The blow-up rate for the critical semilinear wave equation

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$$\begin{cases} u_{tt} = \Delta u + |u|^{p-1}u, \\ \\ u(0) = u_0 \text{ et } u_t(0) = u_1, \end{cases}$$

where

$u(t) : x \in \mathbb{R}^N \to u(x,t) \in \mathbb{R}$,

$u_0 \in \mathrm{H}^1_{\mathrm{loc,u}}(\mathbb{R}^N)$ and $u_1 \in \mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R}^N)$.

$$\|v\|_{\mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R}^N)} = \sup_{a \in \mathbb{R}^N} \left(\int_{|x-a|<1} |v(x)|^2 dx \right)^{1/2}.$$

$$N \geq 2 \text{ and } p = p_c \equiv 1 + \frac{4}{N-1}.$$

Earlier work: $1 < p < p_c$ (Amer. J. Math.).

Critical why?

1- When $p = p_c$, there is a conformal invariance in the equation: if $U(\xi, \tau)$ is defined by

$$U(\xi,\tau) = (|x|^2 - t^2)^{\frac{N-1}{2}} u(x,t), \quad \xi = \frac{x}{|x|^2 - t^2}, \quad \tau = \frac{t}{|x|^2 - t^2},$$

then U satisfies the same equation as u.

2- The subritical case $1 < p < p_c$ has been solved in an earlier work (Amer. J. Math.), where major difficulties to adapt to the ciritical case appeared.

The presentation is done for $1 < p \leq p_c$.

CAUCHY PROBLEM IN $H^1_{loc,u}(\mathbb{R}^N) \times L^2_{loc,u}(\mathbb{R}^N)$

Since $p < \frac{N+2}{N-2}$, it follows from :

- the solution of the Cauchy problem in $H^1 \times L^2(\mathbb{R}^N)$ (Lindblad and Sogge, Shatah and Struwe)

- the finite speed of propagation.

FINITE TIME BLOW-UP SOLUTIONS

Existence:

John, Caffarelli and Friedman, Alinhac, Kichenassamy and Litman

QUESTION (was open before this work)

Evaluate the norm of u, ∇u and u_t in $L^2_{loc,u}(\mathbb{R}^N)$ near the blow-up time T.

A HINT : THE ASSOCIATED ODE

$$u_{tt} = u^p, \quad u(T) = +\infty$$

gives

$$u(t) \sim \kappa(T - t)^{-\frac{2}{p-1}}$$

where $\kappa = \kappa(p)$ is explicitly given.

SELF-SIMILAR VARIABLES

$$w_a(y,s) = (T-t)^{\frac{2}{p-1}} u(x,t),\ y = \frac{x-a}{T-t},\ s = -\log(T-t).$$

Equivalent problem:

For all $y \in \mathbb{R}^N$ and $s \geq -\log T$:

$$\partial_s^2 w + \frac{p+3}{p-1}\partial_s w + 2y.\nabla\partial_s w + \sum_{i,j}(y_i y_j - \delta_{i,j})\partial^2_{y_i y_j} w$$
$$+\frac{2(p+1)}{p-1} y.\nabla w = |w|^{p-1} w - \frac{2(p+1)}{(p-1)^2} w,$$

Equivalent problem in divergence form:

For all $y \in \mathbb{R}^N$ and $s \geq -\log T$:

$$\partial_s^2 w - \frac{1}{\rho}\text{div}\,[\rho\nabla w - \rho(y.\nabla w)y] + \frac{2(p+1)}{(p-1)^2}w - |w|^{p-1}w$$

$$= -\frac{p+3}{p-1}\partial_s w - 2y.\nabla\partial_s w \qquad \text{where } \rho(y) = (1-|y|^2)^\alpha$$

If $p < p_c \equiv 1 + \dfrac{4}{N-1}$, then $\alpha \equiv \frac{2}{p-1} - \frac{N-1}{2} > 0$.

If $p = p_c$, then $\alpha = 0$ and $\rho \equiv 1$.

Th. $(p \le p_c)$ For any sol. u blowing up at time T, for all $s \ge -\log T + 1$ and $a \in \mathbb{R}^N$,

$$\|w_a(s)\|_{H^1(B)} + \|\partial_s w_a(s)\|_{L^2(B)} \le K$$

where $B = B(0,1)$, $K = K(N, p, \|u_0\|, T)$.

