A Liouville theorem for vector valued semilinear heat equations with no gradient structure and applications to blow-up

Tokyo University, December 17, 2009

Hatem ZAAG
LAGA, CNRS UMR 7539
Université Paris 13

joint work with Nejla Nouaili
Université Paris Dauphine

December 17, 2009

The equation

$$\begin{array}{rcl} \partial_t u & = & \Delta u + (1+i\delta)|u|^{p-1}u \\ u(0,x) & = & u_0(x) \in L^\infty(\mathbb{R}^N), \end{array} \qquad (\mathrm{Equ}_\delta)$$

where $u(t) : \mathbb{R}^N \to \mathbb{C}$, $p > 1$ and $\delta \in \mathbb{R}$.

We say that $u(t)$ blows up in finite time T, if $u(t)$ exists for all $t \in [0, T)$ and $\lim_{t\to T} \|u(t)\|_{L^\infty} = +\infty$.

The point a is a blow-up point if and only if there exists $(a_n, t_n) \to (a, T)$ as $n \to +\infty$ such that $|u(a_n, t_n)| \to +\infty$.

Why this equation?

- A submodel of the Ginzburg-Landau equation

$$\partial_t u = (1 + i\beta)\Delta u + (1 + i\delta)|u|^{p-1}u - \gamma u \qquad (1)$$

where β, δ and γ are real (See Masmoudi and Zaag JFA 2008 where a blow-up solution is contructed for equation (1)).

- A lab model for the blow-up problem in parabolic equations with no gradient structure.

Outline of the talk

1. Case $\delta = 0$, $(N-2)p < N+2$
2. Case $\delta \neq 0$
3. Proof of the Liouville theorem case $\delta = 0$
4. Proof of the Liouville theorem case $\delta \neq 0$

Outline of the talk

1. Case $\delta = 0$, $(N-2)p < N+2$
2. Case $\delta \neq 0$
3. Proof of the Liouville theorem case $\delta = 0$
4. Proof of the Liouville theorem case $\delta \neq 0$

Fundamental feature:

Existence of a Lyapunov functional:

$$\frac{d}{dt}E_0(u) = -\int_{\mathbb{R}^N} |\partial_t u|^2 dx$$

where

$$E_0(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1} dx.$$

Remark: From Ball 77, we have $E(u_0) < 0 \Rightarrow u(t)$ blows up in finite time.

Extensive bibliography $\delta = 0$

• Existence of Blow-up solutions? yes, energy method by Levine 1974 and Ball 1977.

• Blow-up rate? Giga-Kohn 1987, Giga, Matsui and Sasayama 2004.

If u blows up at time T, then

$$\forall t \in [0, T),\ \|u(t)\|_{L^\infty} \leq Cv(t),$$

with $v(t) = \kappa(T-t)^{-\frac{1}{p-1}}$, $\kappa = (p-1)^{-\frac{1}{p-1}}$ *and*

$$\begin{cases} v'(t) = & v(t)^p, \\ v(T) = & +\infty. \end{cases}$$

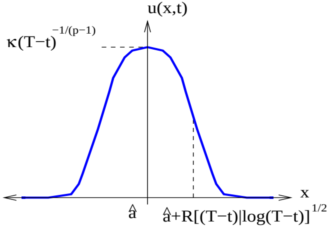
Definition: We say that u is of "type **I**".

- Asymptotic Behavior (Blow-up profile $\delta = 0$)1990' Herrero, Velázquez , Bricmont, Kupiainen, Filippas, Kohn, Liu.

Given a blow-up point a, the (supposed to be generic) profile is the following:

$$u(x,t) \sim (T-t)^{-\frac{1}{p-1}} f_0\left(\left|\frac{x-a}{\sqrt{(T-t)|\log(T-t)|}}\right|\right),$$

$$\text{where } f_0(z) = (p-1+b(p)z)^{-\frac{1}{p-1}}.$$



Remark: If $N = 1$, we know it is generic (Herrero, Velázquez). If $N \geq 2$, open problem.

- Stability of the blow-up profile ($\delta = 0$)

Theorem (Fermanian, Merle, Z. 2000) Consider initial data $\hat{u}_0$, the solution $\hat{u}(x,t)$ of (Equ_0) with blow-up time $\hat{T}$, blow-up point $\hat{a}$ and profile f_0 centered at $(\hat{T}, \hat{a})$.

Then, $\exists \mathcal{V}$ neighborhood of $\hat{u}_0$ s.t. $\forall u_0 \in \mathcal{V}$, $u(x,t)$ the solution of (Equ_0) blows up at time T, at a point a, with the profile f_0 centered at (T, a).

