Configuration categories and embedding spaces

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IST, Lisbon

Alpine Algebraic and Applied Topology Conference
August 2016

Immersion theory (50s/60s)

Theorem (Smale-Hirsch)

Immersion theory is bundle theory. Let M^m and N^n be smooth manifolds with $m < n$. The map

$$\mathrm{imm}(M, N) \to \Gamma(E \to M)$$

is a weak equivalence, where E is the space of triples (x, y, α) where $x \in M$, $y \in N$ and $\alpha : T_x M \to T_y N$ injective linear map.

Examples: $\mathrm{imm}(\mathbb{R}^m, \mathbb{R}^n)$ is the space of linear injective maps from $\mathbb{R}^m$ to $\mathbb{R}^n$, alias $O(n)/O(n-m)$. More generally, $\mathrm{imm}(\mathbb{R}^m, N)$ is $V_m(N)$ the m-frame bundle of N.

Example with boundary: Let N be a manifold with ∂. The space of immersions relative to a fixed immersion on (a neighborhood of) the boundary is $\mathrm{imm}_\partial(D^m, N) \simeq \Omega^m V_m(N \backslash \partial N)$.

Spaces of smooth embeddings

Main theorem (B-Weiss)

Let N be a smooth manifold with boundary of dim $n \geq 5$, or $n \geq 4$ if $N \cong D^n$. Fix an embedding $S^m \to \partial N$. The square

$$\begin{array}{ccc} \mathrm{emb}_\partial(D^m, N) & \longrightarrow & injmap^s_\partial(D^m, N) \\ \downarrow & & \downarrow \\ \mathrm{imm}_\partial(D^m, N) & \longrightarrow & \Omega^m\Gamma \end{array}$$

is homotopy cartesian whenever $n - m \geq 3$.

Here Γ is the space of pairs (y, α) with $y \in N \backslash \partial N$ and α a derived map of operads $E_m \to E_{T_y N}$; $injmap^s_\partial(D^m, N)$ is the union of path components of the space of injective maps ($rel \partial$) that contain a smooth map. The lower hor. map is m-fold loops on the map $V_m(N \backslash \partial N) \to \Gamma$. The right-hand map? Later.

High dimensional knots

Alexander isotopy: $injmap_\partial(D^m, D^n)$ is contractible.

Corollary (earlier variants: Arone-Turchin, Dwyer-Hess and Turchin)

If $n - m \geq 3$, then

$$\mathrm{emb}_\partial(D^m, D^n) \to \mathrm{imm}_\partial(D^m, D^n) \to \Omega^m \mathbb{R}\mathrm{map}(E_m, E_n)$$

is a homotopy fiber sequence of m-fold loop spaces.

If $m = 1$ the right hand-map has a homotopy retraction
$\Rightarrow \mathrm{emb}_\partial(D^1, D^n)$ is also a 2-fold loop space (Salvatore, Sinha).

In fact: (Millett) if N is contractible and $n \geq 5$, then $injmap_\partial(D^m, N)$ is also contractible! So get a similar homotopy fiber sequence in that case.

Configuration categories

Let M^m be a (topological) manifold. Write $\underline{k} = \{1, \ldots, k\}$.

Definition (Andrade)

An object in $\mathrm{con}(M)$ is an embedding $\underline{k} \hookrightarrow M$ for some $k \geq 0$.

A morphism from $x : \underline{k} \hookrightarrow M$ to $y : \underline{\ell} \hookrightarrow M$ is a pair (α, H) where $\alpha : \underline{k} \to \underline{\ell}$ is a map of finite sets and H is a path in $\mathrm{map}(\underline{k}, M)$ from x to $y\alpha$ subject to:

$$H_s(x_i) = H_s(x_j) \text{ for some } s \Rightarrow H_t(x_i) = H_t(x_j) \text{ for all } t > s$$

That is, *when collisions occur, they cannot be undone*.

These are (reversed) exit paths in the stratified space $\mathrm{map}(\underline{k}, M)$.

Nerve: $\mathrm{con}(M)_0$ = space of objects; $\mathrm{con}(M)_1$ = space of morphisms; $\mathrm{con}(M)_2$ = space of 2-composable morphisms; etc.

