Regularity of the blow-up set for the semilinear heat equation

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Motivation: singularities in PDE

Solutions which are regular at $t = 0$, may become “infinite” in finite time T. Example: heat, Schrödinger, wave, generalized KdV, geometric flows, etc...

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Common questions:

- Find the asymptotic behavior(s) near the singularity.
- Discuss their stability.
- Obtain **uniforms** estimates / initial data, etc..
- Understand interactions between regular and singular regions.

The semilinear heat equation

$$\begin{cases} u_t = \Delta u + |u|^{p-1}u, \\ u(0) = u_0, \end{cases}$$

where $u(t) : x \in \mathbb{R}^N \to u(x,t) \in \mathbb{R}$ and

$$1 < p < \frac{N+2}{N-2} \text{ if } N \geq 3.$$

(Critical exponent for the Sobolev injection).

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Rk. This a lab model where one can go far in computations and develop tools for more physical situations.

Generalization :

- A bounded domain,
- $u \in \mathbb{R}^M$,
- Case of the equation

$$u_t = \text{div}(a(x)\nabla u) + f(u)$$

with $a(x) > a_0 > 0$ and $f(u) \sim |u|^{p-1}u$ and $|u| \to \infty$,

- Cases of systems with no gradient structure, like

$$\begin{cases} u_t = \Delta u + v^p, \\ v_t = \Delta v + u^q. \end{cases}$$

The solution of the Cauchy problem exists:

- either on $[0, +\infty)$: there is **global existence**,

- or on $[0, T)$ with $T < +\infty$: there is **finite-time blow-up**.

In this case,

$$\lim_{t \to T} \|u(t)\|_{L^\infty} = +\infty.$$

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A point a is a **blow-up point** if

$$|u(a,t)| \to +\infty \text{ as } t \to T.$$

We denote by $S_u \subset \mathbb{R}^N$ the **blow-up set**, i.e. the set of all blow-up points.

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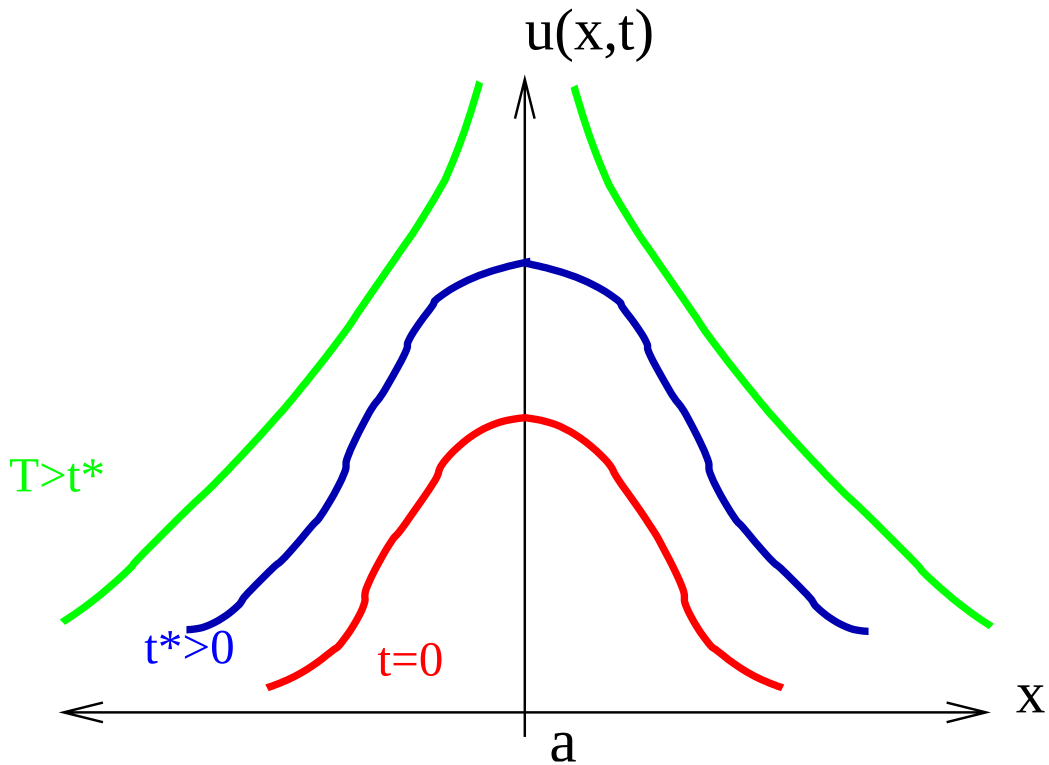
Goal : Study S_u.

