Kahn's realizability problem

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Alpine Algebraic and Applied Topology Conference

Realizability. How to play

- Give you a (abstract) group G
- Give you category $\mathcal{C}$
- Give me back an object X in $\mathcal{C}$ such that $\mathrm{Aut}_{\mathcal{C}}(X) \cong G$

Example 1

- $G = \mathbb{Z}_2$
- $\mathcal{C} = HoTop_*$
- Then, $X = \mathbb{S}^n$

Example 2

- $G = \mathbb{Z}_p$, p odd
- $\mathcal{C} = Groups$,
- Then, $\mathrm{Aut}_{\mathcal{C}}(X) \ncong \mathbb{Z}_p$, $\forall X$

So, finite groups can not, in general, be realized in the category of groups

Are finite groups realizable in $HoTop_*$?

Our problem

Let $\mathcal{E}(X)$= group of homotopy classes of self homotopy-equivalences of X

finite group G

$\Downarrow$ *Realization*

$G \cong \mathcal{E}(X)$ for some X?

Overview

- Proposed by Kahn in the late 60's, appears recurrently in literature
- The only general known procedure to tackle this problem is when $G = Aut(\pi), \pi$ a group. Then $X = K(\pi, n)$, since $\mathcal{E}(X) \cong Aut(\pi)$.
- Approach $\mathcal{E}(X)$ by its distinguished subgroups

$$\mathcal{E}_\sharp(X), \mathcal{E}_*(X), \mathcal{E}^*(X) \dots$$

Example

$\mathbb{Z}_2 \cong \mathcal{E}(S^n)$

$\mathbb{Z}_2 \cong \mathcal{E}(K(\mathbb{Z}_3, n))$ since $Aut(\mathbb{Z}_3) \cong \mathbb{Z}_2$

$\mathbb{Z}_2 \cong \mathcal{E}(X)$ for some 1-connected rational space X [Arkowitz-Lupton'00]

Which finite groups are realizable by simply connected rational spaces?

New perspective

Idea

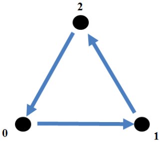
Introduce graphs on the picture

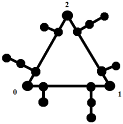
$$\begin{aligned} \text{groups} &\longrightarrow \text{graphs} \\ \text{graphs} &\longrightarrow \text{DGA's} \\ \text{DGA's} &\longrightarrow \text{rational homotopy types} \end{aligned}$$

Theorem (Frucht'39, Realizability in $\mathcal{C} = \mathit{Graphs}$)

Every finite group G is realizable by a finite, connected and simple graph $\mathcal{G}$.

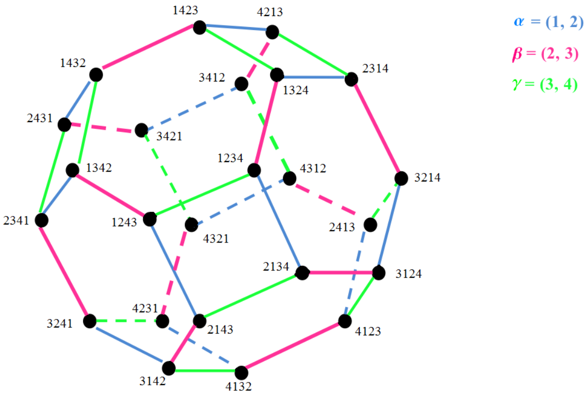
Example 1 ($G = \mathbb{Z}_3$, Cayley graph $\to$ simple graph)





New perspective

Example 2 ($G = \Sigma_4$, Cayley graph $\to$ simple graph)



New perspective

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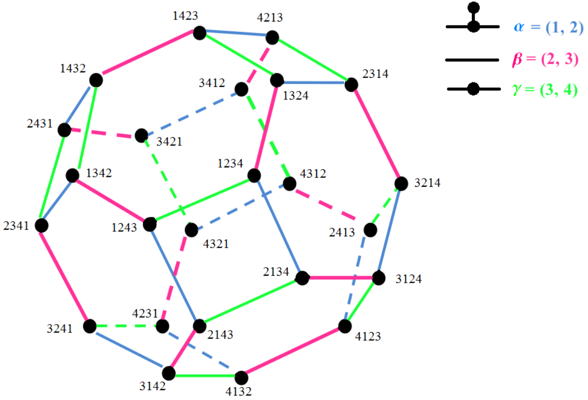

