Free models of enriched T-algebraic theories computed as Kan extensions

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GDR topologie

The tensor algebra

Let k denote a commutative ring. To every k-module A is associated the tensor algebra

$$TA \quad = \quad \bigoplus_{n\in\mathbb{N}} A^{\otimes n}$$

computed as infinite sum of tensorial powers.

Furthermore, this construction is functorial

$$T \quad : \quad k\text{-Mod} \quad \longrightarrow \quad k\text{-Alg}$$

k-algebra as monoid

Recall that a k-algebra M is defined as a k-module equipped with two morphisms,

$$k \xrightarrow{e} M \xleftarrow{m} M \otimes M$$

called unit and multiplication, making the diagrams below commute:

$$\begin{tikzcd}
M \otimes M \otimes M \arrow[r, "m \otimes M"] \arrow[d, "M \otimes m"'] & M \otimes M \arrow[d, "m"] \\
M \otimes M \arrow[r, "m"] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
k \otimes M \arrow[r, "e \otimes M"] \arrow[dr, "\cong"'] & M \otimes M \arrow[d, "m"'] & M \otimes k \arrow[l, "M \otimes e"'] \arrow[dl, "\cong"] \\
& M &
\end{tikzcd}$$

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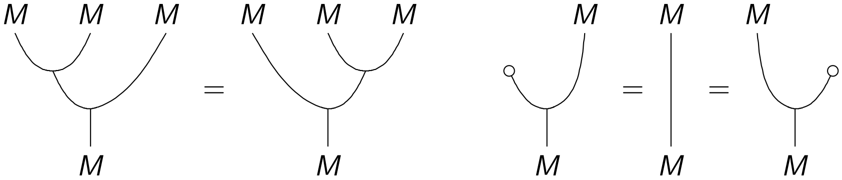


The tensor algebra as a free monoid

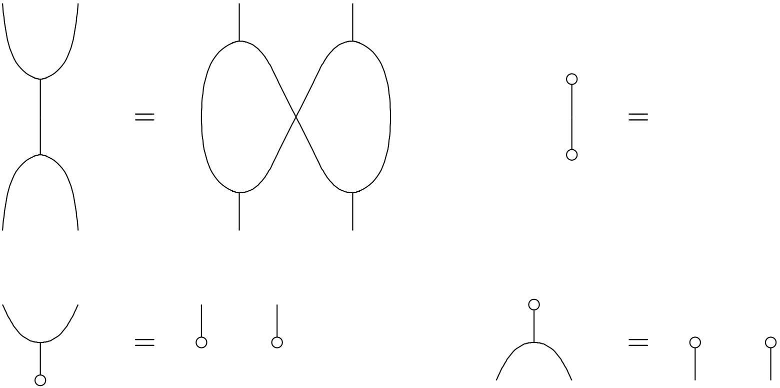
k-algebra = monoid object in the category k-Mod

(k-Mod seen as a monoidal category equipped with the familiar tensor product $\otimes$ of k-modules)

The k-algebra TA is the free monoid object in the category k-Mod

A basic problem in algebra

A k-bialgebra H is a k-module equipped with a k-algebra and a k-cogebra structure, making the bialgebra's compatibility diagrams commute:



A basic problem in algebra

There exists (in general) no free k-bialgebra for a given k-module [Loday]

That is, the forgetful functor

$$\begin{tikzcd} U_{\mathsf{Big}} \quad : \quad k\text{-}\mathsf{Big} \arrow[r] & k\text{-}\mathsf{Mod} \arrow[l, bend left, "\ominus" description] \end{tikzcd}$$

does not have a left adjoint.

A basic problem in algebra

We want to understand more conceptually what distinguishes

- the forgetful functor U_{Alg} which has a left adjoint
- from the forgetful functor U_{Big} which does not have a left adjoint.