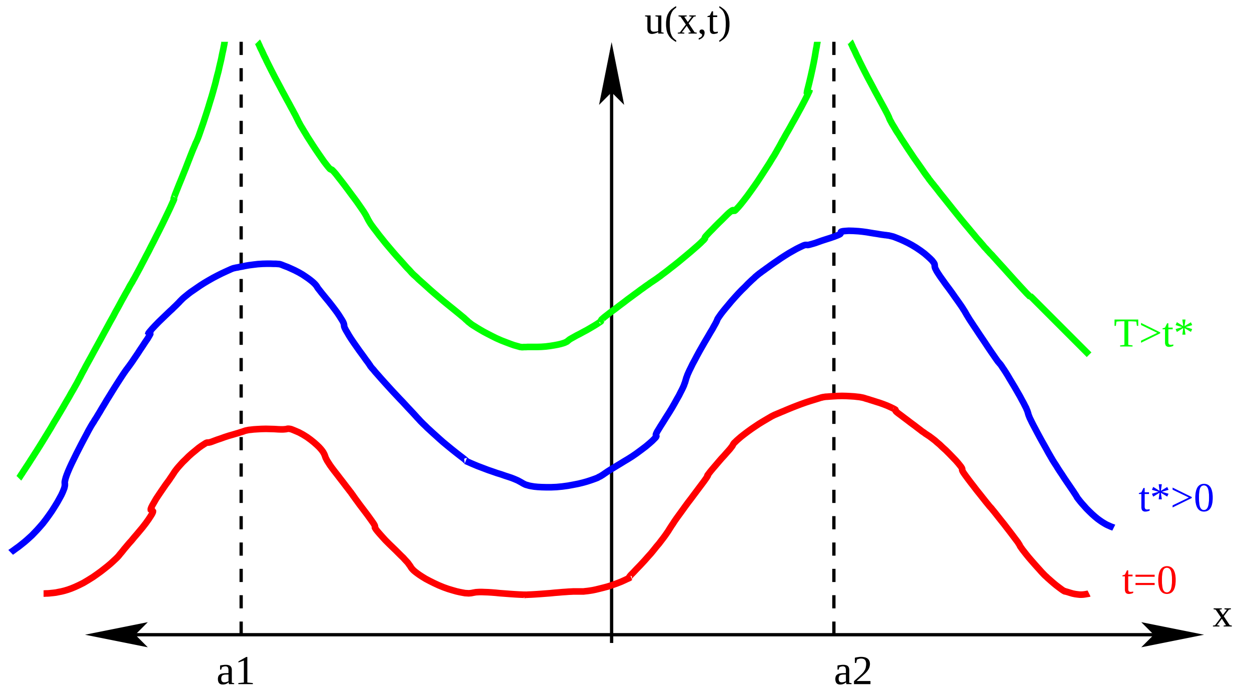
Example 1: Single-point blow-up



Rk. Sorry, this is not a simulation!

Rk. The only blow-up point is a. The other points are called "regular points".