Moreover, $(T, a) \to (\hat{T}, \hat{a})$ as $u_0 \to \hat{u}_0$.

A Liouville Theorem for equation (Equ_0)

Theorem

Assume that u is a solution of (Equ_0) s.t.

$$\forall (x,t) \in \mathbb{R}^N \times (-\infty, T), |u(x,t)| \leq M(T-t)^{-\frac{1}{p-1}}.$$

Then,

$$u \equiv 0 \text{ or } \forall (x,t) \in \mathbb{R}^N \times (-\infty, T),\ u(x,t) = \pm\kappa(T_0 - t)^{-\frac{1}{p-1}},$$

for some $T_0 \geq T$.

Consequences of the Liouville Theorem for equation (Equ_0)

Proposition Consider u a solution of (Equ_0), which blows up at time T.
Then, (i) (L^∞ estimates for u and derivatives)

$$\|u(t)\|_{L^\infty}(T-t)^{\frac{1}{p-1}} \to \kappa \text{ and } \|\nabla^k u(t)\|_{L^\infty}(T-t)^{\frac{1}{p-1}+\frac{k}{2}} \to 0$$

as $t \to T$ for $k = 1$, 2 or 3.

(ii) (Uniform ODE localization) For all $\varepsilon > 0$, there is $C(\varepsilon)$ such that $\forall x \in \mathbb{R}^N$, $\forall t \in [0, T)$,

$$\left|\frac{\partial u}{\partial t}(x,t) - |u|^{p-1}u(x,t)\right| \leq \varepsilon |u(x,t)|^p + C.$$

Other consequences: Regularity of the set of all blow-up points, see Z. 2006.

Outline of the talk

1. Case $\delta = 0$, $(N-2)p < N+2$
2. Case $\delta \neq 0$
3. Proof of the Liouville theorem case $\delta = 0$
4. Proof of the Liouville theorem case $\delta \neq 0$

- What changes? No Lyapunov functional.

- What is known? Existence of a blow-up solution stable/ initial data (constructive method Z. 1998).

- What is unknown? The blow-up rate, the blow-up profile, etc......

- Our approach: Try to prove a Liouville Theorem.

A Liouville theorem for equation (Equ_δ), $\delta \neq 0$

Theorem (Nouaili,Z.)
If $0 < |\delta| \leq \delta_0$ *and*

$$\forall (x,t) \in \mathbb{R}^N \times (-\infty, T) \ |u(x,t)| \leq M(\delta)(T-t)^{-\frac{1}{p-1}}$$

for some $\delta_0 > 0$ *and* $M(\delta) > 0$, *then,*

$$u \equiv 0 \text{ or } \forall (x,t) \in \mathbb{R}^N \times (-\infty, T),\ u(x,t) = \kappa e^{i\theta_0}(T_0 - t)^{-\frac{1+i\delta}{p-1}},$$

for some $T_0 \geq T$ *and* $\theta_0 \in \mathbb{R}$.

Rk. $M(\delta) \to +\infty$ as $\delta \to 0$.

Uniform blow-up estimates

Proposition Consider $0 < |\delta| \le \delta_0$ and u a solution of (Equ_δ) that blows up at time T and satisfies

$$\forall t \in [0,T),\ \|u(t)\|_{L^\infty} \le M(\delta)(T-t)^{-\frac{1}{p-1}}.\ (\text{type } \mathbf{I})$$

Then, (i) (L^∞ estimates for derivatives)

$$\|u(t)\|_{L^\infty}(T-t)^{\frac{1}{p-1}} \to \kappa \text{ and } \|\nabla^k u(t)\|_{L^\infty}(T-t)^{\frac{1}{p-1}+\frac{k}{2}} \to 0$$

as $t \to T$ for $k = 1$, 2 or 3.

(ii) (Uniform ODE localization) For all $\varepsilon > 0$, there is $C(\varepsilon)$ such that $\forall x \in \mathbb{R}^N$, $\forall t \in [0,T)$,

$$\left|\frac{\partial u}{\partial t}(x,t) - (1+i\delta)|u|^{p-1}u(x,t)\right| \le \varepsilon|u(x,t)|^p + C.$$

Proof It follows from the Liouville theorem.

