Convergence to a blow-up profile for the semilinear wave equation

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The equation

$$\begin{cases} u_{tt} = \Delta u + |u|^{p-1}u, \\ u(0) = u_0 \text{ and } u_t(0) = u_1, \end{cases}$$

where
$u(t) : x \in \mathbb{R}^N \to u(x,t) \in \mathbb{R}$,
$u_0 \in \mathrm{H}^1_{\mathrm{loc,u}}(\mathbb{R}^{\mathrm{N}})$ and $u_1 \in \mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R}^{\mathrm{N}})$.

$$\|v\|_{\mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R}^{\mathrm{N}})} = \sup_{a \in \mathbb{R}^N} \left(\int_{|x-a|<1} |v(x)|^2 dx \right)^{1/2}.$$

$$1 < p \text{ and } p \le p_c \equiv 1 + \frac{4}{N-1} \text{ if } N \ge 2.$$

Remark: $p_c \equiv 1 + \frac{4}{N-1} < 1 + \frac{4}{N-2}$, the Sobolev critical exponent.

Why is p_c critical ?

When $p = p_c$, the equation is invariant under the following conformal transformation:

If $U(\xi, \tau)$ is defined by

$$U(\xi,\tau) = (|x|^2 - t^2)^{\frac{N-1}{2}} u(x,t), \quad \xi = \frac{x}{|x|^2 - t^2}, \quad \tau = \frac{t}{|x|^2 - t^2},$$

then U satisfies the same equation as u.

THE CAUCHY PROBLEM IN $\mathrm{H}^1_{\mathrm{loc,u}}(\mathbb{R}^N) \times \mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R}^N)$

This is a consequence of:

- The solution of the Cauchy problem in $H^1 \times L^2(\mathbb{R}^N)$ (Ginibre and Velo, Lindblad and Sogge, Shatah and Struwe)
- The finite speed of propagation.

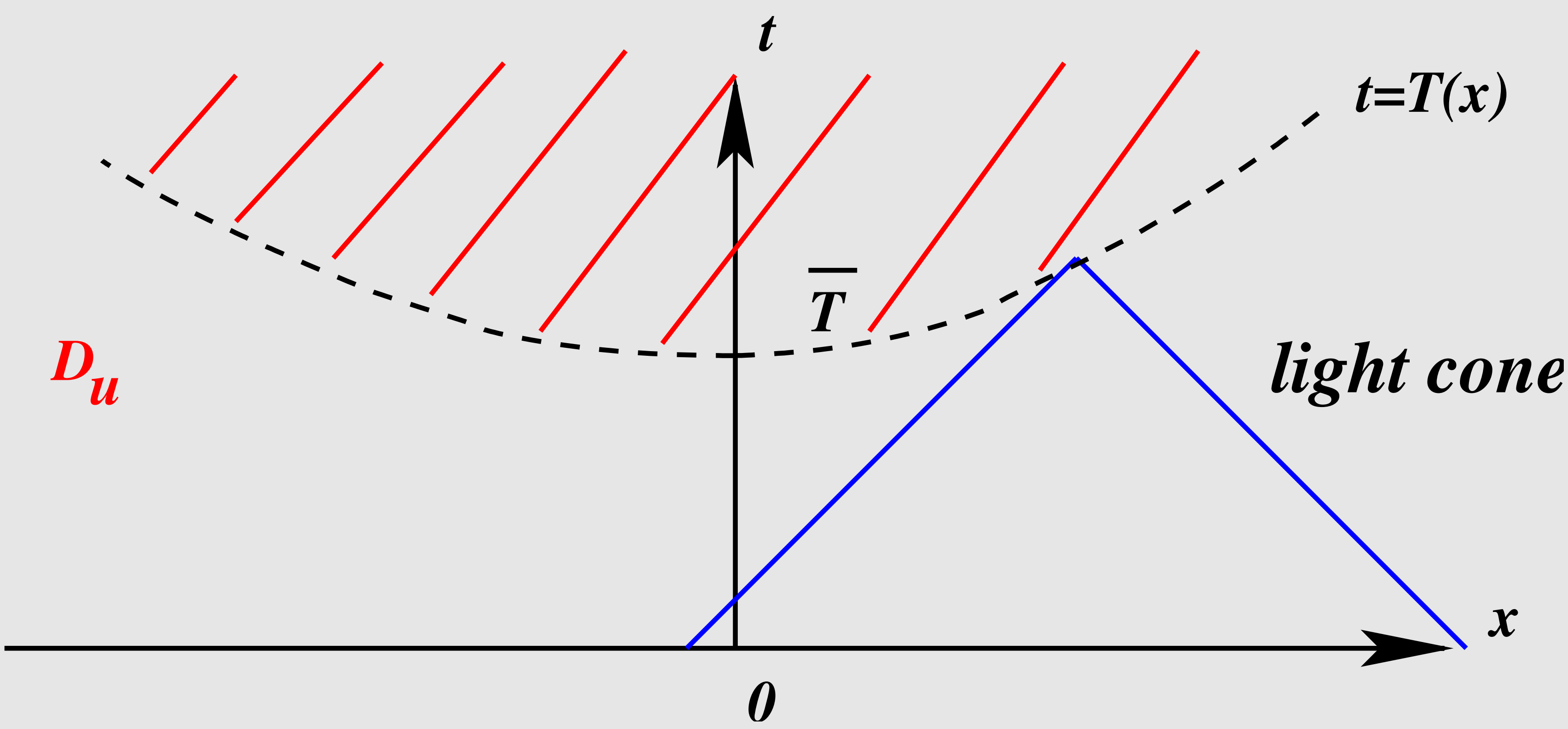
Maximal solution in $\mathrm{H}^1_{\mathrm{loc,u}}(\mathbb{R}^N) \times \mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R}^N)$

