Landweber Flat Real pairs and ER(n)-cohomology.

Nitu Kitchloo, Vitaly Lorman, Steve Wilson.

August, 2016

I. Real Theories:

Everything is localized (possibly completed) at the prime $p = 2$.

We may construct a $\mathbb{Z}/2 = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant model for complex cobordism by retaining the Galois (i.e. complex conjugation) action on the pre-spectrum given by Thom spaces: $\mathrm{BU(k)}^{\gamma_k}$, where BU(k) is the Grassmannian of complex k-planes in $\mathbb{C}^\infty$ supporting the tautological bundle γ_k.

The structure maps are of the form:

$$\Sigma^{(1+\alpha)}\mathrm{BU(k)}^{\gamma_k} \longrightarrow \mathrm{BU(k{+}1)}^{\gamma_{k+1}},$$

where $\Sigma^{(1+\alpha)}$ represents the one point compactification of the representation $1 + \alpha = \mathbb{C}$ (here α is the sign representation).

Notation: $\Sigma^V X := S^V \wedge X$, where S^V is the one-point compactification of a representation V.

Real complex cobordism $\mathbb{MU}$ is defined as the $RO(\mathbb{Z}/2)$ i.e $(\mathbb{Z} \oplus \mathbb{Z}\alpha)$-graded complex cobordism spectrum given by spectrifying the $\mathbb{Z}/2$ pre-spectrum above:

$$\mathbb{MU} := \mathrm{colim}_k \, \Sigma^{-k(1+\alpha)} BU(k)^{\gamma_k},$$

Define bigraded cohomology: $\mathbb{MU}^{a+b\alpha}(X) := [X, \Sigma^{a+b\alpha}\mathbb{MU}]^{\mathbb{Z}/2}$

By construction, the spectrum $\mathbb{MU}$ supports a tautological orientation $\mu \in \mathbb{MU}^{1+\alpha}(\mathbb{CP}^\infty)$. So that:

$$\mathbb{MU}^{*(1+\alpha)}(\mathbb{CP}^\infty) = \mathbb{MU}^{*(1+\alpha)}[[\mu]],$$

$$\mathbb{MU}^{*(1+\alpha)}(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) = \mathbb{MU}^{*(1+\alpha)}[[\mu_1, \mu_2]].$$

This yields a formal group law over $\pi_{*(1+\alpha)}(\mathbb{MU})$ that refines the formal group law of MU. So one obtains classes $v_k \in \mathbb{MU}_{(2^k-1)(1+\alpha)}$ that lift the usual classes $v_k \in MU_{2(2^k-1)}$.

We can now define the $\mathbb{Z}/2$-equivariant versions of the spectra: $\mathbb{BP}$, $\mathbb{BP}\langle n\rangle$ and Real equivariant Johnson-Wilson spectra $\mathbb{E}(n)$:

$$\mathbb{E}(n) := \mathbb{BP}\langle n\rangle[\mathrm{v}_n^{-1}] = \mathbb{BP}[\mathrm{v}_n^{-1}]/\langle \mathrm{v}_{n+1}, \mathrm{v}_{n+2}, \cdots \rangle.$$

These equivariant spectra have been extensively studied by Hu-Kriz. They show, for example $\mathbb{E}(1)$ is equivalent to Atiyah's "real" K-theory KR.

Definition: The real Johnson-Wilson spectrum $\mathrm{ER}(n)$ is defined as the homotopy fixed point spectrum: $\mathbb{E}(n)^{h\mathbb{Z}/2}$.

The (integer graded) homotopy groups of $\mathbb{E}(n)$ and $\mathrm{ER}(n)$ agree:

$$\pi_t(\mathrm{ER}(n)) = \pi_t(\mathbb{E}(n)).$$

For example, $\mathrm{ER}(1)$ is equivalent to usual real K-theory KO.