A few basic properties:

- Reference map to Fin, the category of finite sets.
- Fiberwise complete: Let $\mathrm{con}(M)_1^{he}$ denote the subspace of morphisms which are homotopy invertible. These correspond to isotopies of configurations (underlying map of finite sets is a bijection). So the square

$$\begin{array}{ccc} \mathrm{con}(M)_1^{he} & \longrightarrow & \mathrm{con}(M)_0 \\ \downarrow & & \downarrow \\ \mathrm{Fin}_1^{he} & \longrightarrow & \mathrm{Fin}_0 \end{array}$$

  is homotopy cartesian.
- Functoriality: If $M \hookrightarrow N$ is an injective map, then get $\mathrm{con}(M) \to \mathrm{con}(N)$ over Fin.

The space of morphisms with a fixed target object $y : \underline{\ell} \hookrightarrow M$ is identified with

$$\coprod_{\alpha:\underline{k}\to\underline{\ell},\ k\geq 0} \prod_{i=1}^{\ell} \mathrm{emb}(\alpha^{-1}(i), T_{y_i}M)$$

(using a result of Miller on exit paths in Quinn's homotopically stratified spaces.)

This only depends on the dimension of M and ℓ. So, for $U \subset M$ open, the square

$$\begin{array}{ccc} \mathrm{con}(U)_1 & \longrightarrow & \mathrm{con}(M)_1 \\ \downarrow{\scriptstyle \text{target}} & & \downarrow{\scriptstyle \text{target}} \\ \mathrm{con}(U)_0 & \longrightarrow & \mathrm{con}(M)_0 \end{array}$$

is homotopy cartesian.

Local-to-global

Local-to-global: $\text{con}(-)$ is a homotopy cosheaf wrt open covers $\{U_i \to M\}$ with the property that every finite subset $S \subset M$ is contained in some U_i.

For such a cover, and for every $k \geq 0$, the collection

$$\{\text{emb}(\underline{k}, U_i) \to \text{emb}(\underline{k}, M)\}_{i \in I}$$

forms an open cover. It follows (Dugger-Isaksen) that

$$\underset{[n]\in\Delta}{\text{hocolim}} \coprod_{i_0,\dots,i_n} \text{emb}(\underline{k}, U_{i_0} \cap \cdots \cap U_{i_n}) \to \text{emb}(\underline{k}, M)$$

is a weak equivalence.

The square

$$\begin{array}{ccc}
\underset{[n]\in\Delta}{\text{hocolim}} \coprod_{i_0,\dots,i_n} \text{con}(U_{i_0}\cap\cdots\cap U_{i_n})_1 & \longrightarrow & \text{con}(M)_1 \\
\Big\downarrow \text{target} & & \Big\downarrow \text{target} \\
\underset{[n]\in\Delta}{\text{hocolim}} \coprod_{i_0,\dots,i_n} \text{con}(U_{i_0}\cap\cdots\cap U_{i_n})_0 & \longrightarrow & \text{con}(M)_0
\end{array}$$

is ho. cartesian $\Rightarrow$ top horizontal map is also a weak equivalence.

$\Rightarrow$ same for $\text{con}(M)_k$ for $k \geq 2$.

Relation to operads

Can recover E_n from $\mathrm{con}(\mathbb{R}^n)$.

Roughly, there is a natural zigzag of weak equivalences

$$E_n \xleftarrow{\simeq} \cdots \xrightarrow{\simeq} A(\mathrm{con}(\mathbb{R}^n))$$

where A is some (homotopy invariant) functor from simplicial spaces over $N\mathrm{Fin}$ to (∞) operads.

More details: Let Tree denote the category whose objects are non-empty, finite rooted trees (Moerdijk-Weiss).

Morphisms: Such a tree T freely generates an operad Free(T) with the set of edges as colors and generating operations specified by the vertices. Then set

$$\hom_{\mathrm{Tree}}(S, T) := \{\text{operad maps } \mathrm{Free}(S) \to \mathrm{Free}(T)\}$$

A functor from Tree^{op} to spaces is called a dendroidal space; the nerve $N_d P$ of an operad is given by

$$(N_d P)_T = \hom_{\mathrm{Operads}}(\mathrm{Free}(T), P)$$

.

If P has a single colour:

$$(N_d P)_T = \prod_{v \in T} P(|v|)$$

where v runs over the vertices of T and $|v| =$ set of inputs at v.

Theorem (Cisinski-Moerdijk)

The homotopy theory of dendroidal spaces satisfying Segal + Rezk-type conditions is equivalent to the homotopy theory of operads in spaces.

To relate to configuration categories:
Let simp(Fin) be the category of simplices of $N\mathrm{Fin}$. Objects are (non-empty) strings of maps of finite sets

$$S_0 \to \cdots \to S_k \ ,$$

and morphisms are given by composing maps or inserting identities.

