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Towards topological Hochschild homology of Johnson–Wilson spectra

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We present computations in Hochschild homology that lead to results on the $K(i)$–local behaviour of $\mathrm{THH}(E(n))$ for all $n \geqslant 2$ and $0 \leqslant i \leqslant n$, where $E(n)$ is the Johnson–Wilson spectrum at an odd prime. This permits a computation of $K(i)_*\mathrm{THH}(E(n))$ under the assumption that $E(n)$ is an E_3–ring spectrum. We offer a complete description of $\mathrm{THH}(E(2))$ as an $E(2)$–module in the form of a splitting into chromatic localizations of $E(2)$, under the assumption that $E(2)$ carries an E_∞–structure. If $E(2)$ is admits an E_3–structure, we obtain a similar splitting of the cofiber of the unit map $E(2) \to \mathrm{THH}(E(2))$.

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1 Introduction

The first Johnson–Wilson spectrum $E(1)$ at a prime p is the Adams summand of p–local periodic complex topological K–theory $KU_{(p)}$. McClure and Staffeldt showed that a p–completed connective version of $E(1)$ is an E_∞–ring spectrum [18, Section 9] and Baker and Richter [4, Theorem 6.2] show that $E(1)$ carries a unique E_∞–structure. Thus $\mathrm{THH}(E(1))$ is a commutative $E(1)$–algebra spectrum. McClure and Staffeldt show that the unit map $E(1)_p \to \mathrm{THH}(E(1)_p)$ is a $K(1)$–local equivalence, hence its cofiber $\overline{\mathrm{THH}}(E(1)_p)$ is a rational spectrum. It is easy to calculate the rational homology of $\mathrm{THH}(E(1)_p)$ as

$$H\mathbb{Q}_*\mathrm{THH}(E(1)_p) \cong \mathbb{Q}[v_1^{\pm 1}] \otimes_{\mathbb{Q}} \Lambda_{\mathbb{Q}}(dv_1)$$

using the Bökstedt spectral sequence with E^2–term

$$E^2_{*,*} = \mathsf{HH}^{\mathbb{Q}}_{*,*}(\mathbb{Q}[v_1^{\pm 1}]).$$

There is a map

$$\Sigma^{2p-1} E(1)_p \to \mathrm{THH}(E(1)_p) \to \overline{\mathrm{THH}}(E(1)_p)$$

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that factors through $\Sigma^{2p-1}E(1)_{\mathbb{Q}} \to \overline{\mathrm{THH}}(E(1)_p)$ since $\overline{\mathrm{THH}}(E(1)_p)$ is rational, and that is defined such that the latter map is an equivalence detecting the $H\mathbb{Q}_*E(1)$–summand generated by dv_1. Since the unit map $E(1)_p \to \mathrm{THH}(E(1)_p)$ splits, this yields a splitting [18, Theorem 8.1]

$$\mathrm{THH}(E(1)_p) \simeq E(1)_p \vee \Sigma^{2p-1}E(1)_{\mathbb{Q}}$$

as $E(1)_p$–modules. This computation was also carried out for $KU_{(p)}$ by Ausoni [3], and pushed further to provide formulas for $\mathrm{THH}(KU)$ as a commutative KU–algebra by Stonek [27].

In this paper, we consider the higher Johnson–Wilson spectrum $E(n)$ with coefficient ring

$$E(n)_* = \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n, v_n^{-1}]$$

for an arbitrary value of $n \geqslant 1$ and p an odd prime. Our main motivation is to investigate whether the spectrum $\mathrm{THH}(E(n))$ also splits into copies of $E(n)$ and its lower chromatic localizations, generalizing McClure and Staffeldt's intriguing transchromatic result.

Let $K(i)$ be the i^{th} Morava K–theory at an odd prime. As a first step, we compute the Hochschild homology $\mathsf{HH}_*^{K(i)_*}(K(i)_*E(n))$ of $K(i)_*E(n)$ for $0 \leqslant i \leqslant n$; see Theorem 3.4. We shy away from the prime 2 because Morava K–theory is not homotopy commutative at the prime 2. Theorem 3.4 yields a computation of $K(i)_*\mathrm{THH}(E(n))$ under the modest assumption that $E(n)$ admits an E_3–structure.

We then focus on $E(2)$, and show in Theorem 5.4 that, under the same commutativity assumption, $\mathrm{THH}(E(2))$ sits in a cofiber sequence

$$E(2) \to \mathrm{THH}(E(2)) \to \Sigma^{2p-1}L_1E(2) \vee \Sigma^{2p^2-1}E(2)_{\mathbb{Q}} \vee \Sigma^{2p^2+2p-2}E(2)_{\mathbb{Q}},$$

where $L_1E(2)$ denotes the Bousfield localization of $E(2)$ with respect to $E(1)$. If the unit $E(2) \to \mathrm{THH}(E(2))$ splits, we then get a decomposition of $\mathrm{THH}(E(2))$ into four summands, a higher analogue of McClure and Staffeldt's formula for $\mathrm{THH}(E(1))$.

Remark 1.1 To study $\mathrm{THH}(E(n))$ by means of the Bökstedt spectral sequence, we need sufficient commutativity of $E(n)$. Here we summarize what is known about multiplicative structures on $E(n)$ and related spectra. Basterra and Mandell [7] showed that the Brown–Peterson spectrum BP admits an E_4–structure. The Johnson–Wilson spectra $E(n)$ are built out of the $\mathrm{BP}\langle n\rangle = \mathrm{BP}/(v_i \mid i \geqslant n+1)$ by inverting v_n. In

[15, Theorem 1.1.2], Tyler Lawson shows that the Brown–Peterson spectrum BP and the spectra $\mathrm{BP}\langle n\rangle$ for $n \geqslant 4$ at the prime 2 do not possess an E_{12}–structure. Andrew Senger [25, Theorem 1.2] extends Lawson's result to odd primes p, and shows that BP and the $\mathrm{BP}\langle n\rangle$ (for $n \geqslant 4$) do not have an $E_{2(p^2+2)}$–structure. In particular, the $\mathrm{BP}\langle n\rangle$ are not E_∞–ring spectra at any prime for $n \geqslant 4$. Hence, if $E(n)$ actually possesses an E_∞–structure for $n \geqslant 4$, then this structure does not come from one on $\mathrm{BP}\langle n\rangle$. Richter [20, Proposition 8.2] proves that $E(n)$ at a prime p possesses at least a $(2p-1)$–stage structure. It is unclear how such a structure relates to the E_n–hierarchy, but Barwick conjectures [5, page 1948] that a $(2p-1)$–stage structure corresponds to an A_{2p}^{2p-1}–structure which in turn is a filtration piece of an E_{2p-1}–structure.