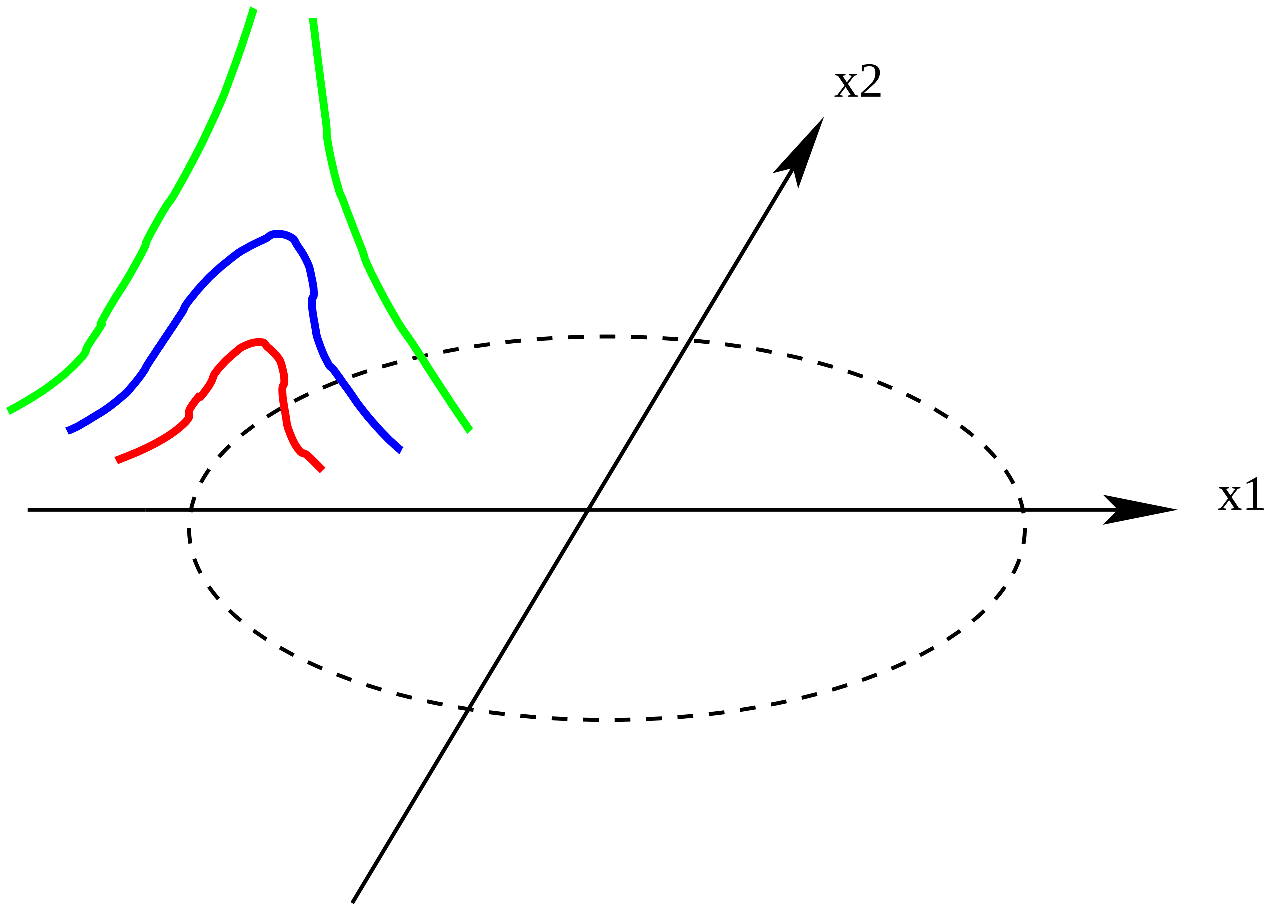
Example 1 bis : Two blow-up points (both isolated)



Rk. This is still not a simulation (by the way, blowing-up at 2 points is *unstable* and hard to get on a computer!)

Rk. Imagine the same picture with k points and in N dimensions.

Example 2: S_u is a sphere (radial sol., picture for $N = 2$).