Outline of the talk

1. Case $\delta = 0$, $(N-2)p < N+2$
2. Case $\delta \neq 0$
3. Proof of the Liouville theorem case $\delta = 0$
   - Part 1: Limits of w as $s \to \pm\infty$
   - Part 2: Trivial cases
   - Part 3: Case when $w_{-\infty} \to \kappa$ as $s \to -\infty$
     - Step 1: Linearization of w near κ as $s \to -\infty$
     - Step 2: The relevant case, $\lambda = 1$
     - Step 3: The irrelevant cases; ii) $\lambda = \frac{1}{2}$ or iii) $\lambda = 0$
4. Proof of the Liouville theorem case $\delta \neq 0$

Let us recall the Liouville Theorem for:

$$\partial_t u = \Delta u + |u|^{p-1} u.$$

Theorem

Assume that u is a solution of (Equ_0) s.t.

$$\forall (x,t) \in \mathbb{R}^N \times (-\infty, T),\ |u(x,t)| \leq M(T-t)^{-\frac{1}{p-1}}.$$

Then,

$$u \equiv 0 \text{ or } \forall (x,t) \in \mathbb{R}^N \times (-\infty, T),\ u(x,t) = \pm\kappa (T_0 - t)^{-\frac{1}{p-1}},$$

for some $T_0 \geq T$.

Statement in selfsimilar variables:

$$w_a(y,s) = (T-t)^{\frac{1}{p-1}} u(x,t),\ y = \frac{x-a}{\sqrt{T-t}},\ s = -\log(T-t),$$

for all $(x,t) \in \mathbb{R}^N \times (-\infty, T)$, the function $w = w_a$ satisfies for all $(y,s) \in \mathbb{R}^N \times \mathbb{R}$:

$$w_s = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{1}{(p-1)} w + |w|^{p-1} w. \qquad (\mathrm{Eq}w_0)$$

Theorem (A Liouville theorem for equation (Eqw_0)) If

$$\|w(y,s)\|_{L^\infty(\mathbb{R}^N\times\mathbb{R},\mathbb{R})} \leq M$$

and w is a solution of (Eqw_0), then

$$w \equiv 0 \text{ or } w \equiv \pm\kappa \text{ or } w = \pm\varphi_0(s-s_0),$$

for some $s_0 \in \mathbb{R}$, and

$$\varphi_0(s) = \kappa(1+e^s)^{-\frac{1}{(p-1)}} \text{ and } \kappa = (p-1)^{-\frac{1}{p-1}}.$$

Outline of the talk

1. Case $\delta = 0$, $(N-2)p < N+2$
2. Case $\delta \neq 0$
3. Proof of the Liouville theorem case $\delta = 0$
   - Part 1: Limits of w as $s \to \pm\infty$
   - Part 2: Trivial cases
   - Part 3: Case when $w_{-\infty} \to \kappa$ as $s \to -\infty$
     - Step 1: Linearization of w near κ as $s \to -\infty$
     - Step 2: The relevant case, $\lambda = 1$
     - Step 3: The irrelevant cases; ii) $\lambda = \frac{1}{2}$ or iii) $\lambda = 0$
4. Proof of the Liouville theorem case $\delta \neq 0$
   - Part 1: Limits of w as $s \to -\infty$
   - Part 2: Case where $w \to 0$ as $s \to -\infty$
   - Part 3: Case where $\inf_{\theta \in \mathbb{R}} \|w(.,s) - \kappa e^{i\theta}\|_{L^2_\rho} \to 0$ as $s \to -\infty$
     - Step 1: Modulation
     - Step 2: Behavior as $s \to -\infty$
     - Step 3: The relevant case $\lambda = 1$
     - Step 4: The irrelevant cases, ii) $\lambda = \frac{1}{2}$ or iii) $\lambda = 0$

A Lyapunov functional in the w variable

$$\mathcal{E}(w) = \int_{\mathbb{R}^N} \left(\frac{1}{2}|\nabla w|^2 + \frac{|w|^2}{2(p-1)} - \frac{|w|^{p+1}}{p+1} \right) \rho(y)dy \text{ with}$$

$$\rho(y) = \frac{e^{-\frac{|y|^2}{4}}}{(4\pi)^{N/2}}.$$

$$\frac{d}{ds}\mathcal{E}(w) = -\int (\partial_s w)^2 \rho(y)dy$$

Consequence: $w_{\pm\infty} = \lim_{s\to\pm\infty} w(y,s)$ exists and is a stationary solution of (Eqw_0). From Giga and Kohn we obtain $w_{\pm\infty} = 0$, $w_{\pm\infty} = \kappa$ or $w_{\pm\infty} = -\kappa$.