At the prime 2, Lawson and Naumann [16] show that there is an E_∞–model of $\mathrm{BP}\langle 2\rangle$ and Hill and Lawson [13] prove that $\mathrm{BP}\langle 2\rangle$ at the prime 3 possesses a model as an E_∞–ring spectrum. With Mathew, Naumann and Noel [17, Theorem A.1] this yields E_∞–structures on the corresponding Johnson–Wilson spectra $E(2)$ at these primes. Current work of Sanath Devalapurkar aims at adapting the arguments used in these results to produce E_∞–models of $\mathrm{BP}\langle 2\rangle$ or $E(2)$ at higher primes.

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2 Rationalized $E(n)$

For $n \geqslant 1$ the homotopy algebra of $L_{K(0)}E(n) = E(n)_{\mathbb{Q}}$ is $\mathbb{Q}[v_1, \dots, v_{n-1}, v_n^{\pm 1}]$ and its algebra of cooperations is

$$\begin{aligned}\pi_*(E(n)_{\mathbb{Q}} \wedge E(n)_{\mathbb{Q}}) &\cong \pi_*E(n)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \pi_*E(n)_{\mathbb{Q}} \\ &\cong \mathbb{Q}[v_1, \dots, v_{n-1}, v_n^{\pm 1}, v_1', \dots, v_{n-1}', {v_n'}^{\pm 1}].\end{aligned}$$

This implies the following result:

Lemma 2.1 *There is a unique E_∞–ring structure on $E(n)_{\mathbb{Q}}$ for all $n \geqslant 1$.*

Proof The obstruction groups for such an E_∞–ring structure on $E(n)_\mathbb{Q}$ are contained in the Gamma cohomology groups of $\pi_*(E(n)_\mathbb{Q} \wedge E(n)_\mathbb{Q})$ as a $\pi_* E(n)_\mathbb{Q}$–algebra [22, Theorem 5.6]. As we work in characteristic zero, Gamma cohomology agrees with André–Quillen cohomology [23, Corollary 6.6]. The algebra

$$\mathbb{Q}[v_1, \dots, v_{n-1}, v_n^{\pm 1}, v_1', \dots, v_{n-1}', {v_n'}^{\pm 1}]$$

is smooth over $\mathbb{Q}[v_1, \dots, v_{n-1}, v_n^{\pm 1}]$ and therefore André–Quillen cohomology is concentrated in cohomological degree zero, where it consists of derivations. The obstructions for existence and uniqueness of an E_∞–ring structure on $E(n)_\mathbb{Q}$ are concentrated in degrees bigger than zero. □

As E_∞–ring structures can be rigidified to commutative ring structures (see eg [12, Section II.3]), we pass to the world of commutative ring spectra from now on.

Topological Hochschild homology of a ring spectrum A can be modelled as the geometric realization of a simplicial spectrum. Using the inclusion of the 1–skeleton, McClure and Staffeldt [18, Section 3] construct a map

$$\sigma\colon \Sigma A \to \mathrm{THH}(A). \tag{2-1}$$

For a commutative ring spectrum A the multiplication maps from $A^{\wedge n+1}$ to A give rise to a map of commutative A–algebra spectra from $\mathrm{THH}(A)$ to A. Composing this map with the map $A \to \mathrm{THH}(A)$ gives the identity, hence we obtain a splitting of A–modules

$$\mathrm{THH}(A) \simeq A \vee \overline{\mathrm{THH}}(A),$$

where $\overline{\mathrm{THH}}(A)$ is the cofiber. The latter spectrum inherits the structure of a nonunital commutative A–algebra. In our case this implies the following result:

Corollary 2.2 *The topological Hochschild homology of $E(n)_\mathbb{Q}$ splits, as an $E(n)_\mathbb{Q}$–module, as*

$$\mathrm{THH}(E(n)_\mathbb{Q}) \simeq E(n)_\mathbb{Q} \vee \overline{\mathrm{THH}}(E(n))_\mathbb{Q}$$

where $\overline{\mathrm{THH}}(E(n))_\mathbb{Q}$ is the cofiber of the unit map

$$E(n)_\mathbb{Q} \to \mathrm{THH}(E(n)_\mathbb{Q}) \simeq \mathrm{THH}(E(n))_\mathbb{Q}.$$

Moreover, the spectrum $\overline{\mathrm{THH}}(E(n))_\mathbb{Q}$ is a nonunital commutative $E(n)_\mathbb{Q}$–algebra.

In the sequel, we follow Ronco [24, Definition E.1] for the definition of étale algebras. It is straightforward to calculate the topological Hochschild homology of $E(n)_\mathbb{Q}$:

Proposition 2.3 *We have*

$$\pi_*\mathrm{THH}(E(n))_{\mathbb{Q}} \cong \mathbb{Q}[v_1,\dots,v_{n-1},v_n^{\pm1}] \otimes \Lambda_{\mathbb{Q}}(dv_1,\dots,dv_n) \tag{2-2}$$

with $|dv_i| = 2p^i - 1$.

Proof The Bökstedt spectral sequence for $\pi_*(\mathrm{THH}(E(n))_{\mathbb{Q}}) \cong H\mathbb{Q}_*\mathrm{THH}(E(n))$ is of the form

$$E^2_{*,*} = \mathsf{HH}^{\mathbb{Q}}_{*,*}(\pi_* E(n)_{\mathbb{Q}}) \Rightarrow \pi_*(\mathrm{THH}(E(n))_{\mathbb{Q}}).$$

As $\mathbb{Q}[v_1,\dots,v_{n-1},v_n^{\pm1}]$ is étale over $\mathbb{Q}[v_1,\dots,v_{n-1},v_n]$ and as $\mathbb{Q}[v_1,\dots,v_{n-1},v_n]$ is smooth, we get

$$\mathsf{HH}^{\mathbb{Q}}_{*,*}(\pi_* E(n)_{\mathbb{Q}}) \cong \mathbb{Q}[v_1,\dots,v_{n-1},v_n^{\pm1}] \otimes \Lambda_{\mathbb{Q}}(dv_1,\dots,dv_n)$$

with dv_i having homological degree one and internal degree $2p^i - 2$. As the Bökstedt spectral sequence is multiplicative and as the algebra generator cannot support any differentials for degree reasons, the spectral sequence collapses at E^2. There are no multiplicative extensions and hence we get the result. □

Remark 2.4 As we work rationally, $\mathrm{THH}(E(n))_{\mathbb{Q}}$ is a commutative $H\mathbb{Q}$–algebra spectrum and hence corresponds to a commutative differential graded $\mathbb{Q}$–algebra (see [26] or [21]).