Rk. Here, all blow-up points are *non isolated* in S_u.

Goal of the talk:

- Study of the blow-up set S_u ($\subset \mathbb{R}^N$).

Two questions arise: **the construction** and **the description**.

The construction : Given a set $\widehat{S} \subset \mathbb{R}^N$, is there a solution $\hat{u}$ of $u_t = \Delta u + |u|^{p-1}u$ that blows up at some finite time T such that $S_{\hat{u}} = \widehat{S}$?

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The answer is YES in the following cases:

- an isolated point (Herrero-Velázquez, Bricmont-Kupiainen, Weissler...),
- k points (Merle),
- a sphere (radial solution, Giga-Kohn).

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- k points (Merle),
- a sphere (radial solution, Giga-Kohn).

In all the other cases, the question remains open (the ellipse for example).

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known information:

- S_u is a closed set (by definition).
- S_u is bounded, if u_0 is small at infinity (Giga-Kohn 1989).
- The Hausdorff dimension of S_u is $\leq N-1$ (Velázquez 1992).

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Open questions: Is S_u locally connected? Is it C^1, C^∞,..?

In this talk, we address the description issue.

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Let us first review the classical approach.

Outline

- *The classical approach*
- The new approach : Liouville Theorem
- Case of an isolated blow-up point (stability / initial data)
- Case of a non isolated blow-up point (regularity of the blow-up set)

The classical Approach

Let u be a solution of $u_t = \Delta u + |u|^{p-1}u$ that blows up at time T and let $a \in S_u$.

Self-similar variables

$$w_a(y,s) = (T-t)^{\frac{1}{p-1}} u(x,t), \quad y = \frac{x-a}{\sqrt{T-t}}, \quad s = -\log(T-t).$$

Study u near the singularity (a,T)

$\Longleftrightarrow$ Study w_a near $y=0$ as $s \to \infty$.

Equation :

For all $s \in [-\log T, +\infty)$ and $y \in \mathbb{R}^N$,

$$\partial_s w_a = \frac{1}{\rho}\text{div}(\rho \nabla w_a) - \frac{w_a}{p-1} + |w_a|^{p-1} w_a$$

with

$$\rho(y) = e^{-\frac{|y|^2}{4}}.$$

Energy (decreasing) :

$$E(w) = \int \rho dy \left(\frac{1}{2}|\nabla w|^2 + \frac{2}{p-1}|w|^2 - \frac{1}{p+1}|w|^{p+1} \right)$$

Uniform bound (Giga-Kohn 1987, Giga, Matsui and Sasayama. 2004)

$$\forall s \geq -\log T, \ \ \frac{1}{C_0} \leq \|w_a(s)\|_{L^\infty} \leq C_0.$$

Convergence in L^2_ρ and L^∞_{loc} (Giga-Kohn)

$$w_a(y,s) \to \pm\kappa \equiv (p-1)^{-\frac{1}{p-1}} \text{ as } s \to +\infty.$$

Rk. (Giga-Kohn) 0, κ and $-\kappa$ are the only stationary solutions.

Rk. $u(a,t) \sim \pm\kappa(T-t)^{-\frac{1}{p-1}}$ as $t \to T$: a local comparison "locale" with the solution of $u' = u^p$.

Rk. Further refinement of the development : Herrero-Velázquez, Bricmont-Kupiainen, Filippas-Kohn.

Problem : the *stability*. The estimates are *too local*: they depend on initial data and on the blow-up point.

If a is isolated in S_u : What happens if we perturb initial data (for u) ?

If a is non isolated : For a given solution $u(x,t)$, how does $w_b(y,s)$ behaves when $b \in S_u$ varies in a neighborhood of a ?

