Fuks & Rokhlin
Beginner's Course in Topology
P. 46-49

7. Spaces of Continuous Maps

1. Let $C(X,Y)$ be the set of all continuous maps of a topological space X into a topological space Y. The set of all maps $\phi \in C(X,Y)$ such that $\phi(A_1) \subset B_1, \ldots, \phi(A_n) \subset B_n$, where $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ are given subsets of X and Y, respectively, is denoted by $C(X, A_1, \ldots, A_n; Y, B_1, \ldots, B_n)$. It may be interpreted as the set of all continuous maps $(X, A_1, \ldots, A_n) \to (Y, B_1, \ldots, B_n)$.

We equip $C(X,Y)$ with the <u>compact-open topology</u>: by definition, this is the topology with the prebase consisting of all sets $C(X,A;Y,B)$ with A compact and B open. Together with $C(X,Y)$, all the sets $C(X, A_1, \ldots, A_n; Y, B_1, \ldots, B_n)$ become topological spaces.

If Y is a point, then $C(X,Y)$ reduces to a point. If X is discrete and consists of the points $x_1, \ldots, x_n$, then $C(X,Y)$ is canonically homeomorphic to the product $Y \times \ldots \times Y$ of n copies of the space Y; this homeomorphism is given by $\phi \to (\phi(x_1), \ldots, \phi(x_n))$.

To each pair of continuous maps $f: X' \to X$ and $g: Y \to Y'$ there corresponds a mapping $C(X,Y) \to C(X',Y')$, given by the rule $\phi \mapsto g \circ \phi \circ f$. This mapping is continuous, and we shall denote it by $C(f,g)$.

2. <u>If Y is a Hausdorff space, then so is</u> $C(X,Y)$.

Indeed, if $\phi, \psi \in C(X,Y)$ and $\phi \neq \psi$, then there is $x \in X$ such that $\phi(x) \neq \psi(x)$. Let U and V be disjoint neighborhoods of the points $\phi(x)$ and $\psi(x)$. Then $C(X,x;Y,U)$ and $C(X,x;Y,V)$ are disjoint neighborhoods of the points ϕ and ψ.

3. <u>If X is compact and Y is metrizable, then $C(X,Y)$ is metrizable. Moreover, if Y is equipped with a metric, then</u> $\mathrm{dist}(\phi,\psi) = \sup_{x \in X} \mathrm{dist}(\phi(x), \psi(x))$ <u>defines a metric on</u> $C(X,Y)$, <u>compatible with its topology.</u>

PROOF. Given $\phi \in C(X,Y)$, the set $\phi(X)$ can be covered by a finite number of balls $U_1, \ldots, U_s$ of an arbitrarily small radius ε (see 1.7.11). It is clear that $W = \cap_{i=1}^{s} C(X, \phi^{-1}(U_i); Y, U_i)$ is a neighborhood of the point ϕ, contained in the ball of radius 2ε

centered at ϕ. Therefore, every ball in $C(X,Y)$ contains a neighborhood of its center.

On the other hand, if $A \subset X$ is compact and $B \subset Y$ is open, with $\phi(A) \subset B$, then $C(X,A;Y,B)$ contains the ball with radius $\mathrm{Dist}(\phi(A), Y \setminus B)$ centered at ϕ (see 1.7.15). Therefore, every neighborhood of ϕ belonging to the prebase considered in 1 contains a ball centered at ϕ.

4. For any topological spaces X and $Y_1,\dots,Y_n$, the space $C(X, Y_1 \times \dots \times Y_n)$ is canonically homeomorphic to the product $C(X,Y_1) \times \dots \times C(X,Y_n)$.

This canonical homeomorphism takes each $\phi \in C(X,Y_1 \times \dots \times Y_n)$ into $(\mathrm{pr}_1 \circ \phi, \dots, \mathrm{pr}_n \circ \phi) \in C(X,Y_1) \times \dots \times C(X,Y_n)$ [cf. 2.4].

5. Let p be a closed partition of the compact Hausdorff space X, and let Y be an arbitrary topological space. Then $C(\mathrm{pr}, \mathrm{id}\, Y) : C(X/p, Y) \to C(X,Y)$ is an embedding.

