Equivariant calculus and the tower of the identity on pointed G-spaces

Emanuele Dotto

University of Bonn

Saas-Almagell, 2016

Overview

Calculus of Functors ($G = 1$)

Let $F\colon \mathscr{C} \to \mathscr{D}$ be a homotopy functor between model categories.

Theorem (Goodwillie)

There is a "Taylor tower" of functors

$$\begin{array}{ccccccccccccc} \dots & \longrightarrow & P_nF & \longrightarrow & P_{n-1}F & \longrightarrow & \dots & \longrightarrow & P_2F & \longrightarrow & P_1F & \longrightarrow & F(*) \\ & \nearrow & & & & & & & & & & & \\ D_nF & & & & & & & & & & & & \end{array}$$

which satisfies:

- $F(X) \simeq \operatorname{holim}_n P_nF(X)$, *sometimes,*
- P_nF *is "n-excisive" (a homology theory when $n = 1$),*
- *For* $\mathscr{C} = \mathscr{D} = \mathrm{Top}_*$ *the layer* $D_nF = \mathrm{hofib}(P_nF \to P_{n-1}F)$ *decomposes as:*

$$D_nF(X) \simeq \Omega^\infty(\partial_nF \wedge X^{\wedge n})_{h\Sigma_n}$$

Where ∂_nF is a spectrum with Σ_n-action (naïve).

This is "Brown representability" for reduced homology theories of degree n.

What Goes Wrong Equivariantly?

Let G be a finite group. Let Top_*^G be the model category of G-spaces and fixed-points equivalences:

Definition

$f\colon X \to Y$ is a w.e. if $f^H\colon X^H \to Y^H$ is a w.e. of spaces for all $H \leq G$.

We can of course set $\mathcal{C} = \mathcal{D} = \mathrm{Top}_*^G$ and take the tower of $F\colon \mathrm{Top}_*^G \to \mathrm{Top}_*^G$. However:

Issues

- The layer is a naïve infinite loop space

$$D_nF(X) \simeq \Omega^\infty(\partial_n F \wedge X^{\wedge n})_{h\Sigma_n}$$

  ($\partial_n F$ is a naïve $G \times \Sigma_n$-spectrum).
- This decomposition holds only when the G-action on X is trivial.

The Case $n = 1$ (Blumberg)

Let $F\colon \mathrm{Top}_*^G \to \mathrm{Top}_*^G$ be reduced: $F(*) \simeq *$. Then

$$P_1 F(X) \simeq \operatorname*{hocolim}_{n\in\mathbb{N}} \Omega^n F(\Sigma^n X)$$

Construction

$$P_G F(X) := \operatorname*{hocolim}_{n\in\mathbb{N}} \Omega^{n\rho_G} F(\Sigma^{n\rho_G} X)$$

where $\rho_G = \mathbb{R}[G]$ is the regular representation of G.

Theorem (Blumberg)

$P_G F(X)$ is the universal "G-linear" approximation of F: $P_G F$ is linear and

$$P_G F(\bigvee_J X) \xrightarrow{\simeq} \prod_J P_G F(X)$$

for every finite G-set J.
It follows that "G-linear functors are equivalent to G-spectra".

Higher Equivariant Excision

Program

1. Formulate G-excision in "cubical terms",
2. Extend this notion to J-excision, for finite G-sets J,
3. Extend the framework from Top_*^G to general "equivariant homotopy theories" (e.g. G-spectra).

Equivariant Homotopy Theory

Let G be a finite group.

Definition (D-Moi/Hill)

A G-model category is a functor $\underline{\mathscr{C}}\colon \mathcal{O}_G^{op} \to ModCat$ where:

- $\mathcal{O}_G = \{$transitive G-sets and G-maps$\}$ is the orbit category of G,
- $ModCat$ is the category of model categories and left and right Quillen functors.

We will further assume that:

- $\underline{\mathscr{C}}(G/H) = \mathscr{C}^H$ is the category of H-objects in some category $\mathscr{C}$ (as 1-categories),
- The functors $\mathscr{C}^H \to \mathscr{C}^K$ are the standard restrictions and conjugations.

This is a homotopy theory "parametrized" by the orbit category of G. [Barwick-D-Glasman-Nardin-Shah] for an ∞-categorical setting.

Example)
- The categories Top^H with the fixed-points model structures,
- The categories Sp^H of orthogonal H-spectra with the H-stable model structures.

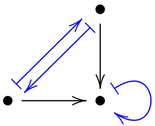
Equivariant Diagrams

Let G be a finite group, I a category with G-action and $\mathscr{C}$ a G-model category.

