Saturated fusion systems as stable retracts of groups

(HKR character theory for fusion systems)

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Outline

1. Motivation: The HKR character map
2. Background on fusion systems and bisets
3. Main theorem and the proof strategy
4. Transfer for free loop spaces

Notes on the blackboard are in red.

Fix a prime p. The HKR character map for Morava E-theory of a finite group was constructed by Hopkins-Kuhn-Ravenel, and generalized by Stapleton, as a map

$$E_n^*(BG) \to C_t \otimes_{L_{K(t)}E_n^0} L_{K(t)}E_n^*(\Lambda_p^{n-t}BG).$$

C_t is of chromatic height t and an algebra over $L_{K(t)}E_n^0$ (and E_n^0).

The r-fold free loop space $\Lambda^r BG$ decomposes as a disjoint union of centralizers:

$$\Lambda^r BG \simeq \coprod_{\substack{\alpha \text{ commuting } r\text{-tuple} \\ \text{in } G \text{ up to } G\text{-conj}}} C_G(\alpha).$$

$\Lambda_p^r BG$ is the collection of components for commuting r-tuples of elements of p-power order.

Theorem (Hopkins-Kuhn-Ravenel, Stapleton)

$$C_t \otimes_{E_n^0} E_n^*(BG) \xrightarrow{\simeq} C_t \otimes_{L_{K(t)} E_n^0} L_{K(t)} E_n^*(\Lambda_p^{n-t} BG).$$

The case $t = 0$ is the original HKR character map.

HKR character theory happens p-locally, so we might replace the finite group G with a saturated fusion system $\mathcal{F}$ at the prime p. We wish to define an HKR character map for $\mathcal{F}$,

$$E_n^*(B\mathcal{F}) \to C_t \otimes_{L_{K(t)}E_n^0} L_{K(t)}E_n^*(\Lambda^{n-t}B\mathcal{F}),$$

so that tensoring with C_t gives an isomorphism

$$C_t \otimes_{E_n^0} E_n^*(B\mathcal{F}) \xrightarrow{\simeq} C_t \otimes_{L_{K(t)}E_n^0} L_{K(t)}E_n^*(\Lambda^{n-t}B\mathcal{F}).$$

A *fusion system* over a finite p-group S is a category $\mathcal{F}$ where the objects are the subgroups $P \leq S$ and the morphisms satisfy:

- $\mathrm{Hom}_S(P,Q) \subseteq \mathcal{F}(P,Q) \subseteq \mathrm{Inj}(P,Q)$ for all $P,Q \leq S$.
- Every $\varphi \in \mathcal{F}(P,Q)$ factors in $\mathcal{F}$ as an isomorphism $P \to \varphi P$ followed by an inclusion $\varphi P \hookrightarrow Q$.

A *saturated* fusion system satisfies a few additional axioms that play the role of Sylow's theorems (e.g. $\mathrm{Inn}(S) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(S))$).

The canonical example of a saturated fusion system is $\mathcal{F}_S(G)$ defined for $S \in \mathrm{Syl}_p(G)$ with morphisms $\mathrm{Hom}_{\mathcal{F}_S(G)}(P,Q) := \mathrm{Hom}_G(P,Q)$ for $P,Q \leq S$.

Example for $D_8 \leq \Sigma_4$: If V_1 consists of the double transpositions in Σ_4, then the fusion system $\mathcal{F} = \mathcal{F}_{D_8}(\Sigma_4)$ gains an automorphism α of V_1 of order 3, and $Q_1 \leq V_1$ becomes conjugate in $\mathcal{F}$ to $Z \leq V_1$.

Each saturated fusion system has an associated classifying space $B\mathcal{F}$, which is not the geometric realization $|\mathcal{F}| \simeq *$. This is due to Broto-Levi-Oliver, Chermak, Glauberman-Lynd.

For a fusion system $\mathcal{F}_S(G)$ realized by a group, we have $B\mathcal{F} \simeq BG^\wedge_p$.