It suffices to show that given a compact subset A of X/p and an open subset B of Y, the set $C(\mathrm{pr}, \mathrm{id}\, Y)[C(X/p,A;Y,B)]$ is open in $C(\mathrm{pr}, \mathrm{id}\, Y)[C(X/p,Y)]$. Since X/p is Hausdorff (see 3.9), A is closed. It follows that $\mathrm{pr}^{-1}(A)$ is closed, and hence compact. Consequently, $C(X, \mathrm{pr}^{-1}(A); Y, B)$ is open in $C(X,Y)$, and it remains to note that

$$C(\mathrm{pr}, \mathrm{id}\, Y)[C(X/p,A;Y,B)] = C(X, \mathrm{pr}^{-1}(A); Y, B) \cap C(\mathrm{pr}, \mathrm{id}\, Y)[C(X/p;Y)].$$

The Mappings $X \times Y \to Z$ and $X \to C(Y,Z)$

6. Suppose that X, Y and Z are topological spaces, and $\phi: X \times Y \to Z$ is continuous. Then the formula $[\phi^{\vee}(x)](y) = \phi(x,y)$ defines a continuous mapping $\phi^{\vee}: X \to C(Y,Z)$.

Let $\psi: X \to C(Y,Z)$ be a continuous mapping, and suppose that Y is Hausdorff and locally compact. Then the formula $\psi^{\wedge}(x,y) = [\psi(x)](y)$ defines a continuous mapping $\psi^{\wedge}: X \times Y \to Z$.

To prove the first assertion, pick a point $x_0 \in X$, a compact set $B \subset Y$, and an open set $C \subset Z$. Then it is enough to exhibit a neighborhood U of x_0 such that $\phi^{\vee}(U) \subset C(Y,B;Z,C)$. For each point $y \in B$ fix neighborhoods U_y and V_y of x_0 and y such that $\phi(U_y \times V_y) \subset C$, and then extract a finite cover $V_{y_1},\dots,V_{y_s}$ of B from the collection $\{V_y\}_{y \in B}$. It is clear that $U = \cap_{i=1}^{s} U_{y_i}$ is a neighborhood of x_0 and that $\phi(U \times B) \subset \cup_{i=1}^{s} \phi(U_{y_i} \times V_{y_i}) \subset C$. It remains to remark that the inclusion $\phi(U \times B) \subset C$ is equivalent to $\phi^\vee(U) \subset C(Y,B;Z,C)$.

To prove the second assertion, pick a point $(x_0,y_0) \in X \times Y$ and a neighborhood W of the point $\psi^\wedge(x_0,y_0)$. Now let us find a neighborhood V of y_0 with compact closure $\mathrm{Cl}\, V$ satisfying $\mathrm{Cl}\, V \subset [\psi(x_0)]^{-1}(W)$ (see 1.7.22), and then a neighborhood U of x_0 satisfying $\psi(U) \subset C(Y,\mathrm{Cl}\, V;Z,W)$. Obviously, $U \times V$ is a neighborhood of the point (x_0,y_0) and $\psi^\wedge(U \times V) \subset W$.

7. <u>The mapping</u> $C(X \times Y,Z) \to C(X,C(Y,Z))$ <u>defined by the rule</u> $\phi \mapsto \phi^\vee$ (<u>see</u> 6) <u>is continuous for any topological spaces</u> X, Y <u>and</u> Z. <u>If</u> X <u>is Hausdorff and</u> Y <u>is Hausdorff and locally compact, then this mapping is a homeomorphism, and its inverse is given by the rule</u> $\psi \mapsto \psi^\wedge$.

