Adaptive mesh refinement techniques for well-balanced schemes for shallow water flows

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Outline

1. Shock capturing schemes for Shallow water flows
2. Adaptive Mesh Refinement
   - Adaptive schemes
   - Grid hierarchy
3. Well-balanced Adaptive techniques
   - Well-balanced schemes
   - Well-balanced AMR
   - Homogeneous discretization for SWE
   - Well-balanced interpolation
4. Numerical results
   - Numerical results
5. Conclusions
Shock capturing schemes for Shallow water flows

Shallow water flow

Shallow water equations (SWE) are obtained from incompressible Navier-Stokes equations by depth-averaging and neglecting some terms:

\[
\begin{align*}
    h_t + \text{div}(hv) & = 0 \\
    (hv)_t + \text{div}(hv \otimes v + \frac{gh^2}{2} I_2) & = -gh \nabla z
\end{align*}
\]

- \( h \equiv \) water depth,
- \( v = (v^x, v^y) \equiv \) depth-averaged velocity,
- \( g \equiv \) gravity acceleration,
- \( z \equiv \) bottom elevation.

To simplify, we do the exposition in 1D:

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\begin{align*}
    h_t + (hv)_x & = 0 \\
    (hv)_t + (hv^2 + \frac{gh^2}{2})_x & = -ghz_x
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Shock capturing schemes

- Use notation:

\[ u = \begin{bmatrix} h \\ hv \end{bmatrix}, \quad f(u) = \begin{bmatrix} hv \\ hv^2 + \frac{gh^2}{2} \end{bmatrix}, \quad s(x, u) = \begin{bmatrix} 0 \\ -ghz_x \end{bmatrix} \]

so that SWE system can be written as:

\[ u_t + f(u)_x = s(x, u). \]

- Nonlinear hyperbolic system ⇒ solutions can develop discontinuities ⇒ use shock capturing schemes:

\[ u_{i}^{n+1} = u_{i}^{n} - \Delta t \left( \frac{\hat{f}_{i+1/2}^{n} - \hat{f}_{i-1/2}^{n}}{\Delta x} - s_{i}^{n} \right), \]

where \( s_{i}^{n}(u(x, t)) \approx s(x_i, u(x_i, t_n)) \) and the numerical fluxes \( \hat{f}_{i+1/2} = \hat{f}(u_{i-s}, \ldots, u_{i+s+1}) \) verify

\[ \left[ \frac{\hat{f}_{i+1/2}^{n} - \hat{f}_{i-1/2}^{n}}{\Delta x} \right] (u(x, t)) \approx f(u)_x(x_i, t_n), \quad x_i = i\Delta x, \quad t_n = n\Delta t \]

and appropriate stability conditions (through upwinding and adding numerical viscosity to comply with entropy conditions).
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   - Well-balanced AMR
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Adaptive schemes

- For $N = 1/\Delta$ and $d$ dimensions, computational cost of scheme is $O(N^{d+1})$, storage is $O(N^d)$, huge to get small errors.
- Numerical errors are not uniformly distributed:
  - larger errors at discontinuities
  - smaller errors at smooth regions
- An Adaptive Scheme, with a smaller $\Delta$ where higher errors occur, would be necessary for $d \geq 2$ and high precision needs.
- Many approaches [Cohen et al., 2003, Müller and Stiriba, 2007] · · · , we briefly review the (Structured) Adaptive Mesh Refinement algorithm, proposed by [Berger and Oliger, 1984] and extended by many authors (Colella, Quirk, · · · ) to FV schemes.
AMR algorithm

- Time evolution for some grid size $\Delta \equiv \Delta x$ and $\Delta t$. 
Want to zoom at Region Of Interest, say by using $\Delta/2$. 
AMR algorithm

- A: use **interpolation** (zoom), but this causes large errors near shocks.
- B: discard results with $\Delta$, start over with $\Delta/2$.
- C: track region of interest through time evolution.
Before going to B plan, notice that solution on $\Omega \times [0, \Delta t]$ (hopefully) depends on solution at Domain of Dependence $\tilde{\Omega} \times \{0\}$ (by hyperbolicity).

Can compute solution at $\Omega \times \{\frac{\Delta t}{2}\}$ (assuming $\Delta/2$ at ROI, same CFL).
How can new DD of region of interest be computed?

