

Algèbres de Batalin-Vilkovisky à homotopie près

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Travail en commun avec Imma Gálvez et Andy Tonks

Overview

- 1 Homotopy algebras and operads
- 2 Definition of homotopy BV-algebra
- 3 Homotopy and deformation theory of algebras

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Associative algebras and homotopy [Stasheff 63]

X, Y homotopy equivalent topological spaces: $X \sim Y$

$X \times X \xrightarrow{\mu} X$ associative product $\xrightarrow{\text{transfer}} Y \times Y \xrightarrow{\nu} Y$

but ν associative **up to homotopy**

\implies “The disadvantage of topological groups and monoids is that they do not live in homotopy theory”

[Mac Lane’s seminar at the University of Chicago 1967]

Homotopy associative algebras [Stasheff 63]

Let (A, d_A) be a dg module. We consider $\text{Hom}(A^{\otimes n}, A)$ with derivative $\partial(f) := d_A \circ f - (-1)^{|f|} f \circ d_{A^{\otimes n}}$

Definition (Homotopy associative algebra or A_∞ -algebra)

Family of operations $\{\mu_n : A^{\otimes n} \rightarrow A\}_{n \geq 2}$, degree $|\mu_n| = n - 2$, such that

$$\sum_{\substack{2 \leq k \leq n-1 \\ 1 \leq i \leq k}} \pm \mu_k \circ \left(\text{id}^{\otimes(i-1)} \otimes \mu_{n-k+1} \otimes \text{id}^{\otimes(k-i)} \right) = \partial(\mu_n)$$

in $\text{Hom}(A^{\otimes n}, A)$ for all $n \geq 2$.

Example: $\mu_2 \circ (\mu_2 \otimes \text{id}) - \mu_2 \circ (\text{id} \otimes \mu_2) = \partial(\mu_3)$

The left hand side relation holds **up to the homotopy μ_n** .

▷ Associative algebra = A_∞ -algebra with $\mu_n = 0$, $n \geq 3$.

Homotopy theory for A_∞ -algebras

Definition (A_∞ -morphism between A_∞ -algebras)

Family of maps $\{f_n : A^{\otimes n} \rightarrow B\}_{n \geq 1}$, of degree $|f_n| = n - 1$, satisfying some condition. **Notation** $f_\bullet : A \rightarrow B$

Proposition

$(A_\infty - \text{algebras}, A_\infty - \text{morphisms})$ forms a category.

▷ Notion of A_∞ -homotopy relation between A_∞ -morphisms.

⇒ [Abstract homotopy theory for \$A_\infty\$ -algebras](#)

Definition (A_∞ -homotopy equivalence)

$f_\bullet : A \rightarrow B$, A_∞ -morphism such that

$\exists g_\bullet : B \rightarrow A$, A_∞ -morphism satisfying

$f \circ g \sim \text{id}_B$ and $g \circ f \sim \text{id}_A$: A_∞ -homotopy equivalence

Homotopy invariant property for A_∞ -algebras

Let $(V, d_V) \xrightarrow{f} (W, d_W)$ be a chain homotopy equivalence.

Theorem (Kadeishvili, Gugenheim, ...)

Any A_∞ -algebra structure on W transfers to an A_∞ -algebra structure on V such that

- *f extends to an A_∞ -morphisms with $f_1 = f$, which is a A_∞ -homotopy equivalence.*

▷ Explicit formula based on planar trees [Kontsevich-Soibelman]

Application: $V = H(A) \rightsquigarrow A = W$

transferred A_∞ -operations on $H(A) =$ **Massey products**

Example: Associative algebra $A = (C^\bullet(X), \cup)$ of singular cochain complex of a space X . \implies the **original Massey product**.

▷ Allow to reconstruct the homotopy class of A in $\text{Ho}(A_\infty\text{-alg})$.

