On the Hurewicz map and Postnikov invariants of $K\mathbb{Z}$

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Abstract. The purpose of this note is to present a calculation of the Hurewicz homomorphism $h : K_*\mathbb{Z} \longrightarrow H_*(GL(\mathbb{Z}); \mathbb{Z})$ on the elements of $K_*\mathbb{Z}$ known to generate direct summands. These results are then used to produce lower bounds for the Postnikov invariants of the space $K\mathbb{Z}$. Under extra hypothesises (compatible with the Quillen-Lichtenbaum conjecture for $\mathbb{Z}$), we give the exact p-primary part of the order of the latter invariants.

1. Introduction

D. Quillen defined, for any integer $n \geq 1$, the higher algebraic K-theory group K_nR of a ring R as the homotopy group $K_nR = \pi_n(BGL(R)^+)$. In this paper, we will calculate the Hurewicz homomorphism

$$h : K_*\mathbb{Z} \longrightarrow H_*(BGL(\mathbb{Z})^+; \mathbb{Z}) \cong H_*(GL(\mathbb{Z}); \mathbb{Z})$$

on elements of $K_*\mathbb{Z}$ that are known to generate direct summands. One motivation for such a calculation is to obtain information on the homotopy type of the space $BGL(\mathbb{Z})^+$, which we will denote in the sequel by $K\mathbb{Z}$. Its weak homotopy type is uniquely determined by its homotopy groups $K_*\mathbb{Z}$ and by its Postnikov invariants, which are related to the Hurewicz homomorphism.

The Hurewicz homomorphism $h : K_*\mathbb{Z} \longrightarrow H_*(GL(\mathbb{Z}); \mathbb{Z})$ has first been used by Borel [7] to calculate the rank of the finitely generated abelian group $K_m\mathbb{Z}$ for all $m \geq 1$: by the Milnor-Moore Theorem, the Hurewicz homomorphism induces an isomorphism from $K_*\mathbb{Z}\otimes\mathbb{Q}$ onto the primitives of $H_*(GL(\mathbb{Z}); \mathbb{Q}) \cong \bigwedge_{\mathbb{Q}}(u_3, u_5, \dots)$, where $|u_i| = 2i - 1$. Hence, if n is an odd integer ≥ 3, the group $K_{2n-1}\mathbb{Z}$ contains an infinite cyclic direct summand which injects in $H_{2n-1}(GL(\mathbb{Z}); \mathbb{Z})$. How? In Theorem 3.2, we show that this injection is far from being split : it is multiplication by $(n - 1)!$ (up to primes that do not satisfy Vandiver's Conjecture from number theory). Theorem 3.2 also gives the Hurewicz homomorphism on all 2-torsion classes of $K_*\mathbb{Z}$, and on the odd torsion classes of $K_*\mathbb{Z}$ corresponding to ImJ. We then apply these results to estimate the order of the Postnikov invariants of the space $K\mathbb{Z}$ (Theorem 4.1).

These calculations are made by comparing the p-adic completion $K\mathbb{Z}_p^\wedge$ of the space $K\mathbb{Z}$ to one of its topological models, called $JK\mathbb{Z}_p^\wedge$ and first defined by M. Bökstedt in [5]. We begin by reviewing some links between these spaces.

2. The model $JK\mathbb{Z}_p^\wedge$ for $K\mathbb{Z}_p^\wedge$

Let ℓ be an odd prime, and define $JK\mathbb{Z}(\ell)$ as the homotopy fibre of the composite map

$$BO \xrightarrow{\Psi^\ell_{\mathbb{R}}-1} BSpin \xrightarrow{c} BSU \tag{2.1}$$

where $\Psi^\ell_{\mathbb{R}}$ is the Adams operation ([1]), and where c is induced by the complexification of vector bundles. The homotopy groups of the space $JK\mathbb{Z}(\ell)$ are given by