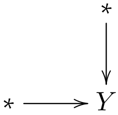
Theorem (D-Moi)

There exists a model category of I-shaped diagrams $X\colon I \to \mathscr{C}$ with "G-action": natural maps $g\colon X_i \longrightarrow X_{gi}$ compatible with the group structure.

Example) Let $G = \mathbb{Z}/2$, and $I = (\bullet \to \bullet \leftarrow \bullet)$ with G-action



If Y is a pointed $\mathbb{Z}/2$-space, the following is a $\mathbb{Z}/2$-equivariant diagram in Top_*:



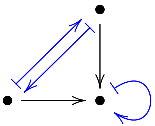
Equivariant Diagrams

Let G be a finite group, I a category with G-action and $\mathscr{C}$ a G-model category.

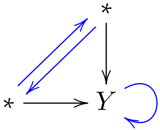
Theorem (D-Moi)

There exists a model category of I-shaped diagrams $X\colon I \to \mathscr{C}$ with "G-action": natural maps $g\colon X_i \longrightarrow X_{gi}$ compatible with the group structure.

Example) Let $G = \mathbb{Z}/2$, and $I = (\bullet \to \bullet \leftarrow \bullet)$ with G-action



If Y is a pointed $\mathbb{Z}/2$-space, the following is a $\mathbb{Z}/2$-equivariant diagram in Top_*:



Equivariant Homotopy Limits and Colimits

Let G be a finite group, I a category with G-action and $\mathscr{C}$ a G-model category.

Theorem (D-Moi)

There are well-behaved homotopy limit and colimit functors

$$\mathrm{holim}, \mathrm{hocolim} \colon \{I\text{-shaped } G\text{-diagrams in } \mathscr{C}\} \longrightarrow \mathscr{C}^G$$

Example) Let $G = \mathbb{Z}/2$ and Y a pointed $\mathbb{Z}/2$-space, then

$$\mathrm{holim}\left(\begin{array}{ccc} & & * \\ & & \downarrow \\ * & \longrightarrow & Y \end{array}\right) = \Omega Y = Map_*(S^1, Y)$$

Equivariant Homotopy Limits and Colimits

Let G be a finite group, I a category with G-action and $\mathscr{C}$ a G-model category.

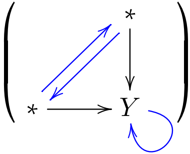
Theorem (D-Moi)

There are well-behaved homotopy limits and colimits functors

$$\mathrm{holim}, \mathrm{hocolim} \colon \{I\text{-shaped } G\text{-diagrams in } \mathscr{C}\} \longrightarrow \mathscr{C}^G$$

Example) Let $G = \mathbb{Z}/2$ and Y a pointed $\mathbb{Z}/2$-space, then

holim



$= \Omega^{\mathrm{sign}} Y = Map_*(S^{\mathrm{sign}}, Y)$

Consequence

This gives a systematic way of incorporating representations into equivariant homotopy theory.

Reformulation of G-Excision

Let J be a finite G-set, and $\mathcal{P}(J)$ the category of (all) subsets of J.
G acts on $\mathcal{P}(J)$ by $g \cdot U = \{g \cdot j \mid j \in U\}$.

Definition (Equivariant cubes)

A J-cube is a diagram $X\colon \mathcal{P}(J) \to \mathscr{C}$ with a G-action.

Let $F\colon \mathscr{C}^G \to \mathscr{D}^G$ be a homotopy functor.

Definition (G-excision)

F is G-excisive if

$$F_*\colon \{G_+\text{-cubes in } \mathscr{C}\} \longrightarrow \{G_+\text{-cubes in } \mathscr{D}\}$$

sends cocartesian cubes to cartesian cubes. (Here $G_+ = G \amalg \{+\}$).

Reformulation of G-Excision

Theorem (D-Moi)

Suppose that $F(*) \simeq *$. The following are equivalent:

- F *is* G*-excisive,*
- F *sends cocartesian* J*-cubes to cartesian* J*-cubes, for every finite* G*-set* J*,*
- $F(X) \simeq \Omega^{\rho_G} F(\Sigma^{\rho_G} X)$*, (that is* $F \simeq P_G F$*),*
- F *is excisive and* $F(\bigvee_J X) \simeq \prod_J F(X)$ *(Blumberg's definition).*

Higher Equivariant Excision

Let J be a finite G-set. Let $F\colon \mathscr{C}^G \to \mathscr{D}^G$ be a homotopy functor.

Definition (J-excision)

F is J-excisive if

$$F_*\colon \{J_+\text{-cubes in } \mathscr{C}\} \longrightarrow \{J_+\text{-cubes in } \mathscr{D}\}$$

sends "strongly cocartesian" cubes to cartesian cubes.