New perspective

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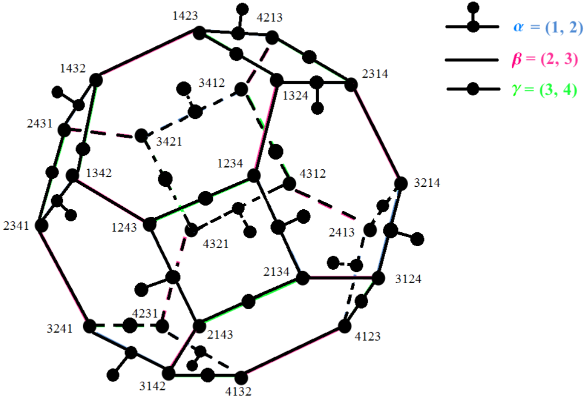

Our problem revisited

Problem 1

Let $\mathcal{G} = (V, E)$ be a finite, simple, connected graph (with more than one vertex). Does there exist a space X such that $\mathrm{Aut}(\mathcal{G}) \cong \mathcal{E}(X)$?

Solving Problem 1

▷ First, restrict ourselves $Graph_{fm} \subset Graph$.

▷ Then, construct

$$A : Graph_{fm} \longrightarrow DGA$$

$$(A_{\mathcal{G}}, d) = \big(\Lambda(x_1, x_2, y_1, y_2, y_3, z) \otimes \Lambda(x_v, z_v | v \in V), d\big)$$

- generators in dimensions: $|x_1| = 8, |x_2| = 10, |y_1| = 33, |y_2| = 35,$ $|y_3| = 37, |z| = 119, |x_v| = 40, |z_v| = 119,$
- differentials:

$$\begin{aligned}
d(x_1) &= 0 & d(y_3) &= x_1 x_2^3 \\
d(x_2) &= 0 & d(x_v) &= 0 \\
d(y_1) &= x_1^3 x_2 & d(z) &= y_1 y_2 x_1^4 x_2^2 - y_1 y_3 x_1^5 x_2 + y_2 y_3 x_1^6 + x_1^{15} + x_2^{12} \\
d(y_2) &= x_1^2 x_2^2 & d(z_v) &= x_v^3 + \textstyle\sum_{[v,w]\in E} x_v x_w x_2^4
\end{aligned}$$

- A is contravariant (morphisms are as expected).

Solving Problem 1

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$$(A_{\mathcal{G}}, d) = (\underbrace{\Lambda(x_1, x_2, y_1, y_2, y_3, z)}_{\text{Homotopically Rigid}} \otimes \underbrace{\Lambda(x_v, z_v | v \in V)}_{\text{Encodes } \mathcal{G}}, d)$$

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- A is contravariant (morphisms are as expected).

Solving Problem 1

Theorem

Let $\mathcal{G}$, $A_{\mathcal{G}}$ defined as previously. Then:

- There exists a split short exact sequence

$$K \to \mathrm{Aut}(A_{\mathcal{G}}) \to \mathrm{Aut}(\mathcal{G})$$

  where K is abelian and torsion-free.

- $A_{\mathcal{G}}$ is an elliptic algebra (hence Poincaré duality) of formal dimension $d = 208 + 80|V|$

- Let $X_{\mathcal{G}}$ the rational elliptic 1-connected space whose Sullivan minimal model is $A_{\mathcal{G}}$. The monoid of self-homotopy classes of $X_{\mathcal{G}}$ is

$$[X_{\mathcal{G}}, X_{\mathcal{G}}] = \{f_0, f_1\} \cup \mathrm{Aut}(\mathcal{G})$$

Solving Problem 1

Theorem

Every finite group G is realized by infinitely many (non homotopically equivalent) rational elliptic spaces X. That is, $G \cong \mathcal{E}(X)$.

