Rational S1 equivariant ring spectra

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Let $\mathcal{O}_{\mathcal{F}} = \prod_{n \geqslant 1} \mathbb{Q}[c_n]$ and $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} = \operatorname{colim}_n \mathcal{O}_{\mathcal{F}}[c_1^{-1}, \ldots, c_n^{-1}]$, with $\deg(c_n) = -2$.

Definition (The algebraic model $\mathcal{A}(\mathbb{T})$)

Let $\mathcal{A}(\mathbb{T})$ be the category whose **objects** are morphisms of $\mathcal{O}_{\mathcal{F}}$-modules of the form

$$\beta \colon M \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathbb{Q}} V$$

such that β is an isomorphism after inverting $\mathcal{E}$.

A **morphism** is a pair (θ, ϕ) which makes the following square commute

$$\begin{array}{ccc}
M & \xrightarrow{\beta} & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathbb{Q}} V \\
\downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathbb{Q}} \phi} \\
M' & \xrightarrow{\beta'} & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathbb{Q}} V'
\end{array}$$

Let $d\mathcal{A}(\mathbb{T})$ be the associated category with differentials.

Theorem (The classification of rational $\mathbb{T}$-spectra)

There is a (zig–zag) of symmetric monoidal Quillen equivalences between rational $\mathbb{T}$-equivariant spectra and $d\mathcal{A}(\mathbb{T})$.

Corollary (Homotopy level)

There is an equivalence of symmetric monoidal triangulated categories between the homotopy category of rational $\mathbb{T}$-equivariant spectra and $\mathrm{Ho}(d\mathcal{A}(\mathbb{T})) = D\mathcal{A}(\mathbb{T})$.

Corollary (Rings and modules)

The categories of rational $\mathbb{T}$-equivariant ring spectra is Quillen equivalent to the category of ring objects in $d\mathcal{A}(\mathbb{T})$.

If E is a rational $\mathbb{T}$-equivariant ring spectrum, then the model category of E-modules is Quillen equivalent to the category of ΘE-modules in $d\mathcal{A}(\mathbb{T})$.

Theorem

Let G be a finite group. The category of rational G-spectra is symmetric monoidally Quillen equivalent to

$$\prod_{(H) \leqslant G} \mathrm{Ch}(\mathbb{Q}[W_G H])$$

Greenlees and May 1992:	homotopy level equivalence
Schwede and Shipley 2003:	Quillen equivalence
B. 2009 and Kedziorek 2014:	symmetric monoidal

Greenlees 1999

$$\mathrm{Ho}(\mathbb{T}\,\mathrm{Sp}_{\mathbb{Q}}) \underset{\triangle}{\simeq} D\mathcal{A}(\mathbb{T}) \Longrightarrow \mathbb{T}\,\mathrm{Sp}_{\mathbb{Q}} \underset{\mathrm{QE}}{\simeq} d\mathcal{A}(\mathbb{T})$$

Shipley 2002

$\rightsquigarrow$ $O(2)$ and $SO(3)$ cases

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BGKS 2016
Monoidal QE

$\rightsquigarrow$

$O(2)$ and $SO(3)$
cases

Corollary (Of Shipley's 2002 paper)

The category of rational $\mathbb{T}$-equivariant spectra is rigid: any model category whose homotopy category is triangulated equivalent to the homotopy category of rational $\mathbb{T}$–spectra is Quillen equivalent to rational $\mathbb{T}$–spectra.

Theorem (B. 2016)

The category of rational $O(2)$-equivariant spectra is Quillen equivalent to an algebraic model.

Theorem (Kędziorek 2016)

The category of rational $SO(3)$-equivariant spectra is Quillen equivalent to an algebraic model.

Let G be group, X a based topological space with G action and let F^* be a cohomology theory.

We need equivariant cohomology theories

- $F^*(X)$ has a G-action.
- This action can be trivial, and is always trivial for $G = \mathbb{T}$.
- There are non-trivial G-spaces X with $F^*(X) = 0$
- such as EG_+ the universal free space, $EG_+/G = BG_+$.
- $E\mathbb{T} = S^\infty \subset \mathbb{C}^\infty$, $B\mathbb{T} = \mathbb{C}P^\infty$.

Examples

- The borel construction: $F^*(X \wedge_G EG_+)$.
- Equivariant K–theory.
- Equivariant cobordism.

For V a representation of G, define S^V as the one-point compactification of V.