- either it exists on $[0, \infty)$ (**global solution**),
- or it exists on $[0, \bar{T})$ (**singular solution**).

Existence of singular solutions

Consequence of ODE techniques and finite speed of propagation (see Levine, Antonini and Merle).

Singular solutions: maximal influence domain



The blow-up curve $t \to T(x)$ is a 1-Lipschitz function (**finite speed of propagation**).

Remark : $\bar{T} = \inf T(x)$ is called **the blow-up time**. For every $x \in \mathbb{R}^N$, there is a “local” blow-up time $T(x)$.

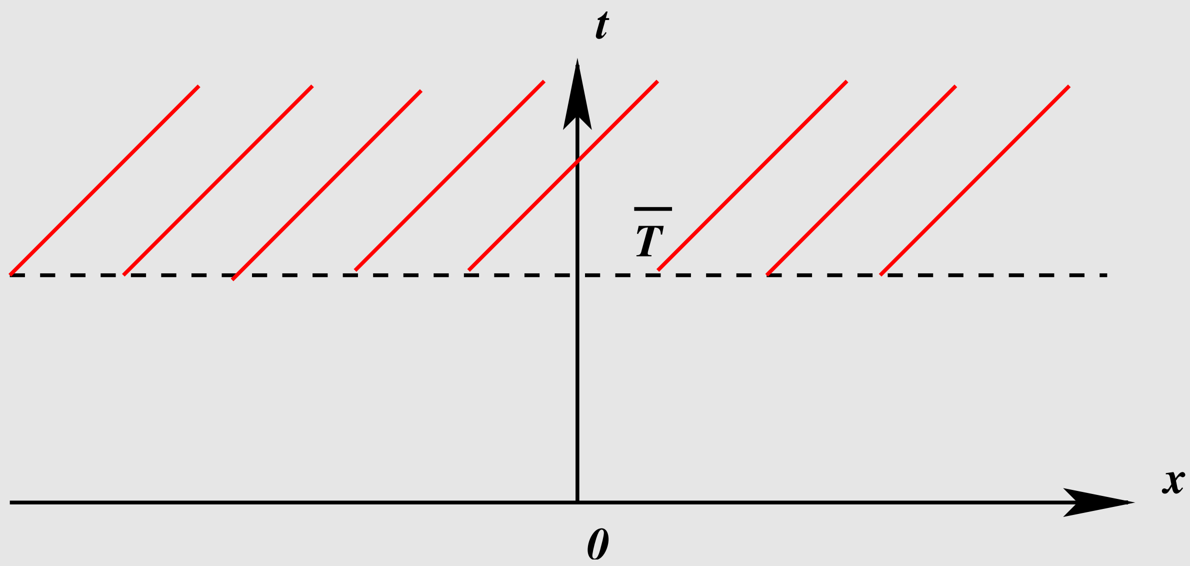
Remark: a comparison with the semilinear heat equation