Rk. From scaling arguments and the solution of the Cauchy Problem in $H^1 \times L^2(\mathbb{R}^N)$, we get for all $s \ge -\log T + 1$,

$$\sup_{a \in \mathbb{R}^N} \|w_a(s)\|_{H^1(B)} + \|\partial_s w_a(s)\|_{L^2(B)} \ge \epsilon_0(N, p) > 0.$$

$\Longrightarrow$ **We are at the good scale**

IN THE ORIGINAL VARIABLES

Th. $(p \leq p_c)$ For any sol. u blowing-up at time T, for any $t \in [T(1-e^{-1}), T)$,

$$\|u\|_{L^2_{loc,u}(\mathbb{R}^N)} \leq K(T-t)^{-\frac{2}{p-1}}$$

$$\left(\|u_t\|_{L^2_{loc,u}(\mathbb{R}^N)} + \|\nabla u\|_{L^2_{loc,u}(\mathbb{R}^N)}\right) \leq K(T-t)^{-\frac{2}{p-1}-1}$$

where $K = K(N, p, \|u_0\|, T)$.

THE ARGUMENTS OF THE PROOF

- Existence of a Lyapunov functional for the equation on w and energy-type estimates.

- Interpolation to gain more regularity.

- Gagliardo-Nirenberg type estimates.

Equivalent problem in divergence form:

For all $y \in \mathbb{R}^N$ and $s \geq -\log T$:

$$\partial_s^2 w - \frac{1}{\rho}\text{div}\,[\rho\nabla w - \rho(y.\nabla w)y] + \frac{2(p+1)}{(p-1)^2}w - |w|^{p-1}w$$

$$= -\frac{p+3}{p-1}\partial_s w - 2y.\nabla\partial_s w \qquad \text{where } \rho(y) = (1-|y|^2)^\alpha$$

If $p < p_c \equiv 1 + \dfrac{4}{N-1}$, then $\alpha \equiv \frac{2}{p-1} - \frac{N-1}{2} > 0$.

If $p = p_c$, then $\alpha = 0$ and $\rho \equiv 1$.

A LYAPUNOV FUNCTIONAL Antonini-Merle

$$E(w) = \int_B \left(\frac{1}{2}(\partial_s w)^2 + \frac{(p+1)}{(p-1)^2}w^2 - \frac{1}{p+1}|w|^{p+1}\right)\rho dy$$

$$+\frac{1}{2}\int_B \left(|\nabla w|^2 - (y.\nabla w)^2\right)\rho dy$$

$$\rho(y) = (1-|y|^2)^\alpha \text{ with } \alpha = \frac{2}{p-1} - \frac{N-1}{2} \geq 0.$$

If $p < p_c$, then $\alpha > 0$.

If $p = p_c$, then $\alpha = 0$ and $\rho \equiv 1$.

If $p > p_c$, then E is not even defined.

Hence, p_c is critical.

Lemma 1 (Monotonicity) *For all s_1 and s_2:*

($p < p_c$, Antonini-Merle),

$$E(w(s_2))-E(w(s_1)) = -2\alpha\int_{s_1}^{s_2}\int_B(\partial_s w)^2(1-|y|^2)^{\alpha-1}dyds.$$

($p = p_c$: degeneracy),

$$E(w(s_2)) - E(w(s_1)) = -\int_{s_1}^{s_2}\int_{\partial B}(\partial_s w(\sigma,s))^2 d\sigma ds.$$

Rk. $2\alpha(1-|y|^2)^{\alpha-1} \to \delta_{\partial B}$ as $p \to p_c$.

Lemma 2 (Blow-up criterion (Antonini-Merle))

If a solution W satisfies $E(W(s_0)) < 0$ for some $s_0 \in \mathbb{R}$, then W blows up in finite time.

BOUNDS ON E

For all $s \geq -\log T$, $s_2 \geq s_1 \geq -\log T$

$$0 \leq E(w(s)) \leq E(w(-\log T)) \leq C_0$$

where $C_0 = C_0(\|u_0\|, T)$.