3 $K(i)_*E(n)$ and $K(i)_*\mathrm{THH}(E(n))$

In the following we assume that p is an odd prime, and that n and i are integers with $1 \leqslant i \leqslant n$. The Hopf algebroid $(\mathrm{BP}_*, \mathrm{BP}_*\mathrm{BP})$ represents the groupoid of strict isomorphisms of p–typical formal group laws [14] (see also [19, Theorem A2.1.27]). There are isomorphisms of graded $\mathbb{Z}_{(p)}$–algebras

$$\mathrm{BP}_* \cong \mathbb{Z}_{(p)}[v_1, v_2, \dots] \quad \text{and} \quad \mathrm{BP}_*\mathrm{BP} \cong \mathrm{BP}_*[t_1, t_2, \dots],$$

where $|v_i| = |t_i| = 2(p^i - 1)$. We use the Araki generators v_i [19, Section A2.2] and by convention $v_0 = p$ and $t_0 = 1$. The i^{th} Morava K–theory $K(i)$ is complex oriented, and its formal group law F_i (the Honda formal group law) corresponds to the map $\mathrm{BP}_* \to K(i)_* = \mathbb{F}_p[v_i^{\pm}]$ sending v_i to v_i and v_k for $k \neq i$ to zero. The p–typical formal group law G_n over $E(n)_*$ comes from the map $\mathrm{BP}_* \to E(n)_*$ that kills all v_i with $i > n$ and inverts v_n. Since $E(n)$ is a Landweber exact homology theory, we obtain an isomorphism

$$K(i)_*E(n) \cong K(i)_* \otimes_{\mathrm{BP}_*} \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} E(n)_*. \tag{3-1}$$

Note that $K(i)_*E(n)$ is trivial for $i > n$ and that the Bousfield class $\langle E(n)\rangle$ of $E(n)$ is $\langle K(0) \vee \cdots \vee K(n)\rangle$.

We first treat the case $i = n$.

The algebra $K(n)_*E(n)$ is isomorphic to $K(n)_*\mathrm{BP}\langle n\rangle$, which is isomorphic to

$$K(n)_* \otimes_{\mathrm{BP}_*} \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} K(n)_*$$

(see for instance [29, page 428]). The latter is known as $\Sigma(n)$. It is of the form

$$\Sigma(n) \cong K(n)_*[t_1, t_2, \dots]/(v_n t_i^{p^n} - v_n^{p^i} t_i \mid i \geqslant 1);$$

see [19, Corollary 6.1.16].

Proposition 3.1 *For all $n \geqslant 1$ the canonical map $E(n) \to \mathrm{THH}(E(n))$ is a $K(n)$–local equivalence.*

Proof If we set

$$C_*^{(k)} := K(n)_*[t_1, \dots, t_k]/(v_n t_i^{p^n} - v_n^{p^i} t_i \mid 1 \leqslant i \leqslant k)$$

then $C_*^{(k)}$ is étale over $K(n)_*$ and $K(n)_*E(n)$ is the directed colimit of the $C_*^{(k)}$.

The $K(n)_*$–Bökstedt spectral sequence for $\mathrm{THH}(E(n))$ has as an E^2–term

$$\mathsf{HH}_*^{K(n)_*}(K(n)_*E(n)) \cong K(n)_*E(n)$$

concentrated in homological degree zero. Thus $K(n)_*\mathrm{THH}(E(n)) \cong K(n)_*E(n)$ and the isomorphism is induced by the map $E(n) \to \mathrm{THH}(E(n))$. Therefore, this map is a $K(n)$–equivalence and thus, $K(n)$–locally, $\mathrm{THH}(E(n))$ is equivalent to $E(n)$. □

We calculate $K(i)_*E(n)$ for $1 \leqslant i \leqslant n-1$ using the following description of morphisms of graded commutative BP_*–algebras from $K(i)_*E(n)$ to some graded commutative ring B_*. For $n = 2$ we had an argument that was rather involved and Paul Goerss suggested the following simpler proof.

We consider the map $g\colon \mathrm{BP}_*\mathrm{BP} \to K(i)_*E(n)$ of graded commutative $\mathbb{Z}_{(p)}$–algebras given by

$$\mathrm{BP}_*\mathrm{BP} \to K(i)_* \otimes_{\mathrm{BP}_*} \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} E(n)_* \cong K(i)_*E(n),$$

which uses the canonical maps $\mathrm{BP}_* \to K(i)_*$ and $\mathrm{BP}_* \to E(n)_*$ and the isomorphism from (3-1). By [19, Theorem A2.1.27], g corresponds to a triple $((\eta_L)_*F_i, (\eta_R)_*G_n, f)$ where $\eta_L\colon K(i)_* \to K(i)_*E(n)$ is the left unit, $\eta_R\colon E(n)_* \to K(i)_*E(n)$ is the right unit and $(\eta_L)_*F_i$ and $(\eta_R)_*G_n$ are the p–typical formal group laws that are given by the corresponding change of coefficients. Here, f is a strict isomorphism between the p–typical formal group laws $(\eta_L)_*F_i$ and $(\eta_R)_*G_n$ over $K(i)_*E(n)$. By [19, Lemma A2.1.26] such a strict isomorphism is always of the form

$$f(x) = \sum_j {}^{(\eta_R)_*G_n} t_j x^{p^j}.$$

The p–series of the Honda formal group law F_i is

$$[p]_{F_i}(x) = v_i x^{p^i}$$

and the same is true for $[p]_{(\eta_L)_*F_i}[x]$ because the left unit just embeds $K(i)_*$ into $K(i)_*E(n)$. The p–series of $(\eta_R)_*G_n$ is

$$[p]_{(\eta_R)_*G_n}(x) = w_1 x^p +_{(\eta_R)_*G_n} \cdots +_{(\eta_R)_*G_n} w_n x^{p^n}$$

for $w_i = \eta_R(v_i)$.