The continuity of the mapping $\phi \mapsto \phi^\vee$ results from the fact that the preimage of $C(X,A;C(Y,Z),C(Y,B;Z,C))$ under this mapping is just $C(X \times Y,A \times B;Z,C)$. Assume that X is Hausdorff and Y is Hausdorff and locally compact. Consider a point $\psi_0 \in C(X,C(Y;Z))$, a compact subset Q of $X \times Y$, a neighborhood W of the set $\psi_0^\wedge(Q)$, and a point $q \in Q$. Now find a neighborhood $U_q \times V_q$ of q such that $\psi_0^\wedge(U_q \times \mathrm{Cl}\, V_q) \subset W$. Since Q is compact, its images $pr_1(Q)$ and $pr_2(Q)$ in X and Y are also compact (see 1.7.8). Moreover, they are Hausdorff spaces together with X and Y, and hence normal (see 1.7.5). Consequently, there exist open subsets U'_q of $pr_1(Q)$ and V'_q of $pr_2(Q)$ such that

$$pr_1(q) \in U'_q, \quad \mathrm{Cl}_{pr_1(Q)} U'_q \subset U_q,$$

and

$$pr_2(q) \in V'_q, \quad \mathrm{Cl}_{pr_2(Q)} V'_q \subset V_q,$$

and it is plain that the intersection $(U'_q \times V'_q) \cap Q$ is open in Q. Being compact, Q can be covered by a finite number of such intersections, say $U'_{q_1} \times V'_{q_1}, \dots, U'_{q_s} \times V'_{q_s}$. Now set

$$T = \bigcap_{i=1}^{s} C(X,\mathrm{Cl}_{pr_1(Q)} U'_{q_i};C(Y,Z),C(Y,\mathrm{Cl}_{pr_2(Q)} V'_{q_i};Z,W)).$$

It is clear that T is a neighborhood of ψ_0 and that the image of T under the mapping $\psi \to \psi^\wedge$ is contained in $C(X \times Y,Q;Z,W)$. We conclude that $\psi \mapsto \psi^\wedge$ is continuous. It is readily see that the mappings $\phi \mapsto \phi^\vee$ and $\psi \mapsto \psi^\wedge$ are inverses of one another.

A Surprising Application

8. <u>Let</u> $f: X \to X'$ <u>be a factorial map. If the space</u> Y <u>is Hausdorff and locally compact, then the map</u> $f \times \text{id}\,Y : X \times Y \to X' \times Y$ <u>is factorial.</u>

One can assume that $X' = X/\text{zer}(f)$ and that f is the projection $X \to X/\text{zer}(f)$. Consider the projection $\text{pr}: X \times Y \to (X \times Y)/(\text{zer}(f) \times \text{zer}(\text{id}\,Y))$. The mapping $\text{pr}^{\vee}: X \to \mathcal{C}(Y, (X \times Y)/(\text{zer}(f) \times \text{zer}(\text{id}\,Y))$ is constant on the elements of the partition $\text{zer}(f)$, and hence it induces continuous mappings

$$\text{fact}\,\text{pr}^{\vee} : X' \to \mathcal{C}(Y, (X \times Y)/(\text{zer}(f) \times \text{zer}(\text{id}\,Y))$$

and

$$(\text{fact}\,\text{pr}^{\vee})^{\wedge} : X' \times Y \to (X \times Y)/(\text{zer}(f) \times \text{zer}(\text{id}\,Y)).$$

It it clear that the second of these mappings is the inverse of the injective factor of $f \times \text{id}\,Y : X \times Y \to X' \times Y$. Thus the injective factor of $f \times \text{id}\,Y : X \times Y \to X' \times Y$ is a homeomorphism.

9. <u>Let</u> $f: X \to X'$ <u>and</u> $g: Y \to Y'$ <u>be factorial maps. If</u> X' <u>and</u> Y <u>are Hausdorff and locally compact, then the map</u> $f \times g: X \times Y \to X' \times Y'$ <u>is factorial.</u>

In fact, one can express $f \times g$ as the composition

$$X \times Y \xrightarrow{f \times \text{id}} X' \times Y \xrightarrow{\text{id} \times g} X' \times Y'$$

and recall that a composition of factorial maps is again factorial.

8. <u>The Case of Pointed Spaces</u>

1. In the sequel, the class of topological spaces equipped with a simple additional structure - a distinguished point (i.e., topological pairs (X,x_0), where x_0 is a point) will play an important role; we call these spaces <u>pointed spaces</u>, and call the distinguished point a <u>base point.</u> The constructions described in the previous subsections must be naturally modified when applied to such spaces. For some of these construction, the modification entails merely the addition of a base point to the resulting space: for example, the quotient space of pointed space (X,x_0) has the natural base point