Let S, T be finite p-groups. An (S, T)-biset is a finite set equipped with a left action of S and a free right action of T, such that the actions commute.

Transitive bisets: $[Q, \psi]_S^T := S \times T/(sq, t) \sim (s, \psi(q)t)$ for $Q \leq S$ and $\psi \colon Q \to T$. Q and ψ are determined up to preconjugation in S and postconjugation in T.

(S, T)-bisets form an abelian monoid with disjoint union. The group completion is the *Burnside biset module* $A(S, T)$, consisting of *"virtual bisets"*, i.e. formal differences of bisets.

The $[Q, \psi]$ form a $\mathbb{Z}$-basis for $A(S, T)$.

Example for D_8 with subgroup diagram. With V_1 as one of the Klein four groups, Q_1 as a reflection contained in V_1, and Z as the centre/half-rotation of D_8, we for example have $[V_1, id] - 2[Q_1, Q_1 \to Z]$ as an element of $A(D_8, D_8)$.

We can compose bisets $\odot\colon A(R,S) \times A(S,T) \to A(R,T)$ given by $X \odot Y := X \times_S Y$ when X, Y are actual bisets.

$A(S,S)$ is the *double Burnside ring* of S.

A special case of the composition formula: $[Q,\psi]_S^T \odot [T,\varphi]_T^R = [Q,\varphi\psi]_S^R$. We can think of (S,T)-bisets as stable maps from BS to BT. $[Q,\psi]_S^T$ is transfer from S to $Q \le S$ followed by the map $\varphi\colon Q \to T$.

Virtual bisets give us all homotopy classes of stable maps between classifying spaces:

Theorem (Segal conjecture. Carlsson, Lewis-May-McClure)

For p-groups S, T:

$$[\Sigma^\infty_+ BS, \Sigma^\infty_+ BT] \approx A(S,T)^\wedge_p \cong \{X \in A(S,T)^\wedge_p \mid |X|/|T| \in \mathbb{Z}\}.$$

If G induces a fusion system on S, we can ask what properties G has as an (S,S)-biset in relation to $\mathcal{F}_S(G)$. Linckelmann-Webb wrote down the essential properties as the following definition:

An element $\Omega \in A(S,S)^\wedge_p$ is said to be $\mathcal{F}$-*characteristic* if

- Ω is left $\mathcal{F}$-stable: $\operatorname{res}_\varphi \Omega = \operatorname{res}_P \Omega$ in $A(P,S)^\wedge_p$ for all $P \leq S$ and $\varphi \in \mathcal{F}(P,S)$.
- Ω is right $\mathcal{F}$-stable.
- Ω is a linear combination of transitive bisets $[Q,\psi]_S^S$ with $\psi \in \mathcal{F}(Q,S)$.
- $|\Omega|/|S|$ is not divisible by p.

G, as an (S,S)-biset, is $\mathcal{F}_S(G)$-characteristic.

Σ_4 as a (D_8, D_8)-biset is isomorphic to

$$\Sigma_4 \cong [D_8, id] + [V_1, \alpha].$$

This biset is $\mathcal{F}_{D_8}(\Sigma_4)$-characteristic. On the other hand, the previous example $[V_1, id] - 2[Q_1, Q_1 \to Z]$ is generated by elements $[Q, \psi]$ with $\psi \in \mathcal{F}$, but it is not $\mathcal{F}$-stable and hence not characteristic.

We prefer a characteristic element that is idempotent in $A(S,S)_p^\wedge$.

Theorem (Ragnarsson-Stancu)

Every saturated fusion system $\mathcal{F}$ over S has a unique $\mathcal{F}$-characteristic idempotent $\omega_\mathcal{F} \in A(S,S)_{(p)} \subseteq A(S,S)_p^\wedge$, and $\mathcal{F}$ can be recovered from $\omega_\mathcal{F}$.

