Homotopy Types of Gauge Groups over 4-manifolds

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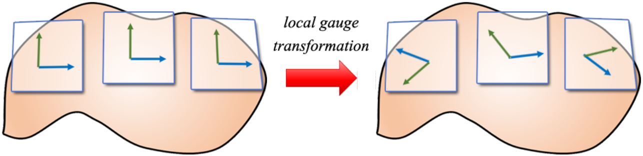
Introduction

What are gauge groups?

- A group of *local gauge transformations*, i.e. pointwise transformations of local frames



Why do we study gauge groups?

- Modern physics: gauge theory
- Mathematics: topology, geometry

Gauge groups

G: simple, simply-connected, compact Lie group
M: orientable, closed, compact 4-manifold

$$\left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{principal } G\text{-bundles over } M \end{array} \right\} \longleftrightarrow [M, BG]$$

$$P_k \longleftrightarrow k \in \mathbb{Z}$$

Definition

The *gauge group* associated to $\pi : P \to M$ is defined by

$$\mathcal{G}(P; M) = \{\phi \in Aut(P) | \phi \text{ is } G\text{-equivariant}\},$$

i.e.

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & P \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ M & = & M \end{array} \qquad \text{and} \qquad \begin{array}{ccc} P & \xrightarrow{\varphi} & P \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} \\ P & \xrightarrow{\varphi} & P \end{array}$$

Some facts about gauge groups

Theorem (Atiyah and Bott, 82)

There is a weak homotopy equivalence

$$B\mathcal{G}(P; M) \simeq \mathrm{Map}_P(M, BG),$$

where $\mathrm{Map}_P(M, BG)$ *is the connected component containing the inducing map of* P.

Theorem (Theriault,10)

If M *is an orientable simply-connected closed 4-manifold, then*

$$\mathcal{G}(P_k; M) \simeq \mathcal{G}(P_k; S^4) \times \prod_{i=1}^{n} \Omega^2 G$$

for the spin case, and the equivalence still holds for non-spin case after localized away from 2.

Sketch of Proof

Cofibration $S^3 \xrightarrow{f} \vee_{i=1}^n S^2 \hookrightarrow M \xrightarrow{q} S^4 \xrightarrow{\Sigma f} \vee_{i=1}^n S^3$ induces

$$\begin{array}{ccccc}
 & & \Omega \mathrm{Map}_k^*(M, BG) & \xrightarrow{\cong} & \mathrm{Map}_k^*(\Sigma M, BG) \\
 & & \downarrow & & \downarrow \\
\mathcal{G}(P_k; S^4) & \xrightarrow{q^*} & \mathcal{G}(P_k; M) & \underset{h'}{\overset{h}{\rightleftarrows}} & \mathrm{Map}_{k'}^*(\vee_{i=1}^n S^3, BG) \\
\| & & \downarrow & & \downarrow \Sigma f^* \\
\mathcal{G}(P_k; S^4) & \longrightarrow & G & \longrightarrow & \mathrm{Map}_k^*(S^4, BG) \\
 & & \downarrow & & \downarrow \\
 & & \mathrm{Map}_k^*(M, BG) & = & \mathrm{Map}_k^*(M, BG)
\end{array}$$

$\Sigma f \simeq * \Rightarrow \quad h'$ exists $\Rightarrow \quad$ homotopy equivalence

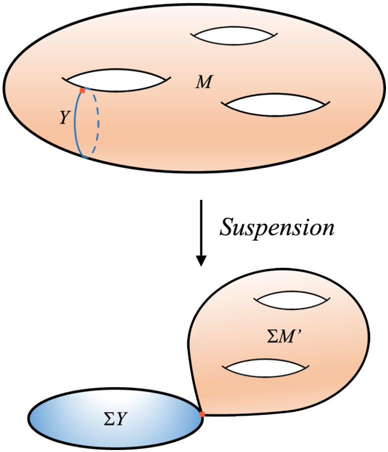
Idea: To find a subcomplex $Y \subset M_3$ such that "Σf does not attach to ΣY".

Theorem

If there is a complex Y of dimension ≤ 3 and a map $\phi : Y \to M$ such that $\Sigma\phi : \Sigma Y \to \Sigma M$ has a left homotopy inverse, then

$$\mathcal{G}(P_k; M) \simeq \mathcal{G}(P_k; M') \times \mathrm{Map}_0^*(Y, G),$$

where $M' = M/\phi(Y)$ and $\mathrm{Map}_0^*(Y, G)$ is the connected component containing the basepoint.



implies that

$$\mathcal{G}(P_k; M) \simeq \mathcal{G}(P_k; M') \times \mathrm{Map}_0^*(Y, G)$$

Example

If $\pi_1(M) = \mathbb{Z}^{*m}$, then $M_3 \simeq (\bigvee^m S^3) \vee (\bigvee^d S^2) \vee (\bigvee^m S^1)$ [Matumoto & Katanaga, 95] and we have:

(1)

$$\Sigma M \simeq S^5 \vee \bigvee_{i=1}^{m} S^4 \vee \bigvee_{j=1}^{d} S^3 \vee \bigvee_{k=1}^{m} S^2$$

$$\mathcal{G}(P_k; M) \simeq \mathcal{G}(P_k; S^4) \times \prod_{i=1}^{m} \Omega_0^3 G \times \prod_{j=1}^{d} \Omega^2 G \times \prod_{k=1}^{m} \Omega G$$

(2)

$$\Sigma M \simeq \Sigma\mathbb{CP}^2 \vee \bigvee_{i=1}^{m} S^4 \vee \bigvee_{j=1}^{d-1} S^3 \vee \bigvee_{k=1}^{m} S^2$$

$$\mathcal{G}(P_k; M) \simeq \mathcal{G}(P_k; \mathbb{CP}^2) \times \prod_{i=1}^{m} \Omega_0^3 G \times \prod_{j=1}^{d-1} \Omega^2 G \times \prod_{k=1}^{m} \Omega G$$

I calculate the homotopy types of ΣM and $\mathcal{G}(P_k; M)$ when

- $\pi_1(M) = \mathbb{Z}^{*m}$;
- $\pi_1(M) = \mathbb{Z}_{p^r}$, where p is an odd prime;
- $\pi_1(M) = (\mathbb{Z}^{*m}) * (*_{j=1}^{n} \mathbb{Z}_{q_j})$, where q_j is a power of an odd prime.

All $\mathcal{G}(P_k; M)$ are related to $\mathcal{G}(P_k; S^4), \mathcal{G}(P_k; \mathbb{CP}^2)$, and some "loop spaces" of G.

Thank you for your attention.