Outline of the talk

1. Case $\delta = 0$, $(N-2)p < N+2$
2. Case $\delta \neq 0$
3. Proof of the Liouville theorem case $\delta = 0$
   - Part 1: Limits of w as $s \to \pm\infty$
   - Part 2: Trivial cases
   - Part 3: Case when $w_{-\infty} \to \kappa$ as $s \to -\infty$
     - Step 1: Linearization of w near κ as $s \to -\infty$
     - Step 2: The relevant case, $\lambda = 1$
     - Step 3: The irrelevant cases; ii) $\lambda = \frac{1}{2}$ or iii) $\lambda = 0$
4. Proof of the Liouville theorem case $\delta \neq 0$
   - Part 1: Limits of w as $s \to -\infty$
   - Part 2: Case where $w \to 0$ as $s \to -\infty$
   - Part 3: Case where $\inf_{\theta \in \mathbb{R}} \|w(.,s) - \kappa e^{i\theta}\|_{L^2_\rho} \to 0$ as $s \to -\infty$
     - Step 1: Modulation
     - Step 2: Behavior as $s \to -\infty$
     - Step 3: The relevant case $\lambda = 1$
     - Step 4: The irrelevant cases, ii) $\lambda = \frac{1}{2}$ or iii) $\lambda = 0$

$$\text{Since } \mathcal{E}(w_{-\infty}) - \mathcal{E}(w_{+\infty}) = \int_{-\infty}^{+\infty} ds \int_{\mathbb{R}} \left|\frac{\partial w}{\partial s}(y,s)\right|^2 \rho dy \geq 0$$

$$\text{and } \mathcal{E}(\kappa) = \mathcal{E}(-\kappa) > 0 = \mathcal{E}(0),$$

we have 2 cases:

- (Trivial)
  $\mathcal{E}(w_{-\infty}) - \mathcal{E}(w_{+\infty}) = 0 \Rightarrow \partial_s w \equiv 0 \Rightarrow w \equiv 0$ or $w \equiv \pm\kappa$.
- (Non trivial)
  $\mathcal{E}(w_{-\infty}) - \mathcal{E}(w_{+\infty}) > 0 \Rightarrow (w_{-\infty}, w_{+\infty}) \equiv (\pm\kappa, 0)$.

Outline of the talk

1. Case $\delta = 0$, $(N-2)p < N+2$
2. Case $\delta \neq 0$
3. Proof of the Liouville theorem case $\delta = 0$
   - Part 1: Limits of w as $s \to \pm\infty$
   - Part 2: Trivial cases
   - Part 3: Case when $w_{-\infty} \to \kappa$ as $s \to -\infty$
     - Step 1: Linearization of w near κ as $s \to -\infty$
     - Step 2: The relevant case, $\lambda = 1$
     - Step 3: The irrelevant cases; ii) $\lambda = \frac{1}{2}$ or iii) $\lambda = 0$
4. Proof of the Liouville theorem case $\delta \neq 0$
   - Part 1: Limits of w as $s \to -\infty$
   - Part 2: Case where $w \to 0$ as $s \to -\infty$
   - Part 3: Case where $\inf_{\theta \in \mathbb{R}} \|w(.,s) - \kappa e^{i\theta}\|_{L^2_\rho} \to 0$ as $s \to -\infty$
     - Step 1: Modulation
     - Step 2: Behavior as $s \to -\infty$
     - Step 3: The relevant case $\lambda = 1$
     - Step 4: The irrelevant cases, ii) $\lambda = \frac{1}{2}$ or iii) $\lambda = 0$

Step 1: Linearization of w near κ as $s \longrightarrow -\infty$

We consider $v(y,s) = w(y,s) - \kappa$.

$$\partial_s v = \mathcal{L}v + f(v), \text{ with } \mathcal{L}v = \Delta v - \frac{1}{2} y \cdot \nabla v + v, \ |f(v)| \leq C|v|^2.$$

$\mathcal{L}$ is self adjoint, $spec(\mathcal{L}) = \{1 - \frac{m}{2} | m \in \mathbb{R}\}$.
The eigenvectors are Hermite polynomials.
As $s \to -\infty$, one of the following cases occurs:

- i) $\lambda = 1$, $w(y,s) = \kappa + C_0 e^s + o(e^s)$, $C_0 \in \mathbb{R}$.
- ii) $\lambda = \frac{1}{2}$, $w(y,s) = \kappa + C_1 e^{s/2} y + o(e^{s/2})$, $C_1 \in \mathbb{R}^*$.
- iii) $\lambda = 0$, $w(y,s) = \kappa - \frac{\kappa}{2ps}(\frac{1}{2}y^2 - 1) + o(\frac{1}{s})$.