Other homotopy algebras

Other examples

- “Homotopy everything (commutative) algebras” or E_∞ -algebras (C_∞ -algebras, \mathbb{K} field of characteristic 0)
[Algebraic Topology, Mandell]
- Homotopy Lie algebras or L_∞ -algebras
[Deformation-Quantization of Poisson manifold, Kontsevich]
- Gerstenhaber algebras up to homotopy or G_∞ -algebras
[Deligne conjecture, Tamarkin]

an operad \mathcal{P}



a category of algebras

Homotopy theory for operads

Theorem (Hinich, Berger-Moerdijk, Spitzweck)

There is a cofibrantly generated model category structure on dg operads transferred from that of dg modules.

an operad $\mathcal{P} \xleftarrow{\sim} \mathcal{P}_\infty$: cofibrant replacement



category of algebras \hookrightarrow category of *homotopy* algebras

Proposition (Berger-Moerdijk)

Let \mathcal{P}_∞ be a *cofibrant* dg operad.

Under some assumptions, \mathcal{P}_∞ -algebra structures transfer through weak equivalences.

▷ Homotopy invariant property for algebras over *cofibrant* operads.

Cofibrant operads

Definition (Quasi-free operad)

Quasi-free operad $(\mathcal{P}, d) :=$ dg operad which is free $\mathcal{P} \cong \mathcal{F}(\mathcal{C})$ when forgetting the differential.

Proposition

$\{\text{projective modules}\} = \{\text{direct summand of free modules}\}$

$\{\text{cofibrant operads}\} = \{\text{retract of quasi-free operads}\}$

\implies free modules are projective

\implies quasi-free operads are cofibrant

\implies Look for **quasi-free resolutions**: $(\mathcal{F}(\mathcal{C}), d) \xrightarrow{\sim} \mathcal{P}$

Quasi-free operads $(\mathcal{F}(\mathcal{C}), d)$

Data: generators \mathcal{C} and differential d

Proposition

Since d is a derivation, it is characterized by its restriction $d|_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{F}(\mathcal{C})$.

$d^2 = 0 \iff$ algebraic structure on \mathcal{C} (homotopy cooperad)

Data:

- the **shape of \mathcal{C}** = underlying \mathbb{S} -module
- $d \iff$ **algebraic structure on \mathcal{C}**

Quasi-free resolution $(\mathcal{F}(V), d) \xrightarrow{\sim} \mathcal{P}$: **operadic syzygies**

General problems

- How to make cofibrant replacements for operads **explicit** ?
 \iff **Explicit** definition of homotopy \mathcal{P} -algebras
- How to make the transferred \mathcal{P}_∞ -algebra structure explicit ?
- How to define \mathcal{P}_∞ -morphism “in general” ?
- Describe the homotopy category of \mathcal{P}_∞ -algebras
- Define the deformation theory of \mathcal{P}_∞ -algebras

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Gerstenhaber algebras

Let (A, d_A) be a dg module.

Definition (Gerstenhaber algebra)

- ▷ symmetric product $\bullet : A \odot A \rightarrow A$, degree 0,
- ▷ skew-symmetric bracket $[,] : A \wedge A \rightarrow A$, degree +1

such that

- ▷ \bullet associative,
- ▷ $[,]$ satisfies the Jacobi identity,
- ▷ \bullet and $[,]$ satisfy the Leibniz relation

$$[- \bullet -, -] = ([-, -] \bullet -) \cdot (23) + (- \bullet [-, -]),$$

Example: Hochschild cohomology $HH(A, A)$ is a Gerstenhaber algebra.

Questions:

- Define the notion of **Gerstenhaber algebra up to homotopy**
- Lift the Gerstenhaber algebra from $\mathrm{HH}(A,A)$ to $\mathrm{CH}(A,A)$
[=Deligne conjecture]
=Question **dual** to that of Massey products.

Gerstenhaber operad

Let \mathcal{G} be the operad of Gerstenhaber algebras.

