* $a \in \mathcal{S}$ *and* $T'(a) = 1$.

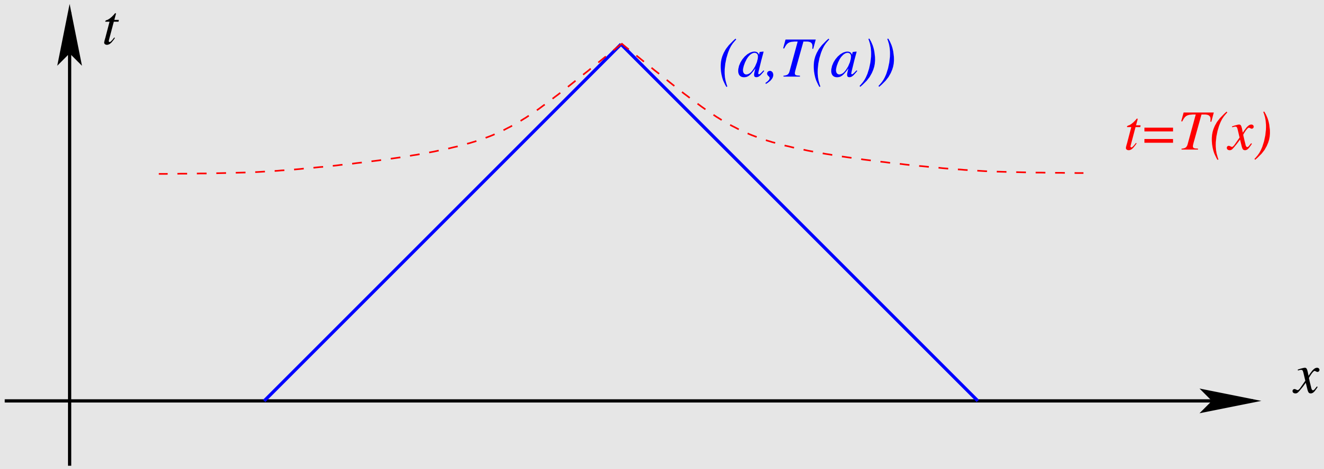
Part 2: The corner property near a *characteristic* point

Th. (the corner property) *If $a \in \mathcal{S}$, then for all x near a,*

$$\frac{1}{C}|x-a||\log|x-a||^{-\gamma(a)} \le T(x) - T(a) + |x-a| \le C|x-a||\log|x-a||^{-\gamma(a)} \quad (1)$$

where

$$\gamma(a) = \frac{(k(a)-1)(p-1)}{2} \text{ with } k(a) \in \mathbb{N}, k(a) \ge 2.$$



Comments

Rk. We recall the result of Caffarelli and Friedman:

If for all $x \in \mathbb{R}$ *and* $t < T(x)$, *we have* $u(x,t) \geq 0$ *and* $\partial_t u \geq (1+\delta_0)|\partial_x u|$ *for some* $\delta_0 > 0$, *then* $\mathcal{R} = \mathbb{R}$.

Here, We improve their criterion:
If for all $x \in [a,b]$ *and* $t < T(x)$, *we have* $u(x,t) \geq 0$, *then* $(a,b) \subset \mathcal{R}$.

Idea of the proof of the regularity in the *non characteristic* case:
The techniques are based on