$$u_t = \Delta u + |u|^{p-1} u$$

with

$$1 < p < 1 + \frac{4}{N-2} \text{ if } N \geq 2.$$

The singular solution is a maximal solution in $C_0(\mathbb{R}^N)$ which exists on $[0, \bar{T})$ where $\bar{T}$ is **the blow-up time** (and the only one). The solution can not be extended beyond $\bar{T}$.



The plan

Part 0- A hidden structure in the problem: **A Lyapunov functional ($N \geq 1$ and $p \leq p_c$).**

Part 1- Issue 1: **The blow-up rate ($N \geq 1$ and $p \leq p_c$).**

Part 2- Issue 2: **Convergence to a blow-up profile ($N = 1$).**

Part 3- Issue 3: C^1 **regularity of the blow-up set ($N = 1$).**

Remark:
- The only obstruction in doing Parts 2 and 3 for $N \geq 2$ is the solution to some elliptic problem. In particular, we use no maximum principle.

Part 0: A hidden structure in the problem ($N \geq 1$ and $p \leq p_c$)

Selfsimilar transformation for all $x_0 \in \mathbb{R}^N$

$$w_{x_0}(y,s) = (T(x_0)-t)^{\frac{2}{p-1}} u(x,t),\ y = \frac{x-x_0}{T(x_0)-t},\ s = -\log(T(x_0)-t).$$

(x,t) in the light cone of vertex $(x_0, T(x_0)) \iff (y,s) \in B(0,1) \times [-\log T(x_0), \infty)$.

Equation on $w = w_{x_0}$: For all $(y,s) \in B(0,1) \times [-\log T(x_0), \infty)$:

$$\partial_s^2 w - \tfrac{1}{\rho}\mathrm{div}\,[\rho \nabla w - \rho (y.\nabla w) y] + \tfrac{2(p+1)}{(p-1)^2} w - |w|^{p-1} w$$

$$= -\tfrac{p+3}{p-1}\partial_s w - 2y.\nabla \partial_s w$$

where $\rho(y) = (1-|y|^2)^\alpha$ and $\alpha \equiv \dfrac{2}{p-1} - \dfrac{N-1}{2} \geq 0.$

A Lyapunov functional (Antonini-Merle)

$$E(w) = \int_B \left(\frac{1}{2}(\partial_s w)^2 + \frac{1}{2}\left(|\nabla w|^2 - (y.\nabla w)^2\right) + \frac{(p+1)}{(p-1)^2} w^2 - \frac{1}{p+1}|w|^{p+1} \right) \rho dy,$$

where $B = B(0,1)$.

Thanks to a Hardy-Sobolev inequality, $E = E(w, \partial_s w)$ is well defined in the energy space

$$\mathcal{H} = \left\{ q \in H^1_{loc} \times L^2_{loc}(B) \mid \|q\|^2_{\mathcal{H}} \equiv \int_B \left(q_1^2 + |\nabla q_1|^2 (1 - y^2) + q_2^2 \right) \rho dy < +\infty \right\}.$$

Properties of the Lyapunov functional E

Lemme 1 (Monotonicity) *For all s_1 and s_2:*
($p < p_c$, Antonini-Merle),

$$E(w(s_2)) - E(w(s_1)) = -2\alpha \int_{s_1}^{s_2} \int_B (\partial_s w)^2 (1 - |y|^2)^{\alpha-1} dy ds.$$