Definition

A G-**equivariant cohomology theory** F_G^* consists of cohomology theories

$$(F_G^V)^* : \mathrm{Ho}(G\,\mathrm{Top}) \to g\,\mathrm{Ab}$$

such that $(F_G^{V \oplus W})^*(S^W \wedge X) \cong (F_G^V)^*(X)$.

The point is that one can think of F_G^* as an $RO(G)$-graded cohomology theory.

Theorem (Equivariant Brown representability)

A G-equivariant cohomology theory F_G^* is represented by a G-spectrum F_G. That is $F_G^*(A) = [\Sigma^\infty A, F_G]_*^G$.

Definition

For G a compact Lie group, a G-**spectrum** X is a collection of based G-spaces $X(V)$ for each finite dimensional real representation V of G, along with structure maps

$$X(V) \wedge S^W \longrightarrow X(V \oplus W)$$

A **morphism** $f{:}X \to Y$ is a collection of equivariant maps

$$f(V){:}X(V) \to Y(V) \quad f(V)(g \cdot x) = g \cdot f(V)(x)$$

commuting with the structure maps. We call this category $G\,\mathrm{Sp}$.

For each V there is an equivalence of categories

$$- \wedge S^V \colon \mathrm{Ho}(G\,\mathrm{Sp}) \xrightarrow{\cong} \mathrm{Ho}(G\,\mathrm{Sp})$$

Example

For a G-space A, let $\Sigma^\infty A$ be the spectrum with $(\Sigma^\infty A)(V) = A \wedge S^V$.

Definition

The **model category of rational G-spectra** $G\,\mathrm{Sp}_{\mathbb{Q}}$ is the category $G\,\mathrm{Sp}$ with weak equivalences those maps f such that $\pi_*^H(f)\otimes\mathbb{Q}$ is an isomorphism for all closed subgroups H of G.

The fibrant objects are those G-spectra X such that the adjoints of the structure maps $X(V)\to\Omega^W X(V\oplus W)$ are weak equivalences of G-spaces and $\pi_n^H(X)$ is rational for each $H\leqslant G$.

$$\pi_n^H(X) := \mathrm{colim}_V\,\pi_n(\Omega^V X(V))^H \cong [S^n\wedge G/H_+, X]^G$$

Theorem (Rational equivariant Brown representability)

An rational G-equivariant cohomology theory F_G^* is represented by a rational G-spectrum F_G.

Definition

The category of 'free' G-spectra or **spectra with a G-action** $\mathrm{Sp}[G]$. Is the category of G-objects and G-equivariant morphisms in Sp.

The weak equivalences of $\mathrm{Sp}[G]$ are those maps which forget to π_*-isomorphisms of non-equivariant spectra.

Theorem (Greenlees and Shipley 2014)

The model category $\mathrm{Sp}_{\mathbb{Q}}[G]$ is Quillen equivalent to the category of torsion $H^*(BN;\mathbb{Q})[W]$–modules. Where N is the identity component of G and $W = G/N$ and BN has a W-action.

General aim

For each compact Lie group G, find a simple algebraic category $\mathcal{A}(G)$ which is symmetric monoidally Quillen equivalent to $G\,\mathrm{Sp}_{\mathbb{Q}}$.

Facts for finite G

- The homotopy category is generated by G/H_+ for varying H.
- $[\Sigma^\infty G/H_+, \Sigma^\infty G/K_+]_*^{G\mathbb{Q}}$ is concentrated in degree zero.
- $\mathbb{S} = \Sigma^\infty S^0$, $[\mathbb{S}, \mathbb{S}]_*^{G\mathbb{Q}} \cong A(G) \otimes \mathbb{Q} \cong \prod_{(H) \leqslant G} \mathbb{Q}$.
- The homotopy category is generated by $e_H \Sigma^\infty G/H_+$ for varying H.

Lemma

There is a symmetric monoidal Quillen equivalence

$$G\,\mathrm{Sp}_{\mathbb{Q}} \underset{\prod}{\overset{\Delta}{\rightleftarrows}} \prod_{(H) \leqslant G} L_{e_H \mathbb{S}} G\,\mathrm{Sp}_{\mathbb{Q}}$$

$L_{e_H \mathbb{S}} G\,\mathrm{Sp}_{\mathbb{Q}}$ is the model category with weak equivalences those f with $e_H \pi_*^K(f) \otimes \mathbb{Q}$ an isomorphism for all $K \leqslant G$.