Presentation:

$$\mathcal{G} = \mathcal{F} \left(\begin{array}{c} \diagdown \quad \diagup \\ \bullet \\ | \end{array}, \begin{array}{c} \diagdown \quad \diagup \\ [,] \\ | \end{array} \right) / (R) = \text{Free operad} / \text{ideal generated by } R$$

$$R = \{ \text{Assoc}(\bullet), \text{Jacobi}([,]), \text{Leibniz}(\bullet, [,]) \}$$

Quadratic presentation:

$$\text{Assoc}(\bullet) = \begin{array}{c} \diagdown \quad \diagup \\ \bullet \\ \diagdown \quad \diagup \\ \bullet \\ | \end{array} - \begin{array}{c} \diagdown \quad \diagup \\ \diagdown \quad \diagup \\ \bullet \\ | \end{array}$$

Quadratic := R writes with 2-vertices trees

Koszul duality theory

Dual notion of operad : **cooperad** [reverse the arrows]

Example: a coalgebra is a cooperad concentrated in arity 1.

Theorem (Ginzburg-Kapranov, Getzler-Jones)

There exist adjoint functors

$$B : \{dg \text{ operads}\} \rightleftarrows \{dg \text{ cooperads}\} : \Omega$$

*called **bar** and **cobar** constructions*

Quadratic operad $\mathcal{P} \xrightarrow{\text{Koszul dual cooperad}} \text{cooperad } \mathcal{P}^i$

with $\mathcal{P}_\infty := \Omega\mathcal{P}^i = \mathcal{F}(s^{-1}\bar{\mathcal{P}}^i) \rightarrow \mathcal{P}$ morphism of dg operads

Definition (Koszul operad)

\mathcal{P} Koszul if $\mathcal{P}_\infty = \Omega\mathcal{P}^i \xrightarrow{\sim} \mathcal{P}$ quasi-isomorphism: **cofibrant replacement**

Applied Koszul duality theory

- Computation of \mathcal{P}^i

Proposition (Ginzburg-Kapranov, Getzler-Jones)

Let $\mathcal{P} = \mathcal{F}(V)/(R)$ be a quadratic operad with $\dim(V) < +\infty$.
 The linear dual \mathcal{P}^{i*} is a quadratic operad and
 the suspension of $\mathcal{P}^{i*} \cong \mathcal{P}^! := \mathcal{F}(V^* \otimes \text{sgn})/(R^\perp)$.

Example: $\mathcal{G}^! = \mathcal{G}$: Koszul auto-dual operad

Method: Compute $\mathcal{P}^!$ and its operadic structure with this formula,
 then dualize everything to get \mathcal{P}^i and its cooperadic structure.

- Methods to prove that \mathcal{P} is Koszul

Proposition (Getzler-Jones, Markl)

The Gerstenhaber operad \mathcal{G} is Koszul.

Homotopy Gerstenhaber algebras or G_∞ -algebras

$$\mathcal{G} = \mathcal{G}^1 = \text{Com} \circ \mathcal{L}ie^1$$

Proposition (Ree 58)

$$\mathcal{L}ie(A) \cong A^{\otimes n} / \text{Shuffles} =: \overline{A^{\otimes n}}$$

Definition-Proposition (Homotopy Gerstenhaber algebra)

$$m_{p_1, \dots, p_n} : \overline{A^{\otimes p_1}} \wedge \dots \wedge \overline{A^{\otimes p_n}} \longrightarrow A, \quad n, p_1, \dots, p_n \geq 1,$$

such that

$$\sum \pm m_{t, p_{j_1}, \dots, p_{j_{n-r}}} \left(\sum m_{q_1, \dots, q_r}^{p_{i_1}, \dots, p_{i_r}} (a_{i_1} \wedge \dots \wedge a_{i_r}) \wedge a_{j_1} \wedge \dots \wedge a_{j_{n-r}} \right) = 0.$$

Applied G_∞ -algebras

Proposition (Getzler-Jones)

Let \mathcal{P} be a Koszul operad. A \mathcal{P}_∞ -algebra structure on $A \iff$ a square-zero derivation on the cofree \mathcal{P}^i -coalgebra $\mathcal{P}^i(A)$.