Lemme 2 (A blow-up criterion) *Consider a solution W such that $E(W(s_0)) < 0$ for some $s_0 \in \mathbb{R}$, then W blows up in finite time $S > s_0$.*

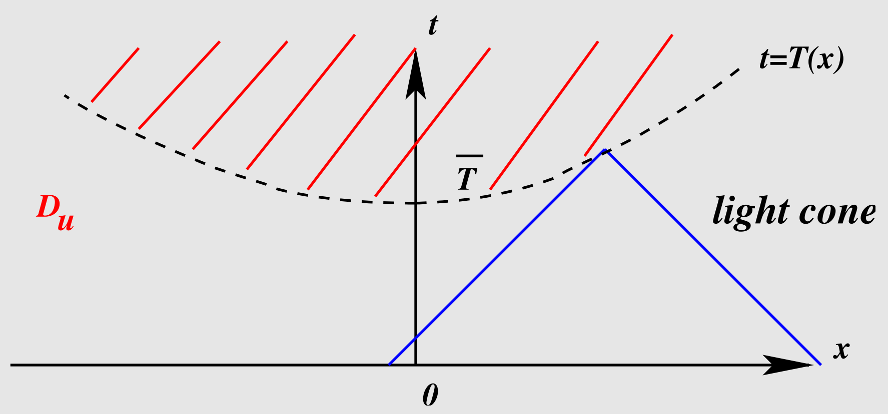
Part 1: The blow-up rate ($N \geq 1$ and $p \leq p_c$)

The heat equation (Giga and Kohn 87, Giga, Matsui and Sasayama 2004)

$$0 < \kappa(p)(\bar{T}-t)^{-\frac{1}{p-1}} \leq \|u(t)\|_{L^\infty} \leq C(\bar{T}-t)^{-\frac{1}{p-1}}.$$

Remark : the blow-up rate is given by the solution of the associated ODE $v' = v^p$, $v(\bar{T}) = +\infty$.

The wave equation: We look for a *local* **blow-up rate** near the singular surface (i.e. near every local blow-up time, $t \to T(x_0)$), in $H^1 \times L^2$ **of the section of the light cone.**



Hint : Is the rate given by the associated ODE $v'' = v^p$?

An upper bound on the blow-up rate in selfsimilar variables

Th. For all $x_0 \in \mathbb{R}^N$ and $s \geq -\log T(x_0) + 1$,

$$\int_B \left(\frac{1}{2}(\partial_s w)^2 + \frac{1}{2}|\nabla w|^2(1-|y|^2) + \frac{(p+1)}{(p-1)^2} w^2 + \frac{1}{p+1}|w|^{p+1} \right) \rho dy \leq K$$

where the constant K depends only on N, p, and an upper bound on $T(x_0)$, $1/T(x_0)$ and $\|(u_0, u_1)\|$.

Getting rid of the weights

Reducing $B = B(0,1)$ to $B_{1/2} = B(0, \frac{1}{2})$, we get:

Cor. For all $x_0 \in \mathbb{R}^N$ and $s \geq -\log T(x_0) + 1$,

$$\int_{B_{1/2}} \left((\partial_s w)^2 + |\nabla w|^2 + w^2 + |w|^{p+1} \right) dy \leq K.$$

Upper bound in the original $u(x,t)$ variables:

Th. sup. For all $x_0 \in \mathbb{R}^N$ and $t \in [\frac{3}{4}T(x_0), T(x_0))$:

$$(T(x_0)-t)^{\frac{2}{p-1}} \frac{\|u(t)\|_{L^2(B(x_0,\frac{T(x_0)-t}{2}))}}{(T(x_0)-t)^{N/2}}$$

$$+(T(x_0)-t)^{\frac{2}{p-1}+1}\left(\frac{\|u_t(t)\|_{L^2(B(x_0,\frac{T(x_0)-t}{2}))}}{(T(x_0)-t)^{N/2}} + \frac{\|\nabla u(t)\|_{L^2(B(x_0,\frac{T(x_0)-t}{2}))}}{(T(x_0)-t)^{N/2}}\right) \le K.$$

What about the lower bound?