BOUNDS ON THE DISSIPATION OF E

For all $s \geq -\log T$, $s_2 \geq s_1 \geq -\log T$

($p < p_c$, supported in a cylinder),

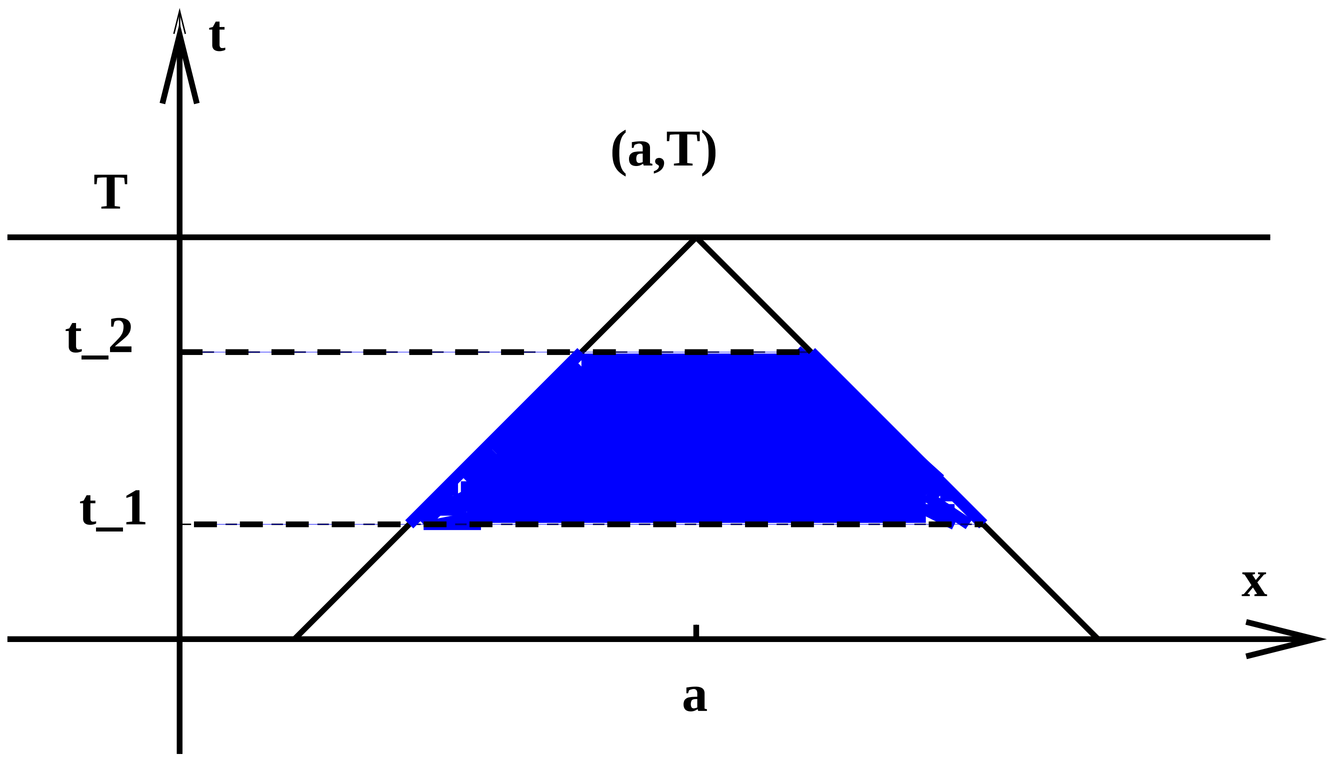
$$\int_{s_1}^{s_2} \int_B (\partial_s w)^2(y,s)(1-|y|^2)^{\alpha-1} dy ds \leq \frac{C_0}{2\alpha},$$

