Galois extensions of motivic ring spectra

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Galois extensions in algebra

Fields

Given L/K an extension of fields, it is Galois if

- it is normal and separable

$\Leftrightarrow$ L is the splitting field of a polynomial with coefficients in K which has no repeated roots

$\Leftrightarrow$ $|Aut(L/K)| = [L : K]$

Rings [Auslander-Goldman, Chase-Harrison-Rosenberg]

A map of rings $R \to S$ is G-Galois if

- $R \to S^G$ and
- $S \otimes_R S \to \mathrm{Hom}(G, S) = \prod_G S \quad \Leftrightarrow \quad S\langle G\rangle \to \mathrm{Hom}_R(S, S)$

are isomorphisms.

Galois extensions in algebra

Galois correspondence

Suppose $R \to S$ is a G-Galois extension.

- For any subgroup $H \subseteq G$, $S^H \to S$ is an H-Galois extension. If H is normal in G, then also $R \to S^H$ is a G/H-Galois extension.
- Suppose given a separable (over R) intermediate extension $R \to T \to S$, and assume S is connected (no non-trivial idempotents). Then $T \to S$ is Galois, with group $H_T = Stab_G(T) = \mathsf{Alg}_T(S, S)$.

Galois extensions in homotopy theory

Definition (Rognes)

A map $\phi : R \to S$ of commutative ring spectra is G-Galois if

- G acts on S and ϕ induces an equivalence $R \simeq S^{hG}$, and
- the natural map $S \wedge_R S \to F(G_+, S)$ is an equivalence.

Galois correspondence [Rognes]

Suppose $R \to S$ is a *faithful* G-Galois extension of commutative ring spectra.

- For any subgroup $H \subseteq G$, $S^{hH} \to S$ is a *faithful* H-Galois extension. If H is normal, then also $R \to S^{hH}$ is G/H-Galois.
- Suppose given a separable (over R) intermediate extension $R \to T \to S$, with S connected, and $T \to S$ *faithful*. Then $T \to S$ is Galois with group $H_T = \pi_0 \mathsf{Alg}_T(S, S)$.

ABSTRACT SETTING FOR GALOIS THEORY

ASSUMPTIONS

Start with $\mathcal{M}$, a locally presentable symmetric monoidal model category with cofibrant unit object. For $A \in \mathsf{Alg}(\mathcal{M})$, and G a dualizable Hopf algebra in $\mathcal{M}$ (eg. finite group), assume Mod_A, $G\mathsf{Mod}_A$, Alg_A, $G\mathsf{Alg}_A$, admit model structures, such that a bunch of natural adjunctions between them are Quillen.

In such a situation, we can define (homotopical) Galois extensions á la Rognes.

ABSTRACT SETTING FOR GALOIS THEORY

DEFINITION

Let $A \to B$ be a map in $G\mathsf{Alg}$, where A has trivial G-action. This is called a G-Galois extension if the induced maps

- $A \to B^{hG}$, and
- $B \wedge_A B \to F(G, B)$

are equivalences.

THEOREM (BHKMS)

If the assumptions hold, the forward Galois correspondence holds, i.e. if $A \to B$ is a G-Galois extension, then for any $H \subseteq G$, $B^{hH} \to B$ is H-Galois.

Motivic spaces

Homotopy theory for schemes/varieties rather than topological spaces

Construction

$$\mathsf{Sm}_k \Rightarrow \mathrm{sPre}(\mathsf{Sm}_k) \Rightarrow \mathrm{sPre}(\mathsf{Sm}_k)_{Nis} \Rightarrow \mathsf{Mot}_k$$

Issue Not all colimits exist in Sm_k

$\Rightarrow$ Formally adjoin colimits

Issue Information about geometry got lost

$\Rightarrow$ Re-enforce Nisnevich covers

Contract $\mathbb{A}^1$ to obtain the category of *motivic spaces*.

MOTIVIC SPECTRA

SOME MOTIVIC SPACES

- Any smooth scheme, via the Yoneda embedding
- Any simplicial set, as a constant presheaf

$\Rightarrow$ Two circles: $\mathbb{G}_m$ and S^1

CONSTRUCTION (MOTIVIC SPECTRA)

Invert $S^{2,1} := \mathbb{P}^1 = S^1 \wedge \mathbb{G}_m$ *in* Mot_k *to obtain* Sp_k.