Examples

- An n-excisive functor is $\underline{n}$-excisive, for the trivial G-set $\underline{n} = \{1, \dots, n\}$,
- Let M be a $\mathbb{Z}[G]$-module.The Dold-Thom construction $M(-)\colon \mathrm{Top}_*^G \to \mathrm{Top}_*^G$ is G-linear,
- Let E be a G-spectrum. $E \wedge (-)\colon \mathrm{Top}_*^G, \mathrm{Sp}^G \to \mathrm{Sp}^G$ is G-linear. In particular the identity on Sp^G is G-linear,
- Let A be a commutative ring, M an A-bimodule. There is a $\mathbb{Z}/2$-spectrum

  $$\mathrm{THR}(A;M) = |[k] \longmapsto HM \wedge (HA)^{\wedge \underline{k}}|$$

  where $\mathbb{Z}/2$ acts on $\underline{k} = \{1, \dots, k\}$ by $i \mapsto k - i + 1$. Then

  $$\mathrm{THR}(A;M(-))\colon \mathrm{Top}_*^{\mathbb{Z}/2} \longrightarrow \mathrm{Sp}^{\mathbb{Z}/2}$$

  is $\mathbb{Z}/2$-excisive.
- Let K be a finite G-set. The norm $(-)^{\wedge K}\colon \mathrm{Sp}^G \to \mathrm{Sp}^G$ is $K \times G = |K| \times G$-excisive.

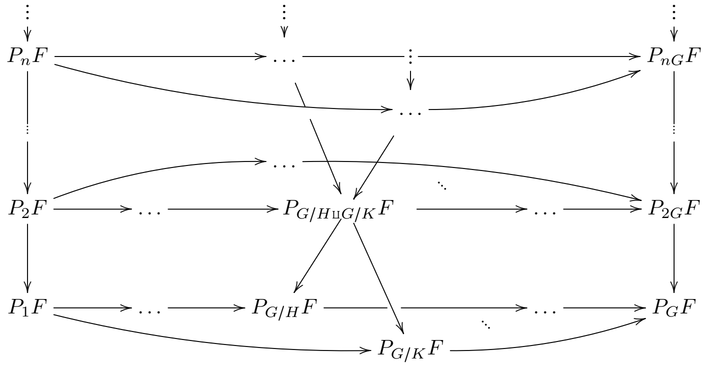
The Equivariant "Tower"

Let $F\colon \mathscr{C}^G \to \mathscr{D}^G$ be a homotopy functor.

Theorem (D)

There are J-excisive approximations $F \to P_J F$, and an essentially unique map $P_J F \to P_K F$ if there is $K \to J$ injective on orbits.

These give a diagram



Basic Properties

Properties

- *Suppose* $F(*) \simeq *$ *and* J *transitive. Then*

$$P_J F(X) \simeq \operatorname*{hocolim}_n \Omega^{nJ} F(\Sigma^{nJ} X)$$

  where nJ *denotes the permutation representation* $\mathbb{R}[nJ]$*,*

- *For every subgroup* $H \leq G$

$$(P_{nG} F)|_H \simeq P_{nH}(F|_H)$$

- *There is a non-equivariant equivalence*

$$(P_J F)|_1 \simeq (P_{|J/G|} F)|_1$$

We think of P_J as an enhancement of $P_{|J/G|}$ that builds in the orbits of J.

Convergence

Q

What kind of convergence should one expect? After all, often enough

$$F \simeq \operatorname*{holim}_n P_n F$$

Consider the "naïve" and "genuine" equivariant stable homotopy monads

$$Q = \Omega^\infty \Sigma^\infty \quad \text{and} \quad Q_G = \Omega^{\infty \rho_G} \Sigma^{\infty \rho_G} \qquad : \mathrm{Top}_*^G \to \mathrm{Top}_*^G$$

Arone-Kankaanrinta: $TotQ^\bullet \simeq \mathrm{holim}_n P_n I$

Carlsson: $TotQ^\bullet \xrightarrow{\simeq} TotQ_G^\bullet$

Then maybe also $\mathrm{holim}_n P_{nG} I \simeq TotQ_G^\bullet \simeq \mathrm{holim}_n P_n I$.

Theorem (D)

Let $J_1 \subsetneq J_2 \subsetneq \dots$, and suppose $F\colon \mathrm{Top}_*^G \to \mathrm{Top}_*^G$ commutes with fixed-points (e.g. $F = I$). Then

$$\operatorname*{holim}_n P_{J_n} F \simeq \operatorname*{holim}_n P_n F$$

Delooping the Layers

By the previous result, sometimes,

$$F \simeq \operatorname{holim}\left(\begin{array}{c} \dots \longrightarrow P_{nG}F \longrightarrow P_{(n-1)G}F \longrightarrow \dots \longrightarrow P_{2G}F \longrightarrow P_G F \\ D_{nG}F \nearrow \end{array} \right)$$

The layer $D_{nG}F$ is nG-excisive and satisfies $P_{kG}D_{nG}F \simeq *$ for $k < n$.