Application: \mathcal{G}_∞ -algebra structures on $A \iff$ square-zero derivations on the cofree Gerstenhaber coalgebra $\mathcal{G}^*(sA)$.

Theorem (Tamarkin-Deligne Conjecture)

There is a \mathcal{G}_∞ -algebra structure on Hochschild cochain complex $CH(A, A)$ which lifts the Gerstenhaber algebra structure on $HH(A, A)$.

Batalin-Vilkovisky algebras or BV-algebras

Definition (Batalin-Vilkovisky algebra)

A Gerstenhaber algebra $(A, \bullet, [,])$ with

▷ **unary operator** $\Delta : A \rightarrow A$, degree +1

such that

▷ $\Delta^2 = 0$,

▷ $[,] =$ obstruction to Δ being a derivation with respect to \bullet

$$[-, -] = \Delta \circ (- \bullet -) - (\Delta(-) \bullet -) - (- \bullet \Delta(-)),$$



▷ Δ derivation with respect to $[,]$

$$\Delta([-, -]) = [\Delta(-), -] + [-, \Delta(-)]$$

BV-operad

Problem: The operad BV is **not** quadratic

$$[-, -] = \Delta \circ (- \bullet -) - (\Delta(-) \bullet -) - (- \bullet \Delta(-)) \text{ writes}$$

1

2

Quadratic Batalin-Vilkovisky algebras or q BV-algebras

Definition (quadratic Batalin-Vilkovisky algebra)

A Gerstenhaber algebra $(A, \bullet, [,])$ with

▷ unary operator $\Delta : A \rightarrow A$, degree $+1$

such that

▷ $\Delta^2 = 0$,

▷ Δ derivation with respect to \bullet

$$\Delta \circ (- \bullet -) - (\Delta(-) \bullet -) - (- \bullet \Delta(-)) = 0,$$

does not imply any more

▷ Δ derivation with respect to $[,]$

$$\Delta([-, -]) = [\Delta(-), -] + [-, \Delta(-)]$$

The operad q BV is **quadratic** by definition.

qBV-operad and Homotopy quadratic BV-algebras

Proposition

The operad qBV is Koszul.

$\implies qBV_\infty := \Omega qBV^i \xrightarrow{\sim} qBV$ quasi-free resolution

Lemma

$$qBV^! \cong \mathbb{K}[\delta] \otimes Com \circ Lie^1 = \mathbb{K}[\delta] \otimes \mathcal{G}^!, \quad |\delta| = 2$$

Definition-Proposition (Homotopy qBV-algebra)

$$m_{p_1, \dots, p_n}^d : \overline{A^{\otimes p_1}} \wedge \dots \wedge \overline{A^{\otimes p_n}} \longrightarrow A, \quad n, p_1, \dots, p_n \geq 1, d \geq 0$$

such that

$$\sum \pm m_{t, p_{j_1}, \dots, p_{j_{n-r}}}^d \left(\sum m_{q_1, \dots, q_r}^{d', p_{i_1}, \dots, p_{i_r}} (a_{i_1} \wedge \dots \wedge a_{i_r}) \wedge a_{j_1} \wedge \dots \wedge a_{j_{n-r}} \right) = 0.$$

Applied Homotopy quadratic BV-algebras

\mathcal{G} -alg \rightsquigarrow qBV -alg

\mathcal{G}_∞ -alg \rightsquigarrow qBV_∞ -alg

Theorem (“Extended Deligne conjecture”)

A associative algebra with a unit 1

$$\Delta : f \in \text{Hom}(A^{\otimes n}, A) \mapsto \sum_{i=1}^n f \circ \left(id^{\otimes(i-1)} \otimes 1 \otimes id^{\otimes(n-i)} \right) \in \text{Hom}(A^{\otimes n-1}, A)$$

defines a qBV -algebra structure on $HH(A, A)$,

which extends the Gerstenhaber algebra structure.

It lifts to a homotopy qBV -algebra structure on $CH(A, A)$.