The strict isomorphism $f(x) = \sum_j {}^{(\eta_R)_*G_n} t_j x^{p^j}$ satisfies

$$[p]_{(\eta_R)_*G_n}(f(x)) = f([p]_{(\eta_L)_*F_i}(x)),$$

and using the above formulas for the p–series, this yields the equality

$$\begin{aligned} w_1(f(x))^p +_{(\eta_R)_*G_n} \cdots +_{(\eta_R)_*G_n} w_n(f(x))^{p^n} &= f(v_i x^{p^i}) \\ &= \sum_j {}^{(\eta_R)_*G_n} t_j (v_i x^{p^i})^{p^j}. \end{aligned} \tag{3-2}$$

Lemma 3.2 *In $K(i)_*E(n)$ the relations $w_r = 0$ for all $1 \leqslant r \leqslant i-1$ and $w_i = v_i$ hold.*

Proof In equality (3-2), the right-hand side starts with the summand $v_i x^{p^i}$ followed by higher powers of x. Looking at the left-hand side, we deduce that $w_1, \dots, w_{i-1} = 0$, and from the coefficient of x^{p^i} we obtain that $w_i = v_i$ in $K(i)_*E(n)$. □

Proposition 3.3 *$K(i)_*E(n)$ is a colimit of étale $K(i)_*[w_{i+1}, \dots, w_n^{\pm 1}]$–algebras for all $1 \leqslant i \leqslant n$.*

Proof In the following we fix i and n. We denote by $B(i,n)_*$ the graded commutative $K(i)_*$–algebra $K(i)_*[w_{i+1}, \dots, w_{n-1}, w_n^{\pm 1}]$. For a given $m \geqslant 1$ consider the graded commutative BP_*–subalgebra $BP_*[t_1,\dots,t_m]$ of BP_*BP, and define the subalgebra

$$B_m = g(BP_*[t_1,\dots,t_m]) \subset K(i)_*E(n).$$

By Lemma 3.2, we deduce that B_m can be written as the quotient

$$B_m = B(i,n)[t_1,\dots,t_m]/\sim,$$

where $\sim$ denotes the relations that the t_r and w_j satisfy in $K(i)_*E(n)$. Note that B_1 is free as a $B(i,n)$–module, and B_{m+1} is free as a B_m–module for all $m \geqslant 1$. Indeed, in each step we adjoin a new polynomial generator x to a graded commutative ring R_* that satisfies relations of the form $x^{p^r} - ux - y$ with a unit $u \in R_*^\times$ and $y \in R_*$. In particular, we have a sequence of subalgebras

$$B(i,n) \subset B_1 \subset \cdots \subset B_m \subset \cdots \subset K(i)_*E(n),$$

and $K(i)_*E(n)$ is the colimit of this sequence.

We prove that B_1 is étale over $B(i,n)_*$ and that for every m, B_m is étale over B_{m-1}. This then yields that the algebras B_m are étale over $B(i,n)_*$ which proves the claim. Thus we have to show that the modules of relative Kähler differentials $\Omega^1_{B_1|B(i,n)_*}$ and $\Omega^1_{B_m|B_{m-1}}$ are trivial for all $m \geqslant 2$. To this end we have to control the Kähler differentials dt_m and we do this now by deriving explicit relations for the t_m that we extract from the equality (3-2).

The first relation for t_m is obtained by looking at the coefficients of $x^{p^{i+m}}$ on the left- and right-hand sides of the equality (3-2).

Let $s \geqslant 2$, let $r, l_1,\dots,l_s$ be natural numbers bigger or equal to 1, and assume that $l_j \neq l_k$ for $j \neq k$. Then, as p^r has a unique representation in base p, it cannot be written as a sum $p^{l_1} + \cdots + p^{l_s}$. This ensures that, for a given $x^{p^{i+r}}$, we only have to consider the coefficient $t_j v_i^{p^j}$ with $i + j = i + r$ coming from the linear term of the $(\eta_R)_*G_n$–sum $\sum_j^{(\eta_R)_*G_n} t_j v_i^{p^j} x^{p^{i+j}}$ and this is $t_r v_i^{p^r}$.

For B_1 we compare the coefficients of $x^{p^{i+1}}$ in (3-2). In this case only the linear terms of the $(\eta_R)_*G_n$–sums contribute something and we obtain (using $w_i = v_i$)

$$v_i t_1^{p^i} + w_{i+1} t_0 = t_1 v_i^p$$

and therefore $t_1 = v_i^{-p}(v_i t_1^{p^i} + w_{i+1})$. This gives that the Kähler differential on t_1 is equal to

$$dt_1 = 0 + v_i^{-p} dw_{i+1}$$

and hence B_1 is étale over $B(i,n)_*$.

We consider now the general case of B_m for $m \geqslant 2$, and study the first relation for t_m given by the coefficients of $x^{p^{i+m}}$ in (3-2).

We know that the formal group law $G_n(x,y)$ is of the form

$$G_n(x,y) = x + y + \sum_{i,j \geqslant 1} a_{i,j} x^i y^j,$$

where the $a_{i,j} \in E(n)_* = \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n^{\pm 1}]$. Equation (3-2) relates power series with coefficients in $K(i)_*E(n)$, hence the coefficients $\overline{a}_{i,j}$ of $(\eta_R)_*G_n$ are now considered in $K(i)_*E(n)$ and are elements of $\mathbb{F}_p[w_i, \dots, w_{n-1}, w_n^{\pm 1}]$. On the left-hand side of (3-2) we get coefficients that involve some polynomials of the $\overline{a}_{i,j}$, some p^{th} powers of the t_j and some expressions in the w_k. For $m+i \leqslant n$ we actually get a coefficient $w_{m+i} t_0^{p^{m+i+0}} = w_{i+m}$.

