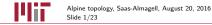
Saturated fusion systems as stable retracts of groups (HKR character theory for fusion systems)

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Outline

- 1 Motivation: The HKR character map
- **2** Background on fusion systems and bisets
- **3** Main theorem and the proof strategy
- **4** Transfer for free loop spaces
- Notes on the blackboard are in red.



Fix a prime p. The HKR character map for Morava E-theory of a finite group was constructed by Hopkins-Kuhn-Ravenel, and generalized by Stapleton, as a map

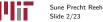
$$E_n^*(BG) \to C_t \otimes_{L_{K(t)}E_n^0} L_{K(t)} E_n^*(\Lambda_p^{n-t}BG).$$

 C_t is of chromatic height t and an algebra over $L_{K(t)}E_n^0$ (and E_n^0).

The *r*-fold free loop space $\Lambda^r BG$ decomposes as a disjoint union of centralizers:

$$\Lambda^r BG \simeq \coprod_{\substack{\alpha \text{ commuting } r-\text{tuple}\\\text{in } G \text{ up to } G-\text{conj}}} C_G(\alpha).$$

 Λ^r_pBG is the collection of components for commuting r-tuples of elements of p-power order.



Theorem (Hopkins-Kuhn-Ravenel, Stapleton)

$$C_t \otimes_{E_n^0} E_n^*(BG) \xrightarrow{\simeq} C_t \otimes_{L_{K(t)} E_n^0} L_{K(t)} E_n^*(\Lambda_p^{n-t}BG).$$

The case t = 0 is the original HKR character map.



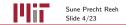
HKR CHARACTER THEORY

HKR character theory happens *p*-locally, so we might replace the finite group G with a saturated fusion system \mathcal{F} at the prime p. We wish to define an HKR character map for \mathcal{F} ,

$$E_n^*(B\mathcal{F}) \to C_t \otimes_{L_{K(t)}E_n^0} L_{K(t)} E_n^*(\Lambda^{n-t}B\mathcal{F}),$$

so that tensoring with C_t gives an isomorphism

$$C_t \otimes_{E_n^0} E_n^*(B\mathcal{F}) \xrightarrow{\simeq} C_t \otimes_{L_{K(t)}E_n^0} L_{K(t)} E_n^*(\Lambda^{n-t}B\mathcal{F}).$$



A fusion system over a finite p-group S is a category \mathcal{F} where the objects are the subgroups $P \leq S$ and the morphisms satisfy:

- $\operatorname{Hom}_{S}(P,Q) \subseteq \mathcal{F}(P,Q) \subseteq \operatorname{Inj}(P,Q)$ for all $P,Q \leq S$.
- Every $\varphi \in \mathcal{F}(P,Q)$ factors in \mathcal{F} as an isomorphism $P \to \varphi P$ followed by an inclusion $\varphi P \hookrightarrow Q$.

A saturated fusion system satisfies a few additional axioms that play the role of Sylow's theorems (e.g. $\operatorname{Inn}(S) \in \operatorname{Syl}_p(\operatorname{Aut}_{\mathcal{F}}(S))$).

The canonical example of a saturated fusion system is $\mathcal{F}_S(G)$ defined for $S \in \operatorname{Syl}_p(G)$ with morphisms $\operatorname{Hom}_{\mathcal{F}_S(G)}(P,Q) := \operatorname{Hom}_G(P,Q)$ for $P, Q \leq S$.

Example for $D_8 \leq \Sigma_4$: If V_1 consists of the double transpositions in Σ_4 , then the fusion system $\mathcal{F} = \mathcal{F}_{D_8}(\Sigma_4)$ gains an automorphism α of V_1 of order 3, and $Q_1 \leq V_1$ becomes conjugate in \mathcal{F} to $Z \leq V_1$.



Each saturated fusion system has an associated classifying space $B\mathcal{F}$, which is not the geometric realization $|\mathcal{F}| \simeq *$. This is due to Broto-Levi-Oliver, Chermak, Glauberman-Lynd.

For a fusion system $\mathcal{F}_S(G)$ realized by a group, we have $B\mathcal{F} \simeq BG_p^{\wedge}$.



Let S, T be finite *p*-groups. An (S, T)-biset is a finite set equipped with a left action of S and a free right action of T, such that the actions commute.

Transitive bisets: $[Q, \psi]_S^T := S \times T/(sq, t) \sim (s, \psi(q)t)$ for $Q \leq S$ and $\psi: Q \to T$. Q and ψ are determined up to preconjugation in S and postconjugation in T.

(S, T)-bisets form an abelian monoid with disjoint union. The group completion is the *Burnside biset module* A(S, T), consisting of "virtual bisets", i.e. formal differences of bisets.

The $[Q, \psi]$ form a \mathbb{Z} -basis for A(S, T).