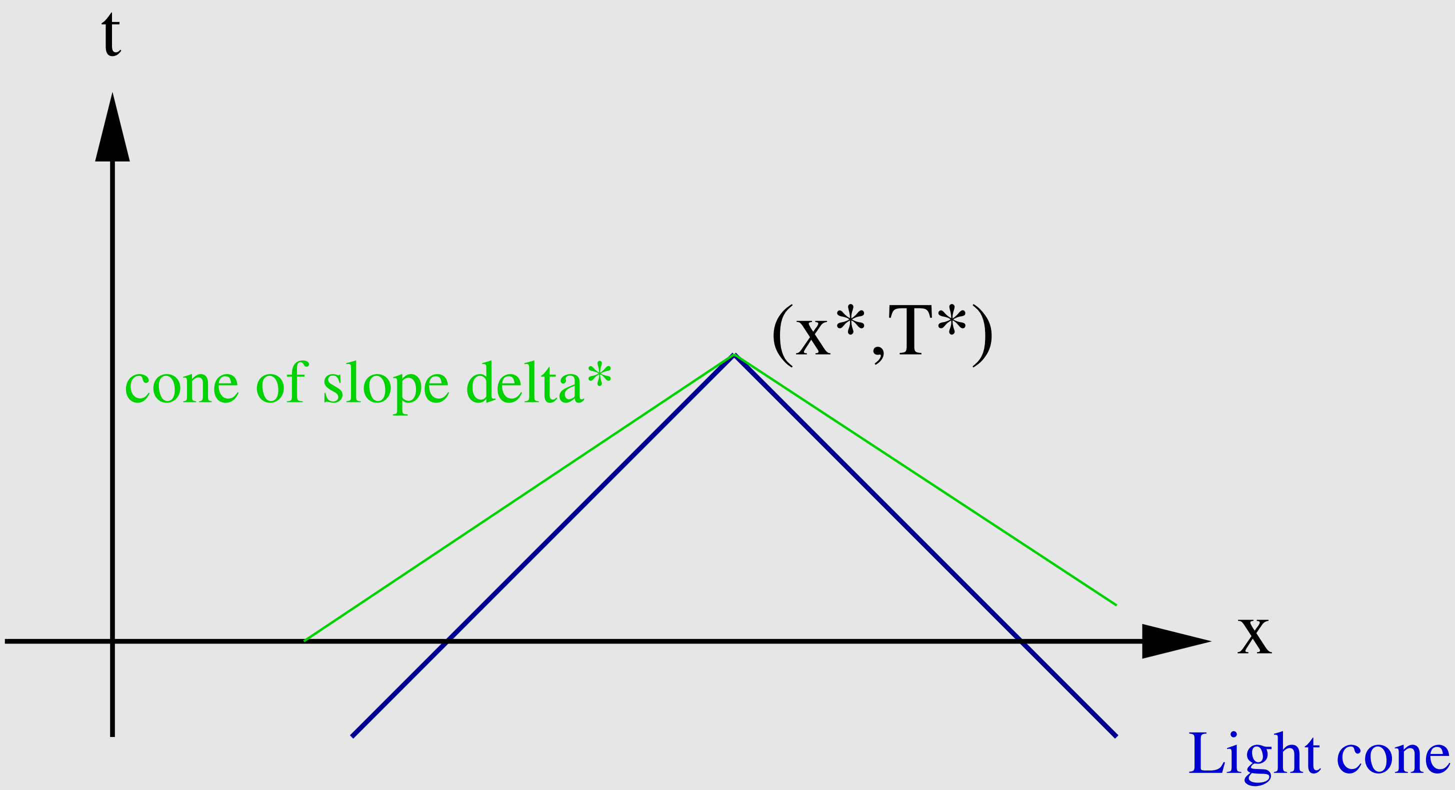
- - a very good understanding of the **behavior of the solution in selfsimilar variables in the energy space** related to the selfsimilar variable (see Part 3 of this talk).
- - a **Liouville Theorem** (see next slide).

Idea of the proof of the regularity in the *characteristic* case: At the end of the talk.