For the fusion system $\mathcal{F} = \mathcal{F}_{D_8}(\Sigma_4)$, the characteristic idempotent takes the form

$$\omega_\mathcal{F} = [D_8, id] + \tfrac{1}{3}[V_1, \alpha] - \tfrac{1}{3}[V_1, id].$$

The characteristic idempotent $\omega_{\mathcal{F}} \in A(S,S)^\wedge_p$ for a saturated fusion system $\mathcal{F}$ defines an idempotent selfmap

$$\Sigma^\infty_+ BS \xrightarrow{\omega_{\mathcal{F}}} \Sigma^\infty_+ BS.$$

This splits off $\Sigma^\infty_+ B\mathcal{F}$ as a direct summand of $\Sigma^\infty_+ BS$.

We have maps $i\colon \Sigma^\infty_+ BS \to \Sigma^\infty_+ B\mathcal{F}$ and $\mathrm{tr}\colon \Sigma^\infty_+ B\mathcal{F} \to \Sigma^\infty_+ BS$ s.t. $i \circ \mathrm{tr} = id_{\Sigma^\infty_+ B\mathcal{F}}$ and $\mathrm{tr} \circ i = \omega_{\mathcal{F}}$.

Each saturated fusion system $\mathcal{F}$ over a p-group S corresponds to the retract $\Sigma^\infty_+ B\mathcal{F}$ of $\Sigma^\infty_+ BS$.

Strategy

- Consider known results for finite p-groups.
- Apply $\omega_{\mathcal{F}}$ everywhere.
- Get theorems for saturated fusion systems, and p-completed classifying spaces.

We consider the HKR character map for p-groups

$$E_n^*(BS) \to C_t \otimes_{L_{K(t)}E_n^0} L_{K(t)}E_n^*(\Lambda^{n-t}BS).$$

By making $\omega_{\mathcal{F}}$ act on both sides in a way that commutes with the character map, we get a character map for $B\mathcal{F}$ and an isomorphism

Theorem (R.-Schlank-Stapleton)

For every saturated fusion system $\mathcal{F}$ we have

$$C_t \otimes_{E_n^0} E_n^*(B\mathcal{F}) \xrightarrow{\simeq} C_t \otimes_{L_{K(t)}E_n^0} L_{K(t)}E_n^*(\Lambda^{n-t}B\mathcal{F}).$$

For $\mathcal{F} = \mathcal{F}_S(G)$ this recovers the theorem for finite groups.

We go further and show that

Theorem (R-S-S)

$E_n^*(BS) \to C_t \otimes_{L_{K(t)}E_n^0} L_{K(t)}E_n^*(\Lambda^{n-t}BS)$ is a natural in BS for all virtual bisets in $A(T,S)_p^\wedge$ and for all p-groups.

Let $\Lambda := \mathbb{Z}/p^k$ for $k \gg 0$. Think of Λ as emulating S^1. The character map can be decomposed as

$$E_n^*(BS) \xrightarrow{ev^*} E_n^*(B\Lambda^{n-t} \times \Lambda^{n-t}BS) \simeq E_n^*(B\Lambda^{n-t}) \otimes_{E_n^*} E_n^*(\Lambda^{n-t}BS)$$
$$\to C_t \otimes_{E_n^0} E_n^*(\Lambda^{n-t}BS) \to C_t \otimes_{L_{K(t)}E_n^0} L_{K(t)}E_n^*(\Lambda^{n-t}BS)$$

The first map is induced by the evaluation map $ev\colon B\Lambda^{n-t} \times \Lambda^{n-t}BS \to BS$.

With the decomposition

$$\Lambda^r BS \simeq \coprod_{\substack{\text{Commuting } r\text{-tuples } \underline{a} \text{ in } S \\ \text{up to } S\text{-conjugation}}} BC_S(\underline{a}),$$

the evaluation map can be described algebraically as $(\mathbb{Z}/p^k)^r \times C_S(\underline{a}) \to S$ given by

$$(t_1, \ldots, t_r, s) \mapsto (a_1)^{t_1} \cdots (a_r)^{t_r} \cdot z.$$

Consider functoriality of the evaluation map ev

$$\begin{array}{ccc} B\Lambda^r \times \Lambda^r BS & \xrightarrow{ev} & BS \\ \downarrow ? & & \downarrow f \\ B\Lambda^r \times \Lambda^r BT & \xrightarrow{ev} & BT \end{array}$$

If f is a map $BS \to BT$ of spaces, then we can just plug in $id \times \Lambda^r(f)$ into the square. However, if f is a stable map, such as $\omega_{\mathcal{F}}$, we can't directly apply $\Lambda^r(-)$ to f.