Example for D_8 with subgroup diagram. With V_1 as one of the Klein four groups, Q_1 as a reflection contained in V_1 , and Z as the centre/half-rotation



of D_8 , we for example have $[V_1, id] - 2[Q_1, Q_1 \rightarrow Z]$ as an element of $A(D_8, D_8)$.

We can compose bisets $\odot: A(R, S) \times A(S, T) \to A(R, T)$ given by $X \odot Y := X \times_S Y$ when X, Y are actual bisets.

A(S, S) is the double Burnside ring of S.

A special case of the composition formula: $[Q, \psi]_S^T \odot [T, \varphi]_T^R = [Q, \varphi \psi]_S^R$. We can think of (S, T)-bisets as stable maps from BS to BT. $[Q, \psi]_S^T$ is transfer from S to $Q \leq S$ followed by the map $\varphi \colon Q \to T$.



Virtual bisets give us all homotopy classes of stable maps between classifying spaces:

Theorem (Segal conjecture. Carlsson, Lewis-May-McClure)

For p-groups S, T:

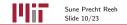
 $[\Sigma^\infty_+BS,\Sigma^\infty_+BT]\approx A(S,T)^\wedge_p\cong \{X\in A(S,T)^\wedge_p\mid |X|/|T|\in\mathbb{Z}\}.$



If G induces a fusion system on S, we can ask what properties G has as an (S, S)-biset in relation to $\mathcal{F}_S(G)$. Linckelmann-Webb wrote down the essential properties as the following definition:

An element $\Omega \in A(S, S)_p^{\wedge}$ is said to be \mathcal{F} -characteristic if

- Ω is left \mathcal{F} -stable: $\operatorname{res}_{\varphi} \Omega = \operatorname{res}_{P} \Omega$ in $A(P, S)_{p}^{\wedge}$ for all $P \leq S$ and $\varphi \in \mathcal{F}(P, S)$.
- Ω is right \mathcal{F} -stable.
- Ω is a linear combination of transitive bisets $[Q, \psi]_S^S$ with $\psi \in \mathcal{F}(Q, S)$.
- $|\Omega|/|S|$ is not divisible by p.



G, as an (S, S)-biset, is $\mathcal{F}_S(G)$ -characteristic.

 Σ_4 as a (D_8, D_8) -biset is isomorphic to

 $\Sigma_4 \cong [D_8, id] + [V_1, \alpha].$

This biset is $\mathcal{F}_{D_8}(\Sigma_4)$ -characteristic. On the other hand, the previous example $[V_1, id] - 2[Q_1, Q_1 \rightarrow Z]$ is generated by elements $[Q, \psi]$ with $\psi \in \mathcal{F}$, but it is not \mathcal{F} -stable and hence not characteristic.



We prefer a characteristic element that is idempotent in $A(S, S)_p^{\wedge}$.

Theorem (Ragnarsson-Stancu)

Every saturated fusion system \mathcal{F} over S has a unique \mathcal{F} -characteristic idempotent $\omega_{\mathcal{F}} \in A(S,S)_{(p)} \subseteq A(S,S)_p^{\wedge}$, and \mathcal{F} can be recovered from $\omega_{\mathcal{F}}$.

For the fusion system $\mathcal{F}=\mathcal{F}_{D_8}(\Sigma_4)$, the characteristic idempotent takes the form

$$\omega_{\mathcal{F}} = [D_8, id] + \frac{1}{3}[V_1, \alpha] - \frac{1}{3}[V_1, id].$$

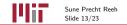


The characteristic idempotent $\omega_{\mathcal{F}} \in A(S, S)_p^{\wedge}$ for a saturated fusion system \mathcal{F} defines an idempotent selfmap

$$\Sigma^{\infty}_{+}BS \xrightarrow{\omega_{\mathcal{F}}} \Sigma^{\infty}_{+}BS.$$

This splits off $\Sigma^{\infty}_{+}B\mathcal{F}$ as a direct summand of $\Sigma^{\infty}_{+}BS$.

We have maps $i: \Sigma^{\infty}_{+}BS \to \Sigma^{\infty}_{+}B\mathcal{F}$ and $\operatorname{tr}: \Sigma^{\infty}_{+}B\mathcal{F} \to \Sigma^{\infty}_{+}BS$ s.t. $i \circ \operatorname{tr} = id_{\Sigma^{\infty}_{+}B\mathcal{F}}$ and $\operatorname{tr} \circ i = \omega_{\mathcal{F}}$.



Each saturated fusion system \mathcal{F} over a *p*-group *S* corresponds to the retract $\Sigma^{\infty}_{+}B\mathcal{F}$ of $\Sigma^{\infty}_{+}BS$.

Strategy

- Consider known results for finite *p*-groups.
- Apply $\omega_{\mathcal{F}}$ everywhere.
- Get theorems for saturated fusion systems, and *p*-completed classifying spaces.