Algebraic theories

An algebraic theory is a category $\mathbb{L}$ with finite products

- objects

$$0, 1, 2, \ldots$$

- categorical product provided by

$$m_1 + \ldots + m_k.$$

An $\mathbb{L}$-model A in a Cartesian category $(\mathbb{C}, \times, \mathbf{1})$ is a finite-product preserving functor $A \quad : \quad \mathbb{L} \to \mathbb{C}$

$$A[m_1 + \ldots + m_k] \quad \longrightarrow \quad A[m_1] \times \ldots \times A[m_k]$$

Examples of algebraic theories

- trivial theory: $\mathbb{L}$, the free category with finite product generated by the category with one object

$$\mathsf{Model}(\mathbb{L},\mathbb{C}) \cong \mathbb{C}$$

- theory of monoids: $\mathbb{M}$, the category whose n-ary operations are the finite words (of arbitrary length) built on an alphabet $[n] = \{1,\dots,n\}$ of n letters

$$\mathsf{Model}(\mathbb{L},\mathbb{C}) \cong \mathit{Mon}(\mathbb{C})$$

Free models as Kan extensions

Any finite-product preserving morphism $f : \mathbb{L}_1 \to \mathbb{L}_2$ defines a forgetful functor by precomposition

$$U_f \quad : \quad \mathrm{Model}(\mathbb{L}_2, \mathbb{C}) \quad \longrightarrow \quad \mathrm{Model}(\mathbb{L}_1, \mathbb{C}).$$

When $\mathbb{C}$ is Cartesian closed and has all small colimits (e.g. *Set*),

$$\boxed{\text{free model } F_f(A) \text{ of } A : \mathbb{L}_1 \to \mathbb{C} \text{ along } f \quad = \quad \text{left Kan extension}}$$

$$\begin{tikzcd}
 & \mathbb{C} & \\
 & \Rightarrow & \\
\mathbb{L}_1 \arrow[uur, "A"] \arrow[rr, "f"'] & & \mathbb{L}_2 \arrow[uul, "F_f A"']
\end{tikzcd}$$

Free models as Kan extensions

The construction is functorial

For example, the free monoid in *Set* is computed as

$$A^* \quad = \quad \coprod_{n \in \mathbb{N}} A^{\times n}.$$

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The magic comes from the fact that the Kan extension always preserves finite product if A and f do.

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The analogy with the tensor algebra is striking

$\Rightarrow$ adapt algebraic theory to linear theory

Linear theory : PRO

Cartesian category $\longrightarrow$ monoidal category

finite-product preserving functor $\longrightarrow$ monoidal functor

Examples of PROs

- trivial PRO: $\mathbb{N}$ = the free monoidal category generated by the category with one object:

$$\mathit{MonCat}(\mathbb{N})(\mathbb{C}) \quad \cong \quad \mathbb{C}$$

- PRO of monoids: Δ = the category of augmented simplices

$$\mathit{MonCat}(\Delta)(\mathbb{C}) \quad \cong \quad \mathit{Mon}(\mathbb{C})$$

The tensor algebra

Let f be the unique monoidal functor from $\mathbb{N}$ to Δ that sends $1 \mapsto 1$

When $\mathbb{C} = k$-Mod, the Kan extension is

$$\mathsf{Lan}_f A \quad : \quad p \quad \mapsto \quad \bigoplus_{n \in \mathbb{N}} \Delta(n, p) \otimes A^{\otimes n}$$

where the k-module $\Delta(n, p) \otimes A^{\otimes n}$ means the direct sum of as many copies of the k-module $A^{\otimes n}$ as there are elements in the hom-set $\Delta(n, p)$.

$$\mathsf{Lan}_f A(1) \quad = \quad TA$$

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$$\mathrm{Lan}_f A(1) \quad = \quad TA$$

Unfortunately, the Kan extension in Cat is not always a Kan extension in *MonCat*.

When is the left Kan extension of a monoidal functor A

along a monoidal functor f, a monoidal left Kan extension?