In the case of a transfer map from S to a subgroup T, there is a reasonable definition of transfer $\Lambda^r(tr_T^S)$ from $\Lambda^r BS$ to $\Lambda^r BT$, which emulates the induction formula for characters of representations. This transfer conjugates

r-tuples in S to r-tuples in $T \leq S$ whenever possible. However $id_{\Lambda^r} \times \Lambda^r(tr_T^S)$ doesn't commute with the evaluation maps.

Theorem (R-S-S)

There is a functor M defined for suspension spectra of p-groups and saturated fusion systems, such that for each stable map $f\colon \Sigma^\infty BS \to \Sigma^\infty BT$ (up to homotopy) the following square commutes:

$$\begin{array}{ccc} B\Lambda^r \times \Lambda^r BS & \xrightarrow{ev} & BS \\ \downarrow{\scriptstyle M(f)} & & \downarrow{\scriptstyle f} \\ B\Lambda^r \times \Lambda^r BT & \xrightarrow{ev} & BT \end{array}$$

$\dashrightarrow$: *Stable maps*

Note: $M(f)$ maps between coproducts of p-groups and fusion systems, so $M(f)$ is a matrix of virtual bisets.

For most stable maps f, it is impossible for $M(f)$ to have the form $id_{(\mathbb{Z}/p^k)^r} \times (?)$. Hence the cyclic factor needs to be used nontrivially.

The free loop space $\Lambda^r B\mathcal{F}$ for a saturated fusion system, also decomposes as a disjoint union of centralizers:

Proposition (Broto-Levi-Oliver)

$$\Lambda^r B\mathcal{F} \simeq \coprod_{\substack{\text{Commuting } r\text{-tuples } \underline{a} \text{ in } S \\ \text{up to } \mathcal{F}\text{-conjugation}}} BC_{\mathcal{F}}(\underline{a})$$

If $\mathbb{AF}_p$ is the category of formal coproducts of p-groups and fusion systems, where maps a matrices of virtual bisets, then M is a functor from $\mathbb{AF}_p$ to itself.

$$
\begin{array}{ccc}
E_n^*(BT) & \xrightarrow{(\mathrm{tr}_T^S)^*} & E_n^*(BS) \\
\downarrow ev^* & & \downarrow ev^* \\
E_n^*(B\Lambda^{n-t} \times \Lambda^{n-t}BT) & \xrightarrow{M(\mathrm{tr}_T^S)^*} & E_n^*(B\Lambda^{n-t} \times \Lambda^{n-t}BS) \\
\wr| & & \wr| \\
E_n^*(B\Lambda^{n-t}) \otimes_{E_n^*} E_n^*(\Lambda^{n-t}BT) & \xrightarrow{\text{not } - \otimes -} & E_n^*(B\Lambda^{n-t}) \otimes_{E_n^*} E_n^*(\Lambda^{n-t}BS) \\
\downarrow & & \downarrow \\
C_t \otimes_{E_n^0} E_n^*(\Lambda^{n-t}BT) & & C_t \otimes_{E_n^0} E_n^*(\Lambda^{n-t}BS) \\
\downarrow & \Lambda^{n-t}(\mathrm{tr}_T^S)^* & \downarrow \\
C_t \otimes_{L_{K(t)}E_n^0} L_{K(t)}E_n^*(\Lambda^{n-t}BT) & \longrightarrow & C_t \otimes_{L_{K(t)}E_n^0} L_{K(t)}E_n^*(\Lambda^{n-t}BS)
\end{array}
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