Chromatic unstable homotopy, plethories, and the Dieudonné correspondence

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Adams-type spectral sequences for computing $[X, Y]$

The classical, homological Adams spectral sequence:

$$\mathrm{Ext}^{**}_{\mathcal{A}_*}(H_*X, H_*Y) \Longrightarrow [X, Y]_*$$

- X, Y spectra
- $\mathcal{A}_*$ dual Steenrod algebra, a commutative $\mathbf{F}_p$-Hopf algebra
- H_*X, H_*Y coefficients in $\mathbf{F}_p$, comodules over A_*
- $\mathrm{Ext}_{\mathcal{A}_*}$ of comodules over a coalgebra
- $[X, Y]_*$ graded homotopy classes, more realistically $[X, Y^{\hat{}}_p]$

Adams-type spectral sequences for computing $[X, Y]$

The stable Adams-Novikov spectral sequence based on a ring spectrum E:

$$\mathrm{Ext}^{**}_{E_*E}(E_*X, E_*Y) \Longrightarrow [X, Y]_*$$

X, Y spectra

E_*E stable cooperations, a Hopf algebroid

E_*X, E_*Y comodules over E_*

Ext_{E_*E} of comodules over a Hopf algebroid

$[X, Y]_*$ graded homotopy classes, more realistically $[X, Y^{\hat{}}_E]$

Hopf algebroids vs. bialgebras

(E_*, E_*E) is a Hopf algebroid if E_*E is a flat E_*-module. A Hopf algebroid is a cogroupoid object in E_*-algebras.
Alternatively: E_*E is an E_*-bimodule with a multiplication

$$E_*E \ {}_{E_*}\!\otimes_{E_*} E_*E \to E_*E \quad \text{(tensor over } E_* \otimes E_*\text{)}$$

and a comultiplication

$$E_*E \to E_*E \otimes_{E_*} E_*E \quad \text{(left-right tensor product).}$$

It is a bialgebra with respect to two different tensor structures, only one of which is symmetric monoidal. This is the point of view that generalizes to the unstable setting.

Adams-type spectral sequences for computing $[X, Y]$

The unstable spectral sequence based on a ring spectrum E:

$$\mathrm{Ext}^{**}_{\mathcal{K}}(E_*X, E_*Y) \Longrightarrow [X, Y]_*$$

- X, Y CW complexes
- $\mathcal{K}$ "unstable cooperations", a "bialgebra" in coalgebras
- E_*X, E_*Y comodules over $\mathcal{K}$ (in coalgebras), free E_*-modules
- $\mathrm{Ext}_{\mathcal{K}}$ some nonlinear derived functor
- $[X, Y]_*$ graded homotopy classes, more realistically $[X, Y^{\hat{}}_E]$. Fringed, i. e. $[X, Y]_{0,1}$ are only sets/groups.

Adams-type spectral sequences for computing $[X, Y]$

There are adjoint functors

$$\Omega^\infty\colon E\text{-module spectra} \leftrightarrows \mathrm{Top}\colon\ E \wedge \Sigma^\infty(-)_+$$

The associated monad $Y \mapsto E(Y) = \Omega^\infty(E \wedge Y_+)$ gives a bar construction

$$(\ Y \to\ )\quad E(Y) \rightrightarrows E(E(Y)) \cdots$$

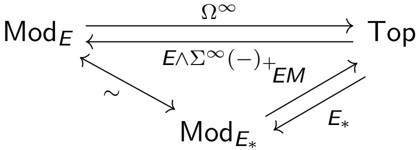
with $\mathrm{Tot}\, E^\bullet(Y) = Y^\wedge_E$.
The Bousfield spectral sequence from applying $[X, -]_*$ to the associated tower gives the unstable Adams spectral sequence we are looking for (Bendersky-Curtis-Miller 1978).

It is not obvious how to algebraically describe the E_2-term.

Adams-type spectral sequences for computing $[X, Y]$

A comonadic point of view

Simplifying assumption: E_* is a graded field, e.g. $E = K(n)$.