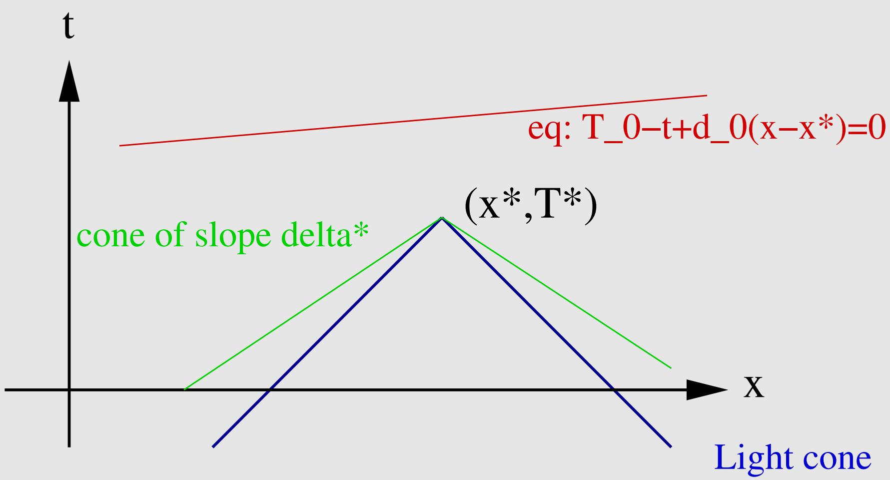
A Liouville Theorem

Th. Consider $u(x,t)$ a solution of $u_{tt} = u_{xx} + |u|^{p-1}u$ such that:

- u is defined in the *infinite* green cone,
- u is less than $(T^* - t)^{-\frac{2}{p-1}}$ (in L^2 average).



A Liouville Theorem



Then,
- either $u \equiv 0$,
- or there exists $T_0 \geq T^*$, $d_0 \in [-\delta_*, \delta_*]$ and $\theta_0 = \pm 1$ such that u is actually defined below the red line by

$$u(x,t) = \theta_0 \kappa_0(p) \frac{(1-d_0^2)^{\frac{1}{p-1}}}{(T_0 - t + d_0(x-x^*))^{\frac{2}{p-1}}}.$$

Remark: u blows up on the red line.

Comments

- The limiting case $\delta^* = 1$ is still open.

The proof:

- The proof has a completely different structure from the proof for the heat equation.
- The proof is based on various energy arguments and on a dynamical result.

Part 3: A Lyapunov functional and the blow-up rate

Selfsimilar transformation for all $x_0 \in \mathbb{R}$

$$w_{x_0}(y,s) = (T(x_0)-t)^{\frac{2}{p-1}} u(x,t),\ y = \frac{x-x_0}{T(x_0)-t},\ s = -\log(T(x_0)-t).$$

(x,t) in the light cone of vertex $(x_0, T(x_0)) \iff (y,s) \in (-1,1) \times [-\log T(x_0), \infty)$.

Equation on $w = w_{x_0}$: For all $(y,s) \in (-1,1) \times [-\log T(x_0), \infty)$:

$$\partial^2_{ss} w - \frac{1}{\rho}\partial_y(\rho(1-y^2)\partial_y w) + \frac{2(p+1)}{(p-1)^2} w - |w|^{p-1} w$$

$$= -\frac{p+3}{p-1}\partial_s w - 2y\partial^2_{sy} w$$

where $\rho(y) = (1-|y|^2)^{\frac{2}{p-1}}$

A Lyapunov functional (Antonini-Merle)

$$E(w) = \int_{-1}^{1} \left(\frac{1}{2} (\partial_s w)^2 + \frac{1}{2} (\partial_y w)^2 (1 - y^2) + \frac{(p+1)}{(p-1)^2} w^2 - \frac{1}{p+1} |w|^{p+1} \right) \rho dy,$$

Thanks to a Hardy-Sobolev inequality, $E = E(w, \partial_s w)$ is well defined in the energy space

$$\mathcal{H} = \left\{ q \in H^1_{loc} \times L^2_{loc}(B) \;\middle|\; \|q\|^2_{\mathcal{H}} \equiv \int_{-1}^{1} \left(q_1^2 + \left(\partial_y q_1\right)^2 (1 - y^2) + q_2^2 \right) \rho dy < +\infty \right\}.$$

Properties of the Lyapunov functional E

Lemma 1 (Monotonicity (Antonini-Merle)) *For all s_1 and s_2:*

$$E(w(s_2)) - E(w(s_1)) = -\frac{4}{p-1}\int_{s_1}^{s_2}\int_{-1}^{1}(\partial_s w)^2(1-|y|^2)^{\frac{2}{p-1}-1}dyds.$$