**Zooming by \((x, t)\)-interpolation**, OK at (supposedly smooth) surrounding band (coarse \(\rightarrow\) fine interpolation)
AMR algorithm

- Recursion $\Rightarrow$ need **nested Grid Hierarchy** (for interpolation), indexed by level $l$ from $l = 0$ (coarsest) to $l = L$ (finest).
- Must synchronize data through GH at same $(x, t)$ (fine $\rightarrow$ coarse project.)
- More (shorter) time steps at finer resolutions (local time stepping).
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**Grid hierarchy**

- **Based on cell averages:** Points in the grid hierarchy (show 1D, 2D obtained by cartesian product): \( x_i^l = (i + \frac{1}{2}) \Delta_0 / 2^l, \ i = 0, \ldots, N_0 2^l - 1 \) (cell centers).

- Since \( \frac{1}{2} (x_{2i}^{l+1} + x_{2i+1}^{l+1}) = x_i^l \), project solution by averaging

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\text{Proj}_{l+1 \rightarrow l}(u_i^{l+1}) = \frac{1}{2} (u_{2i}^{l+1} + u_{2i+1}^{l+1}), \ i = 0, \ldots, N_0 2^l - 1.
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- Usual hierarchy for finite volume schemes [Berger and Oliger, 1984], can be made conservative.
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![Diagram showing grid hierarchy with arrows indicating refinement and coarsening from fine to coarse, and coarse to fine.]
Adaptive Mesh Refinement

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- **Based on cell averages**: Points in the grid hierarchy (show 1D, 2D obtained by cartesian product): $x_i^l = (i + \frac{1}{2})\Delta_0/2^l$, $i = 0, \ldots, N_02^l - 1$ (cell centers).

- Since $\frac{1}{2}(x_{2i}^{l+1} + x_{2i+1}^{l+1}) = x_i^l$, project solution by averaging

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  \text{Proj}_{l+1\rightarrow l}(u^{l+1})_i = u_{2i}^{l+1}, \quad i = 0, \ldots, N_02^l.
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AMR for shallow water flows

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AMR algorithm

- Nested grids as in 2D example with 2 levels. In a time snapshot we have data where marked. All the data is available at level 0.

\[(\text{surrounding band not shown})\]

- AMR algorithm $\equiv$ “time evolution” of grid functions $(u_0^{t_0}, G_0^{t_0}), \ldots, (u_L^{t_L}, G_L^{t_L})$ with data $u_l^{t_l} = (u_{l,i}^{t_l} / i \in G_l^{t_l})$ attached to grid points indexed by subsets $G_l^{t_l}$ and associated to times $t_0 \geq t_1 \geq \cdots \geq t_L$ (coarser levels evolve “faster” to provide interpolation data to finer levels)

$$u_{l,i}^{t_l} \approx \begin{cases} u(x_{l,i}, t_l) & \text{point values} \\ \int_{x_{l,i} - \frac{1}{2}}^{x_{l,i} + \frac{1}{2}} u(x, t_l) \, dx & \text{cell averages} \end{cases}$$
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Index sets $G^t_l$ have to evolve in time to track ROI.

- Coarse cells are marked, including surrounding band (not shown here), by some criterion.
- Marked coarse cells are then grouped into rectangular patches, with the goal of having (relatively) few large patches for efficiency.
- Coarse cells in rectangular patches are finally refined.
Grid adaption

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Criteria for marking for refinement

- Crucial part of algorithm: **decide which cells should be refined** so as salient flow features are contained in properly refined patches.

- Cells are marked by thresholding based on:
  - Large **local truncation errors** [Berger and Oliger, 1984], · · ·:
    - Not easy to implement.
  - Large **gradients** [Quirk, 1996] · · ·
    - Easy, but thresholding is difficult to control (e.g., in rarefactions).
  - Large **interpolation errors** (related to wavelet coefficient thresholding [Cohen et al., 2003], refine cells that cannot be accurately predicted)
    - Relatively easy implementation and thresholding.
    - Need improvement: may be combine with large threshold on derivatives of solution, do statistics of interpolation errors for automatic thresholding.

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Well-balanced schemes

- The convergence of the scheme is usually proved (when possible) through its consistence and stability (this being the harder part).

- When converging to a steady state or dealing with quasi-stationary solutions, the requirement of preserving steady states is plausible.

- When the scheme

\[ u_{i}^{n+1} = u_{i}^{n} - \Delta t \left( \frac{\hat{f}_{i+1/2}^{n} - \hat{f}_{i-1/2}^{n}}{\Delta x} - s_{i}^{n} \right) \]

does so, that is:

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then the scheme is termed well-balanced [Greenberg and Leroux, 1996].

- Special steady state for SWE, water at rest \((h + z = \text{constant}, v = 0)\).