T-algebraic theory

Given a pseudomonad T on Cat, define the bicategory Cat^T

- T-algebraic category = pseudoalgebra of the pseudomonad T,
- T-algebraic functor = pseudoalgebra pseudofunctor,
- T-algebraic natural transformation = pseudoalgebra natural transformation.

A T-algebraic theory is then a small T-algebraic category

Examples of T-algebraic theories

T-algebraic theories	$T\mathbb{A}$
algebraic theories	free category with finite products
linear theories	free monoidal category
symmetric theories	free symmetric monoidal category
braided theories	free braided monoidal category
projective sketches	free category with finite limits

Algebraic distributors at work [Benabou]

The bicategory of distributors consists in

- Categories as 0-cells
- Functors from

  $$\mathbb{A} \times \mathbb{B}^{\mathsf{op}} \quad \longrightarrow \quad \mathit{Set}$$

  as 1-cells, noted

  $$\mathbb{A} \quad \longrightarrow\!\!\!\!\!\!\!\!\;\;\mid\;\; \quad \mathbb{B}$$

- Natural transformations as 2-cells

Right adjoint and Kan extension

Every functor $f : \mathbb{A} \longrightarrow \mathbb{B}$ gives rise to a distributor

$$f_* : \mathbb{A} \longrightarrow\mkern-14mu\mapstochar\mkern14mu \mathbb{B}$$

which as a right adjoint

$$f^* : \mathbb{B} \longrightarrow\mkern-14mu\mapstochar\mkern14mu \mathbb{A}$$

$$\begin{tikzcd}
\mathbb{A} \arrow[r, bend left, "f_*"] \arrow[r, phantom, "\perp"] & \mathbb{B} \arrow[l, bend left, "f^*"]
\end{tikzcd}$$

Right adjoint and Kan extension

The Kan extension of a functor f along a functor j is obtained by

- first composing g_* and f^*
- then taking the representative $\mathsf{Lan}_f(g)$ of $g_* \circ f^*$

$$\mathsf{Dist}(g_* \circ f^*, h_*) \quad \cong \quad \mathsf{Cat}(\mathsf{Lan}_f(g), h)$$

$$\begin{tikzcd}
& \mathbb{C} & \\
\mathbb{L}_1 \arrow[ur, "g_*"] \arrow[rr, "f_*"] \arrow[rr, phantom, shift right=4ex, "\perp"] & & \mathbb{L}_2 \arrow[ul, "g_* \circ f^*"'] \arrow[ul, bend right=60, "\mathsf{Lan}_f(g)"'] \arrow[ll, bend left=50, "f^*"]
\end{tikzcd}$$

The two ingredients of the recipe

Ingredient n°1:

the adjunction
$f_* \dashv f^*$
is T-algebraic

Ingredient n°2:

the T-algebraic distributor
$g_* \circ f^* : \mathbb{A} \quad \longrightarrow\!\!\!\!\!\!\!+ \quad \mathbb{C}$
is represented by a T-algebraic functor

The two ingredients of the recipe

Ingredient n°1:

the adjunction
$f_* \dashv f^*$
is T-algebraic

$\Longrightarrow$ operadicity

Ingredient n°2:

the T-algebraic distributor
$g_* \circ f^* : \mathbb{A} \quad \relbar\joinrel\mapstoarrow \quad \mathbb{C}$
is represented by a T-algebraic functor

$\Longrightarrow$ as the required algebraic colimits

Main result

Hypotheses:

- $f : \mathbb{L}_1 \to \mathbb{L}_2$ is operadic,
- $\mathbb{C}$ is algebraically cocomplete via the adjunction

$$\mathrm{colim} : \overline{\mathbb{C}} \rightleftarrows \mathbb{C} : y$$

- for all morphism $g : \mathbb{L}_1 \to \mathbb{C}$ in Cat, $g_* \circ f^*$ in Dist factorises through y^*.