Lemma 2 (A blow-up criterion) *Consider a solution W such that $E(W(s_0)) < 0$ for some $s_0 \in \mathbb{R}$, then W blows up in finite time $S > s_0$.*

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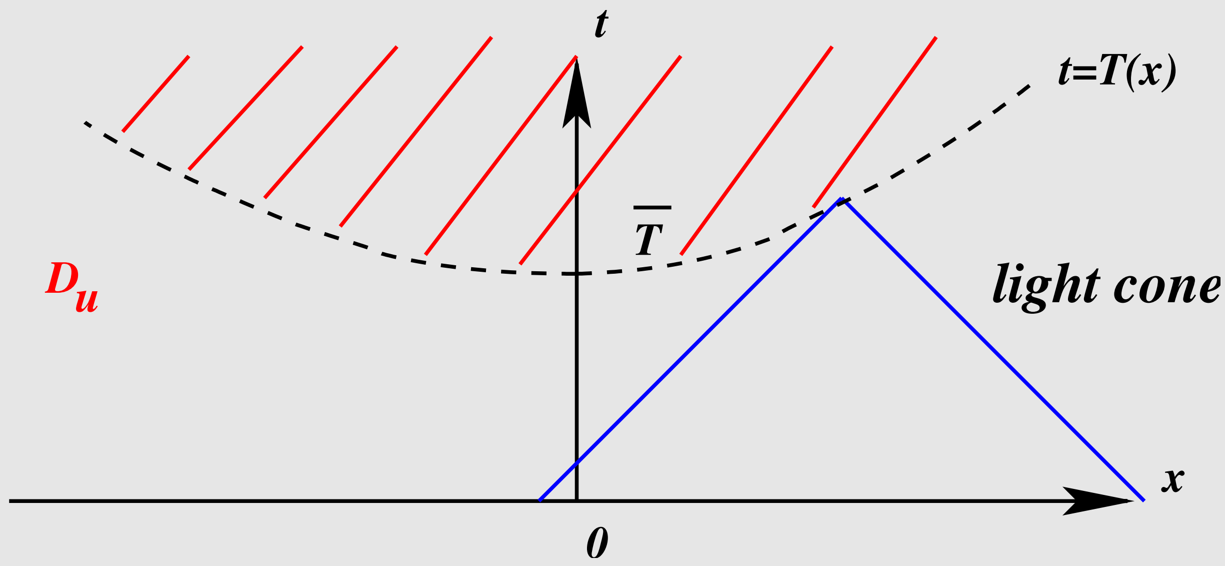
The blow-up rate

We look for a *local* **blow-up rate** near the singular surface (i.e. near every local blow-up time, $t \to T(x_0)$), in $H^1 \times L^2$ **of the section of the light cone.**



Hint: Is the rate given by the associated ODE $v'' = v^p$?

An upper bound on the blow-up rate in selfsimilar variables

Th. For all $x_0 \in \mathbb{R}$ and $s \geq -\log T(x_0) + 1$,

$$\int_{-1}^{1} \left(\frac{1}{2} (\partial_s w)^2 + \frac{1}{2} (\partial_y w)^2 (1 - |y|^2) + \frac{(p+1)}{(p-1)^2} w^2 + \frac{1}{p+1} |w|^{p+1} \right) \rho dy \leq K$$

where the constant K depends only on p and an upper bound on $T(x_0)$, $1/T(x_0)$ and $\|(u_0, u_1)\|$.