Before we get into specific categories of problems, let me say that there are two very broad problems - somewhat vague and general - that most workers agree are very important:

A. Calculate the groups $\mathcal{E}(X)$ explicitly in as many cases as possible, and express the known calculations in the most simple and concrete terms.

B. Develop applications of the group $\mathcal{E}(X)$ to other parts of topology (and mathematics in general).

D. Kahn'90

Applications

Idea (Crowley-Löh, 2015)

Degree theorems "à la Gromov" are strongly related with the existence of inflexible manifolds

Definition (Inflexible manifold)

An oriented closed connected manifold M is inflexible if

$$\{\deg f \mid f : M \to M \text{ continuous}\} \subset \{-1, 0, 1\}$$

Inflexible manifolds are constructed (using rational homotopy theory) in dimensions $64 \cup \{d \cdot k \mid k \in \mathbb{N}, d = 108, 208, 228\}$ ($\equiv 0 \pmod 4$).

Applications

Recall that

- $X_{\mathcal{G}}$ is an elliptic space of formal dimension $d = 208 + 80|V|$ such that

$$[X_{\mathcal{G}}, X_{\mathcal{G}}] = \{f_0, f_1\} \cup \mathrm{Aut}(\mathcal{G})$$

- Therefore, if $X_{\mathcal{G}}$ is the rationalisation of a manifold M, then M is inflexible

But $d \equiv 0 \pmod 4$ so we are in the bad range of the obstruction theory of Barge and Sullivan

Modifying our construction we get ...

Applications

Theorem

For any connected finite graph $\mathcal{G}$, there exist $\widetilde{A}_{\mathcal{G}}, \widetilde{X}_{\mathcal{G}}$ such that:

- $\widetilde{A}_{\mathcal{G}}$ is an elliptic dga of formal dimension $d = 2(208 + 80|V|) - 1$. Since $d \equiv 3 \pmod 4$, $\widetilde{X}_{\mathcal{G}}$ is the rationalization of a d-manifold $M_{\mathcal{G}}$.
- The self-monoid $[\widetilde{X}_{\mathcal{G}}, \widetilde{X}_{\mathcal{G}}] \cong \{f_0, f_1\} \cup \mathrm{Aut}(\mathcal{G})$. Hence $M_{\mathcal{G}}$ is inflexible.

Theorem

For every finite group G, there exist infinitely many inflexible manifolds M_G such that

$$\mathcal{E}((M_G)_{\mathbb{Q}}) \cong G$$

What happens if G acts on a module M?

Realizability level 2. How to play

- Algebraic structure (G, M)

  G is a group, M is a finitely generated $\mathbb{Z}G$-module

- Homotopy invariant $(\mathcal{E}(-), \pi_k(-))$

  $\pi_k(-)$ is a $\mathbb{Z}\mathcal{E}(-)$-module

Problem 2 (realizability of actions)

Is there a finite Postnikov piece X such that the $\mathbb{Z}G$-module M is isomorphic to the $\mathbb{Z}\mathcal{E}(X)$-module $\pi_k(X)$, for some $k \geq 2$?

▷ "Homotopique dual" of the G-Moore spaces problem (Steenrod'60)

▷ It implies realizability of groups

Realizability level 2. How to play

- Algebraic structure (G, V)

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New Perspective

Idea

Introduce Invariant Theory on the picture.