The $\overline{a}_{i,j}$ are in $B(i,n)_*$, so they don't contribute anything to the relative Kähler differentials. The Kähler differentials on the $t_j^{p^k}$ are trivial because we are over $\mathbb{F}_p$. Hence we can express the Kähler differential dt_m up to a factor of $v_i^{p^m} = w_i^{p^m}$ via Kähler differentials in the w_k. As $v_i^{p^m}$ is invertible in $B(i,n)_*$, the relative Kähler differentials $\Omega^1_{B_m|B_{m-1}}$ are trivial for all $m \geqslant 1$. □

Theorem 3.4 *For all $1 \leqslant i \leqslant n$ we have an isomorphism of $K(i)_*E(n)$–algebras*

$$\mathsf{HH}_*^{K(i)_*}(K(i)_*E(n)) \cong K(i)_*E(n) \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}(dw_{i+1}, \dots, dw_n).$$

Proof We have shown that $K(i)_*E(n)$ is the sequential colimit of the B_m. As the $K(i)_*$–algebras B_m are étale over $B(i,n)_*$ and as Hochschild homology commutes with localization, we can rewrite $\mathsf{HH}_*(B_m)$ as

$$\begin{aligned} \mathsf{HH}_*^{K(i)_*}(B_m) &\cong B_m \otimes_{B(i,n)_*} \mathsf{HH}_*^{K(i)_*}(B(i,n)_*) \\ &\cong B_m \otimes_{B(i,n)_*} (B(i,n)_* \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}(dw_{i+1}, \dots, dw_n)) \\ &\cong B_m \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}(dw_{i+1}, \dots, dw_n) \end{aligned}$$

using [28] and the Hochschild–Kostant–Rosenberg theorem. Hochschild homology commutes with colimits, hence we obtain

$$\begin{aligned} \mathsf{HH}_*^{K(i)_*}(K(i)_*E(n)) &\cong \operatorname{colim}_m \mathsf{HH}_*^{K(i)_*}(B_m) \\ &\cong K(i)_*E(n) \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}(dw_{i+1}, \dots, dw_n). \end{aligned}$$ □

Theorem 3.5 *Assume that p is an odd prime and that $E(n)$ is an E_3–ring spectrum. Then, for all $1 \leqslant i \leqslant n$, we have an isomorphism of $K(i)_*E(n)$–algebras*

$$K(i)_*\mathrm{THH}(E(n)) \cong K(i)_*E(n) \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}(dw_{i+1}, \dots, dw_n).$$

Proof We use the Bökstedt spectral sequence [9; 12, Theorem IX.2.9], with E^2–term

$$E^2_{r,s} = \big(\mathsf{HH}_r^{K(i)_*}(K(i)_*E(n))\big)_s,$$

where r denotes the homological and s the internal degree. By a result of Angeltveit and Rognes [1, Proposition 4.3], an E_3–structure on $E(n)$ implies that this spectral is one of commutative $K(i)_*E(n)$–algebras. The multiplicative generators dw_j for $i \leqslant j \leqslant n$ sit in bidegree $(1, 2p^j - 2)$ and hence they cannot carry any nontrivial differentials. Therefore the spectral sequence collapses at the E^2–term. As the abutment is a free graded commutative $K(i)_*E(n)$–algebra, there cannot be any multiplicative extensions. □

Remark 3.6 If $E(n)$ admits an E_2–structure, the Bökstedt spectral sequence is one of $K(i)_*$–algebras by [1, Proposition 4.3]. It therefore collapses since all $K(i)_*$–algebra generators lie in columns 0 and 1. This gives the same formula for $K(i)_*\mathrm{THH}(E(n))$ as a $K(i)_*$–module, but not as a $K(i)_*$–algebra, since there is now room for $K(i)_*$–algebra extensions.

4 Blue-shift for THH($E(n)$)

If we assume that p is an odd prime and that $E(n)$ is an E_∞–ring spectrum, then $\mathrm{THH}(E(n))$ is a commutative $E(n)$–algebra spectrum and the cofiber of the unit map

$$\overline{\mathrm{THH}}(E(n)) = \mathrm{cofiber}\big(E(n) \to \mathrm{THH}(E(n))\big)$$

is a nonunital commutative $E(n)$–algebra spectrum. If $E(n)$ carries an E_3–structure, then by [10, Section 3.3; 6] the morphism $E(n) \to \mathrm{THH}(E(n))$ is an E_2–map. This implies the following useful fact:

Lemma 4.1 *If $E(n)$ is an E_3–spectrum, then $\mathrm{THH}(E(n))$ is an $E(n)$–module spectrum and, in particular, $\mathrm{THH}(E(n))$ is $E(n)$–local.*

Let L_n denote the localization at $E(n)$, and in particular L_0 is the rationalization. Recall that there is a well-known chromatic fracture square

$$\begin{array}{ccc} L_n X & \longrightarrow & L_{K(n)} X \\ \downarrow & & \downarrow \\ L_{n-1} X & \longrightarrow & L_{n-1} L_{K(n)} X \end{array}$$

It is shown for instance in [2, Example 3.3; 8, Proposition 2.2] that the homotopy pullback of

$$\begin{array}{ccc} & & L_{K(n)} X \\ & & \downarrow \\ L_{n-1} X & \longrightarrow & L_{n-1} L_{K(n)} X \end{array}$$

is an $E(n)$–localization of X. The statement in [8, Proposition 2.2] is more general and [2] works out far more general local-to-global statements.

The chromatic square for $\overline{\mathrm{THH}}(E(n))$ is

$$\begin{array}{ccc} \overline{\mathrm{THH}}(E(n)) = L_{K(n)\vee E(n-1)} \overline{\mathrm{THH}}(E(n)) & \longrightarrow & L_{K(n)} \overline{\mathrm{THH}}(E(n)) \\ \downarrow & & \downarrow \\ L_{E(n-1)} \overline{\mathrm{THH}}(E(n)) & \longrightarrow & L_{E(n-1)}\big(L_{K(n)} \overline{\mathrm{THH}}(E(n))\big) \end{array}$$

The $K(n)$–homology of $\overline{\mathrm{THH}}(E(n))$ is zero, since by Proposition 3.1 the unit map is a $K(n)$–equivalence. It follows that the localization $L_{K(n)}\overline{\mathrm{THH}}(E(n))$ is trivial, and hence $L_{E(n-1)}\big(L_{K(n)}\overline{\mathrm{THH}}(E(n))\big)$ is also trivial. Therefore the vertical map on the left-hand side is an equivalence and we obtain a nice example of blue-shift:

Lemma 4.2 *If $E(n)$ is an E_3–spectrum, then the cofiber $\overline{\mathrm{THH}}(E(n))$ is $E(n-1)$–local.*

5 Topological Hochschild homology of $E(2)$

In this section, we discuss in more detail the topological Hochschild homology of $E(2)$, which we will denote by $E = E(2)$ to simplify the notation. As explained in the proof of Lemma 5.1, the computations of Theorem 3.5 for $E(2)$ can be expressed as

$$K(0)_*\mathrm{THH}(E) \cong K(0)_*E \otimes \Lambda_{\mathbb{Q}}(dt_1, dt_2), \tag{5-1}$$

$$K(1)_*\mathrm{THH}(E) \cong K(1)_*E \otimes \Lambda_{\mathbb{F}_p}(dt_1), \tag{5-2}$$

$$K(2)_*\mathrm{THH}(E) \cong K(2)_*E. \tag{5-3}$$

Notice that these computations do not require the assumption that E is an E_3–ring spectrum: for the rational case we have a commutative structure anyhow, while, in the $K(1)$ and $K(2)$ cases, the E^2 page of the Bökstedt spectral sequences is concentrated on columns 0 and 1 (respectively 0).