HOMOTOPY

SHEAVES $\underline{\pi}_{p,q}X(U) = [S^{p-q} \wedge \mathbb{G}_m^q \wedge U_+, X]$

GROUPS $\pi_{p,q}X = \underline{\pi}_{p,q}X(\operatorname{Spec} k)$

EXAMPLES OF MOTIVIC SPECTRA

EILENBERG-MACLANE SPECTRA HA

- We have $\pi_{0,0}\mathsf{HA} = A$, but $\underline{\pi}_{p,q}\mathsf{HA}(U) = H^{-p,-q}(U;A) = H^{-p}(U;A(-q))$ is non-zero in many degrees with $q < 0$, eg.

$$H^{1,1}(U;\mathbb{Z}) = \mathcal{O}^{\times}(U) \qquad H^{2,1}(U;\mathbb{Z}) = \mathrm{Pic}(U)$$

K-THEORY

- KGL is the analogue of complex K-theory
- KO is the analogue of real K-theory,

$$\Sigma^{1,1}\mathsf{KO} \xrightarrow{\eta} \mathsf{KO} \to \mathsf{KGL}$$

- $\mathsf{KT} = \mathsf{KO}[\eta^{-1}]$ is a non-zero analogue of zero, as the motivic η is not nilpotent

MOTIVIC HOMOTOPY WITH GROUP ACTIONS

G is a finite (constant) group

GENUINE EQUIVARIANT MOTIVIC SPACES AND SPECTRA

[Heller-Krishna-Østvær]

$$\mathrm{sPre}(G\mathsf{Sm}_k) \Rightarrow \mathrm{sPre}(G\mathsf{Sm}_k)_{GNis} \Rightarrow \mathsf{Mot}_k^G$$

[Gepner-Heller]
Form spectra by inverting choices of representation spheres.

A VARIANT

Start with $G\mathsf{Sm}_k^{free}$ to get Mot_k^{G-free}.

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Form spectra by inverting choices of representation spheres.

REMARK

The category Mot_k^G is not exactly the category of G-objects $G\mathsf{Mot}_k$ in Mot_k. However, there is a full and faithful monoidal map $\mathsf{Mot}_k^G \to G\mathsf{Mot}_k$ (given by the identity on G-schemes).

CLASSIFYING SPACES

SIMPLICIAL APPROACH

The simplicial $\mathsf{E}_\bullet G$ is in Mot_S^G, but is not a universal free G-space

Example If $k \to L$ is a G-Galois extension of fields, $\operatorname{Spec} L$ is a free G-space in Mot_k^G, but there are no maps $\operatorname{Spec} L \to \mathsf{E}_\bullet G$.

GEOMETRIC APPROACH

Let V be a faithful representation of G. The colimit of

$$\cdots \hookrightarrow \left(\mathbb{A}(V^{\oplus n}) - \bigcup_{1 \neq H \leq G} \mathbb{A}(V^{\oplus n})^H \right) \hookrightarrow \cdots$$

is the universal free G-space $\mathbb{E}G$.

SIMPLICIAL MOTIVIC GALOIS EXTENSIONS

CONSTRUCTION

$$\mathsf{Sm} \xrightarrow{F=G\times -} G\mathsf{Sm} \quad \Rightarrow \quad \mathrm{sPre}(\mathsf{Sm}) \underset{F_*}{\overset{F^*=Lan_F}{\rightleftarrows}} \mathrm{sPre}(G\mathsf{Sm})$$

Pass to localizations, and use $\mathsf{Mot} \underset{F_*}{\overset{F^*}{\rightleftarrows}} \mathsf{Mot}^G$ to right-induce a model structure to $\mathsf{Mot}^G \quad \Rightarrow \quad \mathsf{Mot}^G_{\mathsf{E}_\bullet G}$.

Stabilize $\quad \Rightarrow \quad \mathsf{Sp}^G_{\mathsf{E}_\bullet G}$.

SIMPLICIAL SETTING FOR GALOIS THEORY

We get a corresponding model structure on $\mathsf{Alg}^G_{\mathsf{E}_\bullet G}$, with requisite Quillen adjunctions.