Main result

Then, the forgetful functor

$$U_f \quad : \quad \mathrm{Model}(\mathbb{L}_2, \mathbb{C}) \longrightarrow \mathrm{Model}(\mathbb{L}_1, \mathbb{C})$$

has a left adjoint computed by left Kan extension :

$$\mathrm{Lan}_f \quad : \quad \mathrm{Model}(\mathbb{L}_1, \mathbb{C}) \longrightarrow \mathrm{Model}(\mathbb{L}_2, \mathbb{C}).$$

Moreover, this left Kan extension is computed by

$$\boxed{\mathrm{Lan}_f A \quad = \quad \int^{m \in \mathbb{L}_1} \mathbb{L}_2(fm, n) \otimes A^{\otimes m}}$$

Proarrow equipment [Wood]

A proarrow equipment is a formalisation of the homomorphism of bicategories between Cat and Dist. It consists in a homomorphism of bicategories

$$(-)_* : \mathcal{K} \to \mathcal{M}$$

satisfying the three axioms:

1. The object of $\mathcal{M}$ are those of $\mathcal{K}$ and $(-)_*$ is the identity on objects.
2. $(-)_*$ is locally fully faithful, ie.

$$\mathcal{K}(f, g) \cong \mathcal{M}(f_*, g_*)$$

3. For every arrow f in $\mathcal{K}$, f_* has a right adjoint f^*.

Representative of $\mathcal{M}$ in $\mathcal{K}$

an arrow $g : \mathbb{B} \rightarrow \mathbb{C}$ of $\mathcal{K}$ represents an arrow $f : \mathbb{B} \rightarrow \mathbb{C}$ of $\mathcal{M}$

when

$$\mathcal{M}(f, (-)_*) \quad \cong \quad \mathcal{K}(g, -)$$

Yoneda situation

A morphism $y : \mathbb{C} \rightarrow \overline{\mathbb{C}}$ of $\mathcal{K}$ is in a Yoneda situation if

- y is fully faithful :

$$y^* \circ y_* \quad \cong \quad \mathrm{id}_{\mathbb{C}}$$

- y^* is pseudomonic with respect to $\mathcal{K}$, ie. the functor

$$y^* \circ (-)_* \quad : \quad \mathcal{K}(\mathbb{A}, \overline{\mathbb{C}}) \quad \rightarrow \quad \mathcal{M}(\mathbb{A}, \mathbb{C})$$

  is fully faithful for all objects $\mathbb{A}$ of $\mathcal{K}$

$\overline{\mathbb{C}}$-*cocomplete*

An object $\mathbb{C}$ of $\mathcal{K}$ is

$\overline{\mathbb{C}}$-cocomplete

when there is a Yoneda situation

$$y : \mathbb{C} \to \overline{\mathbb{C}}$$

which as a left adjoint

$$\begin{tikzcd}
\overline{\mathbb{C}} \arrow[r, bend left, "\mathrm{colim}"] \arrow[r, phantom, "\perp"] & \mathbb{C} \arrow[l, bend left, "y"]
\end{tikzcd}$$

Computing the representative

Take a morphism f that can be factorise as

$$\mathbb{B} \xrightarrow{f} \mathbb{C} = \mathbb{B} \xrightarrow{\overline{f}_*} \overline{\mathbb{C}} \xrightarrow{y^*} \mathbb{C}.$$

Then

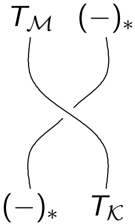
$$colim \circ \overline{f}$$

is a representative of f.