- If a scheme preserves this steady state solution, then the scheme is said to verify the C-property [Bermudez and Vazquez, 1994].
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- Goal: obtain AMR code that preserves steady states (at least water at rest).
- If AMR algorithm should preserve stationary solutions then its ingredients:
  - Single grid solver (basic scheme)
  - Coarse to fine communication (interpolation).
  - Fine to coarse communication (projection).

  should preserve them (mentioned in D. George’s talk) ⇒ need well-balanced interpolation ([Bouchut, 2004]) and projection.

- We apply these adaptive techniques to a scheme introduced in [Donat and Martínez-Gavara, 2011] that satisfies the exact C-property. These techniques are applicable to other well-balanced schemes.

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  - Coarse to fine communication (interpolation).
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  should preserve them (mentioned in D. George’s talk) ⇒ need well-balanced interpolation ([Bouchut, 2004]) and projection.

- We apply these adaptive techniques to a scheme introduced in [Donat and Martínez-Gavara, 2011] that satisfies the exact C-property. These techniques are applicable to other well-balanced schemes.
AMR with well-balanced solver:

Goal: obtain AMR code that preserves steady states (at least water at rest).

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Outline

1. Shock capturing schemes for Shallow water flows

2. Adaptive Mesh Refinement
   - Adaptive schemes
   - Grid hierarchy

3. Well-balanced Adaptive techniques
   - Well-balanced schemes
   - Well-balanced AMR
     - Homogeneous discretization for SWE
     - Well-balanced interpolation

4. Numerical results
   - Numerical results

5. Conclusions
Homogeneous discretization

We build on [Gascón and Corberán, 2001, Caselles-Donat-Haro, 2009, Donat and Martínez-Gavara, 2011]: PDE can be rewritten in “homogeneous” form:

\[
    u_t + f(u)_x = s(x, u) \Leftrightarrow u_t + g[u]_x = 0
\]

where the functional \( g \) (dependent on \( f \) and \( s \)) acts on \( u = u(x, t) \) as:

\[
    g[u](x, t) = f(u(x, t)) - \int_0^x s(r, u(r, t)) \, dr
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We can derive upwind numerical methods for non-homogeneous conservation law from well established techniques for homogeneous conservation laws.
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[Donat and Martínez-Gavara, 2011] propose a **Lax-Wendroff**-type finite differences discretization for $u_t + g[u]_x = 0$, which is hybridized with a first order monotone scheme through **flux-limiting** techniques.

The scheme applied to exact solution $u(x, t)$ is:

$$u_{i}^{n+1} = u_{i}^{n} - \frac{\Delta t}{\Delta x} \left( A_{i}^{n} \Delta g_{i-\frac{1}{2}}^{n} + B_{i}^{n} \Delta g_{i+\frac{1}{2}}^{n} \right)$$

where $G_{i+\frac{1}{2}}$ are numerical fluxes for $g[u]$ and:

$$g_{i}^{n} = g[u](x_{i}, t_{n}) = f(u(x_{i}, t_{n})) - \int_{0}^{x_{i}} s(r, u(r, t_{n}))dr$$

$$\Delta g_{i+\frac{1}{2}}^{n} = g_{i+1}^{n} - g_{i}^{n} = f(u(x_{i+1}, t_{n})) - f(u(x_{i}, t_{n})) + b_{i,i+1}^{n},$$

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Homogeneous discretization

- To get numerical method, need to approximate

$$b^n_{i,i+1} = - \int_{x_i}^{x_{i+1}} s(r, u(r, t_n)) \, dr$$

by some appropriate quadrature rule, $\hat{b}^n_{i,i+1} \approx b^n_{i,i+1}$, so final scheme is

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$$\Delta g^n_{i+\frac{1}{2}} \approx \Delta \hat{g}^n_{i+\frac{1}{2}} := f(u^n_{i+1}) - f(u^n_i) + \hat{b}^n_{i,i+1}.$$ 

- Well balancing is obtained if approximation $\hat{b}^n_{i,i+1} \approx b^n_{i,i+1}$ is exact:

$$f(u(x))_x = s(x, u(x)) \Rightarrow g[u]_x = 0 \Rightarrow g^n_i = g[u](x_i, t_n) = \text{constant} \Rightarrow \hat{g}^n_{i+\frac{1}{2}} = \Delta \hat{g}^n_{i+\frac{1}{2}} = g^n_{i+1} - g^n_i = 0, \forall i \Rightarrow u^{n+1}_i = u^n_i, \forall i$$