$$\pi_n\big(JK\mathbb{Z}(\ell)\big) = \begin{cases} \mathbb{Z}/2 & \text{if } n = 1 \text{ or if } n \equiv 2 \bmod (8), \\ \mathbb{Z}\oplus\mathbb{Z}/2 & \text{if } n \geq 9 \text{ and if } n \equiv 1 \bmod (8), \\ \mathbb{Z}/2(\ell^{\frac{n+1}{2}}-1) & \text{if } n \equiv 3 \bmod (8), \\ \mathbb{Z} & \text{if } n \equiv 5 \bmod (8), \\ \mathbb{Z}/(\ell^{\frac{n+1}{2}}-1) & \text{if } n \equiv 7 \bmod (8), \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Let p be a prime number, and choose $\ell = 3$ if $p = 2$, ℓ a generator of the group of units of $\mathbb{Z}/p^2$ if p is odd. Following Bökstedt [5], let us then call $JK\mathbb{Z}_p^\wedge$ the space $JK\mathbb{Z}(\ell)_p^\wedge$. Here, $X_p^\wedge$ means the p-adic completion of a suitable space or group X. The homotopy group $\pi_n(JK\mathbb{Z}_p^\wedge)$ is isomorphic to $\pi_n\big(JK\mathbb{Z}(\ell)\big)\otimes\mathbb{Z}_p^\wedge$ and can be explicitly computed using (2.1) and the following formulas : if $n \equiv 3, 7 \bmod (8)$ and if ℓ is chosen as above with respect to p, then

$$v_p(\ell^{\frac{n+1}{2}}-1) = \begin{cases} v_p(n+1)+1 & \begin{cases} \text{if } p \neq 2 \text{ and } 2(p-1)|n+1, \\ \text{or if } p = 2 \text{ and } n \equiv 7 \bmod (8), \end{cases} \\ 3 & \text{if } p = 2 \text{ and } n \equiv 3 \bmod (8), \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

Here v_p denotes the p-adic valuation.

Bökstedt showed that there is a map $\phi : K\mathbb{Z}_2^\wedge \longrightarrow JK\mathbb{Z}_2^\wedge$ which, after looping once, is a homotopy retraction (Theorem 2 of [5]). The recent calculation (in [18] and [15]) of the 2-primary part of $K_*\mathbb{Z}$ implies that the map

$$\phi : K\mathbb{Z}_2^\wedge \xrightarrow{\simeq} JK\mathbb{Z}_2^\wedge \tag{2.4}$$

is actually a homotopy equivalence.

When p is odd, the homotopy groups of $JK\mathbb{Z}_p^\wedge$ are isomorphic to direct summands of $(K_*\mathbb{Z})_p^\wedge$ (see [7] and [14]). If p is a regular prime, the Quillen-Lichtenbaum conjecture asserts that $K\mathbb{Z}_p^\wedge$ and $JK\mathbb{Z}_p^\wedge$ have same homotopy groups (see [10], Corollary 2.3), while if p is irregular, there are p-torsion classes in $(K_*\mathbb{Z})_p^\wedge$ which do not appear in the homotopy groups of $JK\mathbb{Z}_p^\wedge$ (see [16]). It is not known in whole generality whether the group-level splitting

$$(K_*\mathbb{Z})_p^\wedge \;\cong\; \pi_*(JK\mathbb{Z}_p^\wedge) \;\oplus\; \ldots$$

can be induced by a space level retraction $K\mathbb{Z}_p^\wedge \longrightarrow JK\mathbb{Z}_p^\wedge$ or not. However, it follows from the work of Quillen and Dwyer-Mitchell that this is the case when p is a Vandiver prime (Proposition 2.5), that is when p is an odd prime that does not divide the class number $h^+(\mathbb{Q}(\zeta_p))$ of the maximal real subfield of the cyclotomic field $\mathbb{Q}(\zeta_p)$. It is a conjecture by Vandiver that all primes verify this condition, and it is known to be true for $p < 4'000'000$ (see [17], page 158).

Proposition 2.5. *If p is a Vandiver prime, then $JK\mathbb{Z}_p^\wedge$ is a retract of $K\mathbb{Z}_p^\wedge$.*