Pseudomonad in a proarrow equipment

A pseudomonad T in a proarrow equipment $(-)_* : \mathcal{K} \to \mathcal{M}$ is given by

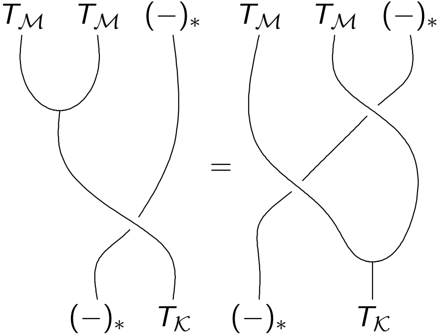
- a pseudomonad $T_{\mathcal{K}}$ on $\mathcal{K}$
- a pseudomonad $T_{\mathcal{M}}$ on $\mathcal{M}$
- a pseudo natural transformation $h : T_{\mathcal{M}} \circ (-)_* \to (-)_* \circ T_{\mathcal{K}}$ noted

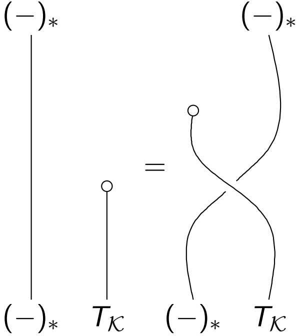


making $((-)_*, h)$ be a map of pseudomonads from $T_{\mathcal{K}}$ to $T_{\mathcal{M}}$,

Pseudomonad in a proarrow equipment

Comes from a notion of morphism between pseudomonads introduced by Nicola Gambino.





Operadicity

A $T_{\mathcal{K}}$-algebraic morphism f of $\mathcal{K}$ is operadic

when its right adjoint f^* in $\mathcal{M}$ is $T_{\mathcal{M}}$-algebraic

$$\begin{tikzcd}
\mathbb{A} \arrow[rr, bend left, "f_* \text{ is } T_{\mathcal{M}}\text{-algebraic}"] \arrow[rr, phantom, "\perp"] & & \mathbb{B} \arrow[ll, bend left, "f^* \text{ is } T_{\mathcal{M}}\text{-algebraic}"]
\end{tikzcd}$$

Recall that f^* is always a lax $T_{\mathcal{M}}$-algebraic morphism.

Algebraic colimits

An $\overline{\mathbb{C}}$-cocomplete object $\mathbb{C}$ of $\mathcal{K}$ with the adjunction

$$\boxed{\text{colim} : \begin{tikzcd}\overline{\mathbb{C}} \arrow[r, shift left] & \mathbb{C} \arrow[l, shift left]\end{tikzcd} : y}$$

is algebraically $\overline{\mathbb{C}}$-cocomplete

when

colim,y and y^* are algebraic

Main result

Hypotheses:

- $f : \mathbb{L}_1 \to \mathbb{L}_2$ in $\mathcal{K}$ is operadic,
- $\mathbb{C}$ is algebraically cocomplete via the adjunction

$$\text{colim} : \overline{\mathbb{C}} \rightleftarrows \mathbb{C} : y$$

- for all morphism $g : \mathbb{L}_1 \to \mathbb{C}$ in $\mathcal{K}$, $g_* \circ f^*$ in $\mathcal{M}$ factorises through y^*.

Main result

Then, the forgetful functor

$$U_f \quad : \quad \mathrm{Model}(\mathbb{L}_2, \mathbb{C}) \to \mathrm{Model}(\mathbb{L}_1, \mathbb{C})$$

has a left adjoint computed by left Kan extension :

$$\mathrm{Lan}_f \quad : \quad \mathrm{Model}(\mathbb{L}_1, \mathbb{C}) \to \mathrm{Model}(\mathbb{L}_2, \mathbb{C}).$$

When the proarrow equipment is $(-)_* : \mathrm{Cat} \to \mathrm{Dist}$, this left Kan extension is computed by

$$\boxed{\mathrm{Lan}_f A \quad = \quad \int^{m \in \mathbb{L}_1} \mathbb{L}_2(fm, n) \otimes A^{\otimes m}}$$