Let $\mathcal{K}$ be the **co**monad $E_* \circ EM$ on Mod_{E_*}. For any space X, E_*X is a $\mathcal{K}$-comodule.

Theorem (BCM 78 for certain connective E, Bendersky-Thompson 00 for nonconnective E)

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{K}\text{-comodules}}(E_*X, E_*Y)$$

Example

For $E = H\mathbf{F}_p$, $\mathcal{K}$ is the free unstable algebra functor on a graded vector space (rather, its linear dual).

This defines the spectral sequence and identifies its E_2-term. But

- It forces us to use the cobar construction – unsuitable for daily use!
- It does not exhibit the bialgebraic structure we saw in the stable case

Aim: Give an algebraic description of the object $\mathcal{P}$ for which $\mathcal{K}$ is the cofree construction.

History, previous work

Aim: Give an algebraic description of the object $\mathcal{P}$ for which $\mathcal{K}$ is the cofree construction.

Boardman-Johnson-Wilson 95: $\mathcal{P}$ is $E_*(\underline{E}_*)$. This has the following structure:

- An E_*-coalgebra because $\underline{E}_n$ are spaces
- A Hopf algebra because $\underline{E}_n$ are infinite loop spaces, furthermore a morphism $E_* \to \mathrm{Hom}_{\mathrm{Hopf}}(\mathcal{P}, \mathcal{P})$
- A *Hopf ring* because $\underline{E}_*$ is a ring space
- An *"enriched" Hopf ring* by including the action of $E^*(\underline{E}_*)$ in an ad-hoc way.

Problem: what is the (co)composition actually defined on?

History, previous work

A similar algebraic structure has appeared in algebra: *Tall-Wraith monoids* (**Tall-Wraith 70**), a.k.a. *plethories*, studied extensively by Borger-Wieland 2005.

Problem: this works for *co*homology $E^*(\underline{E}_*)$, and a profinite topology has to be taken into account.

Stacey-Whitehouse 09: give a description of a "completed" version of plethories when E_* is a field (and some more general situations)

B 14: "formal plethories": definition that works when E_* is a Prüfer domain, informed by algebraic geometry.

But these structures are complex and unwieldy.

Formal groups

(but not what you think)

From now on, assume that $k = E_*$ is a *perfect* graded field, i.e.:

- k_0 is perfect
- If k has period l and of characteristic p then $(l, p) = 1$.

Definition

A formal scheme is a directed colimit of functors represented by finite-dimensional k-algebras.

$$\begin{aligned} \mathrm{Top} &\to \text{formal schemes}/k \\ X &\mapsto \mathrm{Spf}\, E^*(X) = \operatorname*{colim}_{F \subset X \text{ finite}} \mathrm{Spec}\, E^*(F) \\ X \text{ double loop space} &\mapsto \text{formal (abelian) group } /k \end{aligned}$$

Formal groups

For simplicity, assume even grading, so that everything is commutative.

Lemma (Fontaine)

Every formal group G over perfect k splits naturally

$$G = G^c \times G^{\text{ét}}, \quad \textit{where}$$

- G^c *is connected, i. e.* $G^c(k') = 0$ *for all extensions* k' *of* k
- $G^{\text{ét}}$ *is étale.*

There is an equivalence

$$\begin{aligned} \textit{étale group schemes} \quad &\leftrightarrow \quad \textit{discrete } \mathrm{Gal}(k)\textit{-modules} \\ G \quad &\mapsto \quad \operatorname*{colim}_{k<k'} G(k') \\ \operatorname{Spf} \operatorname{map}_{\mathrm{Gal}(k)}(M, \bar{k}) \quad &\longleftarrow \quad M \end{aligned}$$

Formal groups

If A is an abelian group, considered as a trivial $\mathrm{Gal}(k)$-module, then

$$\operatorname{Spf}\operatorname{map}_{\mathrm{Gal}(k)}(A, \bar{k}) = \operatorname{Spf}\operatorname{map}(A, k) = \underline{A}$$

is the constant group scheme.