($p = p_c$, supported in the boundary of the cylinder),

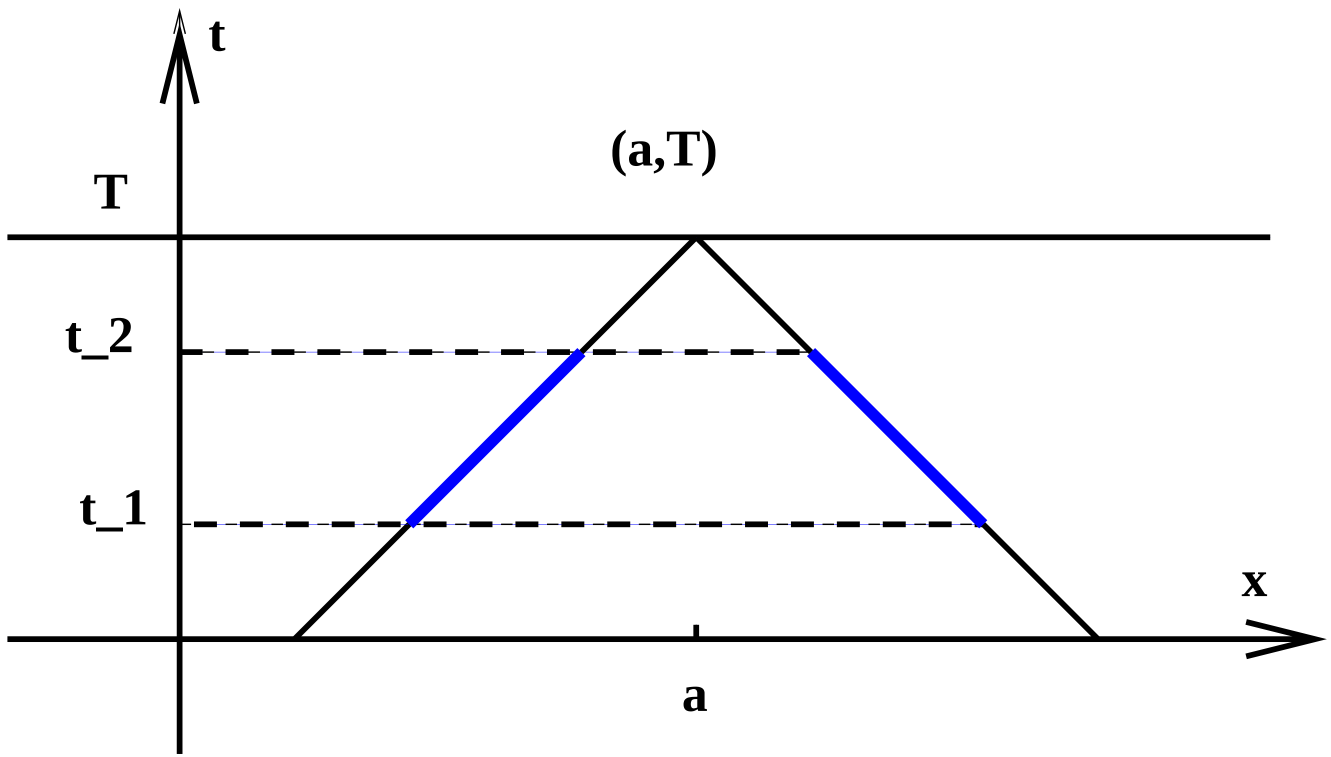
$$\int_{s_1}^{s_2} \int_{\partial B} (\partial_s w(\sigma,s))^2 d\sigma ds \leq C_0.$$

This *degeneracy* is a major difficulty in adapting the subcritical case to the critical.