- If x_0 is a non characteristic point: there is a lower bound easy to derive.
- If x_0 is a characteristic point: still open.

A non formal definition:
- x_0 is **characteristic** if the local Lipschitz constant (“the slope”) of the singular surface is 1.
- x_0 is *non* **characteristic** if the local Lipschitz constant (“the slope”) of the singular surface is *strictly less* than 1.

Thus, we focus on the case of **non characteristic** points.

Case of a non characteristic point

Using a covering property coming from w_x (replacing $(x_0, T(x_0))$ by (x, T) where x is close to x_0 and T is close to $T(x_0)$), we get:

Th. n. car. If x_0 is non characteristic, then $\forall t \in [\frac{3}{4}T(x_0), T(x_0))$,

$$0 < \epsilon_0(N,p) \le \quad (T(x_0)-t)^{\frac{2}{p-1}} \frac{\|u(t)\|_{L^2(B(x_0,T(x_0)-t))}}{(T(x_0)-t)^{N/2}}$$

$$+(T(x_0)-t)^{\frac{2}{p-1}+1} \quad \left(\frac{\|u_t(t)\|_{L^2(B(x_0,T(x_0)-t))}}{(T(x_0)-t)^{N/2}} + \frac{\|\nabla u(t)\|_{L^2(B(x_0,T(x_0)-t))}}{(T(x_0)-t)^{N/2}} \right) \le K$$

where K depends also on $\delta(x_0) < 1$, the local Lipschitz constant.

Proof of the lower bound: it needs the fact that x_0 is non characteristic. By contradiction, if we are below some $\epsilon_0(N,p)$, then, we can extend the solution beyond $T(x_0)$.

Idea of the proof of the upper bound

- Selfsimilar transformation and existence of a Lyapunov functional
- Interpolation to gain regularity
- Gagliardo-Nirenberg estimates.

Remark: The critical case $p = p_c$ is degenerate, therefore, we need more ideas.

Part 2: Convergence to a blow-up profile in selfsimilar variables ($N = 1$)

Take $x_0 \in \mathbb{R}$ **non characteristic**. We know that $\|w_{x_0}(s)\|_{H^1 \times L^2(B)}$ is bounded.

Question: Does $w_{x_0}(y, s)$ have a limit or not, as $s \to \infty$ (that is as $t \to T(x_0)$).

Remark: In the context of Hamiltonian systems, **this question is delicate**, and there is no natural reason for such a convergence, since **the wave equation is time reversible.**

See for similar difficulty and approach, results for

- the **critical KdV** (Martel and Merle),
- **NLS** (Merle and Raphaël).

Stationary solutions when $N = 1$.

We look for solutions of

$$\frac{1}{\rho}\left(\rho(1-y^2)w'\right)' - \frac{2(p+1)}{(p-1)^2}w + |w|^{p-1}w = 0.$$

We work in $\mathcal{H}_0$, the (**stationary energy space**) defined by

$$\mathcal{H}_0 = \{r \in H^1_{loc}(-1,1) \mid \|r\|^2_{\mathcal{H}_0} \equiv \int_{-1}^{1}\left(r'^2(1-y^2)+r^2\right)\rho dy < +\infty\}.$$

Prop. Take $N = 1$. Consider $w \in \mathcal{H}_0$ a stationary solution. Then, either $w \equiv 0$ or there exist $d \in (-1,1)$ and $\omega = \pm 1$ such that $w(y) = \omega\kappa(d,y)$ where

$$\forall (d,y) \in (-1,1)^2,\ \ \kappa(d,y) = \kappa_0\frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dy)^{\frac{2}{p-1}}} \text{ and } \kappa_0 = \left(\frac{2(p+1)}{(p-1)^2}\right)^{\frac{1}{p-1}}.$$

Remark: We have 3 connected components.

Stationary solutions if $N \geq 2$

If $N \geq 2$, we have no classification, unfortunately.
This is the only obstruction in generalizing our results to $N \geq 2$.