Observation: If X is a space, $E^*(X)/\mathrm{nil} \cong E^*(\pi_0 X)$ has trivial $\mathrm{Gal}(k)$-action. In particular, when X is a commutative, associative grouplike H-space, $(\operatorname{Spf} E^*(X))^{\text{ét}}$ is always constant.

Definition

A *cohomological formal group* is a formal group with constant étale part.

The category of cohomological formal groups

Theorem (B)

The category of cohomological formal groups

- *is abelian*
- *has all colimits and limits, and directed colimits are exact*
- *has a generator* and *a cogenerator*
- *is well-powered and co-well-powered.*

Corollary (Freyd)

Any functor from cohomological formal groups to another category $\mathcal{C}$ that preserves limits, has a left adjoint. Any functor that preserves colimits has a right adjoint.

In particular, the forgetful functor to formal schemes has a left adjoint Fr.

Tensor products of formal groups

Corollary (Goerss 1999)

The category of cohomological formal groups is closed monoidal with respect to a tensor product $\otimes$ classifying bilinear maps of formal groups.

Example

$\underline{A} \otimes \underline{B} \cong \underline{A \otimes B}$

Example

$\mathrm{Fr}(X \times Y) \cong \mathrm{Fr}(X) \otimes \mathrm{Fr}(Y)$

In characteristic 0, every connected formal group is the zero component of a free formal group (Milnor-Moore), so this is nearly a complete description.

Formal l-algebra schemes

Definition

Let l be a graded commutative ring. A formal l-algebra scheme is a formal group A with $\underline{l} \to A$, $A \otimes A \to A$ making it into a functor from k-algebras to l-algebras.

Example

If F is a commutative ring spectrum, $\operatorname{Spf} E^* \underline{F}_*$ is a cohomological F_*-algebra scheme.

Lemma

Composition gives:

$$\begin{aligned} \{f.\ l\text{-alg schemes}/k\} \times \{f.\ \text{schemes}/k\} &\to \{f.\ \text{schemes}/l\} \\ \{f.\ l\text{-alg schemes}/k\} \times \{f.\ k\text{-alg schemes}/k\} &\to \{f.\ l\text{-alg schemes}/k\} \end{aligned}$$

Formal plethories

Lemma

Composition gives:

$$\begin{aligned} \{\text{f. } l\text{-alg schemes}/k\} \times \{\text{f. schemes}/k\} &\to \{\text{f. schemes}/l\} \\ \{\text{f. } l\text{-alg schemes}/k\} \times \{\text{f. } k\text{-alg schemes}/k\} &\to \{\text{f. } l\text{-alg schemes}/k\} \end{aligned}$$

In particular, composition $\circ$ is a non-symmetric monoidal structure on k-algebra schemes over k, and schemes over k are tensored over it.

Definition

A **formal plethory** is a cohomological formal k-algebra scheme P with a comonoid structure

$$P \to P \circ P, P \to \mathrm{id}$$

A (left) comodule over a formal plethory P is a formal scheme X with a coaction $X \to P \circ X$.

Thus: algebra for $\otimes$, coalgebra for $\circ$.

Formal plethories

Example

Spf $E^*\underline{E}_*$ is a formal plethory, and Spf $E^*(X)$ is a comodule over it for any space X.

Theorem (2014, "nonlinear Künneth theorem")

For E, F commutative ring spectra, E_* a graded field, and X a space, there is an isomorphism

$$\operatorname{Spf}(E^*(F(X)) = \operatorname{Spf} E^*(\Omega^\infty(F \wedge X)_+) \cong \operatorname{Spf} E^*\underline{F}_* \circ \operatorname{Spf} F^* X$$

Corollary

The functor $C \mapsto \operatorname{Spf} C^*$ from coalgebras to formal schemes extends to an equivalence between $\mathcal{K}$-coalgebras and Spf $E^*\underline{E}_*$-comodules. Thus

$$E_2 = \operatorname{Ext}_{\operatorname{Spf} E^*\underline{E}_*\text{-comod}}(\operatorname{Spf} E^*(S^*), \operatorname{Spf} E^*(X)) \Rightarrow \pi_* X_{\hat{E}}.$$