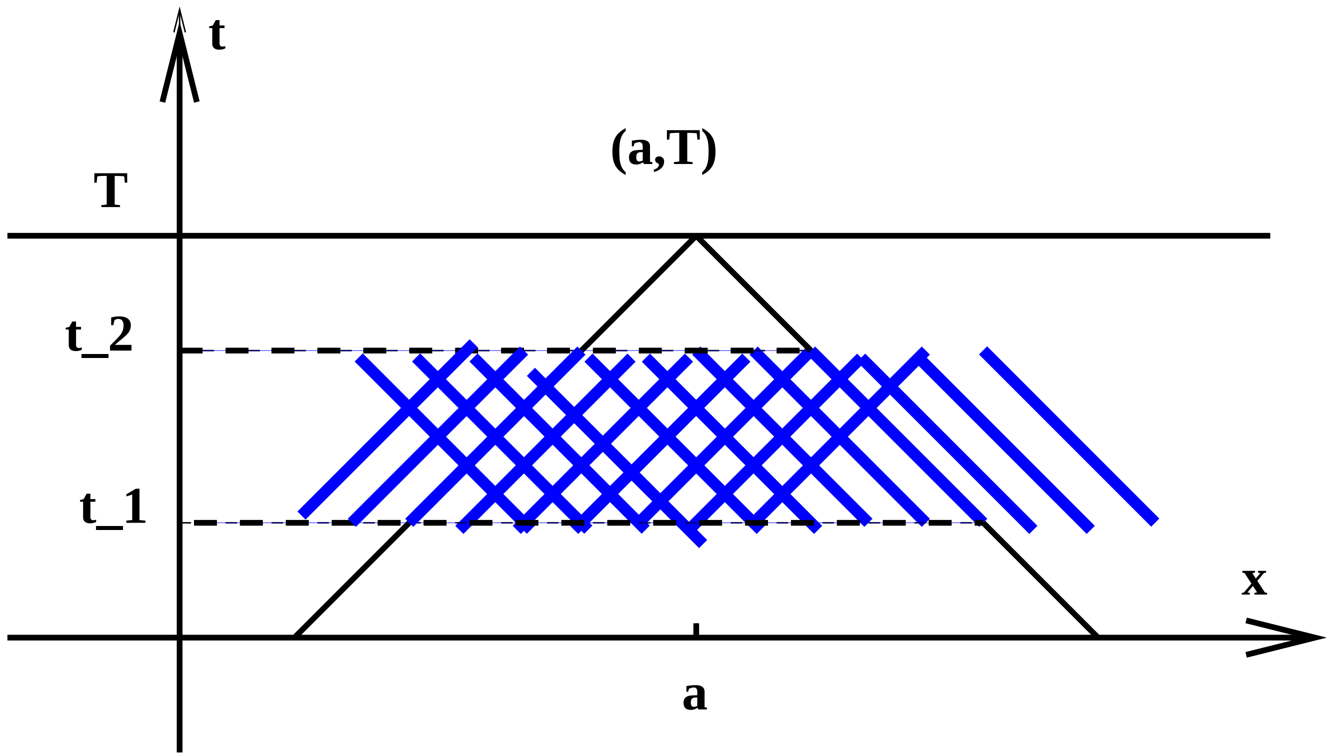
$(p < p_c)$ **SUPPORT OF THE DISSIPATION IN THE** (x,t) **VAR.** (remember $w = w_a$): In the interior of the light cone with vertex (a, T).



($p = p_c$) **SUPPORT OF THE DISSIPATION IN THE** (x, t) **VAR.** (remember $w = w_a$): On the EDGE of the light cone with vertex (a, T).



Since all is uniform in a, we move the a, and then integrate in a...



We recover an estimate in the interior of the light cone, between t_1 and t_2 (like the subcritical case).

More precisely $(p = p_c)$,

Proposition 1 *For all* $a \in \mathbb{R}^N$ *and* $s_2 \geq s_1 \geq -\log T$,

$$\int_{s_1}^{s_2} \int_B \partial_s w_a(y,s)^2 dyds \leq C_0.$$

Even better: **NON CONCENTRATION OF** $\int (\partial_s w)^2$

Proposition 2 *For any ball* $B(b, r_0) \subset B(0,3)$ *with* $r_0 < 1$,

$$\int_{s_1}^{s_2} \int_{B(b,r_0)} \partial_s w_a(y,s)^2 dyds \leq C_0 r_0.$$

With this adaptation, we concentrate (from now on) on the subcritical case. Recall Bounds on E and its dissipation:

For all $s \geq -\log T$, $s_2 \geq s_1 \geq -\log T$

$$0 \leq E(w(s)) \leq E(w(-\log T)) \; \leq \; C_0,$$

$$\int_{s_1}^{s_2} \int_B (\partial_s w)^2 (y,s)(1-|y|^2)^{\alpha-1} dy ds \; \leq \; \frac{C_0}{2\alpha},$$

where $C_0 = C_0(\|u_0\|, T)$.