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The new approach: Liouville (or rigidity) theorem (Merle, Z.)

$$1 < p < \frac{N+2}{N-2}.$$

Consider $u(x,t)$ a solution of $u_t = \Delta u + |u|^{p-1}u$ such that

$$\forall (x,t) \in \mathbb{R}^N \times (-\infty, T), \;\; |u(x,t)| \le C(T-t)^{-\frac{1}{p-1}}.$$

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Then,

either $u \equiv 0$,

of there exists $T^* \geq T$ such that

$$\forall (x,t) \in \mathbb{R}^N \times (-\infty, T), \;\; u(x,t) = \kappa(T^* - t)^{-\frac{1}{p-1}}.$$

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Rk. This result yields blow-up estimates which are *uniform* (with respect to initial data, blow-up point, etc...)

Generalization

- Critical exponent $p = \frac{N+2}{N-2}$.

- Same equation with $u \in \mathbb{R}^M$ (there is still a Lyapunov functional).

- Case of systems with no gradient structure, like

$$\begin{cases} u_t = \Delta u + v^p, \\ v_t = \Delta v + u^q, \end{cases}$$

with p and q subcritical and close to each other.

Cor. (Merle-Z.) If u is a solution of $u_t = \Delta u + |u|^{p-1}u$ that blows up at time T, then $\forall \epsilon > 0$, $\exists C_\epsilon > 0$ such that $\forall (x,t) \in \mathbb{R}^N \times [0,T)$,

$$|\Delta u| = |u_t - |u|^{p-1}u| \leq \epsilon |u|^p + C_\epsilon$$

where C_ϵ depends only on ϵ and on bounds on T and $\|u_0\|$.

Rk. Localization property for the equation. The interaction due to Δu is controlled by a local term $\epsilon|u|^p$ and a uniform constant C_ϵ.

Rk. If $u \geq 0$, then

$$u^p(1-\epsilon) - C_\epsilon \leq u_t \leq u^p(1+\epsilon) + C_\epsilon.$$

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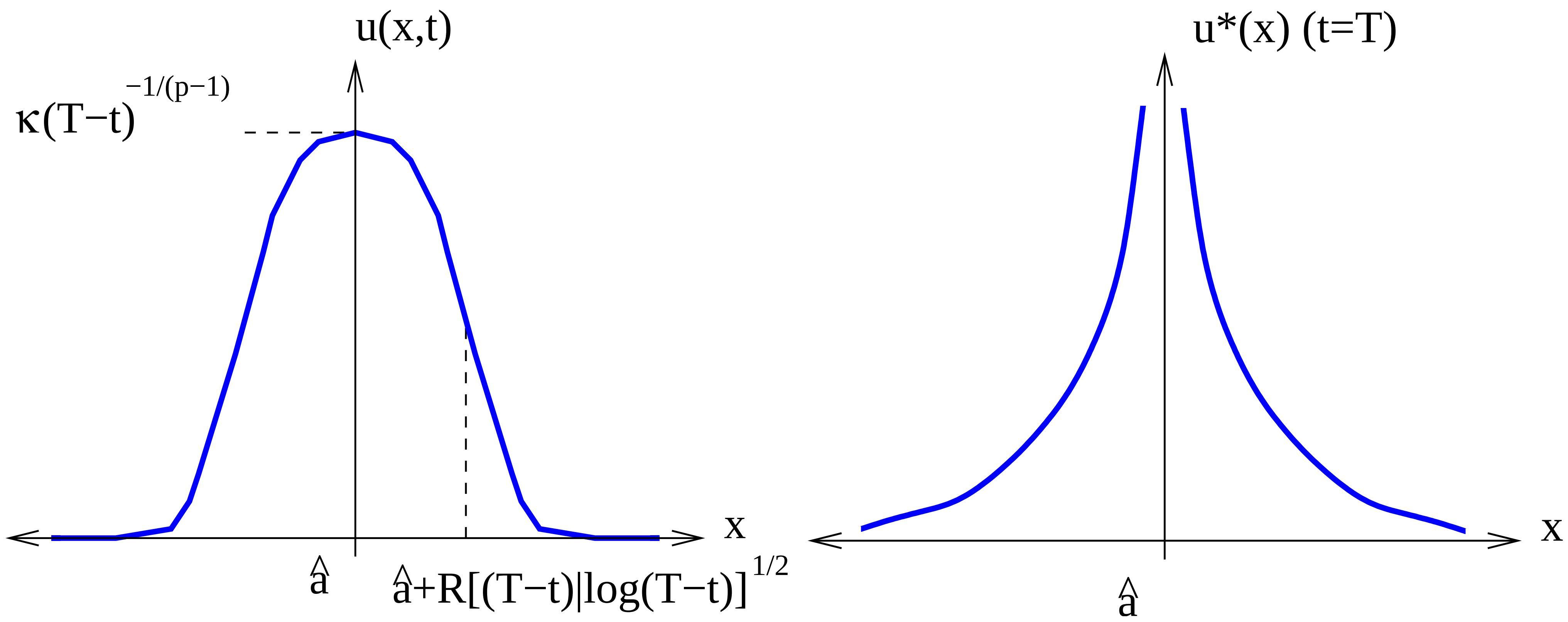
Case of an isolated blow-up point $\hat{a}$