A simplicial space over the nerve of Fin, $X \to N\mathrm{Fin}$, is the same as a functor $\mathrm{simp}(\mathrm{Fin})^{\mathrm{op}} \to \mathit{spaces}$. There are maps

$$\mathrm{simp}(\mathrm{Fin}) \xrightarrow{\psi} \mathrm{Tree}^{rc} \xrightarrow{\iota} \mathrm{Tree}$$

where Tree^{rc} is the subcategory of trees with no leaves and root-preserving maps.

Note: $(\iota\psi)^* N_d E_n \simeq \mathrm{con}(\mathbb{R}^n)$.

Have:

$$\mathrm{PSh}(\mathrm{simp}(\mathrm{Fin})) \underset{\psi_*}{\overset{\psi^*}{\leftrightarrows}} \mathrm{PSh}(\mathrm{Tree}^{rc}) \underset{\iota^*}{\overset{\iota_!}{\rightleftarrows}} \mathrm{PSh}(\mathrm{Tree})$$

(left adjoints on top)

For X an operad (dendroidal space) with a single color such that $X(0)$ and $X(1)$ are contractible, the (co)unit maps

$$X \leftarrow \mathbb{L}\iota_!\mathbb{R}\iota^*X \to \mathbb{L}\iota_!(\mathbb{R}\psi_*\mathbb{L}\psi^*\iota^*X)$$

are weak equivalences.

Theorem (B-Weiss)

Let P and Q be operads with contractible spaces of 0 and 1-arity operations. Then

$$\psi^*\iota^* : \mathbb{R}\mathrm{map}_{Operads}(P, Q) \to \mathbb{R}\mathrm{map}_{\mathrm{Fin}}(\psi^*\iota^* P, \psi^*\iota^* Q)$$

is a weak equivalence.

In particular, for every m, n, the map

$$\psi^*\iota^* : \mathbb{R}\mathrm{map}(E_m, E_n) \to \mathbb{R}\mathrm{map}_{\mathrm{Fin}}(\mathrm{con}(\mathbb{R}^m), \mathrm{con}(\mathbb{R}^n))$$

is a weak equivalence.

Local configurations

Definition

The local configuration category of M is the overcategory of $\mathrm{con}(M)$ over the subspace of objects consisting of configurations of cardinality 1.

I.e. $\mathrm{con}^{\mathrm{loc}}(M)_0 \subset \mathrm{con}(M)_1$ consisting of morphisms over $\underline{k} \to \underline{1}$, $k \geq 0$.

Properties:

- reference map to Fin, fiberwise complete
- $\mathrm{con}^{\mathrm{loc}}(-)$ is "functorial" with respect to *local* embeddings.
- local-to-global: $\mathrm{con}^{\mathrm{loc}}(-)$ is a homotopy cosheaf with respect to all open covers

Parametrized version: $\mathbb{R}\mathrm{map}_{\mathsf{Fin}}(\mathrm{con}^{\mathrm{loc}}(M), \mathrm{con}^{\mathrm{loc}}(N))$ is identified with the section space of a fibration $E \to M$ where the fiber over $x \in M$ is

$$\{(y, \alpha) : y \in N, \alpha \text{ a derived operad map } E_{T_x M} \to E_{T_y N}\}$$

A homotopy pullback square

Theorem (B-Weiss)

There is a commutative square

$$\begin{array}{ccc} \mathrm{emb}(M,N) & \longrightarrow & \mathbb{R}\mathrm{map}_{\mathsf{Fin}}(\mathrm{con}(M),\mathrm{con}(N)) \\ \downarrow & & \downarrow \\ \mathrm{imm}(M,N) & \longrightarrow & \mathbb{R}\mathrm{map}_{\mathsf{Fin}}(\mathrm{con}^{\mathrm{loc}}(M),\mathrm{con}^{\mathrm{loc}}(N)) \end{array}$$

which is homotopy cartesian whenever $n - m \geq 3$.

Proof: manifold functor calculus (and so it relies on the multiple disjunction lemmas of Goodwillie-Klein).

∂ version

Fix an embedding of a collar of ∂M into a collar of ∂N.

Theorem (B-Weiss)

There is a commutative square

$$\begin{array}{ccc} \mathrm{emb}_\partial(M,N) & \longrightarrow & \mathbb{R}\mathrm{map}^\partial_{\mathrm{Fin}_*}(\mathrm{con}(M), \mathrm{con}(N)) \\ \downarrow & & \downarrow \\ \mathrm{imm}_\partial(M,N) & \longrightarrow & \mathbb{R}\mathrm{map}^\partial_{\mathrm{Fin}_*}(\mathrm{con}^{\mathrm{loc}}(M), \mathrm{con}^{\mathrm{loc}}(N)) \end{array}$$

which is homotopy cartesian whenever $n - m \geq 3$.