Of course, we already know that $\pm\kappa(d, \omega.y)$ is an $\mathcal{H}_0$ stationary solution for any $|d| < 1$ and $\omega \in \mathbb{R}^N$ with $|\omega| = 1$.

Now, back to $N = 1$.

Part 2A: Blow-up profile near a non characteristic point

Th. There exist $C_0 > 0$ and $\mu_0 > 0$ such that
if x_0 is **non characteristic**, then there exist $d_\infty(x_0) \in (-1,1)$, $\omega^*(x_0) = \pm 1$ and $s^*(x_0) \geq -\log T(x_0)$ such that for all $s \geq s^*(x_0)$,

$$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - \omega^* \begin{pmatrix} \kappa(d_\infty, \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq C_0 e^{-\mu_0(s-s^*)}$$

where the energy space

$$\mathcal{H} = \left\{ q \in H^1_{loc} \times L^2_{loc}(-1,1) \mid \|q\|^2_{\mathcal{H}} \equiv \int_{-1}^{1} \left(q_1^2 + (q_1')^2 (1-y^2) + q_2^2\right) \rho dy < +\infty \right\}.$$

Remark: We have exponentially fast convergence.

Difficulties of the proof of convergence

- ▷ - The set of non zero stationary solutions is made up of non isolated solutions (one parameter family):
  $\longrightarrow$ we need **modulation theory**.
- ▷ - The linearized operator around a non zero stationary solution **is non self-adjoint**:
  $\longrightarrow$ we need to use dispersive properties coming from the Lyapunov functional to control the negative part of the spectrum.

Part 2B: Convergence for a *characteristic* point

Th. If $x_0 \in \mathbb{R}$ is **characteristic**, then, there exist $N(x_0) \in \mathbb{N}$, $\omega_i^* = \pm 1$ and continuous $d_i(s) \in (-1,1)$ for $i = 1, ..., N$ such that:
(i)

$$\|w_{x_0}(s) - \sum_{i=1}^{N(x_0)} \omega_i^* \kappa(d_i(s), \cdot)\|_{\mathcal{H}} \to 0 \text{ as } s \to \infty,$$

and

$$\left| \frac{1}{2} \log\left(\frac{1+d_i(s)}{1-d_i(s)}\right) - \frac{1}{2} \log\left(\frac{1+d_j(s)}{1-d_j(s)}\right) \right| \to \infty \text{ for } i \neq j$$

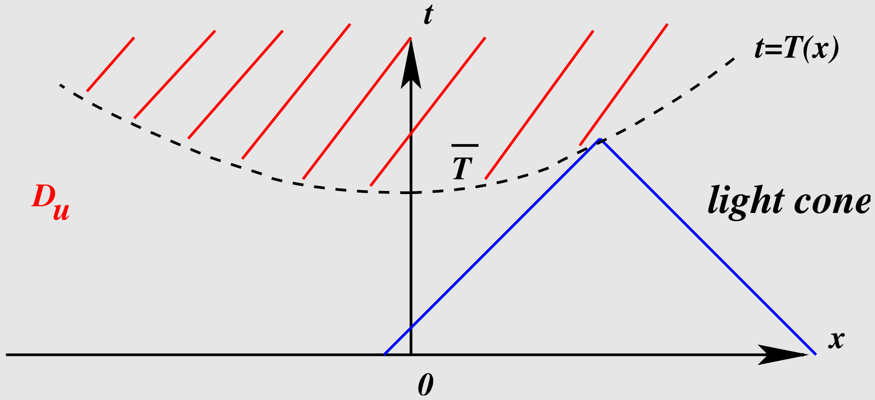
as $s \to \infty$,
(ii)

$$E(w_{x_0}(s)) \to N(x_0) E(\kappa_0) \text{ as } s \to \infty.$$

Remark: As $s \to \infty$, w_{x_0} becomes like a **decoupled** sum of stationary solutions (“solitons”).