Convergence is in L^2_ρ and uniformly on compact sets.

Step 2: The relevant case, $\lambda = 1$

$$\text{If } \varphi^*(s) = \begin{cases} = & \kappa \text{ if } C_0 = 0, \\ = & \varphi_0(s - s_0) = \kappa(1 + e^{s-s_0})^{-\frac{1}{p-1}}, \text{ if } C_0 < 0, \\ = & \tilde{\varphi}(s - s_0) = \kappa(1 - e^{s-s_0})^{-\frac{1}{p-1}}, \text{ if } C_0 > 0, \end{cases}$$

with $s_0 = -\log\left(\frac{(p-1)}{\kappa}|C_0|\right)$, then φ^* is a solution of (Eqw_0) with the same expansion of w as $s \to -\infty$.
If $V = w - \varphi^*$, then $\|V(y,s)\|_{L^2_\rho} = O(e^{3/2s})$.
Since $\frac{3}{2} > 1 = \max\{\lambda \in spec(\mathcal{L})\}$, then $V \equiv 0$.
Because $w_{+\infty} = 0$, we get $\varphi^* = \varphi_0(s - s_0)$.

$$w(y,s) = \varphi(s - s_0) = \kappa(1 + e^{s-s_0})^{-\frac{1}{p-1}}, \text{ for some } s_0 \in \mathbb{R}.$$

Step 3: The irrelevant cases; ii) $\lambda = \frac{1}{2}$ or iii) $\lambda = 0$

Merle-Zaag (Blow-up criterion). Let W a solution of (Eqw_0), such that

$$\left(\int |W(y,s_0)|^2 \rho(y) dy\right)^{\frac{p+1}{2}} > 2\frac{p+1}{p-1}\mathcal{E}(W(.,s_0)), \qquad (Is_0)$$

for some $s_0 \in \mathbb{R}$. Then W blows-up at some time $S > s_0$.

In case ii) and iii) one can find a_0 and s_0 such that (Is_0) is true with $W(y,s_0) = w_{a_0}(y,s_0) = w(y + a_0 e^{s_0/2}, s_0)$.

Then, there exists $S > s_0$, such that w_{a_0} blows up at S, contradiction because w $(w(y,s) = w_{a_0}(y - a_0 e^{s/2}, s))$ is bounded.

Outline of the talk

1. Case $\delta = 0$, $(N-2)p < N+2$
2. Case $\delta \neq 0$
3. Proof of the Liouville theorem case $\delta = 0$
4. Proof of the Liouville theorem case $\delta \neq 0$
   - Part 1: Limits of w as $s \to -\infty$
   - Part 2: Case where $w \to 0$ as $s \to -\infty$
   - Part 3: Case where $\inf_{\theta\in\mathbb{R}} \|w(.,s) - \kappa e^{i\theta}\|_{L^2_\rho} \to 0$ as $s \to -\infty$
     - Step 1: Modulation
     - Step 2: Behavior as $s \to -\infty$
     - Step 3: The relevant case $\lambda = 1$
     - Step 4: The irrelevant cases, ii) $\lambda = \frac{1}{2}$ or iii) $\lambda = 0$

What changes?

No Lyapunov functional:

- No Lyapunov functional to get the limits as $s \to \pm\infty$.
- No blow-up criterion to rule out the irrelevant cases.

Let us recall the Liouville Theorem for:

$$\partial_t u = \Delta u + (1 + i\delta)|u|^{p-1} u.$$

Theorem (Nouaili, Z.)
If $0 < |\delta| \leq \delta_0$ and u is a solution of (Equ_δ) satisfying

$$\forall (x,t) \in \mathbb{R}^N \times (-\infty, T) \ |u(x,t)| \leq M(\delta)(T-t)^{-\frac{1}{p-1}}$$

for some $\delta_0 > 0$ and $M(\delta) > 0$, then,

$$u \equiv 0 \text{ or } \forall (x,t) \in \mathbb{R}^N \times (-\infty, T),\ u(x,t) = \kappa e^{i\theta_0}(T_0 - t)^{-\frac{1+i\delta}{p-1}},$$

for some $T_0 \geq T$ and $\theta_0 \in \mathbb{R}$.