Proof. If p is an odd prime, the space $BSU_p^\wedge$ splits as a product $BSU_p^\wedge \simeq BO_p^\wedge \times B(SU/SO)_p^\wedge$, thus induces a splitting $JK\mathbb{Z}_p^\wedge \simeq (\mathrm{F}\Psi_{\mathbb{C}}^\ell)_p^\wedge \times (SU/SO)_p^\wedge$, where $(\mathrm{F}\Psi_{\mathbb{C}}^\ell)_p^\wedge$ is the p-adic completion of the homotopy fibre $\mathrm{F}\Psi_{\mathbb{C}}^\ell$ of $\Psi_{\mathbb{C}}^\ell - 1 : BU \longrightarrow BU$, or equivalently the homotopy fibre of $\Psi_{\mathbb{R}}^\ell - 1 : BO_p^\wedge \longrightarrow BO_p^\wedge$ (because of the above choice of ℓ). However, the space $\mathrm{F}\Psi_{\mathbb{C}}^\ell$ is homotopy equivalent to $K\mathbb{F}_\ell$, and the reduction map $K\mathbb{Z}_p^\wedge \longrightarrow (K\mathbb{F}_\ell)_p^\wedge$ is a retraction according to [14].

On the other hand, W. Dwyer and S. Mitchell proved in [11], Theorem 9.3 and Example 12.2, that if p is a Vandiver prime, then $(U/O)_p^\wedge$ is a retract of $K\mathbb{Z}[\frac{1}{p}]_p^\wedge$. The space $(SU/SO)_p^\wedge$ is the universal cover of $(U/O)_p^\wedge$ and, by the localization exact sequence, $K\mathbb{Z}_p^\wedge$ is the universal cover of $K\mathbb{Z}[\frac{1}{p}]_p^\wedge$. This implies that $(SU/SO)_p^\wedge$ is a retract of $K\mathbb{Z}_p^\wedge$. The product of the above retractions

$$K\mathbb{Z}_p^\wedge \longrightarrow (\mathrm{F}\Psi_{\mathbb{C}}^\ell)_p^\wedge \times (SU/SO)_p^\wedge \simeq JK\mathbb{Z}_p^\wedge$$

is then itself a retraction. □

3. The Hurewicz homomorphism for $K\mathbb{Z}$

Let us choose for all odd $n \geq 3$ a representative $b_n \in K_{2n-1}\mathbb{Z}$ of a generator of $K_{2n-1}\mathbb{Z}/(\text{Torsion}) \cong \mathbb{Z}$, thus obtaining a decomposition $K_{2n-1}\mathbb{Z} \cong \langle b_n \rangle \oplus T_{2n-1}$, where T_{2n-1} is the (finite) torsion subgroup of $K_{2n-1}\mathbb{Z}$. Since the homomorphism $h : K_{2n-1}\mathbb{Z} \longrightarrow H_{2n-1}(GL(\mathbb{Z});\mathbb{Z})$ is injective after rationalization, there exists a generator v_n of an infinite cyclic summand of $H_{2n-1}(GL(\mathbb{Z});\mathbb{Z})$ and an integer $\mu_n > 0$ such that $h(b_n) \equiv \mu_n v_n$ modulo torsion elements. Equivalently, we may define μ_n as the order of the torsion subgroup of the cokernel of the homomorphism $h : K_{2n-1}\mathbb{Z} \longrightarrow H_{2n-1}(GL(\mathbb{Z});\mathbb{Z})/\{\text{Torsion}\}$. On the other hand, if $n \geq 1$, it is known that $K_n\mathbb{Z}$ contains the following finite cyclic groups as direct summands :

$$\begin{cases} \mathbb{Z}/2 & \text{if } n \equiv 1, 2 \bmod (8), \\ \mathbb{Z}/16 & \text{if } n \equiv 3 \bmod (8), \\ \mathbb{Z}/2^{v_2(n+1)+1} & \text{if } n \equiv 7 \bmod (8), \\ \mathbb{Z}/p^{v_p(n+1)+1} & \text{if } p \text{ is an odd prime and if } 2(p-1)|n+1. \end{cases} \tag{3.1}$$

We know, because of the equivalence $\phi : K\mathbb{Z}_2^\wedge \xrightarrow{\simeq} JK\mathbb{Z}_2^\wedge$ and of (2.1), that this is all the 2-torsion there is in $K_*\mathbb{Z}$. The odd torsion direct factors in (3.0) are given by [14] (see proof of Proposition 2.5). Let us choose a generator $\omega_{2,n}$ of the 2-torsion subgroup of $K_n\mathbb{Z}$ whenever $n \equiv 1, 2, 3, 7 \bmod (8)$, and a generator $\omega_{p,n}$ of the p-torsion subgroup of $K_n\mathbb{Z}$ given by (3.0) whenever p is an odd prime with $2(p-1)|n+1$.