Lemma 5.1 *For $i = 1, 2$, there exist classes $\lambda_i \in \mathrm{THH}_{2p^i-1}(E)$ with the following properties. Under the Hurewicz homomorphism*,

(a) *the class λ_i maps to $dt_i \in K(0)_{2p^i-1}\mathrm{THH}(E)$ for $i = 1, 2$;*

(b) *the class λ_1 maps to $dt_1 \in K(1)_{2p^2-1}\mathrm{THH}(E)$.*

Proof We use McClure and Staffeldt's computation of $\mathrm{THH}_*(\mathrm{BP})$ in [18, Remark 4.3], which has been validated by the proof [7] that BP admits an E_4–structure. We briefly recall the computation. The integral, rational and mod p homology of BP are given as

$$H\mathbb{Z}_*\mathrm{BP} \cong \mathbb{Z}_{(p)}[t_i \,|\, i \geqslant 1], \quad K(0)_*\mathrm{BP} \cong \mathbb{Q}[t_i \,|\, i \geqslant 1] \quad \text{and} \quad H\mathbb{F}_{p*}\mathrm{BP} \cong \mathbb{Z}[\bar{\xi}_i \,|\, i \geqslant 1],$$

where the class $t_i \in H\mathbb{Z}_{2p^i-1}\mathrm{BP}$ maps to $\bar{\xi}_i$ under mod (p) reduction [19, Proof of Theorem 5.2.8] and to the class with same name t_i under rationalization. The associated Bökstedt spectral sequences collapse, providing isomorphisms

$$H\mathbb{Z}_*\mathrm{THH}(\mathrm{BP}) \cong H\mathbb{Z}_*\mathrm{BP} \otimes \Lambda_{\mathbb{Z}_{(p)}}(dt_i \,|\, i \geqslant 1),$$
$$K(0)_*\mathrm{THH}(\mathrm{BP}) \cong K(0)_*\mathrm{BP} \otimes \Lambda_{\mathbb{Q}}(dt_i \,|\, i \geqslant 1),$$
$$H\mathbb{F}_{p*}\mathrm{THH}(\mathrm{BP}) \cong H\mathbb{F}_{p*}\mathrm{BP} \otimes \Lambda_{\mathbb{F}_p}(d\bar{\xi}_i \,|\, i \geqslant 1),$$

with $dx = \sigma_*(x)$, where $\sigma\colon \Sigma\mathrm{BP} \to \mathrm{THH}(\mathrm{BP})$ is the map given in (2-1). There is an isomorphism

$$\mathrm{THH}_*(\mathrm{BP}) \cong \mathrm{BP}_* \otimes \Lambda_{\mathbb{Z}_{(p)}}(\lambda_i \,|\, i \geqslant 1),$$

and the Hurewicz homomorphism

$$\mathrm{THH}_*(\mathrm{BP}) \to H\mathbb{Z}_*\mathrm{THH}(\mathrm{BP})$$

is an inclusion mapping λ_i to dt_i. In particular, the classes dt_i (integral and rational) and $d\overline{\xi}_i$ are spherical: they are the image of λ_i under the Hurewicz homomorphism mapping from $\mathrm{THH}_*(\mathrm{BP})$. For $i \geqslant 1$, let us define

$$\lambda_i \in \mathrm{THH}_{2p^i-1}(E)$$

as the image of the class with same name under the natural map

$$\mathrm{THH}_*(\mathrm{BP}) \to \mathrm{THH}_*(E).$$

In the rational case, we have

$$\eta_R(v_i) \equiv \alpha_i t_i$$

modulo decomposables in $K(0)_*\mathrm{BP}$, where $\alpha_i \in \mathbb{Q}$ is a unit. We deduce that

$$K(0)_*E \cong \mathbb{Q}[t_1, t_2][\eta_R(v_2)^{-1}]$$

and the Bökstedt spectral sequence recovers

$$K(0)_*\mathrm{THH}(E) \cong K(0)_*E \otimes \Lambda_{\mathbb{Q}}(dt_1, dt_2).$$

By naturality, comparing with the case of BP, we deduce that the Hurewicz homomorphism $\mathrm{THH}_*(E) \to K(0)_*\mathrm{THH}(E)$ maps λ_i to dt_i.

For $K(1)_*$–homology, we argue similarly, using the commutative square

$$\begin{array}{ccc} \mathrm{THH}_*(\mathrm{BP}) & \longrightarrow & K(1)_*\mathrm{THH}(\mathrm{BP}) \\ \downarrow & & \downarrow \\ \mathrm{THH}_*(E) & \longrightarrow & K(1)_*\mathrm{THH}(E) \end{array}$$

We have $K(1)_*\mathrm{BP} \cong K(1)_*[t_i \mid i \geqslant 1]$, and the Bökstedt spectral sequence yields

$$K(1)_*\mathrm{THH}(\mathrm{BP}) \cong K(1)_*\mathrm{BP} \otimes \Lambda_{\mathbb{F}_p}(dt_i \mid i \geqslant 1).$$

Comparing the Bökstedt spectral sequences for $H\mathbb{Z}_*\mathrm{THH}(\mathrm{BP})$ and $K(1)_*\mathrm{THH}(\mathrm{BP})$, we deduce that the class $\lambda_1 \in \mathrm{THH}_*(\mathrm{BP})$ maps to $dt_1 \in K(1)_*\mathrm{THH}(\mathrm{BP})$. Recall that

$$K(1)_*E = K(1)_*[t_i \mid i \geqslant 1][\eta_R(v_2)^{-1}]/(\eta_R(v_j) \mid j \geqslant 3)$$

is a colimit of étale algebras over $K(1)_*[w_2, w_2^{-1}]$, where

$$w_2 = \eta_R(v_2) = v_1^p t_1 - v_1 t_1^p.$$

In particular, $dw_2 = v_1^p dt_1$, and the Bökstedt spectral sequence provides the formula given above for $K(1)_*\mathrm{THH}(E)$. Now obviously $dt_1 \in K(1)_*\mathrm{THH}(\mathrm{BP})$ maps to $dt_1 \in K(1)_*\mathrm{THH}(E)$. This implies assertion (b) of the lemma. □

Remark 5.2 The above proof does not require the map $\mathrm{BP} \to E(n)$ to be an E_3–map.