Operadicity

When the proarrow equipment is $(-)_* : \mathsf{Cat} \to \mathsf{Dist}$, operadicity means that

$$\int^{h \in T(\mathbb{L}_1)} \mathbb{L}_1(m, [h]) \otimes T(\mathbb{L}_2)(Tf(h), n) \quad \longrightarrow \quad \mathbb{L}_2(fm, [n])$$

is an isomorphism

Operadicity

$$\boxed{\text{operadicity} \quad = \quad \text{tree decomposition property}}$$

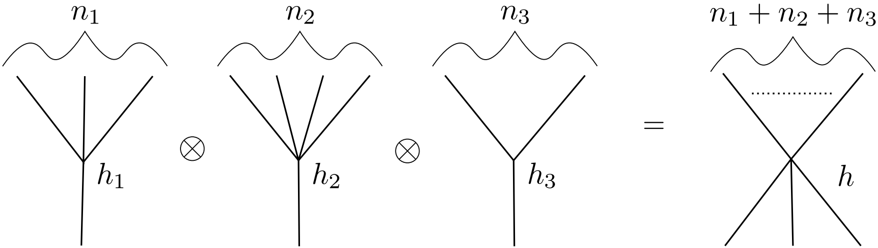
Operadicity for linear theories

When T is the pseudomonad for monoidal category, the isomorphism becomes

$$\int^{h_1 \in \mathbb{L}_1} \cdots \int^{h_k \in \mathbb{L}_1} \mathbb{L}_1(h, h_1 + \cdots + h_k) \times \mathbb{L}_2(h_1, n_1) \times \cdots \times \mathbb{L}_2(h_k, n_k)$$

$$\longrightarrow \quad \mathbb{L}_2(h, n_1 + \cdots + n_k)$$

Operadicity for linear theories



Operadicity for linear theories

This terminology "operadic" is justifies by the fact:

Every map of operads f between two operads
$\mathbb{L}_1$ and $\mathbb{L}_2$ (seen as monoidal categories)
is operadic

Operadicity for algebraic theories or projective sketches

The pseudomonads for algebraic theories and projective sketches share the property that

$$\mathbb{A}(m, [p]) \quad \cong \quad T\mathbb{A}(\eta_{\mathbb{A}}(m), p)$$

where p is a finite diagram.

This isomorphism comes from that fact that there are as many morphisms from m to the limit of p as cones from m to p.

When T is the pseudomonad
for algebraic theories or projective sketches,
every T-algebraic functor is operadic.

Factorisation system of Cat [Street, Walters]

- **E** : the classe of final functors
- **M** : the classe of discrete fibrations

Any diagram $F : J \to \mathbb{C}$ may be seen as the presheaf φ given by the decomposition

$$J \xrightarrow{F} \mathbb{C} \quad = \quad J \xrightarrow{F_1} \mathrm{Elt}\varphi \xrightarrow{F_2} \mathbb{C}$$

where F_1 is a final functor and F_2 is a discrete fibration.

Algebraically cocomplete

When the proarrow equipment is $(-)_* : \mathsf{Cat} \to \mathsf{Dist}$,

$$\boxed{\begin{array}{c}\text{algebraic cocompleteness} = \text{colimits under some class } \mathcal{F} \\ \text{commute with the } T\text{-algebraic structure}\end{array}}$$

Algebraically cocomplete : linear theories

When T is the pseudo-monad for monoidal categories, one chooses a subcategory of the category of presheaves

$$\overline{\mathbb{C}} \hookrightarrow \widehat{\mathbb{C}}$$

closed under the Day's tensor product

$$\varphi_1 \otimes_{\overline{\mathbb{C}}} \varphi_2 : b \mapsto \int^{a_1, a_2 \in \mathbb{C}} \mathbb{C}(b, a_1 \otimes_{\mathbb{C}} a_2) \otimes \varphi_1(a_1) \otimes \varphi_2(a_2)$$

Algebraically cocomplete : linear theories

This is the case for example when the class $\mathcal{F}$ is closed under product