Getting rid of the weights

Reducing $(-1,1)$ to $(-\frac{1}{2}, \frac{1}{2})$, we get:

Cor. For all $x_0 \in \mathbb{R}$ and $s \geq -\log T(x_0) + 1$,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left((\partial_s w)^2 + (\partial_y w)^2 + w^2 + |w|^{p+1} \right) dy \leq K.$$

Upper bound in the original $u(x,t)$ variables

Th. sup. *For all* $x_0 \in \mathbb{R}$ *and* $t \in [\frac{3}{4}T(x_0), T(x_0))$*:*

$$(T(x_0)-t)^{\frac{2}{p-1}} \frac{\|u(t)\|_{L^2(B(x_0,\frac{T(x_0)-t}{2}))}}{(T(x_0)-t)^{1/2}}$$

$$+(T(x_0)-t)^{\frac{2}{p-1}+1}\left(\frac{\|u_t(t)\|_{L^2(B(x_0,\frac{T(x_0)-t}{2}))}}{(T(x_0)-t)^{1/2}}+\frac{\|\partial_x u(t)\|_{L^2(B(x_0,\frac{T(x_0)-t}{2}))}}{(T(x_0)-t)^{1/2}}\right) \leq K.$$

Rk. We have a lower bound of the same size when x_0 is non characteristic (see Part 4 on profiles near a non characteristic point).

Idea of the proof of the upper bound

- Selfsimilar transformation and existence of a Lyapunov functional
- Interpolation to gain regularity
- Gagliardo-Nirenberg estimates.

Part 4: Asymptotic behavior at a *non characteristic* point

Take $x_0 \in \mathbb{R}$ **non characteristic.** Using a covering argument for x near x_0, we obtain that $\|(w_{x_0}(s), \partial_s w_{x_0}(s))\|_{H^1\times L^2(-1,1)}$ is bounded.

Question: Does $w_{x_0}(y,s)$ have a limit or not, as $s \to \infty$ (that is as $t \to T(x_0)$).

Remark: In the context of Hamiltonian systems, **this question is delicate**, and there is no natural reason for such a convergence, since **the wave equation is time reversible.**

See for similar difficulty and approach, results for

- the **critical KdV** (Martel and Merle),
- **NLS** (Merle and Raphaël).

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Stationary solutions.

We look for solutions of

$$\frac{1}{\rho}\left(\rho(1-y^2)w'\right)' - \frac{2(p+1)}{(p-1)^2}w + |w|^{p-1}w = 0.$$

We work in $\mathcal{H}_0$, the (**stationary energy space**) defined by

$$\mathcal{H}_0 = \{r \in H^1_{loc}(-1,1) \mid \|r\|^2_{\mathcal{H}_0} \equiv \int_{-1}^{1} \left(r'^2(1-y^2) + r^2\right) \rho dy < +\infty\}.$$

Prop. Consider a stationary solution in $\mathcal{H}_0$. Then, either $w \equiv 0$ or there exist $d \in (-1,1)$ and $e = \pm 1$ such that $w(y) = e\kappa(d,y)$ where

$$\forall (d,y) \in (-1,1)^2, \;\; \kappa(d,y) = \kappa_0 \frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dy)^{\frac{2}{p-1}}} \text{ and } \kappa_0 = \left(\frac{2(p+1)}{(p-1)^2}\right)^{\frac{1}{p-1}}.$$

Remark: We have 3 connected components. $E(0) = 0 < E(\pm\kappa(d)) = E(\kappa_0)$.

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Blow-up profile near a non characteristic point

Th. *There exist* $C_0 > 0$ *and* $\mu_0 > 0$ *such that*
if x_0 *is* **non characteristic**, *then there exist* $d(x_0) \in (-1,1)$, $e(x_0) = \pm 1$ *and* $s^*(x_0) \geq -\log T(x_0)$ *such that:*
(i) For all $s \geq s^*(x_0)$,

$$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - e(x_0) \begin{pmatrix} \kappa(d(x_0),\cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq C_0 e^{-\mu_0(s-s^*)}$$

and $E(w_{x_0}(s) \to E(\kappa_0)$ *where the energy space*

$$\mathcal{H} = \left\{ q \in H^1_{loc} \times L^2_{loc}(-1,1) \mid \|q\|^2_{\mathcal{H}} \equiv \int_{-1}^{1} \left(q_1^2 + (q_1')^2 (1-y^2) + q_2^2 \right) \rho dy < +\infty \right\}.$$

(ii) $d(x_0) = T'(x_0)$.
Rk. We have exp. fast convergence (hence, C^{1,μ_0} regularity of $\mathcal{R}$, see Nouaili).
Rk. $\|w_{x_0}(y,s) - e(x_0)\kappa(d(x_0),y)\|_{L^\infty(-1,1)} \to 0$.
Rk. The parameter of the profile $d(x_0)$ has a geometrical interpretation $(T'(x_0))$.