SIMPLICIAL MOTIVIC GALOIS EXTENSIONS

PROPOSITION [BHKMS]

In $\mathsf{Mot}^G_{\mathsf{E}_\bullet G}$ and $\mathsf{Sp}^G_{\mathsf{E}_\bullet G}$, a map f is an equivalence if and only if $f \wedge \mathsf{E}_\bullet G_+$ is an equivalence before the $\mathsf{E}_\bullet G$-localization.
Fibrant replacement is formed by

$$X \to \mathrm{Map}(\mathsf{E}_\bullet G_+, \tilde{X}),$$

where $\tilde{X}$ is fibrant before the $\mathsf{E}_\bullet G$-localization.

DEFINITION

A map $A \to B$ of motivic rings is a simplicial G-Galois extension if

- $A \to F(\mathsf{E}_\bullet G_+, B)^G =: B^{h_s G}$, and
- $B \wedge_A B \to F(G_+, B)$

are equivalences.

THE EILENBERG-MACLANE EXAMPLE

THEOREM [BHKMS]

A map $R \to S$ of commutative rings is a G-Galois extension if and only if the induced map $HR \to HS$ on motivic Eilenberg-Maclane spectra is a simplicial G-Galois extension.

PROOF.

Difficulties arise because $\pi_{*,*}\mathsf{HR}$ *is not concentrated in one degree.*

If $R \to S$ is G-Galois, S is an invertible $R[G]$-module

- $\Rightarrow$ $\mathsf{HS} \wedge_{\mathsf{HR}} \mathsf{HS} \simeq \mathsf{H}(S \otimes_R S) \simeq \prod_G \mathsf{HS}$
- $\Rightarrow$ $\underline{\pi}_{*,*}\mathsf{HS} \cong \underline{\pi}_{*,*}\mathsf{HR} \otimes_R S$
- $\Rightarrow$ $\mathsf{HR} \simeq \mathsf{HS}^{h_s G}$.

□

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PROOF.

Difficulties arise because $\pi_{*,*}\mathsf{HR}$ *is not concentrated in one degree.*
If $\mathsf{HR} \to \mathsf{HS}$ is G-Galois,

$$\Rightarrow \prod_G \pi_{0,0}\mathsf{HS} = \pi_{0,0}(\mathsf{HS} \wedge_{\mathsf{HR}} \mathsf{HS}) \cong S \otimes_R S$$

$$\Rightarrow \pi_{0,0}(\mathsf{HS}^{h_s G}) \cong \pi_{0,0}\mathsf{HR}.$$

□

THE K-THEORY SEMI-EXAMPLE

THEOREM [HU-KRIZ-ORMSBY]

Assume we work over a field k of characteristic zero. The simplicial homotopy fixed points of KGL are equivalent to KO

- if the 2-cohomological dimension of $k[i]$ is finite, and
- after 2-completion,

but not in general.

CONSEQUENCE

$\mathsf{KO} \to \mathsf{KGL}$ is a simplicial Galois extension only in a setting as above, but not in general.

GEOMETRIC MOTIVIC GALOIS EXTENSIONS

CONSTRUCTION

$$G\mathsf{Sm}^{free} \overset{i}{\hookrightarrow} G\mathsf{Sm} \quad \Rightarrow \quad \mathrm{sPre}(G\mathsf{Sm}^{free}) \underset{i_*}{\overset{i^*=Lan_i}{\rightleftarrows}} \mathrm{sPre}(G\mathsf{Sm})$$

Pass to localizations, and use $\mathsf{Mot}^{G-free} \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathsf{Mot}^{G}$ to right-induce a model structure to $\mathsf{Mot}^G \quad \Rightarrow \quad \mathsf{Mot}^G_{\mathbb{E}G}$.

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are equivalences.

THE K-THEORY EXAMPLE

THEOREM [HELLER, BHKMS]

Assume our base is a field of characteristic different from 2. Then

$$\mathsf{KO} \to \mathsf{KGL}$$

is a geometric C_2-Galois extension.

PROOF.

Difficulty is that η is not nilpotent motivically.
We know $KO = KGL^{C_2}$; to get $\mathsf{KO} \simeq \mathsf{KGL}^{h_g C_2}$, we show that $F(\widetilde{\mathbb{E}}C_2, \mathsf{KGL})^{C_2} \simeq *$, using $\widetilde{\mathbb{E}}C_2 \simeq S^0[e_{\mathbb{P}^1_-}^{-1}] \simeq S^0[e_{S^1_-}^{-1}, e_{\mathbb{G}_m^-}^{-1}]$.
After inverting $e_{S^1_-}$ in KGL, $e_{\mathbb{G}_m^-}^{-1}$ must be nilpotent. □