GOAL Prove that w, ∇w and $\partial_s w$ are bounded in $L^2(B(0,1))$. Since they appear in E (with a weight):

$$E(w) = \int_B \left(\frac{1}{2}(\partial_s w)^2 + \frac{(p+1)}{(p-1)^2} w^2 - \frac{1}{p+1}|w|^{p+1}\right)\rho dy$$

$$+\frac{1}{2}\int_B \left(|\nabla w|^2 - (y.\nabla w)^2\right)\rho dy,$$

it is enough to bound $\int w^{p+1}\rho$.

Rk. We get rid of the weights through a covering argument.

Rk. Since we have an average (in time) estimate on $\partial_s w$,

$$\int_{s_1}^{s_2} \int_B (\partial_s w)^2(y,s)(1-|y|^2)^{\alpha-1} dyds \le C_0,$$

we will look for average (in time) estimates of the terms in the functional E.

CONTROL OF SPACE-TIME INTEGRALS.

First, we integrate $E(w)$ between s_1 and s_2 :

$$\int_{s_1}^{s_2} E(w(s))ds$$

$$= \int_{s_1}^{s_2} \int_B \left(\frac{1}{2}(\partial_s w)^2 + \frac{(p+1)}{(p-1)^2} w^2 - \frac{1}{p+1}|w|^{p+1} \right) \rho dy$$

$$+ \ \frac{1}{2} \int_{s_1}^{s_2} \int_B \left(|\nabla w|^2 - (y.\nabla w)^2 \right) \rho dy$$

Remark that $\iint w^{p+1}$ controls all the other terms in $\int E$.

2nd IDENTITY

We multiply the w-equation by $w\rho$, integrate on $B \times (s_1, s_2)$, IBP and use the definition of $\int E$ to write :

$$\frac{(p-1)}{2(p+1)} \int_{s_1}^{s_2} \int_B |w|^{p+1} \rho dy = \int_{s_1}^{s_2} E(w(s)) ds$$

$$+ \int_{s_1}^{s_2} \int_B \left(-(\partial_s w)^2 \rho - \partial_s w y.\nabla w \rho - \partial_s w w y.\nabla \rho \right) dy ds$$

$$+ \frac{1}{2} \left[\int_B \left(w \partial_s w + \left(\frac{p+3}{2(p-1)} - N \right) w^2 \right) \rho dy \right]_{s_1}^{s_2}.$$

Proposition 3 *($p \leq p_c$) For all $a \in \mathbb{R}^N$ and $s \geq -\log T + 1$,*

$$\int_s^{s+1} \int_B |w|^{p+1} \rho dy ds \leq C(C_0, N, p, T).$$

Rk. There is a weight...

PROOF ($p < p_c$) : We will control all terms on the RHS of the previous identity by

$$\frac{C}{\epsilon} + C\epsilon \int_{s_1}^{s_2} \int_B |w|^{p+1} \rho dy ds$$

and then take ϵ small.

Back to $p \le p_c$

Corollary 1 *For all* $a \in \mathbb{R}^N$ *and* $s \ge -\log T + 1$,

$$\int_s^{s+1} \int_{B_{1/2}} \left((\partial_s w_a)^2 + |\nabla w_a|^2 + |w_a|^{p+1} + |w_a|^2 \right) dy ds \le C$$

where $B_{1/2} \equiv B(0, 1/2)$, $C = C(N, p, \|u_0\|, T)$.

Rk.

If $p < p_c$, we first get estimates on $B \equiv B(0,1)$ with the weight ρ.

If $p = p_c$, we directly get estimates on B.

We are ready for the proof of the Thoerem that I recall here :

Th. $(p \le p_c)$ For any sol. u blowing up at time T, for all $s \ge -\log T + 1$ and $a \in \mathbb{R}^N$,

$$\int_B \left((\partial_s w_a)^2 + |\nabla w_a|^2 + |w_a|^2 \right) dy \le K$$

where $B = B(0,1)$, $K = K(N, p, \|u_0\|, T)$.

Step 1 : ($p \le p_c$) Control of $\int_B |w_a(y,s)|^2 dy$

We start from

$$\int_s^{s+1} \int_{B_{1/2}} \left((\partial_s w_a)^2 + |w_a|^2 \right) dy ds \le C.$$

We first get rid of $\int ds$, and then extend the integration in space to B.