The fibrant objects are the fibrant objects of $G\,\mathrm{Sp}_{\mathbb{Q}}$ such that $X \to e_H X$ is a weak equivalence in $G\,\mathrm{Sp}_{\mathbb{Q}}$.

The model category $L_{e_H\mathbb{S}} G\,\mathrm{Sp}_{\mathbb{Q}}$ is generated by $e_H \Sigma^\infty G/H_+$. The self maps of this generator are very simple:

$$F(e_H G/H_+, e_H G/H_+)^G \simeq W_G H_+$$

Lemma

The Morita-type Quillen adjunction below is a Quillen equivalence.

$$L_{e_H\mathbb{S}} G\,\mathrm{Sp}_{\mathbb{Q}} \underset{F(e_H G/H_+,-)^G}{\overset{e_H G/H_+ \wedge \varepsilon^*(-)}{\rightleftarrows}} F(e_H G/H_+, e_H G/H_+)^G\text{-mod}\,\mathrm{Sp}_{\mathbb{Q}} \simeq \mathrm{Sp}_{\mathbb{Q}}[W_G H]$$

By work of Shipley 2007,

$$\mathrm{Sp}_{\mathbb{Q}}[W_G H] \underset{\mathrm{QE}}{\simeq} \mathrm{Ch}(\mathbb{Q}[W_G H])$$

Theorem (Kedziorek 2014)

Let G be a finite group, $N = N_G H$ and $W_G H = N_G H/H$. There are symmetric monoidal Quillen equivalences

$$L_{e_H^G \mathbb{S}} G\,\mathrm{Sp}_{\mathbb{Q}} \underset{F_N(G,-)}{\overset{i^*}{\rightleftarrows}} L_{e_H^N \mathbb{S}} N\,\mathrm{Sp}_{\mathbb{Q}} \underset{(-)^H}{\overset{\varepsilon^*}{\leftrightarrows}} \mathrm{Sp}_{\mathbb{Q}}[W_G H] \;\simeq \mathrm{Ch}(\mathbb{Q}[W_G H])$$

Recap

- Split the category using the Burnside ring.
- Take fixed points of each piece.
- Use algebraicization / Shipleyfication.
- Simple as $W_G H$ finite (all homotopy is in degree zero).

Facts for $\mathbb{T}$

- The homotopy category is generated by $\mathbb{T}/H_+$ for varying H.
- $[\Sigma^\infty \mathbb{T}/H_+, \Sigma^\infty \mathbb{T}/K_+]_*^{\mathbb{T}\mathbb{Q}}$ is not concentrated in degree zero.
- $[\mathbb{S}, \mathbb{S}]_*^{\mathbb{T}\mathbb{Q}} \cong A(G) \otimes \mathbb{Q} \cong \mathbb{Q}$.

We decompose the sphere spectrum $\mathbb{S} = \Sigma^\infty S^0$. Let $\mathcal{F}$ be the family of finite subgroups of $\mathbb{T}$. There is a cofibre sequence of based $\mathbb{T}$-spaces

$$E\mathcal{F}_+ \to S^0 \to \tilde{E}\mathcal{F}$$

There is a pullback square of $\mathbb{T}$-spectra

$$\begin{array}{ccc} \mathbb{S} & \longrightarrow & DE\mathcal{F}_+ \\ \downarrow & & \downarrow \\ \tilde{E}\mathcal{F} & \longrightarrow & DE\mathcal{F}_+ \wedge \tilde{E}\mathcal{F} \end{array}$$

Caution: while $\tilde{E}\mathcal{F} = S^{\infty V} = \mathrm{colim}_{V^{\mathbb{T}}=0} S^V$ is a ring spectrum, it is not a commutative ring $\mathbb{T}$–equivariant orthogonal spectrum.