The class $\lambda_1 \in \mathrm{THH}_{2p-1}(E)$ of Lemma 5.1 corresponds to a map $\lambda_1\colon S^{2p-1} \to \mathrm{THH}(E)$. Smashing with E, using the E–module structure of $\mathrm{THH}(E)$ (assuming an E_3–structure on E), and composing with the cofiber $\mathrm{THH}(E) \to \overline{\mathrm{THH}}(E)$ of the unit, we obtain a map

$$j_1\colon \Sigma^{2p-1}E \cong E \wedge S^{2p-1} \to E \wedge \mathrm{THH}(E) \to \mathrm{THH}(E) \to \overline{\mathrm{THH}}(E).$$

In the same fashion, we obtain a map $j_2\colon \Sigma^{2p^2-1}E \to \overline{\mathrm{THH}}(E)$ corresponding to the class λ_2.

Lemma 5.3 *The map j_1 factors through a map*

$$\bar{j}_1\colon \Sigma^{2p-1}L_1E \to \overline{\mathrm{THH}}(E)$$

that is a $K(1)_*$–isomorphism, and whose cofiber $C(\bar{j}_1)$ is a rational spectrum.

Proof Recall from Lemma 4.2 that the cofiber $\overline{\mathrm{THH}}(E)$ of the unit map is $E(1)$–local. In particular, the map j_1 factors through a map

$$\bar{j}_1\colon \Sigma^{2p-1}L_1E \to \overline{\mathrm{THH}}(E).$$

The localization map $E \to L_1E$ is a $K(1)_*$–isomorphism, and therefore so are the induced maps $\ell\colon \mathrm{THH}(E) \to \mathrm{THH}(L_1E)$ and $\bar{\ell}\colon \overline{\mathrm{THH}}(E) \to \overline{\mathrm{THH}}(L_1E)$, by convergence of the $K(1)$–based Bökstedt spectral sequence. Hence, to prove the claim, it suffices to show that the composition

$$\Sigma^{2p-1}L_1E \xrightarrow{\bar{j}_1} \overline{\mathrm{THH}}(E) \xrightarrow{\bar{\ell}} \overline{\mathrm{THH}}(L_1E) \tag{5-4}$$

is a $K(1)_*$–isomorphism. The $K(1)$–based Bökstedt spectral sequence for L_1E is identical to the one of E, computed above as

$$E^2_{*,*} = K(1)_*E \otimes \Lambda_{\mathbb{F}_p}(dt_1) \Rightarrow K(1)_*\mathrm{THH}(E),$$

where $K(1)_*E$ is in filtration degree zero and $K(1)_*E\{dt_1\}$ is in filtration degree 1, and where all differentials are zero. By definition of the map j_1, if $1 \in K(1)_0E$ is the unit, then $j_{1*}(\Sigma^{2p-1}1)$ is represented modulo lower filtration by the permanent cycle dt_1 in $E^2_{1,*}$. Since this is a spectral sequence of $K(1)_*E$–modules, the composition (5-4) induces a map in $K(1)$ homology that is represented modulo lower filtration by the isomorphism $\Sigma^{2p-1}K(1)_*E \to E^2_{1,*} = K(1)_*E\{dt_1\}$ sending a class $\Sigma^{2p-1}w$ to $w\,dt_1$. It is therefore a $K(1)_*$–isomorphism, proving the claim.

Now we consider the cofiber $C(\overline{\jmath}_1)$ of $\overline{\jmath}_1$, sitting in an exact triangle

$$\Sigma^{2p-1}L_1E \xrightarrow{\overline{\jmath}_1} \overline{\mathrm{THH}}(E) \xrightarrow{k} C(\overline{\jmath}_1) \xrightarrow{\delta} \Sigma^{2p}L_1E. \tag{5-5}$$

Since $\overline{\jmath}_1$ is a $K(1)_*$–isomorphism, we know that $K(1)_*C(\overline{\jmath}_1) = 0$, and since $\overline{\mathrm{THH}}(E)$ and thus $C(\overline{\jmath}_1)$ are $E(1)$–local, we deduce (as in Lemma 4.2) that $C(\overline{\jmath}_1)$ is $E(0)$–local (ie rational). □

We now define a map $\lambda_{12}\colon L_0S^{2p^2-2p-2} \to C(\overline{\jmath}_1)$ as a composition over the cofibers,

$$L_0S^{2p^2-2p-2} \to L_0\mathrm{THH}(E) \to L_0\overline{\mathrm{THH}}(E) \to C(\overline{\jmath}_1),$$

where the first map above realizes the class $dt_1dt_2 \in K(0)_*\mathrm{THH}(E)$. Smashing λ_{12} with E and using the module structure, we obtain a map

$$j_{12}\colon \Sigma^{2p^2-2p-2}L_0E \to C(\overline{\jmath}_1).$$

Similarly, λ_2 induces a map

$$j_2\colon \Sigma^{2p^2-1}L_0E \to C(\overline{\jmath}_1).$$

Theorem 5.4 *Let p be an odd prime such that $E = E(2)$, the second Johnson–Wilson spectrum at p, is an E_3–ring spectrum. Then the map $j_2 \vee j_{12}$ lifts to a map*

$$\overline{\jmath}_2 \vee \overline{\jmath}_{12}\colon \Sigma^{2p^2-1}L_0E \vee \Sigma^{2p^2-2p-2}L_0E \to \overline{\mathrm{THH}}(E)$$

and the sum β of $\overline{\jmath}_1$, $\overline{\jmath}_2$ and $\overline{\jmath}_{12}$ is a weak equivalence of E–modules

$$\beta\colon \Sigma^{2p-1}L_1E \vee \Sigma^{2p^2-1}L_0E \vee \Sigma^{2p^2+2p-2}L_0E \to \overline{\mathrm{THH}}(E).$$