The Alexander trick for configuration categories

Theorem (B-Weiss)

$injmap_\partial(D^m, D^n) \simeq \mathbb{R}\mathrm{map}^\partial_{\mathsf{Fin}_*}(\mathrm{con}(D^m), \mathrm{con}(D^n)).$

That is, the restriction map

$$\mathbb{R}\mathrm{map}_{\mathsf{Fin}_*}(\mathrm{con}(D^m), \mathrm{con}(D^n)) \to \mathbb{R}\mathrm{map}_{\mathsf{Fin}_*}(\mathrm{con}(D^m \backslash 0), \mathrm{con}(D^n \backslash 0))$$

is a weak homotopy equivalence. As opposed to the usual Alexander trick, this is difficult!

Combine with the operadic description to get the main theorem for disks. The argument for a general N is deduced from the case of disks, through smoothing theory.

Smoothing theory

Let emb^{TOP} and imm^{TOP} denote the spaces of (locally flat) topological embeddings and immersions, respectively.

Theorem (Morlet, Lashof)

Let $m, n \geq 5$. The commutative square

$$\begin{array}{ccc} \mathrm{emb}(M,N) & \longrightarrow & \mathrm{emb}^{TOP}(M,N) \\ \downarrow & & \downarrow \\ \mathrm{imm}(M,N) & \longrightarrow & \mathrm{imm}^{TOP}(M,N) \end{array}$$

is homotopy cartesian.

Sketch proof of the main theorem for an arbitrary target N: Let $e : D^m \to N$ be a smooth embedding extending $\partial D^m \to \partial N$. Take a normal tube around e, i.e.

$$f : D^m \times D^{n-m} \hookrightarrow N$$

such that $f^{-1}(\partial N) = \partial D^m \times D^{n-m}$. By smoothing theory and the main theorem for disks (...),

$$\begin{array}{ccc} \mathit{injmap}^s_\partial(D^m, D^n) & \longrightarrow & \mathbb{R}\mathrm{map}^\partial_{\mathsf{Fin}}(\mathrm{con}(D^m), \mathrm{con}(D^n)) \\ \downarrow & & \downarrow \\ \mathit{injmap}^s_\partial(D^m, N) & \longrightarrow & \mathbb{R}\mathrm{map}^\partial_{\mathsf{Fin}}(\mathrm{con}(D^m), \mathrm{con}(N)) \end{array}$$

is homotopy cartesian. Top horizontal map is a weak equivalence, so the lower horizontal map is also weak equivalence over the basepoint component determined by f. Now vary e.

Application to spaces of homeomorphisms

Let $TOP(n)$ denote the top. group of homeomorphisms of $\mathbb{R}^n$ and $TOP(n, m)$ the subgroup of those homeomorphisms which fix $\mathbb{R}^m$ pointwise. Let $TOP(n)/TOP(n, m)$ denote the homotopy fiber of

$$BTOP(n) \to BTOP(n, m) .$$

There is a diagram of m-fold loop maps:

$$\begin{array}{ccccc} \mathrm{emb}_\partial(D^m, D^n) & \longrightarrow & \Omega^m O(n)/O(n, m) & \longrightarrow & \Omega^m TOP(n)/TOP(n, m) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{emb}_\partial(D^m, D^n) & \longrightarrow & \Omega^m O(n)/O(n, m) & \longrightarrow & \Omega^m \mathbb{R}\mathrm{map}(E_m, E_n) \end{array}$$

The top sequence is a homotopy fiber sequence by Morlet, Lashof, Lees...

Conclusion: the map

$$TOP(n)/TOP(n,m) \to \mathbb{R}\mathrm{map}(E_m, E_n)$$

is an iso on π_i for $i > m$.

Speculations

Question: Is the map

$$TOP(n)/TOP(n,m) \to \mathbb{R}\mathrm{map}(E_m, E_n)$$

an almost weak equivalence (ho. fibers contractible or empty)?

Dwyer: Is the map

$$TOP(n) \to \mathbb{R}Aut^h(E_n)$$

a weak equivalence?

Calculate $\pi_*\mathbb{R}\mathrm{map}(E_m, E_n)$? Rationally, recent work by Fresse-Turchin-Willwacher (via graph complexes and Kontsevich formality).