- G acts on $\mathbb{Q}[V]$: for $g \in G$, $p \in \mathbb{Q}[V]$, $(gp)(v) = p(g^{-1}v)$.
- G-invariant function: $p \in \mathbb{Q}[V]$ such that for all $g \in G$, $gp = p$.
- The invariant ring $\mathbb{Q}[V]^G$: all the G- invariant functions in $\mathbb{Q}[V]$

(Characterization of finite $G \leq GL(V)$, Hilbert, Noether)

Let V be a finitely generated and faithful $\mathbb{Q}G$-module. Then, there exists algebraic forms $p_1, \ldots, p_r \in \mathbb{Q}[V]^G$ such that, for $f \in GL(V)$

$$f \in G \quad \text{if and only if} \quad p_i \circ f = p_i,\ \forall i$$

we modify those algebraic forms

Solving Problem 2

Lemma

There exist a family $\mathcal{Q} = \{q_0, q_1, \dots q_r, q_{r+1}\} \subset \mathbb{Q}[V]^G$ where

1. $q_0 = \sum_{1}^{N} \underset{\neq 0}{\lambda_j} v_j^2$, for a good choice of basis of V^* ($N = dim_{\mathbb{Q}} V$),
2. $\deg(q_i) < \deg(q_{i+1})$ for all i,
3. $q_{r+1} = (q_0)^s$ for $s \gg N$

such that G is the orthogonal group $O(\mathcal{Q}) \leq GL(V)$.

Definition (Realizable family of forms)

A family of algebraic forms $\mathcal{Q} \subset \mathbb{Q}[v_1, \dots, v_N]$ verifying 1, 2 and 3.

For an arbitrary realizable family, and any $n > \; deg(q_{r+1}) \dots$

Solving Problem 2

$$\mathcal{M}_{(\mathcal{Q},n)} = \Big(\Lambda(x_1, x_2, y_1, y_2, y_3, z, v_j \mid j = 1, \ldots, N), d\Big)$$

$$\begin{aligned}
\deg x_1 &= 8, & d(x_1) &= 0 \\
\deg x_2 &= 10, & d(x_2) &= 0 \\
\deg y_1 &= 33, & d(y_1) &= x_1^3 x_2 \\
\deg y_2 &= 35, & d(y_2) &= x_1^2 x_2^2 \\
\deg y_3 &= 37, & d(y_3) &= x_1 x_2^3 \\
\deg v_j &= 40, & d(v_j) &= 0 \\
\deg z &= 80n + 39, & d(z) &= \sum_{i=1}^{r+1} q_i x_1^{10n+5-5\deg(q_i)} + q_0(x_1^{10n-5} + x_2^{8n-4}) \\
& & &+ x_1^{10(n-1)}(y_1 y_2 x_1^4 x_2^2 - y_1 y_3 x_1^5 x_2 + y_2 y_3 x_1^6) \\
& & &+ x_1^{10n+5} + x_2^{8n+4}.
\end{aligned}$$

Solving Problem 2

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& & &+ x_1^{10n+5} + x_2^{8n+4}.
\end{aligned}$$

Codifies the action

Solving Problem 2

Theorem

$$\mathcal{E}(\mathcal{M}_{(\mathcal{Q},n)}) \cong O(\mathcal{Q})$$

Corollary

Let G be a finite group, and V a finitely generated faithful $\mathbb{Q}G$-module. Then, there exists a Postnikov piece X such that, for some $k \geq 2$,

$$\big(G, V\big) \cong \big(\mathcal{E}(X), \pi_k X\big)$$

Example (realization of infinite groups)

Let $\mathcal{O}(m; k) < GL_{m+k}(\mathbb{R})$ preserving:

$$q_0 = x_1^2 + x_2^2 + \ldots + x_m^2 - x_{m+1}^2 - \ldots - x_{m+k}^2.$$

The family $\mathcal{Q} = \{q_0, (q_0)^{m+k+1}\} \subset \mathbb{Q}[x_1, \ldots, x_{m+k}]$ is realizable. Then,

- $O(\mathcal{Q})$ can be realized by infinitely many (rational) spaces.
- $O(\mathcal{Q}) \cong \mathcal{O}(m; k)(\mathbb{Q})$, which is an infinite group for $m \geq 2$.