Blow-up profile (Herrero-Velázquez, Bricmont-Kupiainen, Merle
Z).



$$u(x,t) \sim (T-t)^{-\frac{1}{p-1}} f\left(\left|\frac{x-\hat{a}}{\sqrt{(T-t)|\log(T-t)|}}\right|\right)$$

and $\forall x \neq \hat{a}$, $u(x,t) \to u^*(x)$ as $t \to T$ and

$$u^*(x) \sim U(|x-\hat{a}|) \text{ as } x \to \hat{a}$$

where

$$f(z) = \left(p-1+b(p)z^2\right)^{-\frac{1}{p-1}} \text{ et } U(z) = \left(\frac{b(p)}{2}\frac{z^2}{|\log z|}\right)^{-\frac{1}{p-1}}.$$

Rk. The profile is radial (it is a function of $|x-\hat{a}|$).

Rk. This is the generic profile (proved in dim. 1 by Herrero and Velázquez).

Th. This behavior is stable with respect to initial data (the solution blows up at only one point with the same profile).

Rk. Two proofs:

- A geometrical approach, with construction of a stable manifold near the limiting profile (Merle-Z.).

- A dynamical system approach Fermanian, Merle, Z.)

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Case of non isolated blow-up points ($C^0 \Longrightarrow C^1$)

Th (Regularity of the blow-up set). ($N = 2$) Consider u a solution of $u_t = \Delta u + |u|^{p-1}u$ and $\hat{a}$ a non isolated blow-up in S_u such that:

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1/ ($S_u \supset$ Continuum)

$\exists a \in C((-1,1), \mathbb{R}^2)$, $a(0) = \hat{a}$ and $\operatorname{Im} a \subset S_u$.

2/ ($\hat{a}$ is not an endpoint).

3/ (A "reasonable" technical condition) $(T-t)^{\frac{1}{p-1}} u(\hat{a}, t) \to (p-1)^{-\frac{1}{p-1}}$ qd $t \to T$ avec la vitesse la plus lente.

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Conclusion: Locally near $\hat{a}$, S_u is the graph of a C^1 function.