We want to decompose $\mathbb{T}\,\mathrm{Sp}_{\mathbb{Q}}$ using that decomposition of the sphere. For this we need some model category technology. We want to describe a pullback of model categories

$$\begin{array}{ccc} \mathbb{T}\,\mathrm{Sp}_{\mathbb{Q}} & \longrightarrow & DE\mathcal{F}_+\text{-mod} \\ \downarrow & & \downarrow \\ L_{\tilde{E}\mathcal{F}}\mathbb{T}\,\mathrm{Sp}_{\mathbb{Q}} & \longrightarrow & L_{DE\mathcal{F}_+\wedge \tilde{E}\mathcal{F}}DE\mathcal{F}_+\text{-mod} \end{array}$$

Definition

Define the category $S^\bullet$-mod to have **objects** the quintuples (X, f, Y, g, Z) where $X \in \mathbb{T}\,\mathrm{Sp}_{\mathbb{Q}}$, Y and $Z \in DE\mathcal{F}_+$-mod and $f{:}X \wedge DE\mathcal{F}_+ \to Y$ and $g{:}Z \to Y$ are maps in $DE\mathcal{F}_+$-mod.

Morphisms are triples that make the obvious squares commute. A map is a weak equivalence if each component is a weak equivalence. The cofibrations are defined objectwise.

There is a Quillen adjunction as below, but it is not a Quillen equivalence.

$$\mathbb{T}\,\mathrm{Sp}_{\mathbb{Q}} \underset{\text{pullback}}{\overset{S^\bullet \wedge -}{\rightleftarrows}} S^\bullet\text{--mod}$$

The derived unit map is essentially

$$X \to \mathrm{pullback}(X \wedge \tilde{E}\mathcal{F} \to X \wedge \tilde{E}\mathcal{F} \wedge DE\mathcal{F}_+ \leftarrow X \wedge DE\mathcal{F}_+)$$

and hence is a weak equivalence. It follows that the left adjoint $S^\bullet \wedge -$ is full and faithful. We just need to make it essentially surjective.

For this we use a cellularisation (right Bousfield localisation).

Let K be the set of “cells”: $\{S^\bullet \wedge (\mathbb{T}/C_n)_+ \mid n \geqslant 1\} \cup \{S^\bullet \wedge (\mathbb{T}/\mathbb{T})_+\}$.

Definition

The model category K–cell–$S^\bullet$–mod has the same fibrations as $S^\bullet$–mod. The cofibrant objects are those built from the objects of K using homotopy colimits. The weak equivalences are those maps $f\colon M \to N$ such that for each $k \in K$

$$[k, M]^{S^\bullet} \xrightarrow{\cong} [k, N]^{S^\bullet}$$

This is essentially replacing Ho($S^\bullet$–mod) by the full subcategory generated by the images of K.

Theorem

There is a Quillen equivalence

$$\mathbb{T}\,\mathrm{Sp}_{\mathbb{Q}} \underset{pullback}{\overset{S^\bullet \wedge -}{\rightleftarrows}} K\text{–cell–}S^\bullet\text{–mod}$$

Lemma

Taking fixed points induces Quillen equivalences:

$$\begin{array}{rcl} L_{\tilde{E}\mathcal{F}}\mathbb{T}\,\mathrm{Sp}_{\mathbb{Q}} & \simeq & \mathrm{Sp}_{\mathbb{Q}} \\ DE\mathcal{F}_{+}\text{-mod} & \simeq & (DE\mathcal{F}_{+})^{\mathbb{T}}\text{-mod} \\ L_{DE\mathcal{F}_{+}\wedge\tilde{E}\mathcal{F}}DE\mathcal{F}_{+}\text{-mod} & \simeq & L_{(DE\mathcal{F}_{+}\wedge\tilde{E}\mathcal{F})^{\mathbb{T}}}(DE\mathcal{F}_{+})^{\mathbb{T}}\text{-mod} \end{array}$$

We define a new diagram of model categories $S^{\bullet}_{top}$ using the right hand side of the above.

Theorem

There is a Quillen equivalence

$$S^{\bullet}\text{-mod} \underset{(-)^{\mathbb{T}}}{\overset{\longleftarrow}{\longrightarrow}} S^{\bullet}_{top}\text{-mod}$$

Lemma

Taking fixed points induces Quillen equivalences:

$$\begin{array}{rcl} L_{\tilde{E}\mathcal{F}} \mathbb{T}\,\mathrm{Sp}_{\mathbb{Q}} & \simeq & \mathrm{Sp}_{\mathbb{Q}} \\ DE\mathcal{F}_+ -\mathrm{mod} & \simeq & (DE\mathcal{F}_+)^{\mathbb{T}} -\mathrm{mod} \\ L_{DE\mathcal{F}_+ \wedge \tilde{E}\mathcal{F}} DE\mathcal{F}_+ -\mathrm{mod} & \simeq & L_{(DE\mathcal{F}_+ \wedge \tilde{E}\mathcal{F})^{\mathbb{T}}} (DE\mathcal{F}_+)^{\mathbb{T}} -\mathrm{mod} \end{array}$$

We define a new diagram of model categories $S^{\bullet}_{top}$ using the right hand side of the above.

