

The André-Quillen (co)homology of an algebra over an operad¹

Introduction

0.1. Introduction

- 1945 : Hochschild cohomology for associative algebras

Theorem.

$$HH^*(A, M) \cong \text{Ext}_{A \otimes A^{op}}^{*-1}(A, M)$$

- 1948 : Chevalley-Eilenberg cohomology for Lie algebras

Theorem.

$$H_{CE}^*(\mathcal{G}, M) \cong \text{Ext}_{\mathcal{U}(\mathcal{G})}^{*-1}(\mathbb{K}, M)$$

0.2. Introduction

- What about commutative algebras ?
- Can we always write the cohomology of commutative algebras as an Ext-functor ?

0.3. Overview

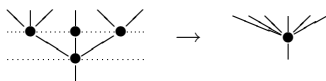
Contents

§1. André-Quillen cohomology of an algebra over an operad

1.1. Operad An **operad** is an algebraic structure which encodes the notion of composition. That is a set of operations



endowed with a composition

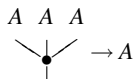


Paradigm : The endomorphism operad

$$\text{End}_V := \{\text{Hom}(V^{\otimes n}, V)\}_{n \in \mathbb{N}}$$

1.2. Algebra and module

- An **algebra over the operad** \mathcal{P} is a vector space A endowed with maps



satisfying compatibility relations with the operad structure.

- An **A-module over the operad** \mathcal{P} is a vector space M endowed with maps



satisfying compatibility relations with the operad and the algebra structures.

1.3. André-Quillen cohomology In a model categories setting, we derive the functor of **\mathcal{P} -derivations from A to M** .

Definition (Hinich, Goerss and Hopkins). Let $R \xrightarrow{\sim} A$ be a cofibrant resolution.

$$H_{\mathcal{P}}^*(A, M) := H^*(\text{Hom}(\mathbb{L}_{R/A}, M)),$$

where $\mathbb{L}_{R/A}$ is the **cotangent complex**.

How to make the cotangent complex explicit ?

1.4. Koszul duality theory We use operadic resolutions given by the **Koszul duality theory** to make the cotangent complex explicit.

- $\mathcal{P} \circ \mathcal{B}(\mathcal{P})(A) \xrightarrow{\sim} A$ **augmented bar construction,**
- $\mathcal{P} \circ \mathcal{P}^i(A) \xrightarrow{\sim} A$ **Koszul complex,**
- $\Omega(\mathcal{C}) \circ \mathcal{C}(A) \xrightarrow{\sim} A$ **coaugmented cobar construction.**

More generally, we consider resolutions of the following type

$$\mathcal{P} \circ \mathcal{C}(A) \xrightarrow{\sim} A.$$

In this case,

$$\mathbb{L}_{R/A} \cong A \otimes^{\mathcal{P}} \mathcal{C}(A) = \bigoplus \begin{array}{c} A \quad \mathcal{C}(A) \quad A \quad A \quad A \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \end{array} / \sim$$

§2. Study of the cotangent complex

2.1. Study of the cotangent complex

- Let $\mathcal{P} \circ \mathcal{C}(A) \xrightarrow{\sim} A$ be a cofibrant resolution of the \mathcal{P} -algebra A . When we apply the cotangent complex functor, we get a map

$$\mathbb{L}_{R/A} = A \otimes^{\mathcal{P}} \mathcal{C}(A) \rightarrow \mathbb{L}_{A/A} = \Omega_{\mathcal{P}}(A),$$

where $\Omega_{\mathcal{P}}(A)$ is the module of Kähler differentials of A .

- When this map is a quasi-isomorphism, we get that the cohomology can be written as an Ext-functor.
- Since we consider only functorial cofibrant resolution, we can define a complex $\mathcal{L}_{\mathcal{P}}$ (depending only on the operad) such that

$$\mathbb{L}_{R/A} \cong \mathcal{L}_{\mathcal{P}} \circ_{\mathcal{P}} A.$$

2.2. Study of the cotangent complex

Theorem. The following properties are equivalent

- (P_0) The André-Quillen cohomology is an Ext-functor;
- (P_1) the cotangent complex is quasi-isomorphic to the module of Kähler differentials for any algebra A ;
- (P_2) the cotangent complex is quasi-isomorphic to the module of Kähler differentials for any vector space A^{tr} ;
- (P_3) the functorial cotangent complex $\mathbb{L}_{\mathcal{P}} := \mathcal{L}_{\mathcal{P}} \circ_{\mathcal{P}} I$ is acyclic;
- (P_4) the module of obstructions $\mathbb{O}_{\mathcal{P}}$ is acyclic.

¹Joan Millès. This document was automatically generated from the slides of a talk I gave in Paris on 30/10/2008. It may look odd in certain places.

2.3. Applications

Proposition. *The functorial cotangent complex \mathbb{L}_{Com} is not acyclic. More precisely,*

$$H_2(\mathbb{L}_{Com}) \neq 0.$$

Proposition. *The functorial cotangent complex \mathbb{L}_{Perm} is not acyclic. More precisely,*

$$H_2(\mathbb{L}_{Perm}) \neq 0.$$

2.4. The case of homotopy algebras Let \mathcal{P} be a Koszul operad and let $\mathcal{P}_\infty := \Omega(\mathcal{P}^i)$.

A **homotopy algebra** is a \mathcal{P}_∞ -algebra.

Theorem. *For any \mathcal{P}_∞ -algebra A and any A -module M , we have*

$$H_{\mathcal{P}_\infty}^\bullet(A, M) \cong \text{Ext}_{A \otimes_{\mathcal{P}_\infty} \mathbb{K}}^\bullet(\Omega_{\mathcal{P}_\infty}(A), M).$$

2.5. The case of homotopy algebras The maps of operads $\mathcal{P}_\infty \rightarrow \mathcal{P}$ endows any \mathcal{P} -algebra A with a structure of \mathcal{P}_∞ -algebra.

Theorem. *For any A -module M over \mathcal{P} , we have*

$$H_{\mathcal{P}}^\bullet(A, M) = H_{\mathcal{P}_\infty}^\bullet(A, M).$$

2.6. The case of homotopy algebras We observe that, in certain cases ($\mathcal{P} = Com, Perm, Poiss$), the André-Quillen cohomology cannot always be written as an Ext-functor over the enveloping algebra $A \otimes^{\mathcal{P}} \mathbb{K}$. However, it can always be written as an Ext-functor over the enveloping algebra $A \otimes^{\mathcal{P}_\infty} \mathbb{K}$.

Corollary. *Let \mathcal{P} be a Koszul operad. We have*

$$H_{\mathcal{P}}^\bullet(A, M) \cong \text{Ext}_{A \otimes_{\mathcal{P}_\infty} \mathbb{K}}^\bullet(\Omega_{\mathcal{P}_\infty}(A), M).$$

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