Theorem 3.2. *The Hurewicz homomorphism $h : K_*\mathbb{Z} \longrightarrow H_*(GL(\mathbb{Z}); \mathbb{Z})$ has the following properties :*
a) *If $p = 2$ or if p is a Vandiver prime, and if $n \geq 3$ is odd, then*

$$v_p(\mu_n) = v_p\big((n-1)!\big).$$

b) *If p is an odd prime and if $(p, n) \neq (p, 2p-3), (3, 11)$, then $\omega_{p,n}$ belongs to the kernel of h. The image $h(\omega_{p,2p-3})$ generates a direct summand of order p of $H_{2p-3}(GL(\mathbb{Z}); \mathbb{Z})$, and $h(\omega_{3,11})$ is of order 3 in a direct summand of order 9 of $H_{11}(GL(\mathbb{Z}); \mathbb{Z})$.*
c) *If $n \neq 1, 2, 3, 7, 15$, then $\omega_{2,n}$ belongs to the kernel of h. If $n = 1$ or 2, then $K_n\mathbb{Z} \cong \mathbb{Z}/2$ and $h : K_n\mathbb{Z} \longrightarrow H_n(GL(\mathbb{Z}); \mathbb{Z})$ is an isomorphism. The image $h(\omega_{2,3})$ generates the 2-torsion subgroup of $H_3(GL(\mathbb{Z}); \mathbb{Z})$, which is of order 8. The image $h(\omega_{2,7})$ is of order 8 in a cyclic direct summand of order 16 of $H_7(GL(\mathbb{Z}); \mathbb{Z})$, and $h(\omega_{2,15})$ is of order 2 in a cyclic direct summand of order 32 of $H_{15}(GL(\mathbb{Z}); \mathbb{Z})$.*

To prove Theorem 3.2 we will need the equivalence (2.3), Proposition 2.5, as well as a computation of the Hurewicz homomorphism h' for $JK\mathbb{Z}(\ell)$. The next Lemma is the main ingredient of this computation.

For any $n \geq 2$, let us choose a generator ε_n of $\pi_{2n-1}(SU) \cong \mathbb{Z}$. Recall that there is an isomorphism of algebras $H_*(SU; \mathbb{Z}) \cong \bigwedge_{\mathbb{Z}}(x_2, x_3, \dots)$. Here x_i is a primitive class of degree $2i-1$, defined as the dual of the class $e_i = \sigma(c_i) \in H^{2i-1}(SU; \mathbb{Z})$, where σ is the cohomology suspension and $c_i \in H^{2i}(BSU; \mathbb{Z})$ is the ith Chern class. The Hurewicz homomorphism for SU was calculated by Douady ([9], Théorème 6), and is given by the rule

$$\varepsilon_n \longmapsto \pm(n-1)!\ x_n. \tag{3.3}$$

By looping the fibration

$$JK\mathbb{Z}(\ell) \xrightarrow{f} BO \xrightarrow{g} BSU, \tag{3.4}$$

where g is the composition (2.0), we get a map $\partial : SU \longrightarrow JK\mathbb{Z}(\ell)$ having the following properties.

Lemma 3.5. *Let ℓ be an odd prime and n an integer ≥ 2. The image of the element $x_n \in H_{2n-1}(SU; \mathbb{Z})$ under the homomorphism*

$$\partial_* : H_{2n-1}(SU; \mathbb{Z}) \longrightarrow H_{2n-1}(JK\mathbb{Z}(\ell); \mathbb{Z})$$

generates a direct summand in $H_{2n-1}(JK\mathbb{Z}(\ell); \mathbb{Z})$. This summand is of infinite order if n is odd, and of order $(\ell^n - 1)$ if n is even.

Proof. Suppose first n is odd. The integral cohomology algebra of SU is given by an isomorphism $H^*(SU;\mathbb{Z}) \cong \bigwedge_{\mathbb{Z}}(e_2, e_3, \dots)$, where e_i is the dual class of x_i. We must show that the class e_n is in the image of the homomorphism $\partial^* : H^*(JK\mathbb{Z}(\ell);\mathbb{Z}) \longrightarrow H^*(SU;\mathbb{Z})$.