Definition

Φ is J-homogeneous if it is J-excisive and $P_K\Phi \simeq *$ for every G-subset $K \subsetneq J$.

Theorem

Let $\Phi\colon \mathrm{Top}_*^G \to \mathrm{Top}_*^G$ *be* J*-homogeneous. Then*

$$\Phi \simeq \Omega^{\infty J}\widehat{\Phi}$$

for some $\widehat{\Phi}\colon \mathrm{Top}_*^G \to \mathrm{Sp}^G$*. In particular*

$$D_{nG}F \simeq \Omega^{\infty\rho_G}\widehat{D_{nG}F}$$

Digression: Equivariant Deloopings

Q

Given $X \in \mathrm{Top}_*^G$, how does one prove that $X \simeq \Omega^{\rho_G} Y$?

For $G = 1$, construct a fiber sequence $X \to E \to Y$ with $E \simeq *$, or equivalently

$$\begin{array}{ccc} X & \to & E \\ \downarrow & & \downarrow \\ E & \to & Y \end{array}$$

homotopy cartesian, with $E \simeq *$.

Construction

In general, define a homotopy cartesian G_+-cube Z with

- $Z_\varnothing = X$
- $Z_{G_+} = Y$
- $Z_U \simeq_{G_U} *$ *for every* $\varnothing \neq U \subsetneqq G_+$

Then

$$X \simeq \operatorname*{holim}_{\varnothing \neq U \subset G_+} Z_U \simeq \Omega^{\rho_G} Y$$

Homotopy Orbits

Let Λ be a finite group, $\mathcal{R}$ a collection of subgroups of Λ. Let $\overline{E}\mathcal{R}$ be a pointed Λ-space s.t.

$$(\overline{E}\mathcal{R})^{\Gamma} \simeq \begin{cases} S^0 & \text{if } \Gamma \in \mathcal{R} \\ * & \text{if } \Gamma \notin \mathcal{R} \end{cases}$$

Suppose $\Lambda = G \times \Sigma_k$, and that $\mathcal{R}$ contains only graphs of group homomorphisms $\rho\colon H \to \Sigma_k$, for $H \leq G$.

Construction

We let $(-)_{h\mathcal{R}}\colon \mathrm{Sp}^{G\times\Sigma_k} \to \mathrm{Sp}^G$ *be the homotopy* $\mathcal{R}$*-orbits functor:*

$$E_{h\mathcal{R}} := E \wedge_{\Sigma_k} \overline{E}\mathcal{R}.$$

Example: Symmetric Indexed Powers

Let $\mathcal{F}_k$ be the collection of graph subgroups of $G \times \Sigma_k$. For $n \in \mathbb{N}$ we let

$$\mathcal{F}_k(n) = \begin{cases} \{graph(\rho\colon H \to \Sigma_k) \mid (\rho^* k)/H = n-1\} & \text{if } n < k \\ \{graph(\rho\colon H \to \Sigma_k) \mid (\rho^* n)/H = n-1 \text{ or } \rho = 1\} & \text{if } n = k \\ \varnothing & \text{if } n > k \end{cases}$$

Theorem (D)

There is an equivalence of functors $\mathrm{Sp}^G \to \mathrm{Sp}^G$

$$D_{nG}(X^{\wedge k})_{h\mathcal{F}_k} \simeq (X^{\wedge k})_{h\mathcal{F}_k(n)}$$

The Identity Functor

Let $I\colon \mathrm{Top}_*^G \to \mathrm{Top}_*^G$ be the identity functor,
$\mathcal{F}_k$ the family of all graphs $H \to \Sigma_k$, for $H \leq G$,
T_k the partition complex of $\{1, \dots, k\}$ (with the trivial G-action).

Theorem (D)

$$D_{nG} I(X) \simeq \Omega^{\infty G} \bigvee_{k=n}^{n|G|} \big(\mathrm{Map}_*(T_k, \mathbb{S}_G) \wedge X^{\wedge k} \big)_{h\mathcal{F}_k(n)}$$

Remark

$$\Phi^H \bigvee_{k=n}^{n|G|} \big(\mathrm{Map}_*(T_k, \mathbb{S}_G) \wedge X^{\wedge k} \big)_{h\mathcal{F}_k(n)} \simeq \bigvee_{\substack{[H \circlearrowleft K] \\ K/H = n-1 \\ \text{or } K=n}} \Phi^H (\mathrm{Map}_*(T_K, \mathbb{S}_H) \wedge X^{\wedge K})_{hAut_K}$$

where T_K is the partition complex of the H-set K.

The End

Thank you!