Algebraically cocomplete : linear theories

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$$\begin{tikzcd}
I \times J \arrow[r, "\text{final}"] \arrow[dr, "F \times G"'] & \mathrm{Elt}\varphi \times \mathrm{Elt}\psi \arrow[r, "\text{final}"] \arrow[d, "\text{discrete}"', "\text{fibration}"] & \mathrm{Elt}(\varphi \otimes \psi) \arrow[d, "\text{discrete}"', "\text{fibration}"] \\
& \mathbb{C} \times \mathbb{C} \arrow[r, "\otimes"'] & \mathbb{C}
\end{tikzcd}$$

Algebraically cocomplete : linear theories

$\mathbb{C}^{\bullet}$ is the restriction of the category of presheaves to presheaves having a colimit in $\mathbb{C}$

$$\begin{tikzcd}
\mathbb{C} \arrow[r, bend left, "y"] \arrow[r, phantom, "\perp"] & \mathbb{C}^{\bullet} \arrow[l, bend left, "\mathsf{colim}"] \arrow[r, hook] & \widehat{\mathbb{C}}
\end{tikzcd}$$

$\overline{\mathbb{C}}$ is the restriction of the category of presheaves to presheaves having an algebraic colimit in $\mathbb{C}$

$$\begin{tikzcd}
\mathbb{C} \arrow[r, bend left, "y"] \arrow[r, phantom, "\perp"] & \overline{\mathbb{C}} \arrow[l, bend left, "\mathsf{colim}"] \arrow[r, hook] & \mathbb{C}^{\bullet} \arrow[r, hook] & \widehat{\mathbb{C}}
\end{tikzcd}$$

Observe that $\overline{\mathbb{C}}$ and $\widehat{\mathbb{C}}$ are equipped with $\otimes_{\mathsf{Day}}$ but not necessarily $\mathbb{C}^{\bullet}$.

A diagram as a colimit of diagrams

Let $\mathbb{C}$ be a monoidal category and $\overline{C}$ be the restriction of the category of presheaves to presheaves having an algebraic colimit in $\mathbb{C}$

If every J-indexed diagram has an algebraic colimit, then $\overline{\mathbb{C}}$ is closed under J-indexed colimits.

Free monoid: the Dubuc construction

$\mathbb{C}$ is an monoidal category with colimits for which

- coequalisers commute with the tensor product
- sequential colimits commute with the tensor product

Then we can compute the free monoid on pointed object

Free monoid: the Dubuc construction

- $\mathbb{L}_1 = \Delta_{\mathsf{face}}$: the category of augmented simplices and injective maps

  theory of pointed objects

- $\mathbb{L}_2 = \Delta$: the category of augmented simplices

  theory of monoids

$$\begin{tikzcd}
 & \mathbb{C} & \\
\Delta_{\mathsf{face}} \arrow[ur, "A^{\bullet}"] \arrow[rr, hook, "f \text{ operadic}"'] & & \Delta \arrow[ul, "\mathrm{Lan}\, A"']
\end{tikzcd}$$

Free monoid: the Dubuc construction

In practice, we have to show that

all the diagrams defining the Kan extension in Dist

live in $\overline{\mathbb{C}}$

Free monoid: the Dubuc construction

Coequalisers commute with the tensor product in $\mathbb{C}$.

Thus, the presheaf φ_n associated to the diagram

$$\begin{tikzcd}
1 \arrow[r] & A \arrow[r, shift left] \arrow[r, shift right] & A^{\otimes 2} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \cdots \arrow[r, shift left=4] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[r, shift right=4] & A^{\otimes n}
\end{tikzcd}$$

lives in $\overline{\mathbb{C}}$ for every n.