- For SWE, suitable $\hat{b}^n_{i,i+1}$ can be defined to get exact C-property for wet and wet/dry beds. The exactness of $\hat{b}^n_{i,i+1}$ heavily relies on the scheme being based on point-values.
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C-property preserving interpolation: cell-averages

- In cell-based grid hierarchy, projection is given by \( h_{i+\frac{1}{2}} = \frac{1}{2}(h_i + h_{i+1}) \), where indexes indicate the point the data is attached to.
- If \( h_i = h(x_i) \) correspond to a water at rest solution, does \( h_{i+\frac{1}{2}} = \frac{1}{2}(h_i + h_{i+1}) \) correspond to point values (at \( x_{i+\frac{1}{2}} \)) of the solution?
- If it were so, from \( h(x) = \eta - z(x) \) we get
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  h_{i+\frac{1}{2}} = h(x_{i+\frac{1}{2}}) = \eta - z(x_{i+\frac{1}{2}}),
  \]
  but
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  z(x_i) + z(x_{i+1}) = z \left( \frac{x_i + x_{i+1}}{2} \right), \forall i,
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  which does not hold for general \( z \Rightarrow \) Projection not OK for point values
- Projection OK if \( h_i \) are cell-averages of stationary solution, but then underlying scheme should preserve them (OK for well-balanced schemes as in Carlos Parés’ course, not OK for our scheme).
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C-property preserving interpolation: point-values

- For point value grid hierarchy, the projection from level $l + 1$ to level $l$ is given by copying values with even indexes, corresponding to the same point-values, so this projection is automatically well-balanced.

- **Well-balanced** interpolation (related to hydrostatic reconstruction [Audusse-Bouchut-Bristeau-Klein-Perthame, 2004], appears in Carlos Pare’s course and Professor Valiani’s talk): if we only want to preserve water at rest solutions, given interpolator $I((w_i); x)$ (i.e., $I((w_i); x_j) = w_j$), and

$$V(x, \begin{bmatrix} h \\ q \end{bmatrix}) = \begin{bmatrix} h + z(x) \\ q \end{bmatrix}, \quad V(x, \cdot)^{-1} \begin{bmatrix} \eta \\ q \end{bmatrix} = \begin{bmatrix} \eta - z(x) \\ q \end{bmatrix}$$

then we can define an interpolator by

$$\tilde{I}((u_i); x) = V(x, \cdot)^{-1}(I((V_i); x)), \quad V_i = V(x_i, u_i)$$

(i.e., interpolate total heights, then subtract bottom height).

- $I$ preserves constants $\Rightarrow \tilde{I}$ preserves water at rest.
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\[
u(x) \text{ is solution of PDE } \iff V(x, u(x)) \text{ is constant at regions of smoothness + jump conditions}.
\]

\( V(x, u) \equiv \text{equilibrium variables} \), which are for SWE:
\[
V(x, \begin{bmatrix} h \\ hv \end{bmatrix}) = \begin{bmatrix} \frac{v^2}{2} + g(h + z(x)) \\ hv \end{bmatrix}
\]

If \( V(x, \cdot) \) is bijective onto some relevant range then we can define an interpolator that preserves equilibrium variables by:
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For SWE, \( V(x, \cdot) \) is not injective, but could select, as in [Bouchut and Morales de Luna, 2010], appropriate branch of inverse (helped here by the fact that interpolation takes place at smooth regions).

Could get well-balanced interpolation in the cell-average sense by using techniques that Carlos Parés showed in his course.
General well-balanced interpolation

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Tests setup

- Based on code developed by A. Baeza for cell-based AMR.
- We use point-value-based grid hierarchy, with well-balanced interpolation based on linear interpolation.
- Refinement criterion: mark cells to refine when interpolation error exceeds some relative error $\texttt{rtol}$ with respect to the maximal interpolation error at each level.
**Test for stationary 1D solutions**

- Water at rest solution of total height=12, bottom topography below. Solution at $T = 200$.
- Have used rtol=$10^{-1}$, $N_0 = 50$, and eight levels ($L = 7$, $N_7 = 6400$) to obtain:

  \[
  \|h + z - 12\|_\infty = 1.06 \cdot 10^{-14} \quad \text{and} \quad \|v\|_\infty = 3.36 \cdot 10^{-14} \Rightarrow \text{C-property OK to double precision.}
  \]

with a CPU speedup $\approx 11.5$. 