Our solutions to Problem 1 and Problem 2 depend on:

- ▷ A very specific homotopically rigid algebra. It is not unique:

  For a fixed $k > 4$, define $\mathcal{M}_k = \Big(\Lambda(x_1, x_2, y_1, y_2, y_3, z), d\Big)$

$$
\begin{aligned}
\deg x_1 &= 5k-2, & d(x_1) &= 0\\
\deg x_2 &= 6k-2, & d(x_2) &= 0\\
\deg y_1 &= 21k-9, & d(y_1) &= x_1^3 x_2\\
\deg y_2 &= 22k-9, & d(y_2) &= x_1^2 x_2^2\\
\deg y_3 &= 23k-9, & d(y_3) &= x_1 x_2^3\\
\deg z &= 15k^2-11k+1, & d(z) &= x_1^{3k-12}(x_1^2 y_2 y_3 - x_1 x_2 y_1 y_3 + x_2^2 y_1 y_2)\\
& & &\quad + x_1^{\frac{6k-2}{2}} + x_2^{\frac{5k-2}{2}}.
\end{aligned}
$$

  Theorem $[\mathcal{M}_k, \mathcal{M}_k] = \{0, 1\}$

- ▷ Rational homotopy theory (finite type over $\mathbb{Q}$, not over $\mathbb{Z}$).

Realizability. An integral approach

Following our approach for $\mathbb{Q}$

- Find an integral homotopically rigid space.
- Find a functor from a combinatorial category to integral spaces.

Idea

Introduce Toric Topology in the picture

Homotopically rigid space

$$\mathbb{H}P^\infty \simeq BS^3$$

Definition (Degree)

For $f : \mathbb{H}P^\infty \to \mathbb{H}P^\infty$, if $\deg(\Omega f : S^3 \to S^3) = k$, we say that $\deg(f) = k$.

(Feder-Gitler, Sullivan)

Self-maps of $\mathbb{H}P^\infty$ have either degree zero or any odd square integer.

(Classification Theorem, Mislin)

Self-maps of $\mathbb{H}P^\infty$ are classified up to homotopy by their degree.

Corollary

$\mathcal{E}(\mathbb{H}P^\infty) = \{1\}$

Polyhedral product functor

Let K be a simplicial complex on a set V of vertices, $v_1, \ldots, v_n$.
Let $(X, *)$ be a pointed space.

Definition(Bucthstaber-Panov, Bahri-Bendersky-Cohen-Gitler, Notbohm-Ray)

- For $\sigma \subset V$ face of K, the $\sigma-$power of X is:

$$X^\sigma = \{(x_1, \ldots, x_n) \in X^n \mid x_i = * \text{ if } v_i \notin \sigma\}$$

- The polyhedral product is the (homotopy) colimit of the diagram:

$$\begin{array}{rcl} X^K : CAT(K) & \to & Top_* \\ \sigma & \mapsto & X^\sigma \end{array}$$

By abuse of notation, we will also denote by X^K:

$$hocolim\, X^K \simeq colim\, X^K = \bigcup_{\sigma \in K} X^\sigma \subseteq X^n$$

Polyhedral product functor, examples

Example 1

$$\begin{array}{rcll} X^{\Delta[n-1]} & \simeq & X^n & \text{the } n\text{-fold product} \\ X^{\partial\Delta[n-1]} & \simeq & T^nX & \text{the fat wedge} \\ X^{\emptyset} & \simeq & * & \text{the trivial space} \end{array}$$

Example 2 (Davis-Januszkiewicz space)

For $X = BS^1$, $(BS^1)^K \simeq DJ(K)$ where $H^*(DJ(K);\mathbb{Z}) \underbrace{\cong \mathbb{Z}[K]}_{\text{face ring of } K}$.

Recall that: $\mathbb{Z}[K] = S_{\mathbb{Z}}(V)/\underbrace{(v_U : U \notin K)}_{\text{square free monomials}}$.