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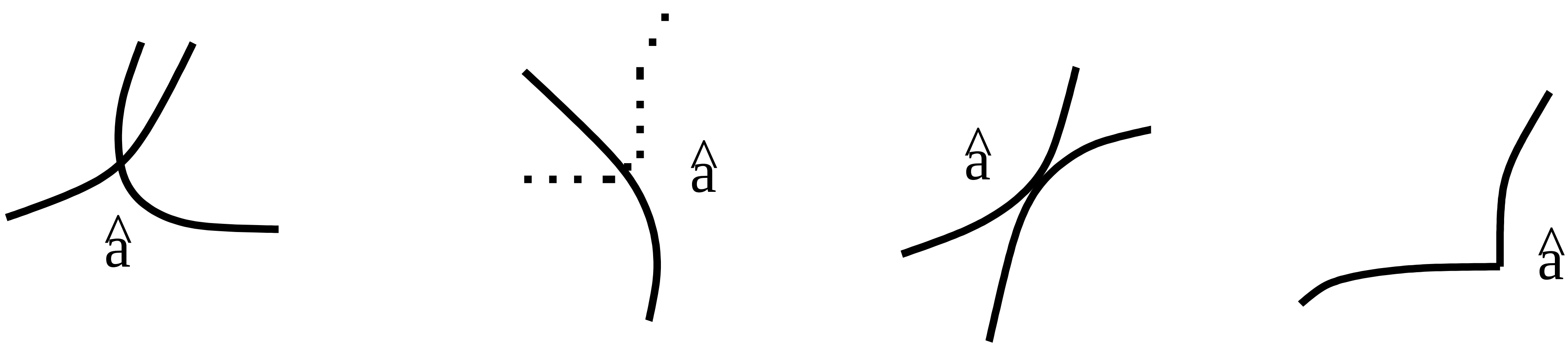
Conclusion: Locally near $\hat{a}$, S_u is the graph of a C^1 function.

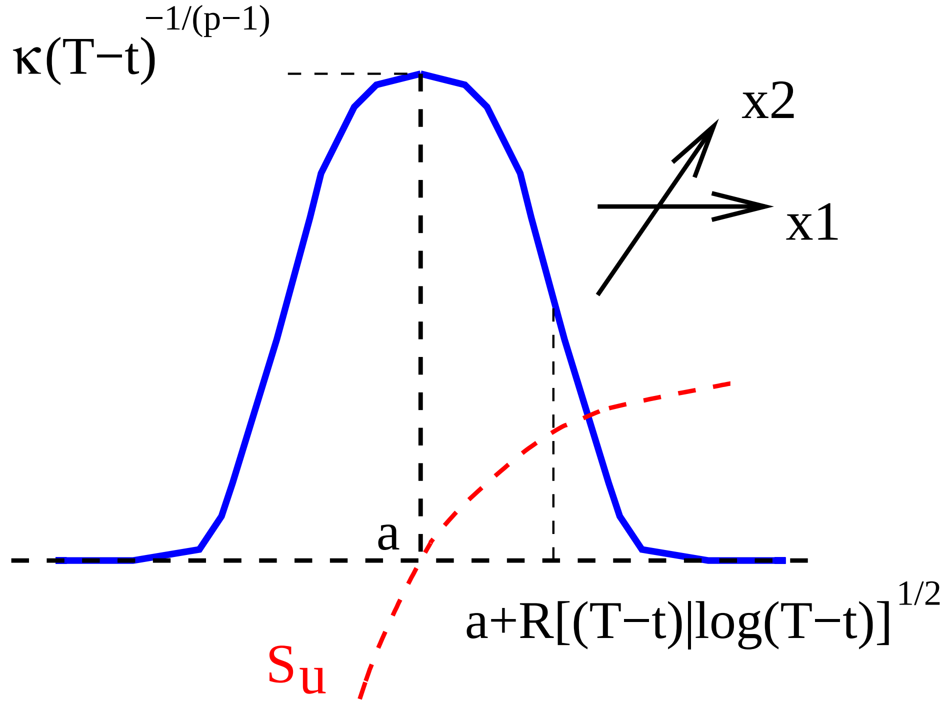
Rk. Valid in any dimension.

Rk. If $\operatorname{codim} S_u = 1$, then S_u is $C^{1,\frac{1}{2}}$.