Corollary

The functor $C \mapsto \operatorname{Spf} C^*$ from coalgebras to formal schemes extends to an equivalence between $\mathcal{K}$-coalgebras and $\operatorname{Spf} E^*\underline{E}_*$-comodules. Thus

$$E_2 = \operatorname{Ext}_{\operatorname{Spf} E^*\underline{E}_*\text{-comod}}(\operatorname{Spf} E^*(S^*), \operatorname{Spf} E^*(X)) \Rightarrow \pi_* X^{\hat{}}_E.$$

+ Fully algebraic description of E_2
– Cobar construction still the only obvious resolution – what are injective $\operatorname{Spf} E^*\underline{E}_*$-comodules??

From now on, $\mathrm{char}(k) = p > 0$. Semi-classically:

Theorem

Let l be a $\mathbf{Z}_p$-algebra. Then there is an equivalence of abelian categories

$$\left\{ \begin{matrix} \text{discrete cohomological} \\ l\text{-module schemes}/k \end{matrix} \right\} \xrightarrow{D} \left\{ \begin{matrix} \text{Modules } M \text{ over} \\ \mathcal{R} = W(k) \otimes_{\mathbf{Z}_p} l\langle F, V\rangle/(FV - p) \\ \text{s. t. } M = M_0 \oplus M_c \text{ where} \\ V|_{M_0} = \mathrm{id},\ V|_{M_c} \text{ nilpotent} \end{matrix} \right\}$$

Remark: If one drops the cohomologicality requirement, V has finite order on M_0 instead of order 1.

The Dieudonné correspondence

Example

$$D(\underline{A}) = A \quad \text{with } V = \mathrm{id},\ F = p$$

Example

$$D(\hat{\mathbb{G}}_a) = k[V^{\pm 1}]/k[V] = \langle \bullet \xleftarrow{V} \bullet \xleftarrow{V} \cdots \rangle, \quad F = 0$$

Example (and Proposition)

For the free formal group $\mathrm{Fr}(X)$ on a scheme $X = \operatorname{Spec} A$, A finite:

$$D(\mathrm{Fr}(X)) = \mathrm{Hom}(CW(A), CW(k))$$

$CW(A)$ = p-typical "co-Witt vectors" are to $W(A)$ what $\mathbf{Z}/p^\infty$ is to $\mathbf{Z}_p$.

The tensor product

Theorem (Goerss '99, Buchstaber-Lazarev '07)

Given two formal I-modules M, N,

$$D(M \otimes N) \cong D(M) \boxtimes D(N),$$

where

$$A \boxtimes B = \mathcal{R} \otimes_{W(k)\langle V\rangle} (A \otimes B)/\sim,$$

$$Fx \otimes Va \otimes b \sim x \otimes a \otimes Fb, \quad Fx \otimes a \otimes Vb \sim x \otimes Fa \otimes b$$

In particular, formal I-algebra schemes correspond to $\boxtimes$-algebras A with unit $I \to A$.

The evaluation product

Theorem (B)

The evaluation product

$$\{\textit{cohomological formal } l\textit{-modules}\} \times \{k\textit{-algebras}\} \xrightarrow{\text{ev}} \{l\textit{-modules}\}$$

satiesfies $G(A) = D(G) \circ A$*, where*

$$M \circ A = \mathrm{Tor}^{W(k)}(M, CW(A))^{F,V} = \ker \begin{pmatrix} \mathrm{Tor}(F, \mathrm{id}) - \mathrm{Tor}(\mathrm{id}, V) \\ \mathrm{Tor}(V, \mathrm{id}) - \mathrm{Tor}(\mathrm{id}, F) \end{pmatrix}$$

Note: $\circ$ is linear on the left, but not on the right.