Theorem

There is a Quillen equivalence by the cellularisation principle [Greenlees and Shipley 2013]

$$K -\mathrm{cell}- S^{\bullet} -\mathrm{mod} \underset{(-)^{\mathbb{T}}}{\overset{\longleftarrow}{\longrightarrow}} K^{\mathbb{T}} -\mathrm{cell}- S^{\bullet}_{top} -\mathrm{mod}$$

Using the work of Shipley we can again get an algebraic version of the category.

Theorem

There is a diagram of model categories

$$S_t^\bullet = (\Theta DE\mathcal{F}_+^{\mathbb{T}})\text{–mod} \to L_A(\Theta DE\mathcal{F}_+^{\mathbb{T}})\text{–mod} \leftarrow \mathrm{Ch}(\mathbb{Q}))$$

such that the model categories below are symmetric monoidally Quillen equivalent.

$$K_t^{\mathbb{T}}\text{–cell–} S_t^\bullet\text{–mod} \simeq K^{\mathbb{T}}\text{–cell–} S_{top}^\bullet\text{–mod}$$

We know that we have isomorphisms of *commutative* rings

$$H_*(\Theta DE\mathcal{F}_+^{\mathbb{T}}) \cong \pi_*^{\mathbb{T}}(DE\mathcal{F}_+) \cong \mathcal{O}_{\mathcal{F}} = \prod_{n\geqslant 1} \mathbb{Q}[c_n]$$

hence $\Theta DE\mathcal{F}_+^{\mathbb{T}} \simeq \mathcal{O}_{\mathcal{F}}$ by a formality argument.

We now have the diagram of model categories

$$\mathcal{O}_{\mathcal{F}}\text{-mod} \to L_A\mathcal{O}_{\mathcal{F}}\text{-mod} \leftarrow \mathrm{Ch}(\mathbb{Q})$$

We know that A is a ring object and

$$H_*(A) \cong \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} = \mathrm{colim}_n\, \mathcal{O}_{\mathcal{F}}[c_1^{-1}, \ldots, c_n^{-1}]$$

but we do not know that that A is commutative. This ring is not formal! However the map $\mathcal{O}_{\mathcal{F}} \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$ is formal and this suffices to show that

$$L_A\mathcal{O}_{\mathcal{F}}\text{-mod} \simeq \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}\text{-mod}$$

This sequence of Quillen equivalences takes the set of cells $K_t^{\mathbb{T}}$ to a set of cells K_a.

We have shown that the model category of rational $\mathbb{T}$–spectra is symmetric monoidally Quillen equivalent to

$$K_a\text{–cell–}(\mathcal{O}_{\mathcal{F}}\text{–mod} \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}\text{–mod} \leftarrow \mathrm{Ch}(\mathbb{Q}))\text{–mod}$$

Call this category $d\hat{\mathcal{A}}$. An object is an $\mathcal{O}_{\mathcal{F}}$-module M, a $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$-module N and a rational chain complex V with maps

$$\mathcal{E}^{-1}M \rightarrow N \leftarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathbb{Q}} V$$

Recall that $d\mathcal{A}(\mathbb{T})$ is the category whose objects are morphisms of $\mathcal{O}_{\mathcal{F}}$-modules of the form

$$\beta\colon M \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathbb{Q}} V$$

such that β is an isomorphism after inverting $\mathcal{E}$.

We can include $d\mathcal{A}(\mathbb{T})$ into $d\hat{\mathcal{A}}$ by defining $N = \mathcal{E}^{-1}M$. This gives an adjunction between the two categories.

We can use another formality argument to identify the cells, and hence we can show that the cellularisation has exactly the effect of requiring that the structure maps of $d\hat{A}$ are homology isomorphisms.

Theorem

The model categories $d\hat{\mathcal{A}}$ *and* $d\mathcal{A}(\mathbb{T})$ *are Quillen equivalent.*

Theorem (The classification of rational $\mathbb{T}$-spectra)

There is a (zig–zag) of symmetric monoidal Quillen equivalences between rational $\mathbb{T}$*-equivariant spectra and* $d\mathcal{A}(\mathbb{T})$*.*