Some impossible cases for the blow-up set



Th. (The blow-up profile)

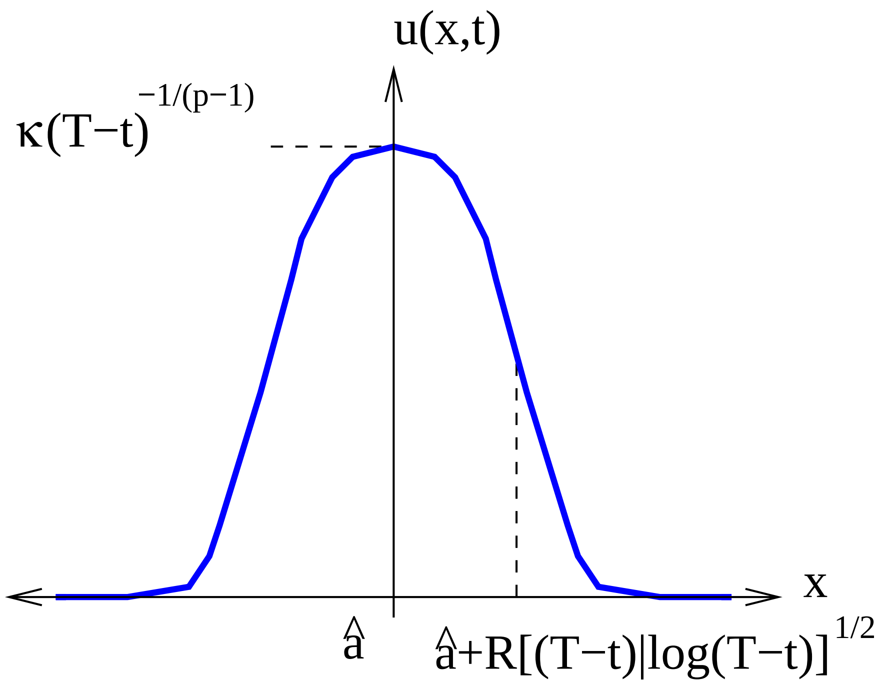


$$u(x,t) \sim (T-t)^{-\frac{1}{p-1}} f\left(\frac{dist(x,S_u)}{\sqrt{(T-t)|\log(T-t)|}}\right)$$

where $f(z) = \left(p - 1 + b(p)z^2\right)^{-\frac{1}{p-1}}$.

Rk. Only the one-dimensional variable $dist(x, S_u)$ (orthogonal to S_u) is responsible of the size of u at blow-up.

Rk. f is the generic profile in dimension 1.



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where $f(z) = \left(p-1+b(p)z^2\right)^{-\frac{1}{p-1}}$.

Rk. In this case $|x - \hat{a}| = dist(x, S_u)$.

Hence, in all cases (isolated points or not),

$$u(x,t) \sim (T-t)^{-\frac{1}{p-1}} f\left(\frac{dist(x, S_u)}{\sqrt{(T-t)|\log(T-t)|}}\right)$$

$\Longrightarrow$ **Universality**.

Th (Universality) (Z.) : Under a non degeneracy condition at some $\hat{a} \in S_u$, the blow-up set is (locally near $\hat{a}$) :

- Either an isolated point (of dimension 0),
- Or a C^1 manifold of dimension 1,... $N-1$.

$$u(x,t) \sim (T-t)^{-\frac{1}{p-1}} f\left(\frac{dist(x,S_u)}{\sqrt{(T-t)|\log(T-t)|}}\right)$$

and $\forall x \notin S_u$, $u(x,t) \to u^*(x)$ as $t \to T$ and

$$u^*(x) \sim U\left(dist(x,S_u)\right) \text{ and } dist(x,S_u) \to 0$$

where $f(z) = \left(p-1+b(p)z^2\right)^{-\frac{1}{p-1}}$ and $U(z) = \left[\frac{b(p)}{2}\frac{z^2}{|\log z|}\right]^{-\frac{1}{p-1}}$

"is" the generic profile in dimension 1.

Most recent contribution (preprint 2004)

If the blow-up set is of co-dimension 1, then it is in fact C^2, and we can explicitly compute its curvature (which is a geometric invariant).

In one word, $C^0 \Longrightarrow C^2$.