Idea: Add a suitable inner differential $d_1 : qBV^i \rightarrow qBV^i$
and consider $\Omega(qBV^i, d_1) \xrightarrow{? \sim ?} BV$.

Koszul duality theory revisited

Let $\mathcal{P} = \mathcal{F}(V)/(R)$ be a quadratic **and linear** presentation.
 Let $q : \mathcal{F}(V) \twoheadrightarrow \mathcal{F}(V)^{(2)}$ the quadratic projection
 and $q\mathcal{P} := \mathcal{F}(V)/(qR)$, the quadratic analogue of \mathcal{P} .

Definition (Quadratic-Linear Koszul operad)

$\mathcal{P} = \mathcal{F}(V)/(R)$ is a Koszul operad if

- $R \cap V = \{0\} \iff V$ is “minimal”
- $(R \otimes V + V \otimes R) \cap \mathcal{F}(V)^{(2)} \subset R \cap \mathcal{F}(V)^{(2)} \iff R$ is “maximal”
- $q\mathcal{P}$ is Koszul

R maximal corresponds to \Downarrow in the example of BV.

Quadratic Koszul operad are also Koszul in the above definition.

The inner differential

Lemma

- $R \cap V = \{0\} \implies R = \text{Graph}(\varphi : qR \rightarrow V)$.
- $(R \otimes V + V \otimes R) \cap \mathcal{F}(V)^{(2)} \subset R \cap \mathcal{F}(V)^{(2)} \iff \varphi$ induces a square-zero coderivation d_φ on $q\mathcal{P}^i$.

Proof.

$$q\mathcal{P}^i \twoheadrightarrow \mathcal{F}^c(sV)$$

$\exists!$ coderivation on $\mathcal{F}^c(sV)$ which extends

$$\mathcal{F}^c(sV) \twoheadrightarrow s^2 qR \xrightarrow{s^{-1}\varphi} sV$$

It squares to 0 and descends to $q\mathcal{P}^i$ iff

$$(R \otimes V + V \otimes R) \cap \mathcal{F}(V)^{(2)} \subset R \cap \mathcal{F}(V)^{(2)} \quad \square$$

Cofibrant resolutions for Koszul operads

Definition (Koszul dual dg cooperad)

$$\mathcal{P}^i := (q\mathcal{P}^i, d_\varphi)$$

Theorem

When \mathcal{P} is a quadratic-linear Koszul operad, then

$$\mathcal{P}_\infty := \Omega \mathcal{P}^i \xrightarrow{\sim} \mathcal{P}$$

Theorem

The operad BV is Koszul

$$\implies BV_\infty := \Omega BV^i = \Omega(qBV^i, d_\varphi) \xrightarrow{\sim} BV$$

Homotopy Batalin-Vilkovisky algebras

$$qBV^1 \cong \mathbb{K}[\delta] \otimes Com \circ Lie^1 = \mathbb{K}[\delta] \otimes \mathcal{G}^1$$

Proposition

$d_\varphi(\delta^d \otimes L_1 \odot \cdots \odot L_t) = \sum_{i=1}^t \pm \delta^{d-1} \otimes L_1 \odot \cdots \odot L'_i \odot L''_i \odot \cdots \odot L_t$,
 where $L'_i \odot L''_i = \text{image of } L_i \text{ under the decomposition map of the cooperad } Lie^{1*}$.