Free monoid: the Dubuc construction

As sequential colimits commute with the tensor product in $\mathbb{C}$, the sequential colimit of the presheaves φ_n

$$\begin{tikzcd}
1 \arrow[r] & A \arrow[r, shift left=1] \arrow[r, shift right=1] & A^{\otimes 2} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \cdots \arrow[r, shift left=4] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[r, shift right=4] & A^{\otimes n} \arrow[r, dotted, dash, shift left=4] \arrow[r, dotted, dash, shift left=2] \arrow[r, dotted, dash] \arrow[r, dotted, dash, shift right=2] \arrow[r, dotted, dash, shift right=4] & \cdots
\end{tikzcd}$$

$$=$$

$$\Delta_{\mathrm{face}} \xrightarrow{A^{\bullet}} \mathbb{C}$$

lives in $\overline{\mathbb{C}}$

Free monoid: the Vallette construction

$\mathbb{C}$ is an monoidal category with colimits for which

- reflexive coequalisers commute with the tensor product
- sequential colimits commute with the tensor product

Then we can compute the free monoid on pointed object

Free monoid: the Vallette construction

Recipe : replace the pair

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & A^{\otimes 2} \end{tikzcd}$$

with the reflexive pair (having the same coequaliser)

$$\begin{tikzcd} A \oplus A^{\otimes 2} \arrow[r, shift left, "f \oplus A^{\otimes 2}"] \arrow[r, shift right, "g \oplus A^{\otimes 2}"'] & A^{\otimes 2} \arrow[l, bend right=50, "i_2"'] \end{tikzcd}$$

and apply the same construction.

Free commutative monoid

$\mathbb{C}$ is an symmetric monoidal category with colimits for which

- coequalisers commute with the tensor product
- coproducts commute with the tensor product

Then we can compute the free commutative monoid

Free commutative monoid

First, we coequalise the permutation on $A^{\otimes n}$

Then we take the coproduct of the coequalisers

$$\boxed{TA = \bigoplus_{n \in \mathbb{N}} A^{\otimes n} / \sim}$$

Free exponential : the recipe

In semantics, many exponentials can be computed by the following recipe

1. Take an object A of the category,

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$$A^\bullet \quad \longrightarrow \quad 1$$

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Free exponential : the recipe

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   This object is often $A \,\&\, 1$

3. Compute the free commutative comonoid $!A$ on this pointed object. Given by the limit of the diagram

$$\begin{tikzcd}
1 & (A \,\&\, 1) \arrow[l, "i_2"'] & (A \,\&\, 1)^2 \arrow[l, shift right, "\mathrm{id}_{(A\&1)} \otimes i_2"'] \arrow[l, shift left, "i_2 \otimes \mathrm{id}_{(A\&1)}"] \arrow[loop above, "\sigma"] & \cdots (A \,\&\, 1)^n \cdots \arrow[l, shift right=2, "\mathrm{id}_{(A\&1)^2} \otimes i_2"'] \arrow[l] \arrow[l, shift left=2, "i_2 \otimes \mathrm{id}_{(A\&1)^2}"] \arrow[loop above, "\sigma"]
\end{tikzcd}$$

Free exponential : the recipe

By composition of adjunctions, we have computed

the free exponential on A.

Free exponential : coherent spaces

The free affine object on A is just

$$A \,\&\, 1$$

the coherent space A and extra element (which will be seen as the empty clique).

Free exponential : coherent spaces

The limit (equalizer) A_n of the diagram

$$\begin{tikzcd}
(A \,\&\, 1)^n & (A \,\&\, 1)^n \arrow[l, shift right=2, "\mathrm{id}_{(A\&1)^{n-2}} \otimes \sigma"'] \arrow[l] \arrow[l, shift left=2, "\sigma \otimes \mathrm{id}_{(A\&1)^{n-2}}"]
\end{tikzcd}$$

is the usual multiclique with at most n elements of A.

Free exponential : coherent spaces

The free exponential $!_e A$ is thus given by the limit of

$$\begin{tikzcd} A_0 = 1 & A_1 = (A \,\&\, 1) \arrow[l, "i_2"'] & A_2 \arrow[l] & \cdots A_n \cdots \arrow[l] \end{tikzcd}$$

This is the usual set of finite multicliques.

Free exponential : Conway games

The same construction applies in game semantics,

e.g. the category of Conway games.