Consider the homotopy commutative diagram

$$\begin{array}{ccccccc} (1) & & SU & \xrightarrow{\partial} & JK\mathbb{Z}(\ell) & \xrightarrow{f} & BO \\ \varphi\downarrow & & \downarrow = & & \downarrow & & \downarrow g=(\Psi^\ell_{\mathbb{C}}-1)c \\ (2) & & SU & \longrightarrow & PBSU & \longrightarrow & BSU \end{array} \tag{3.6}$$

whose rows (1) and (2) are homotopy fibrations. Here $PBSU \longrightarrow BSU$ is the path fibration. For $i = 1$ or 2, let us call $(E^{*,*}_*(i), d^i_*)$ the Serre spectral sequence for $H^*(-;\mathbb{Z})$ associated to the fibration (i). The fibration morphism φ induces a morphism between these spectral sequences, which we will denote by $\varphi^{*,*}_*$. It suffices to verify that the element e_n in $E^{0,2n-1}_2(1) \cong H^{2n-1}(SU;\mathbb{Z})$ is a permanent cycle. Since the cohomology suspension $\sigma : H^{2n}(BSU;\mathbb{Z}) \longrightarrow H^{2n-1}(SU;\mathbb{Z})$ maps the n-th Chern class c_n to e_n, e_n is transgressive in $(E^{*,*}_*(2), d^2_*)$ and by naturality belongs to $E^{0,2n-1}_{2n}(1)$. Now $E^{2n,0}_{2n}(1)$ is a quotient of $H^{2n}(BO;\mathbb{Z})$, which contains only elements of order 2 since n is odd (see [6], Theorem 24.7 page 86). On the other hand, one can show by induction on n that $d^1_{2n}(e_n)$ is equal to $\varphi^{2n,0}_{2n}(c_n) = (\ell^n - 1)c^*(c_n)$ in $E^{2n,0}_{2n}(1)$, so is divisible by 2. Hence e_n is a permanent cycle in $E^{*,*}_*(1)$.

If n is even, we work with the Serre spectral sequences $(E^*_{*,*}(i), d^*_i)$ for $H_*(-;\mathbb{Z})$ of the fibrations (i=1,2) of diagram (3.5). Using the homology suspension, one verifies that there is a primitive generator $p_n \in H_{2n}(BSU;\mathbb{Z}) = E^2_{2n,0}(2)$ that transgresses to $x_n \in H_{2n-1}(SU) = E^2_{0,2n-1}(2)$ at the $2n$-th stage. Since n is even, there is in $H_{2n}(BO;\mathbb{Z})$ an element $\bar{p}_n$ (the n-th Pontryagin class) that verifies $c_*(\bar{p}_n) = p_n$ (see [8], equation 61, page 19). Now $(\Psi^\ell_{\mathbb{C}} - 1)_*(p_n) = (\ell^n - 1)p_n$, so by naturality, the class $\bar{p}_n \in H_{2n}(BO;\mathbb{Z}) = E^2_{2n,0}(1)$ is transgressive in the spectral sequence $(E^*_{*,*}(1), d^*_1)$ and transgresses to $(\ell^n - 1)x_n$. It follows that $\partial_*(x_n)$ is of order $\ell^n - 1$ in $H_{2n-1}(JK\mathbb{Z}(\ell);\mathbb{Z})$. To verify that $\partial_*(x_n)$ indeed generates a direct summand, it is enough to check that the dual class of x_n in $H^{2n-1}\big(SU;\mathbb{Z}/(\ell^n - 1)\big)$ is in the image of $\partial^* : H^{2n-1}\big(JK\mathbb{Z}(\ell);\mathbb{Z}/(\ell^n - 1)\big) \longrightarrow H^{2n-1}\big(SU;\mathbb{Z}/(\ell^n - 1)\big)$. This can be proven using again a Serre spectral sequence argument of the same flavour as above. □