Statement in selfsimilar variables:

$$w_a(y,s) = (T-t)^{\frac{1+i\delta}{p-1}} u(x,t),\ y = \frac{x-a}{\sqrt{T-t}},\ s = -\log(T-t),$$

for all $(x,t) \in \mathbb{R}^N \times (-\infty, T)$, the function $w = w_a$ satisfies for all $(y,s) \in \mathbb{R}^N \times \mathbb{R}$:

$$w_s = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{1+i\delta}{(p-1)} w + (1+i\delta)|w|^{p-1} w. \quad (\mathrm{Eq}w_\delta)$$

Theorem(A Liouville theorem for equation (Eqw_δ)) If $0 < |\delta| \leq \delta_0$ and w is a solution of (Eqw_δ) s.t.

$$\|w(y,s)\|_{L^\infty(\mathbb{R}^N \times \mathbb{R}, \mathbb{C})} \leq M(\delta),$$

then,

$$w \equiv 0 \text{ or } w \equiv \kappa e^{i\theta_0} \text{ or } w = \varphi_\delta(s - s_0) e^{i\theta_0},$$

for some $\theta_0 \in \mathbb{R}$ and $s_0 \in \mathbb{R}$, where

$$\varphi_\delta(s) = \kappa(1 + e^s)^{-\frac{(1+i\delta)}{(p-1)}} \text{ and } \kappa = (p-1)^{-\frac{1}{p-1}}.$$

Outline of the talk

1. Case $\delta = 0$, $(N-2)p < N+2$
2. Case $\delta \neq 0$
3. Proof of the Liouville theorem case $\delta = 0$
   - Part 1: Limits of w as $s \to \pm\infty$
   - Part 2: Trivial cases
   - Part 3: Case when $w_{-\infty} \to \kappa$ as $s \to -\infty$
     - Step 1: Linearization of w near κ as $s \to -\infty$
     - Step 2: The relevant case, $\lambda = 1$
     - Step 3: The irrelevant cases; ii) $\lambda = \frac{1}{2}$ or iii) $\lambda = 0$
4. Proof of the Liouville theorem case $\delta \neq 0$
   - Part 1: Limits of w as $s \to -\infty$
   - Part 2: Case where $w \to 0$ as $s \to -\infty$
   - Part 3: Case where $\inf_{\theta\in\mathbb{R}} \|w(.,s) - \kappa e^{i\theta}\|_{L^2_\rho} \to 0$ as $s \to -\infty$
     - Step 1: Modulation
     - Step 2: Behavior as $s \to -\infty$
     - Step 3: The relevant case $\lambda = 1$
     - Step 4: The irrelevant cases, ii) $\lambda = \frac{1}{2}$ or iii) $\lambda = 0$

(Stationary solution) Consider $w \in L^\infty(\mathbb{R}^N)$ a stationary solution of (Eqw_δ). Then, $w \equiv 0$ or there exists $\theta_0 \in \mathbb{R}$ such that $w \equiv \kappa e^{i\theta_0}$.

Remark: The proof is trivial and much easier than the case $\delta = 0$.

To get the limits, we have no Lyapunov functional.
Fortunately, a perturbation method used by Andreucci, Herrero and Velázquez, works here and yields the following:

Proposition If $0 < |\delta| \leq \delta_0$ and w is a solution of (Eqw_δ) satisfying for all $(y,s) \in \mathbb{R}\times\mathbb{R}$, $|w(y,s)| \leq M(\delta)$ for some δ_0 and $M(\delta)$, then, as $s \to -\infty$

$$\begin{array}{ll} \text{either} & (i)\ \|w(.,s)\|_{L^2_\rho} \to 0 \\ \text{or} & (ii)\ \inf_{\theta\in\mathbb{R}} \|w(.,s) - \kappa e^{i\theta}\|_{L^2_\rho} \to 0. \end{array}$$

Outline of the talk

1. Case $\delta = 0$, $(N-2)p < N+2$
2. Case $\delta \neq 0$
3. Proof of the Liouville theorem case $\delta = 0$
   - Part 1: Limits of w as $s \to \pm\infty$
   - Part 2: Trivial cases
   - Part 3: Case when $w_{-\infty} \to \kappa$ as $s \to -\infty$
     - Step 1: Linearization of w near κ as $s \to -\infty$
     - Step 2: The relevant case, $\lambda = 1$
     - Step 3: The irrelevant cases; ii) $\lambda = \frac{1}{2}$ or iii) $\lambda = 0$
4. Proof of the Liouville theorem case $\delta \neq 0$
   - Part 1: Limits of w as $s \to -\infty$
   - Part 2: Case where $w \to 0$ as $s \to -\infty$
   - Part 3: Case where $\inf_{\theta \in \mathbb{R}} \|w(.,s) - \kappa e^{i\theta}\|_{L^2_\rho} \to 0$ as $s \to -\infty$
     - Step 1: Modulation
     - Step 2: Behavior as $s \to -\infty$
     - Step 3: The relevant case $\lambda = 1$
     - Step 4: The irrelevant cases, ii) $\lambda = \frac{1}{2}$ or iii) $\lambda = 0$