- Scheme gives approximated solution such that $\|h + z - 12\|_\infty = 1.06 \cdot 10^{-14}$ and $\|v\|_\infty = 3.36 \cdot 10^{-14} \Rightarrow \text{C-property OK to double precision.}$
Test for stationary 1D solutions

- Same setup, but without well balanced interpolation:

\[ \| h + z - 12 \|_\infty = 5.31 \cdot 10^{-2} \]
\[ \| v \|_\infty = 2.16 \cdot 10^{-14} \Rightarrow \text{loss of exact C-property}. \]
Test for non stationary 1D solutions

- Dam break problem with square bump bottom topography.
- Solution at $T = 15$. Have used $\text{rtol}=10^{-3}$, $N_0 = 50$, and eight levels ($L = 7$, $N_7 = 6400$) to obtain:

  \[ \| h_{AMR} - h_{fixed} \|_1 = 1.44 \cdot 10^{-4}, \| v_{AMR} - v_{fixed} \|_1 = 1.47 \cdot 10^{-4} \]

  with CPU speedup $\approx 14.04$.

- Scheme gives approximated solution such that
Test for stationary 2D solutions

([LeVeque, 1998]) Water at rest, total height = 1 and bottom:

Have used rtol=10^{-1}, N_0 = 25, and 4 levels (L = 3, N_3 = 200), T = 0.1 to obtain:
\[ \| h + z - 1 \|_\infty = 1.11 \cdot 10^{-15}, \| v^x \|_\infty = 3.52 \cdot 10^{-15}, \| v^y \|_\infty = 3.88 \cdot 10^{-15} \Rightarrow \text{C-property OK to double precision.} \]

CPU speedup=3.96
Numerical results

Test for non-stationary 2D solutions

- Circular dam break problem ([Castro-Fernández-Nieto-Ferreiro-García-Rodríguez-Parés, 2009]). Have used $rtol=10^{-1}$, $N_0 = 100$, and 5 levels ($L = 4$, $N_4 = 1600$), $T = 0.25$

- CPU speedup = 5.22

- $\|h_{AMR} - h_{fixed}\|_1 = 8.33 \cdot 10^{-4}$, $\|v^x_{AMR} - v^x_{fixed}\|_1 = 1.5 \cdot 10^{-3}$, $\|v^y_{AMR} - v^y_{fixed}\|_1 = 1.4 \cdot 10^{-3}$, difference of mass $\approx 7 \cdot 10^{-4}$. 

\[ T = 0 \quad \text{and} \quad T = 0.25 \]
In grid hierarchy, lighter color means finer resolution.
Conclusions

- We have presented a technique for obtaining well-balanced point-value-based adaptive mesh refinement schemes for shallow water equations.
- We have seen some of the difficulties for getting well-balanced adaptive mesh refinement schemes for SWE based on cell-averages.
- We have tested the scheme with Donat&Martinez-Gavara homogenized SWE solver and we have obtained an adaptive scheme with the exact C-property.

Future research

- We are working on its parallelization and extension to deal with dry zones.
- Possibility of getting an adaptive scheme that preserves more stationary solutions if underlying scheme does so.
- Comparison of present code with AMR without well-balanced interpolation
- Comparison of present code with AMR with cell-average-based AMR.
A fast and stable well-balanced scheme with hydrostatic reconstruction for shallow water flows.  

Adaptive mesh refinement techniques for high-order shock capturing schemes for multi-dimensional hydrodynamic simulations.  

Adaptive mesh refinement for hyperbolic partial differential equations.  

Logically rectangular finite volume methods with adaptive refinement on the sphere.  
*Phil. Trans. R. Soc. A*, 367:4483–4496.

Upwind methods for hyperbolic conservation laws with source terms.  

Nonlinear stability of finite volume methods for hyperbolic conservation laws and well-balanced schemes for sources.

A subsonic-well-balanced reconstruction scheme for shallow water flows.

Flux-gradient and source-term balancing for certain high resolution shock-capturing schemes.

High order extensions of roe schemes for two-dimensional nonconservative hyperbolic systems.

Fully adaptive multiresolution finite volume schemes for conservation laws.

A hybrid second order scheme for shallow water flows. to appear in APNUM.

Construction of second-order TVD schemes for nonhomogeneous hyperbolic conservation laws.

INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, 66(8):1000–1018.

A well-balanced scheme for the numerical processing of source terms in hyperbolic equations.

Balancing source terms and flux gradients in high-resolution godunov methods: the quasi-steady wave-propagation algorithm.
Fully adaptive multiscale schemes for conservation laws employing locally varying time stepping.


A parallel adaptive grid algorithm for computational shock hydrodynamics.