Part 3: The blow-up curve ($N = 1$)

Known facts:
- The blow-up set is $\Gamma = \{(x, T(x))\} \subset \mathbb{R}^2$.
- By definition, Γ is 1 Lipschitz. No more results.
- Notation: $I_0 \subset \mathbb{R}$ is the set of non characteristic points.
- $I_0 \neq \emptyset$ (indeed, $\bar{x}$ such that $T(\bar{x}) = \min_{x\in\mathbb{R}} T(x)$ is non characteristic).



Questions:
- Do we have $I_0 = \mathbb{R}$?
- If no answer or if not, is I_0 an open set ?

Regularity of the blow-up curve

Th. The set of non characteristic points I_0 **is open** and $T(x)$ **is** C^1 **on that set.** Moreover,

$$\forall x_0 \in I_0, \;\; T'(x_0) = d(x_0) \in (-1,1)$$

where $d(x_0)$ is such that for all $s \geq s^*$,

$$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - \omega^* \begin{pmatrix} \kappa(d(x_0),\cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq C_0 e^{-\mu_0(s-s^*)}.$$

Remark: We have a geometrical interpretation of the parameter $d(x_0)$.

Conjecture: $I_0 = \mathbb{R}$?

Comments

Remark: **Caffarelli and Friedman** proved that $T'(x) = d(x)$ for $N \leq 3$, under strong hypothesis on the nonlinearity and the initial data (which guarantee that $I_0 = \mathbb{R}$).
They heavily rely on the **positivity of the fundamental solution for** $N \leq 3$ (no hope to generalize their techniques to $N \geq 4$).

Idea of the proof:
The techniques are based on

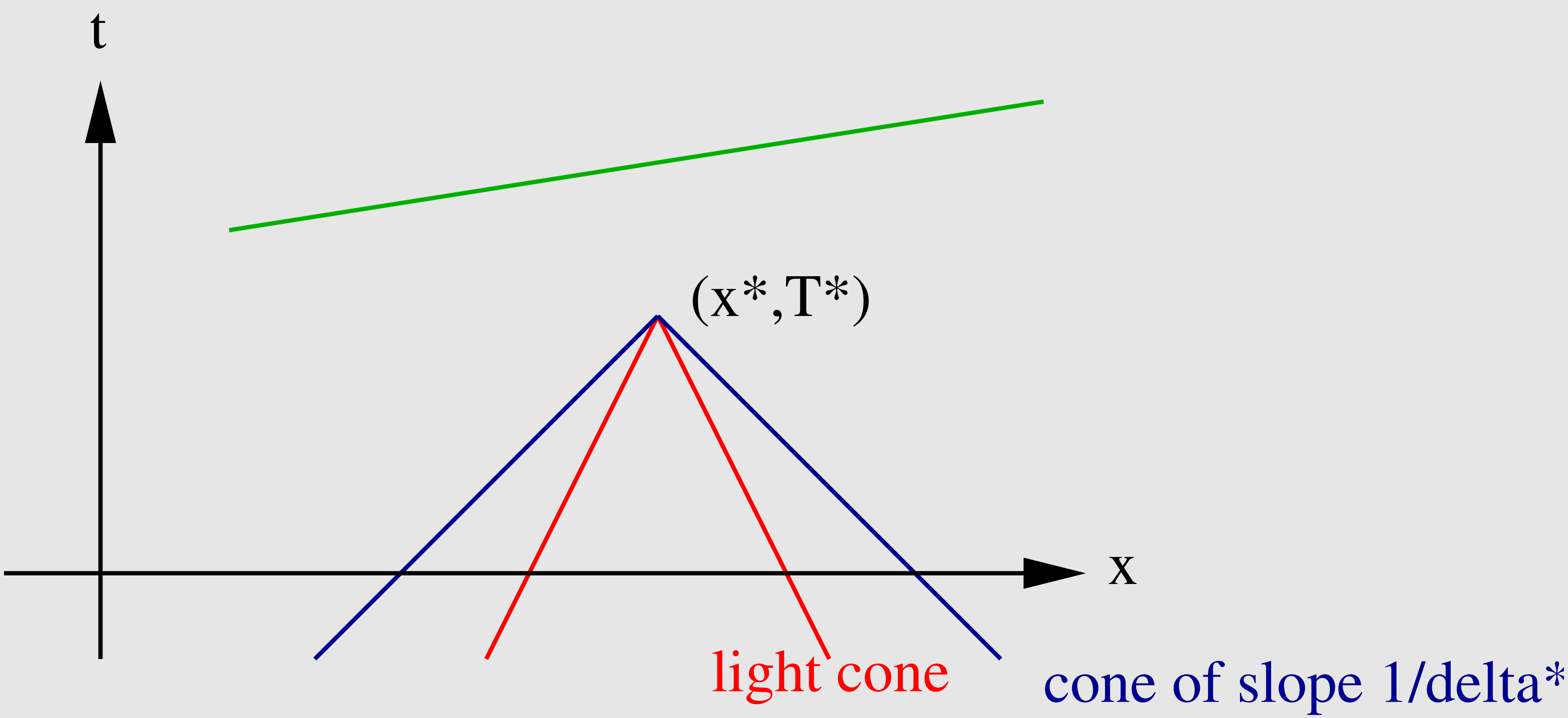
- - a very good understanding of the **dynamics of the equation in selfsimilar variables in the energy space**,
- - a **Liouville Theorem** (see next slide).