The evaluation product, simplified

If I is an $\mathbf{F}_p$-algebra, there is a simpler description.

Theorem

If I is an $\mathbf{F}_p$-algebra, $M \circ A \cong M \otimes_F^V W(A)$, where

$$M \otimes_F W(A) = M \otimes W(A)/(Fm \otimes a - m \otimes Va)$$

and

$$M \otimes_F^V W(A) = \ker(V \otimes \mathrm{id} - \mathrm{id} \otimes F) \text{ on } M \otimes_F W(A)$$

The evaluation product, simplified

$$M \otimes_F W(A) = M \otimes W(A)/(Fm \otimes a - m \otimes Va)$$

in more explicit terms:
Define polynomials $c_i(x, y)$ inductively by

$$x^{p^n} + y^{p^n} = c_0(x, y)^{p^n} + pc_1(x, y)^{p^{n-1}} + \cdots + p^n c_n(x, y)$$

$$c_0(x, y) = x + y, \quad c_1(x, y) = \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} x^i y^{p-i}, \ldots$$

Then $M \otimes_F W(A)$ is generated by symbols (m, a) modulo left linearity and

$$(m, a) + (m, b) \sim \sum_{i=0}^{\infty} (F^i m, c_i(a, b))$$

The composition product

Theorem (B)

Given a two k-algebra scheme F and G, $D(G \circ F) = D(F) \circ D(G)$. *Here,*

$$M \circ N = (M \otimes_F^V N),$$

where $M \otimes_F N$ *is generated by* (m, n) *modulo left linearity and*

$$(m, n) + (m, n') = (m, n + n') + (Fm, Vc_1(n, n')) + \cdots ,$$

$$(M \otimes_F^V N) = \ker(V \otimes \mathrm{id} - \mathrm{id} \otimes V(-)^p).$$

The $\mathcal{R}$*-module structure on* $M \circ N$ *is given by* $V(m, n) = (m, Vn)$ *and* $F(m, n) = (m, Fn) + (Fm, n^p)$. *The multiplication is componentwise.*

The plethory for $K(1)$, $p > 2$

Classical stable computation:

$$K(n)_*(K(n)) = P(b_1, b_2, \dots)/(b_i^{p^n} - v_n^? b_i) \otimes \bigwedge(a_0, \dots, a_{n-1}).$$

Make this an $\mathcal{R}$-module by defining $F = 0$, $V(a_i) = a_{i-1}$, $V(b_i) = b_{i-1}$.
Then it becomes a $(\boxtimes, \circ)$-bialgebra.
Let $P = D(\operatorname{Spf} K(n)^* \underline{K(n)}_*)$ be the plethory for $K(n)$ under the Dieudonné correspondence.
Stabilization: $P \to K(n)_* K(n)$ is *surjective* [Kuhn,Wilson].

Theorem

There is a short exact sequence of $(\boxtimes, \circ)$-bialgebras

$$k \to k[e]/(e^{2p-1} - e) \to P \to K(1)_* K(1) \to k.$$

Theorem

There is a short exact sequence of $(\boxtimes, \circ)$*-bialgebras*

$$k \to k[e]/(e^{2p-1} - e) \to P \to K(1)_* K(1) \to k.$$

More precisely, $P \cong k[e]/(e^{2p-1} - e) \otimes K(1)_* K(1)$ *as algebras,*

- $|e| = (1,1)$, $|a_0| = (2,1)$, $|b_i| = (2p^i, 2)$
- $V(b_i) = b_{i-1}$, $V(b_1) = e^2 =: b_0$, $V(e) = V(a_0) = 0$
- $F(a_0) = (1 - v_1^{-1} e^{2p-2} v_1) a_0$, $F(b_i) = 0 = F(e)$
- $\psi(e) = e \circ e$, $\psi(a_0) = a_0 \circ e_1 + e_1^2 \circ a_0$, $\sum_{n \geq 0}^F \psi(b_n) = \sum_{i,j \geq 0}^F b_i^{p^j} \circ b_j$