Definition-Proposition (BV_∞ -algebra)

$$m_{p_1, \dots, p_n}^d : \overline{A^{\otimes p_1}} \wedge \cdots \wedge \overline{A^{\otimes p_n}} \longrightarrow A, \quad n, p_1, \dots, p_n \geq 1, d \geq 0$$

s.t.

$$\sum \pm m_{t, p_{j_1}, \dots, p_{j_{n-r}}}^d \left(\sum m_{q_1, \dots, q_r}^{d', p_{i_1}, \dots, p_{i_r}} (a_{i_1} \wedge \cdots \wedge a_{i_r}) \wedge a_{j_1} \wedge \cdots \wedge a_{j_{n-r}} \right) \\ + \sum \pm m_{p_1, \dots, p'_r, p_r - p'_r, \dots, p_n}^{d-1} (a_1 \wedge \cdots \wedge \overline{a_{p'_r}^r} \wedge \overline{a_{p_r - p'_r}^r} \wedge \cdots \wedge a_n) = 0$$

Applied Homotopy Batalin-Vilkovisky algebras

\mathcal{G} -alg \rightsquigarrow BV-alg

\mathcal{G}_∞ -alg \rightsquigarrow BV_∞ -alg

Theorem (Ginzburg, Tradler, Menichi)

A: Frobenius algebra or Calabi-Yau

Hochschild cohomology $HH(A, A)$ is a BV-algebra

Theorem (Cyclic Deligne conjecture)

This structure lifts to a BV_∞ -algebra structure on $CH(A, A)$.

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General theory

Generalized Koszul duality theory

- includes **unary** operations [V. Ph.D. Thesis]
- **no restriction** on the homological degree of \mathcal{P}
- works for **properads**, i.e. operations with several outputs
- **PBW Theorem**: $q\mathcal{P} \cong \text{gr}\mathcal{P}$ for the operad \mathcal{P}

Homotopy theory for \mathcal{P}_∞ -algebras

When \mathcal{P} is a **Koszul** operad

- Define the notion of \mathcal{P}_∞ -morphism and \mathcal{P}_∞ -homotopy
- Category of \mathcal{P}_∞ -algebras = Category of **fibrant** objects of \mathcal{P}^i -coalgebras
 \implies understand the homotopy category $\text{Ho}(\mathcal{P}_\infty\text{-alg})$
- **Transfer theorem** for \mathcal{P}_∞ -algebras with explicit formula
 \implies **Massey products**

Example: **BV_∞ -algebras**

Deformation theory

\mathcal{P} be a Koszul operad and (A, d) dg module

Proposition (Deligne philosophy on deformation theory)

\exists dg Lie algebra $(\mathfrak{g}_{\mathcal{P}_\infty}, [,], \partial)$ such that
 Maurer-Cartan elements $\iff \mathcal{P}_\infty$ -algebra structures on A

Definition (Cohomology of \mathcal{P}_∞ -algebras=tangent homology)

Given a \mathcal{P}_∞ -algebra structure $\alpha \in \text{MC}(\mathfrak{g})$
twisted dg Lie algebra $\mathfrak{g}_{\mathcal{P}_\infty}^\alpha := (\mathfrak{g}, [,], \partial + [-, \alpha])$

$H(\mathfrak{g}^\alpha) =$ **Obstructions** to deformations of \mathcal{P}_∞ -algebra structures

Proposition

$\mathfrak{g}_{BV_\infty} \cong \mathfrak{g}_{\mathcal{G}_\infty} \otimes \mathbb{K}[[\delta]]$ isomorphism of Lie algebras

Obstruction theory

In general \mathfrak{g} is triangulated [Sullivan]

Proposition

When \mathcal{P} is a Koszul operad, \mathfrak{g} and \mathfrak{g}^α are *graded* by an extra weight.

\implies the Maurer-Cartan equation splits with respect to this weight.

Theorem (Generalized Lian-Zuckerman conjecture)

For any \mathbb{N} -graded topological vertex algebra A , such that for each conformal weight the fermionic grading is finite,

\exists *BV $_\infty$ -algebra* structure on A which lifts the BV-algebra structure of Lian-Zuckerman on $H(A)$.

Relative obstruction theory

Let α be a \mathcal{G}_∞ -algebra structure on A .

Theorem

The obstructions to lift α to a BV_∞ -algebra structure live in

$$H_{-2n}(\mathfrak{g}\mathcal{G}_\infty), \quad n \geq 1$$

negative 2-periodic cohomology of the \mathcal{G}_∞ algebra A

Thank you for your attention !