A Liouville theorem (N=1)

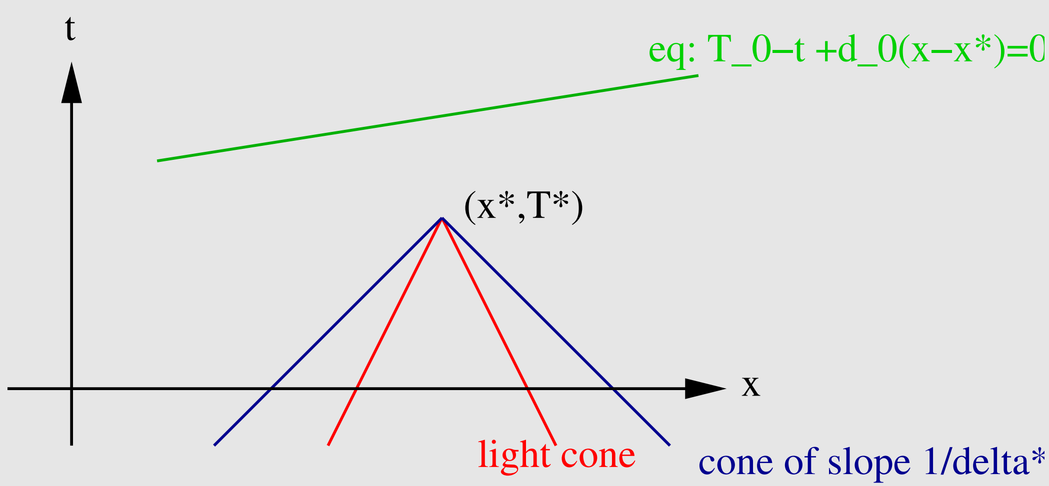
Behind this regularity result, there is another hidden structure in the equation:

Th. Consider $u(x,t)$ a solution of $u_{tt} = u_{xx} + |u|^{p-1}u$ such that:

- u is defined in the *infinite* blue cone,
- u is less than $(T^* - t)^{-\frac{2}{p-1}}$ (in L^2 average).



A Liouville Theorem



Then,
- either $u \equiv 0$,
- or there exists $T_0 \geq T^*$, $d_0 \in [-\delta_*, \delta_*]$ and $\theta_0 = \pm 1$ such that u is actually defined below the green line by

$$u(x,t) = \theta_0 \kappa_0(p) \frac{(1-d_0^2)^{\frac{1}{p-1}}}{(T_0 - t + d_0(x-x^*))^{\frac{2}{p-1}}}.$$

Remark: u blows up on the green line.

Comments

- The limiting case $\delta^* = 1$ is still open.
- $N \geq 2$: we expect the result to be valid. The only obstruction comes from the classification of stationary solutions.

The proof:

- The proof has a completely different structure from the proof for the heat equation.
- The proof is based on various energy arguments and on a dynamical result.