Difficulties of the proof of convergence

- ▷ The set of non zero stationary solutions is made up of non isolated solutions (one parameter family):
  $\longrightarrow$ we need **a modulation technique**.
- ▷ The linearized operator around a non zero stationary solution **is non self-adjoint**:
  $\longrightarrow$ we need to use dispersive properties coming from the Lyapunov functional to control the negative part of the spectrum.

Part 5: Asymptotic behavior at a *characteristic* point

Th. If $x_0 \in \mathbb{R}$ is **characteristic**, then, there exist $k(x_0) \geq 2$, $e(x_0) = \pm 1$ and continuous $d_i(s) = -\tanh \zeta_i(s)$ for $i = 1, ..., k$ such that:
(i)

$$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - e(x_0) \sum_{i=1}^{k(x_0)} (-1)^i \begin{pmatrix} \kappa(d_i(s), \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \to 0 \text{ as } s \to \infty,$$

(ii) Introducing

$$\bar{w}_{x_0}(\xi, s) = (1 - y^2)^{\frac{1}{p-1}} w_{x_0}(y, s) \text{ with } y = \tanh \xi \text{ and } \zeta_i(x_0) = -\tanh^{-1} d_i(s),$$

we get

$$\|\bar{w}_{x_0}(\xi, s) - e(x_0) \sum_{i=1}^{k(x_0)} (-1)^i \cosh^{-\frac{2}{p-1}}(\xi - \zeta_i(s))\|_{H^1 \cap L^\infty(\mathbb{R})} \to 0 \text{ as } s \to \infty,$$

Part 5: Asymptotic behavior at a *characteristic* point (cont.)

(iii) For all $i = 1, ..., k(x_0)$ and s large enough,

$$\left(i - \frac{(k(x_0)+1)}{2}\right)\frac{(p-1)}{2}\log s - C_0 \le \zeta_i(s) \le \left(i - \frac{(k(x_0)+1)}{2}\right)\frac{(p-1)}{2}\log s + C_0.$$

(iv) $E(w_{x_0}(s)) \to k(x_0)E(\kappa_0)$ as $s \to \infty$.

Rk.

- As $s \to \infty$, w_{x_0} becomes like a **decoupled** sum of *equidistant* stationary solutions (“solitons”), with *alternate* signs.
- In the ξ variable, half of the solitons go to $-\infty$, and the other half to $+\infty$.
- The main difficulty in the proof is to prove that $k(x_0) \ge 2$ (the case $k(x_0) = 0$ is harder to eliminate).
- The $\zeta_i(s)$ satisfy a Toda system:

$$\frac{1}{c_1}\zeta_i'(s) = e^{-\frac{2}{p-1}(\zeta_i - \zeta_{i-1})} - e^{-\frac{2}{p-1}(\zeta_{i+1} - \zeta_i)} + R_i \text{ with } R_i = o\left(\sum_{j=1}^{k-1} e^{-\frac{2}{p-1}(\zeta_{j+1}-\zeta_j)}\right) \text{ as } s \to \infty.$$

The energy behavior

Defining

$$k(x_0) = 1 \text{ when } x_0 \in \mathcal{R},$$

we get the following:

Cor.
(i) *For all* $x_0 \in \mathbb{R}$ *and* $s \geq -\log T(x_0)$, *we have*

$$E(w_{x_0}(s)) \geq k(x_0)E(\kappa_0).$$

(ii) **(An energy criterion for non characteristic points)** *If for some* $x_0 \in \mathbb{R}$ *and* $s_0 \geq -\log T(x_0)$, *we have*

$$E(w_{x_0}(s_0)) < 2E(\kappa_0),$$

then $x_0 \in \mathcal{R}$.