Two Remarks:

(1) Let $\lambda = 2^{n+2}(2^{n-1} - 1) + 1$. Then there is a nilpotent class:

$$\eta \in \pi_\lambda(\mathbb{E}(n)) = \pi_\lambda(\mathrm{ER}(n)), \qquad 2\eta = \eta^{2^{n+1}-1} = 0.$$

So for example, for $n = 1$, we have $\lambda = 1$ and : $\eta \in \pi_1(\mathrm{KO})$.

(2) There is an invertible class $\mathrm{y} \in \pi_{\lambda+\alpha}(\mathbb{E}(n))$ lifting $\mathrm{v}_n^{(2^n-1)}$.
So we may shift cohomology classes to integral degree:

$$\mathbb{E}(n)^{k(1+\alpha)}(\mathrm{X}) \longrightarrow \mathbb{E}(n)^{k(1-\lambda)}(\mathrm{X}), \qquad \mathrm{z} \mapsto \hat{\mathrm{z}} := \mathrm{y}^k \mathrm{z}.$$

In particular, $\mathrm{v}_i \in \mathbb{E}(n)_{(2^i-1)(1+\alpha)}$ have integral shifts: $\hat{\mathrm{v}}_i$

$$\hat{\mathrm{v}}_i \in \mathbb{E}(n)_{(2^i-1)(1-\lambda)} = \mathrm{ER}(n)_{(2^i-1)(1-\lambda)}, \qquad i \leq n.$$

In the example of $n = 1$, we have: $\hat{\mathrm{v}}_0 = 2, \quad \hat{\mathrm{v}}_1 = 1$. For general n, the classes $\hat{\mathrm{v}}_i$ will typically have nonzero grading.

II. The Bockstein Spectral Sequence $E_r(X)$:

Theorem (KW): There is a fibration of $ER(n)$-module spectra:

$$\Sigma^{\lambda} ER(n) \xrightarrow{\cup \eta} ER(n) \longrightarrow E(n).$$

Multiplication by η generates a tower, and gives rise to a first and fourth quadrant spectral sequence of $ER(n)^*$-modules called the Bockstein spectral sequence:

$$E_r(X)^{i,j} \Rightarrow ER(n)^{j-i}(X), \quad |d_r| = (r, r+1).$$

The E_1-term is given by:

$$E(X)_1^{i,j} = E(n)^{i\lambda + j - i}(X), \quad d_1(z) = v_n^{-(2^n - 1)}(1 - \sigma)(z),$$

where σ is complex conjugation acting on $E(n)^*(X)$. Also,

$$d_{2^{k+1}-1}(v_n^{-2^k}) = \hat{v}_k \eta^{2^{k+1}-1} v_n^{-2^{n+k}}, \quad |\eta| = (1, -\lambda + 1).$$

Three Facts:

(1) Since $\eta^{2^{n+1}-1} = 0$, the spectral sequence collapses at $E_{2^{n+1}}(X)$. In other words:

$$E_{2^{n+1}}(X) = E_\infty(X).$$

(2) For $X = pt$, the coefficients $ER(n)^*$ are a subquotient of

$$\frac{\mathbb{Z}_{(2)}[\eta, \hat{v}_1, \ldots, \hat{v}_{n-1}, v_n^{\pm 1}]}{\langle 2\eta, \eta^{2^{n+1}-1}, \eta^{2^{k+1}-1}\hat{v}_k \rangle}.$$

(3) The invertible class $v_n^{2^{n+1}}$ survives and generates the periodicity of $ER(n)$. In other words, $ER(n)$ is $2^{n+2}(2^n - 1)$-periodic.

Internal structure of the BSS:

Notice that there is an Algebraic map:

$$\varphi : E(n)_{2*} = \mathbb{Z}_{(2)}[v_1, \dots, v_n, v_n^{-1}] \longrightarrow ER(n)_{(1-\lambda)*}, \quad v_i \mapsto \hat{v}_i.$$

This map scales the degrees of classes by the factor $(1-\lambda)/2$. The Bockstein spectral sequence for $X = pt$, is a spectral sequence of finitely presented $E(n)_*E(n)$-comodules under the map φ.