Conjecture

For a simplicial complex K,

$$\mathcal{E}((BS^3)^K) \cong Aut(K)$$

Example 1

For $K = \Delta[n-1]$

$$\mathcal{E}\big((BS^3)^n\big) \underset{\text{(Iwase)}}{\cong} \Sigma_n$$

Example 2

For $X = BS^1$, $K = \Delta[n-1]$

$$\mathcal{E}\big((BS^1)^{\Delta[n-1]}\big) \cong GL(n,\mathbb{Z}) \ncong \Sigma_n \cong Aut\big(\Delta[n-1]\big)$$

Solving Conjecture

Let K be a simplicial complex

Proposition

$$\mathcal{E}\big((BS^3)^K\big)/\mathcal{E}^*\big((BS^3)^K\big) \cong Aut(K)$$

Proof

- First, show $H^*((BS^3)^K;\mathbb{Z}) \cong \mathbb{Z}[K]$ with generators in degree 4.
- Then, identify $\mathcal{E}\big((BS^3)^K\big)/\mathcal{E}^*\big((BS^3)^K\big)$ to the image of

$$\begin{array}{rcl} \psi : \mathcal{E}((BS^3)^K) & \to & Aut\big(H^4((BS^3)^K;\mathbb{Z})\big) \\ f & \mapsto & H^4(f;\mathbb{Z}) \end{array}$$

- Finally, the entries of $M_f \in GL(n,\mathbb{Z})$ induced by $H^4(f;\mathbb{Z})$ are non negative integers (degrees of self-maps of BS^3). Then M_f and $M_{f^{-1}}$ are permutation matrices, and $Im\,\psi = Aut(K)$. □

Solving Conjecture

Theorem

Let K be a simplicial complex of dimension 1. Then

$$\mathcal{E}^*\left((BS^3)^K\right) \cong \{1\}$$

Proof (techniques of Dwyer-Mislin, Jackowski-McClure-Oliver, Nothbom-Ray)

Fix notation $X = BS^3$.

▷ Step1 We have:

$$\begin{array}{ccccccc} [X^K, X^K] & \overset{\text{injection}}{\rightsquigarrow} & [X^K, X^n] & \overset{\{\pi_j\}_1^n}{\rightsquigarrow} & [X^K, X] & \overset{\text{injection}}{\rightsquigarrow} & \prod_p [X^K, X_p^\wedge] \\ f & \rightsquigarrow & f & \rightsquigarrow & \{f_j\}_1^n & \rightsquigarrow & \{f_{j\,p}^{\wedge} \mid p\}_1^n \end{array}$$

we also have, for a face σ of K:

$$[X^\sigma, X] \underset{(Iwase)}{\cong} \{\underbrace{(0, 0, \ldots, a_i, 0)}_{\dim\sigma+1} \mid a_i = 0 \text{ or } a_i \text{ odd square}\}$$

Solving Conjecture

▷ Step 2 We then have, for every $j = 1, \ldots, n$, for every p prime:

$$\begin{array}{ccc} \mathcal{E}^*(X^K) & \rightsquigarrow & \left\{[X^\sigma, X_p^\wedge] \mid \sigma \in CAT(K)\right\} \\ f & \rightsquigarrow & f_j^\sigma \simeq_p \begin{cases} \pi_j & \text{if } v_j \in \sigma \\ * & \text{if } v_j \notin \sigma \end{cases} \end{array}$$

Is there $f \not\simeq Id_{X^K}$ inducing the same family?

▷ Step 3 The obstruction for the unicity lies in $\lim^i \Pi_i{}^p$ for

$$\begin{array}{rcl} \Pi_i{}^p : CAT^{op}(K) & \to & \mathcal{A}b \\ \sigma & \mapsto & \pi_i(map(X^\sigma, X_p^\wedge)_{f_j^\sigma}) \end{array}$$

that can be computed as the cohomology of a cochain complex

$$N^n(\Pi_i{}^p) = \prod_{\sigma_0 \to \sigma_1 \to \cdots \to \sigma_n} \Pi_i{}^p(\sigma_n)$$

As dim K = 1, $N^{\geq 3}(\Pi_i{}^p) = 0$, $N^2(\Pi_2{}^p) = 0$, and $H^1(N^*(\Pi_1{}^p)) = 0$. □

Solving Conjecture

Corollary 1

Let K be a simplicial complex of dimension 1. Then

$$\mathcal{E}\big((BS^3)^K\big) \cong Aut(K)$$

Corollary 2

Every finite group is realizable by infinitely many integral spaces.