If $h(s) \equiv \int_{\mathbb{R}} |w(y,s)|^2 \rho(y)dy$, then

$$h'(s) \leq -\frac{2}{p-1} h(s) + 2\int_{\mathbb{R}} |w(y,s)|^{p+1} \rho(y)dy.$$

Using the regularizing effect of equation $(\mathrm{Eq}w_\delta)$, we derive the following delay estimate, for some positive $s*$ and C

$$\forall s \in \mathbb{R},\ h'(s) \leq -\frac{2}{p-1} h(s) + C(M)h(s-s*)^{\frac{p+1}{2}}.$$

Using $h(s) \to 0$ as $s \to -\infty$ and delay ODE techniques, we have for some $\varepsilon > 0$ small enough,

$$\forall \sigma \in \mathbb{R},\ \forall s \geq \sigma + s*,\ h(s) \leq \varepsilon e^{-\frac{2(s-\sigma)}{p-1}},$$

Fixing s and letting $\sigma \to -\infty$, we get $w \equiv 0$.

Outline of the talk

1. Case $\delta = 0$, $(N-2)p < N+2$
2. Case $\delta \neq 0$
3. Proof of the Liouville theorem case $\delta = 0$
   - Part 1: Limits of w as $s \to \pm\infty$
   - Part 2: Trivial cases
   - Part 3: Case when $w_{-\infty} \to \kappa$ as $s \to -\infty$
     - Step 1: Linearization of w near κ as $s \to -\infty$
     - Step 2: The relevant case, $\lambda = 1$
     - Step 3: The irrelevant cases; ii) $\lambda = \frac{1}{2}$ or iii) $\lambda = 0$
4. Proof of the Liouville theorem case $\delta \neq 0$
   - Part 1: Limits of w as $s \to -\infty$
   - Part 2: Case where $w \to 0$ as $s \to -\infty$
   - Part 3: Case where $\inf_{\theta\in\mathbb{R}} \|w(.,s) - \kappa e^{i\theta}\|_{L^2_\rho} \to 0$ as $s \to -\infty$
     - Step 1: Modulation
     - Step 2: Behavior as $s \to -\infty$
     - Step 3: The relevant case $\lambda = 1$
     - Step 4: The irrelevant cases, ii) $\lambda = \frac{1}{2}$ or iii) $\lambda = 0$

Step 1: Modulation

We introduce $\theta(s)$ and v such that

$$w(y,s) = e^{i\theta(s)}(v(y,s)+\kappa),\ \forall s \le s_1,\ \int (\,\mathrm{Im}\,(v) - \delta\,\mathrm{Re}\,(v))\rho = 0.(*)$$

$$\partial_s v = \tilde{\mathcal{L}} v - i\theta_s(v+\kappa) + G, \text{ where}$$

$\tilde{\mathcal{L}} v = \Delta v - \frac{1}{2} y \nabla v + (1+i\delta)v_1$, $|G(v)| \le C|v|^2$.
$spec(\tilde{\mathcal{L}}) = \{1 - \frac{m}{2} | m \in \mathbb{R}\}$ its eigenvectors are given by $\{(1+i\delta)h_m, ih_m | n \in \mathbb{N}\}$ and h_m are Hermite polynomials.

The choice of $\theta(s)$ $(*)$ kills one neutral mode.

Step 2: Behavior as $s \to -\infty$

- $\lambda = 1$, with eigenfunction $(1+i\delta)h_0(y) = (1+i\delta)$.
- $\lambda = 1/2$, with eigenfunction $(1+i\delta)h_1(y) = (1+i\delta)y$.
- $\lambda = 0$, with two eigenfunctions $(1+i\delta)h_2(y) = (1+i\delta)(y^2-2)$

and $ih_0(y) = i$ (killed by the choice of $\theta(s)$ $(*)$).
We have one of the following cases as $s \to -\infty$:

(i) $w(y,s) = \{\kappa + (1+i\delta)C_0 e^s\}e^{i\theta_0} + o(e^{\frac{3}{2}s})$, $C_0 \in \mathbb{R}$
(ii) $w(y,s) = \{\kappa + (1+i\delta)C_1 e^{s/2} y\}e^{i\theta_0} + o(e^{s/2})$, $C_1 \in \mathbb{R}^*$,
(iii) $w(y,s) = e^{i\theta_0}\{\kappa - (1+i\delta)\frac{\kappa}{4(p-\delta^2)s}(y^2-2) - i\frac{(1+\delta^2)\delta\kappa^2}{2(p-\delta^2)^2}\frac{1}{s}\} + o(\frac{1}{|s|})$.