Corollary (KW): Let M be a Landweber flat $E(n)^*$-module, and let (E_r, d_r) denote the Bockstein spectral sequence for $X = pt$. Then $(M \otimes_\varphi E_r, id \otimes d_r)$ is a spectral sequence of $ER(n)^*$-modules converging to $M \otimes_\varphi ER(n)^*$.

The goal now is to identify those spaces X, so that we may model $E_r(X)$ as $M \otimes_\varphi E_r(pt)$ for a suitable subalgebra of permanent cycles: $M \subseteq ER(n)^*(X)$. Such spaces are surprisingly common.

III. The Projective Property and LFRP:

Definition: A pointed $\mathbb{Z}/2$-space Z is called *Projective* if $H_*(Z, \mathbb{Z})$ is of finite type, and Z is homeomorphic to a space of the form $\bigvee_l (\mathbb{CP}^\infty)^{k_l}$ for some sequence k_l.

A $\mathbb{Z}/2$-equivariant H-space Y is said to have the *Projective Property* if there exists a projective space Z endowed with an equivarinat map $f : Z \longrightarrow Y$, such that $H_*(Y, \mathbb{Z}/2)$ is generated as an algebra by the image of f.

Eamples of spaces with projective propery:

$$\underline{\mathbb{MU}}_{k(1+\alpha)}, \quad \underline{\mathbb{BP}}_{k(1+\alpha)}, \quad \underline{\mathbb{BP}}\langle n\rangle_{k(1+\alpha)} \quad \text{for } k < 2^{n+1}.$$

Theorem (KW): If Y is a space with the projective property, then the map ρ given by forgetting the equivariant structure:

$$\rho : \mathbb{E}(n)^{*(1+\alpha)}(Y) \longrightarrow E(n)^{2*}(Y),$$

is an isomorphism of $\mathbb{MU}^{*(1+\alpha)}$-algebras.

The above theorem, along with the shift isomorphism yields:
Corollary (KW): If Y is a space with the projective property, then we have an isomorphism:

$$\varphi : E(n)^{2*}(Y) \longrightarrow ER(n)^{*(1-\lambda)}(Y).$$

Definition (LFRP): Let X be a (non-equivariant) space such that $E(n)^*(X)$ is Landweber flat. Assume that there exists a space Y with the projective property equipped with a map: $X \longrightarrow Y^{\mathbb{Z}/2}$ such that the composite map: $\iota : X \longrightarrow Y^{\mathbb{Z}/2} \longrightarrow Y$ is surjective in $E(n)$, and that the natural map:

$$\iota^*\varphi : E(n)^{2*}(Y) \longrightarrow ER(n)^{*(1-\lambda)}(X),$$

factors through $E(n)^{2*}(X)$. Then we call the pair (X, Y), a *Landweber Flat Real Pair*. One can show that the factorization: $E(n)^{2*}(X) \longrightarrow ER(n)^{*(1-\lambda)}(X)$ is injective. Call its image $\hat{E}(n)^*(X)$. We treat the case $n = 1$ separately.

IV. The Main theorem and Examples:

Theorem (KLW): Assume that (X, Y) is a LFRP. Let $\hat{E}(n)^*(X) \subseteq ER(n)^{*(1-\lambda)}(X)$ denote the (injective) image of the above factorization. Then there is an isomorphism of algebras:

$$ER(n)^* \otimes \hat{E}(n)^*(X) \longrightarrow ER(n)^*(X),$$

where the tensor product is being taken over $\hat{E}(n)^*(pt)$.

Two Remarks:

(1) The ring $\hat{E}(n)^*(X)$ is abstractly isomorphic to $E(n)^*(X)$ with a rescaling of degrees and so the above theorem shows that $ER(n)^*(X)$ is obtained from $E(n)^*(X)$ by a subtle base change.