Let $g(s) = \left(\int_{B_{1/2}} w_a(y,s)^2 dy \right)^{\frac{1}{2}}$. We write

$$\|g\|_{L^2(s,s+1)} = \left(\int_s^{s+1} \int_B w^2 dy ds\right)^{\frac{1}{2}} \leq C$$

$$\begin{aligned}\|\partial_s g\|^2_{L^2(s,s+1)} &= \int_s^{s+1} ds \frac{\left(\int_{B_{1/2}} w \partial_s w dy\right)^2}{4 \int_{B_{1/2}} w^2 dy} \\ &\leq \frac{1}{4} \int_{s_1}^{s_2} ds \int_{B_{1/2}} (\partial_s w)^2 dy \leq C_0.\end{aligned}$$

Hence, $g \in H^1(s, s+1)$. From the Sobolev injection in one dimension, $g \in L^\infty(s, s+1)$, i.e.

$$\int_{B_{1/2}} |w_a|^2 dy \leq C.$$

Now, we extend the integration to B.

We take $a = 0$. Since

$$\forall b \in \mathbb{R}^N, \int_{|y|<\frac{1}{2}} |w_b(y,s)|^2 dy \leq C$$

$$w_b(y,s) = w_0(y + be^s, s),$$

then $(z = y + be^s)$

$$\forall b \in \mathbb{R}^N, \int_{|z-be^s|<\frac{1}{2}} |w_0(z,s)|^2 dz \leq C$$

$+$ covering, this yields $\int_{|z|<1} |w_0(z,s)|^2 dz \leq C$.

Step 2 : ($p \le p_c$) Control of $w_a(s)$ in L^r_{loc}

Proposition 4

For all $s \ge -\log T + 1$ *and* $a \in \mathbb{R}^N$,

$$\int_B |w_a(y,s)|^{\frac{p+3}{2}} dy \le C$$

where $B = B(0,1)$.

Proof : Follows from $\int w^2$ and $\iint w^{p+1}$ by interpolation ($H^1 \subset L^\infty$ in one dimension).

Step 3 : ($p \le p_c$) Control of the gradient in $L^2_{loc,u}$

Lemma 3 *For all* $s \ge -\log T + 1$ *et* $a \in \mathbb{R}^N$,

$$\int_B |w_a|^{p+1} \le C \left(\int_B |w_a|^{\frac{p+3}{2}} dy \right)^{\gamma} \left(\int_B |\nabla w_a|^2 dy \right)^{\beta},$$

where $\gamma(p,N) > 0$ *and*

if $p < p_c$, $\beta = \beta(p,N) \in [0,1)$

if $p = p_c$, $\beta = 1$.

Proof : Gagliardo-Nirenberg.

Proposition 5 *For all* $s \geq -\log T + 1$ *and* $a \in \mathbb{R}^N$,

$$\int_B |\nabla w_a(y,s)|^2 dy \leq C.$$

Formal proof : If all the weights were equal to 1, we would have from the functional E :

$$\int_B |\nabla w_a|^2 dy \leq C + \int_B |w_a|^{p+1} dy$$

+ Gagliardo Nirenberg

$$\int_B |w_a|^{p+1} \leq C \left(\int_B |w_a|^{\frac{p+3}{2}} dy \right)^{\gamma} \left(\int_B |\nabla w_a|^2 dy \right)^{\beta}.$$

If $p < p_c$, then $\beta < 1$ and we get the conclusion.

If $p = p_c$, then $\beta = 1$. we can conclude only if $\int_B |w_a|^{\frac{p+3}{2}} dy$ is small.

This is possible if we replace B by $B(b, r_0)$ for small r_0. Indeed,
remember the **NON CONCENTRATION** result:

Proposition 6 *For any ball* $B(b, r_0) \subset B(0, 3)$ *with* $r_0 < 1$,

$$\int_{s_1}^{s_2} \int_{B(b,r_0)} \partial_s w_a(y, s)^2 dyds \le C_0 r_0.$$

Corollary 2

$$\int_{B(b,r_0)} |w_a(y, s)|^{\frac{p+3}{2}} dy \le C_0 \sqrt{r_0}.$$

We then cover the unit ball B by balls of radius r_0 with overlapping less than some $C(N)$.