We consider the HKR character map for p-groups

$$E_n^*(BS) \to C_t \otimes_{L_{K(t)}} E_n^0 L_{K(t)} E_n^*(\Lambda^{n-t}BS).$$

By making $\omega_{\mathcal{F}}$ act on both sides in a way that commutes with the character map, we get a character map for $B\mathcal{F}$ and an isomorphism

Theorem (R.-Schlank-Stapleton)

For every saturated fusion system \mathcal{F} we have

$$C_t \otimes_{E_n^0} E_n^*(B\mathcal{F}) \xrightarrow{\simeq} C_t \otimes_{L_{K(t)}E_n^0} L_{K(t)} E_n^*(\Lambda^{n-t}B\mathcal{F}).$$

For $\mathcal{F} = \mathcal{F}_S(G)$ this recovers the theorem for finite groups.



We go further and show that

Theorem (R-S-S)

$$E_n^*(BS) \to C_t \otimes_{L_{K(t)}E_n^0} L_{K(t)}E_n^*(\Lambda^{n-t}BS)$$
 is a natural in BS for all virtual bisets in $A(T, S)_p^{\wedge}$ and for all p-groups.

Let $\Lambda := \mathbb{Z}/p^k$ for $k \gg 0$. Think of Λ as emulating S^1 . The character map can be decomposed as

$$E_n^*(BS) \xrightarrow{ev^*} E_n^*(B\Lambda^{n-t} \times \Lambda^{n-t}BS) \simeq E_n^*(B\Lambda^{n-t}) \otimes_{E_n^*} E_n^*(\Lambda^{n-t}BS) \rightarrow C_t \otimes_{E_n^0} E_n^*(\Lambda^{n-t}BS) \rightarrow C_t \otimes_{L_{K(t)}E_n^0} L_{K(t)}E_n^*(\Lambda^{n-t}BS)$$

The first map is induced by the evaluation map $ev \colon B\Lambda^{n-t} \times \Lambda^{n-t}BS \to BS.$



With the decomposition

$$\Lambda^r BS \simeq \coprod_{\substack{ \text{Commuting r-tuples \underline{a} in S} \\ \text{up to S-conjugation}} BC_S(\underline{a}),$$

the evaluation map can be described algebraically as $(\mathbb{Z}/p^k)^r \times C_S(\underline{a}) \to S$ given by

$$(t_1,\ldots,t_r,s)\mapsto (a_1)^{t_1}\cdots (a_r)^{t_r}\cdot z.$$



Consider functoriality of the evalutation map ev

$$B\Lambda^{r} \times \Lambda^{r}BS \xrightarrow{ev} BS$$

$$\downarrow ? \qquad \qquad \downarrow f$$

$$B\Lambda^{r} \times \Lambda^{r}BT \xrightarrow{ev} BT$$

If f is a map $BS \to BT$ of spaces, then we can just plug in $id \times \Lambda^r(f)$ into the square. However, if f is a stable map, such as $\omega_{\mathcal{F}}$, we can't directly apply $\Lambda^r(-)$ to f.

In the case of a transfer map from S to a subgroup T, there is a reasonable definition of transfer $\Lambda^r(tr_T^S)$ from $\Lambda^r BS$ to $\Lambda^r BT$, which emulates the induction formula for characters of representations. This transfer conjugates



r-tuples in S to r-tuples in $T \leq S$ whenever possible. However $id_{\Lambda^r} \times \Lambda^r(tr_T^S)$ doesn't commute with the evaluation maps.



Theorem (R-S-S)

There is a functor M defined for suspension spectra of p-groups and saturated fusion systems, such that for each stable map

 $f: \Sigma^{\infty}BS \to \Sigma^{\infty}BT$ (up to homotopy) the following square commutes:

 $B\Lambda^{r} \times \Lambda^{r}BS \xrightarrow{ev} BS$ $\downarrow M(f) \qquad \qquad \downarrow f$ $B\Lambda^{r} \times \Lambda^{r}BT \xrightarrow{ev} BT$

 $\dots \rightarrow$: Stable maps

Note: M(f) maps between coproducts of p-groups and fusion systems, so M(f) is a matrix of virtual bisets.

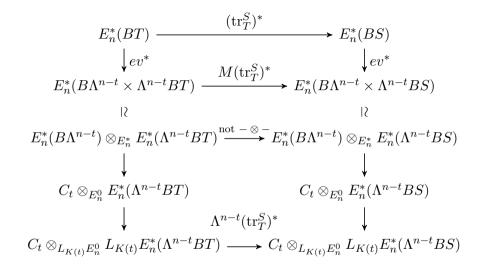


For most stable maps f, it is impossible for M(f) to have the form $id_{(\mathbb{Z}/p^k)^r} \times (?)$. Hence the cyclic factor needs to be used nontrivially. The free loop space $\Lambda^r B\mathcal{F}$ for a saturated fusion system, also decomposes as a disjoint union of centralizers:



If \mathbb{AF}_p is the category of formal coproducts of p-groups and fusion systems, where maps a matrices of virtual bisets, then M is a functor from \mathbb{AF}_p to itself.







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