Convergence takes place in L^2_ρ and uniformly on compact sets.

Step 3: The relevant case, $\lambda = 1$

We do exactly as in case $\delta = 0$.

$$\text{If } \varphi^*(s) = \begin{cases} = & \kappa e^{i\theta_0} \text{ if } C_0 = 0, \\ = & \varphi_\delta(s-s_0) = \kappa e^{i\theta_0}(1+e^{s-s_0})^{-\frac{1+i\delta}{p-1}}, \text{ if } C_0 < 0, \\ = & \tilde{\varphi}_\delta(s-s_0) = \kappa e^{i\theta_0}(1-e^{s-s_0})^{-\frac{1+i\delta}{p-1}}, \text{ if } C_0 > 0, \end{cases}$$

with $s_0 = -\log\left(\frac{(p-1)}{\kappa}|C_0|\right)$ and $\theta_0 \in \mathbb{R}$. Then φ^* is a solution of $(\mathrm{Eq}w_\delta)$ with the same expansion of w as $s \to -\infty$.

If $V = w - \varphi^*$, then $\|V(y,s)\|_{L^2_\rho} = O(e^{3/2s})$.

Since $\frac{3}{2} > 1 = \max\{\lambda \in spec(\mathcal{L})\}$, then $V \equiv 0$.

Because w is bounded, we get $\varphi^* \not\equiv \tilde{\varphi}_\delta$, hence $w(y,s) = \kappa e^{i\theta_0}$ or

$$w(y,s) = \varphi_\delta(s-s_0)e^{i\theta_0} = \kappa e^{i\theta_0}(1+e^{s-s_0})^{-\frac{1+i\delta}{p-1}}, \text{ for some } s_0 \in \mathbb{R}.$$

Step 4: The irrelevant cases, ii) $\lambda = \frac{1}{2}$ or iii) $\lambda = 0$

No blow-up criterion ? Our source of inspiration is Velázquez's work.
We extend the convergence in ii) and iii) from $|y| < R$ to larger regions to find singular profiles.

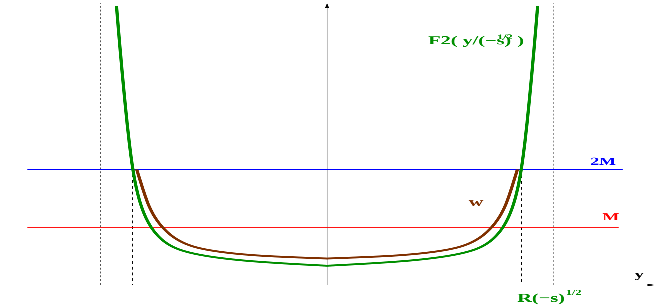
$$ii) f_1(\xi) = \kappa(1 - C_1\kappa^{-p}\xi)^{-\frac{(1+i\delta)}{(p-1)}} \text{singular for } \xi = R_1(p)$$

$$\lim_{s\to-\infty} \sup_{|y|\le Re^{-s/2}} \left| w(y,s) - f_1(ye^{s/2}) \right| = 0, \text{ where } R < R_1(p).$$

$$iii) f_2(\xi) = \kappa\left(1 - \frac{(p-1)}{4(p-\delta^2)}\xi^2\right)^{-\frac{(1+i\delta)}{p-1}} \text{ singular for } \xi = R_2(p).$$

$$\lim_{s\to-\infty} \sup_{|y|\le R\sqrt{-s}} \left| w(y,s) - f\left(\frac{y}{\sqrt{-s}}\right) \right| = 0 \text{ where } R < R_2(p).$$

A picture for the case iii) $\lambda = 0$



Here, we choose $R = R(M)$ such that $|f_2(\frac{R}{\sqrt{-s}})| = 2M$, where $\|w\|_{L^\infty(\mathbb{R}^N\times\mathbb{R})} \leq M = M(\delta)$ (*). Then, for $|s|$ large enough,

$$|w(R\sqrt{-s},s) - f_2(\frac{R}{\sqrt{-s}})| \leq \frac{M}{2}, \quad |w(R\sqrt{-s},s)| \geq |f_2(\frac{R}{\sqrt{-s}})| - \frac{M}{2} = \frac{3M}{2}.$$

Contradiction with (*).