(2) The Künneth theorem holds:

$$ER(n)^*(X_1 \times X_2) = ER(n)^*(X_1) \hat{\otimes} ER(n)^*(X_2),$$

where the completed tensor product is over $ER(n)^*$.

Examples of LFRP (X, Y):

$$X = K(\mathbb{Z}, 2m+1), \quad Y = \underline{\mathbb{BP}}\langle 2m-1\rangle_{(2^{2m}-1)(1+\alpha)}$$

$$X = K(\mathbb{Z}/2^q, 2m), \quad Y = \underline{\mathbb{BP}}\langle 2m-1\rangle_{(2^{2m}-1)(1+\alpha)}$$

$$X = K(\mathbb{Z}/2, m), \quad Y = \underline{\mathbb{BP}}\langle m-1\rangle_{(2^{m}-1)(1+\alpha)}$$

$$X = BO, \quad Y = \underline{\mathbb{BP}}\langle 1\rangle_{(1+\alpha)} \cong BU$$

$$X = BSO, \quad Y = \underline{\mathbb{BP}}\langle 1\rangle_{2(1+\alpha)} \cong BSU$$

$$X = BSpin, \quad Y = \underline{\mathbb{BP}}\langle 1\rangle_{2(1+\alpha)} \cong BSU$$

$$X = \widetilde{BSpin}, \quad Y = \underline{\mathbb{BP}}\langle 1\rangle_{3(1+\alpha)} \cong BU\langle 6\rangle$$

$\widetilde{BSpin}$ is the fiber of $p_1 : BSpin \longrightarrow K(\mathbb{Z}, 4)$.

We can be more explicit in some cases, for example:

$$\mathrm{ER}(n)^*(\mathrm{BO}) = \mathrm{ER}(n)^*[[\hat{\mathrm{c}}_1, \ldots,]]/\langle \hat{\mathrm{c}}_i - \hat{\mathrm{c}}_i^* \rangle \sim \mathrm{E}(n)^*(\mathrm{BU})/\langle \mathrm{c}_i - \mathrm{c}_i^* \rangle.$$

In general $\hat{\mathrm{E}}(n)^*(\mathrm{X})$ is a regraded quotient of $\mathrm{E}(n)^*(\mathrm{Y})$.

All the previous examples tie into short exact sequences of completed algebras.

Definition: A sequence of complete, augmented topological R algebras:

$$\mathrm{A} \longrightarrow \mathrm{B} \longrightarrow \mathrm{C},$$

is a called *SES of completed algebras* if the following is a SES of R-modules:

$$0 \longrightarrow \mathrm{B} \hat{\otimes} \mathrm{I}(\mathrm{A}) \longrightarrow \mathrm{B} \longrightarrow \mathrm{C} \longrightarrow 0,$$

where I(A) denotes the augmentation ideal of A, and the completed tensor product is taken over R.

Theorem (KLW): The following are SES of completed $ER(n)^*$-algebras:

$$ER(n)^*(K(\mathbb{Z}/2,1)) \longrightarrow ER(n)^*(BO) \longrightarrow ER(n)^*(BSO),$$

$$ER(n)^*(K(\mathbb{Z}/2,2)) \longrightarrow ER(n)^*(BSO) \longrightarrow ER(n)^*(BSpin),$$

$$ER(n)^*(BSpin) \longrightarrow ER(n)^*(\widetilde{BSpin}) \longrightarrow ER(n)^*(K(\mathbb{Z},3)),$$

$$ER(n)^*(K(\mathbb{Z}/2,3)) \longrightarrow ER(n)^*(\widetilde{BSpin}) \longrightarrow ER(n)^*(BO\langle 8\rangle).$$

Two Remarks:

(1) All the above SES's are induced by topological connective covers.

(2) The ring $ER(n)^*(K(\mathbb{Z}/2,3))$ is trivial if $n < 3$, so we notice:

$$ER(n)^*(\widetilde{BSpin}) = ER(n)^*(BO\langle 8\rangle), \quad n \leq 2.$$