Proof The composition $\delta \circ (j_2 \vee j_{12})$ is trivial, so that $j_2 \vee j_{12}$ lifts to a map $\overline{\jmath}_2 \vee \overline{\jmath}_{12}$:

$$\begin{array}{ccccc}
 & & \Sigma^{2p^2-1}L_0E \vee \Sigma^{2p^2+2p-2}L_0E & & \\
 & \swarrow^{\overline{\jmath}_2 \vee \overline{\jmath}_{12}} & \downarrow{\scriptstyle j_2 \vee j_{12}} & \searrow^{\simeq *} & \\
\overline{\mathrm{THH}}(E) & \xrightarrow{\quad k \quad} & C(\overline{\jmath}_1) & \xrightarrow{\quad \delta \quad} & \Sigma^{2p}L_1E
\end{array}$$

Indeed, $\Sigma^{2p} L_1 E$ fits in the chromatic fracture pullback diagram

$$\begin{array}{ccc} \Sigma^{2p} L_1 E & \longrightarrow & \Sigma^{2p} L_{K(1)} E \\ \downarrow & & \downarrow \\ \Sigma^{2p} L_0 E & \longrightarrow & \Sigma^{2p} L_0(L_{K(1)} E) \end{array}$$

The composition of $\delta \circ (j_2 \vee j_{12})$ with the left vertical map to $\Sigma^{2p} L_0 E$ is trivial, since it factors over the composition

$$L_0 \overline{\mathrm{THH}}(E) \to L_0 C(\bar{j}_1) \to \Sigma^{2p} L_0 E$$

of two consecutive maps in the ($E(0)$–localized) cofiber sequence (5-5). The composition of $\delta \circ (j_2 \vee j_{12})$ with the top map to $\Sigma^{2p} L_{K(1)} E$ is trivial as well; indeed, there is no nontrivial map from a $K(1)$–acyclic to a $K(1)$–local spectrum. This finishes the proof that $\delta \circ (j_2 \vee j_{12})$ is trivial and that the lift exists. We now define β as the sum

$$\beta = \bar{j}_1 \vee \bar{j}_2 \vee \bar{j}_{12} \colon \Sigma^{2p-1} L_1 E \vee \Sigma^{2p^2-1} L_0 E \vee \Sigma^{2p^2+2p-2} L_0 E \to \overline{\mathrm{THH}}(E).$$

Finally, we claim that β is a $K(0)_*$–isomorphism: this is analogous to the proof above that $\bar{j}_1$ is a $K(1)_*$–isomorphism, working this time with the $K(0)$–based Bökstedt spectral sequence. Since β is a $K(0)_*$– and a $K(1)_*$–isomorphism of $E(1)$–local spectra, it is a weak equivalence. □

Assume now that in addition to E being an E_3–ring spectrum, the unit map $E \to \mathrm{THH}(E)$ splits in the homotopy category (this holds for example if E is an E_∞–ring spectrum). We then have a weak equivalence of E–modules $E \vee \overline{\mathrm{THH}}(E) \to \mathrm{THH}(E)$. On the other hand, summing β with the identity of E gives a weak equivalence

$$\mathrm{id} \vee \beta \colon E \vee \Sigma^{2p-1} L_1 E \vee \Sigma^{2p^2-1} L_0 E \vee \Sigma^{2p^2+2p-2} L_0 E \to E \vee \overline{\mathrm{THH}}(E).$$

This implies the following corollary of Theorem 5.4:

Corollary 5.5 *Assume that p is an odd prime, and that the second Johnson–Wilson spectrum $E = E(2)$ admits an E_3–structure. If the unit map $E \to \mathrm{THH}(E)$ splits in the homotopy category, then the maps above provide a weak equivalence of E–modules*

$$E \vee \Sigma^{2p-1} L_1 E \vee \Sigma^{2p^2-1} L_0 E \vee \Sigma^{2p^2+2p-2} L_0 E \to \mathrm{THH}(E).$$

Remark 5.6 Corollary 5.5 implies that

- the 2^0 summand of $K(2)_* E$ in $K(2)_* \mathrm{THH}(E)$ indexed by 1,

- the 2^1 summands of $K(1)_*E$ in $K(1)_*\mathrm{THH}(E)$ indexed by 1 and dt_1, and
- the 2^2 summands of $K(0)_*E$ in $K(0)_*\mathrm{THH}(E)$ indexed by 1, dt_1, dt_2 and $dt_1 dt_2$

assemble, in $\mathrm{THH}(E)$, into

- the 2^0 summand E indexed by 1 and detected by $K(0)_*$, $K(1)_*$ and $K(2)_*$,
- the 2^1-2^0 summand L_1E indexed by dt_1 and detected by $K(0)_*$ and $K(1)_*$, and
- the 2^2-2^1 summands L_0E indexed by dt_2 and dt_1dt_2 and detected by $K(0)_*$.

Bruner and Rognes [11] obtain very similar computations for $K(i)_*\mathrm{THH}(\mathrm{tmf})$ for $i=0,1,2$, where tmf denotes the connective spectrum of topological modular form.

We can picture the summands of $\mathrm{THH}(E)$ in a 2–dimensional cube of local pieces (up to suspensions, where $E=L_2E$):

	1	dt_1
1	E	L_1E
dt_2	L_0E	L_0E

We conjecture that this picture extends to describe a decomposition of $\mathrm{THH}(E(n))$ into 2^n summands, with summands placed in an n–dimensional cube, where the i^{th} edge has two coordinates 1 and dt_i. We formulate this as follows:

Conjecture 5.7 If p is an odd prime such that $E(n)$ is a sufficiently commutative S–algebra, then $\mathrm{THH}(E(n))$ decomposes as a sum of 2^n factors, namely 2^{n-i-1} suspended copies of $L_iE(n)$ for each $0\leqslant i\leqslant n-1$ plus one copy of $E(n)$. More precisely, the $L_iE(n)$–summands are indexed by the 2^{n-i-1} monomial generators

$$\omega\in\Lambda_{\mathbb{Q}}(dt_1,\dots,dt_{n-i-1})\{dt_{n-i}\}\subset K(0)_*\mathrm{THH}(E(n)),$$

and the summand corresponding to such a monomial ω is $\Sigma^{|\omega|}L_iE(n)$.

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