Remarks 3.7.
a) Lemma 3.5, together with (3.2), allows one to compute the Hurewicz homomorphism h' of $JK\mathbb{Z}(\ell)$ on all elements of $\pi_*\big(JK\mathbb{Z}(\ell)\big)$ that are in the image of $\partial_* : \pi_*(SU) \longrightarrow \pi_*\big(JK\mathbb{Z}(\ell)\big)$. The only elements of $\pi_*\big(JK\mathbb{Z}(\ell)\big)$ that are not in this image are the 2-torsion elements in dimensions n with $n \equiv 1, 2 \bmod (8)$.
b) A very similar argument to the one of the proof of Lemma 3.5 implies that, for any prime ℓ, the connecting map $\partial : SU \longrightarrow \mathrm{F}\Psi^\ell_\mathbb{C}$ has the following property: for any integer $n \geq 2$, the image of the element $x_n \in H_{2n-1}(SU;\mathbb{Z})$ under the homomorphism $\partial_* : H_{2n-1}(SU;\mathbb{Z}) \longrightarrow H_{2n-1}(\mathrm{F}\Psi^\ell_\mathbb{C};\mathbb{Z})$ generates a direct summand in $H_{2n-1}(\mathrm{F}\Psi^\ell_\mathbb{C};\mathbb{Z})$ of order $(\ell^n - 1)$.
c) Notice that if $n \equiv 2 \bmod (4)$, the element $\partial_*(\varepsilon_n)$ is of order $2(\ell^n - 1)$ in $\pi_{2n-1}\big(JK\mathbb{Z}(\ell)\big)$, while $\partial_*(x_n)$ is of order $(\ell^n - 1)$ in $H_{2n-1}(JK\mathbb{Z}(\ell);\mathbb{Z})$.

Proof of Theorem 3.2.

a) Let $p = 2$ and $\ell = 3$, or let p be a Vandiver prime and ℓ an odd prime that generates the units of $\mathbb{Z}/p^2$. We compare the Hurewicz homomorphisms of $K\mathbb{Z}$ and $JK\mathbb{Z}(\ell)$ by means of the following commutative diagram. Let us call $\psi : JK\mathbb{Z}^\wedge_p \longrightarrow K\mathbb{Z}^\wedge_p$ the inclusion as a summand given by Proposition 2.5, and choose an integer $k > \max\big\{v_p(\mu_n), v_p\big((n-1)!\big)\big\} + v_p(T)$, where T is the largest order of any p-torsion element in $K_{2n-1}\mathbb{Z}$ or $H_{2n-1}(K\mathbb{Z};\mathbb{Z})$.

$$\begin{array}{ccccccc}
\pi_*\big(JK\mathbb{Z}(\ell)\big) & \xrightarrow{\alpha_{JK\mathbb{Z}(\ell)}} & \pi_*(JK\mathbb{Z}^\wedge_p;\mathbb{Z}/p^k) & \xrightarrow{\psi_*} & \pi_*(K\mathbb{Z}^\wedge_p;\mathbb{Z}/p^k) & \xleftarrow{\alpha_{K\mathbb{Z}}} & \pi_*(K\mathbb{Z}) \\
\downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle \bar{h}'} & & \downarrow{\scriptstyle \bar{h}} & & \downarrow{\scriptstyle h} \\
H_*(JK\mathbb{Z}(\ell);\mathbb{Z}) & \xrightarrow{\gamma_{JK\mathbb{Z}(\ell)}} & H_*(JK\mathbb{Z}^\wedge_p;\mathbb{Z}/p^k) & \xrightarrow{\psi_*} & H_*(K\mathbb{Z}^\wedge_p;\mathbb{Z}/p^k) & \xleftarrow{\gamma_{K\mathbb{Z}}} & H_*(K\mathbb{Z};\mathbb{Z})
\end{array}$$

Here $\pi_*(-,\mathbb{Z}/p^k)$ and $\bar{h}$, $\bar{h}'$ are the mod p^k homotopy groups and Hurewicz maps (see Chapter 3 of [13]). The map α_X given in the diagram is the composite

$$\pi_*(X) \longrightarrow \pi_*(X) \otimes \mathbb{Z}/p^k \hookrightarrow \pi_*(X;\mathbb{Z}/p^k) \cong \pi_*(X^\wedge_p;\mathbb{Z}/p^k)$$

and the map γ_X is defined in a similar way. The assertion is proven by inspection of this diagram, using our knowledge of $h' : \pi_{2n-1}\big(JK\mathbb{Z}(\ell)\big) \longrightarrow H_{2n-1}(JK\mathbb{Z}(\ell);\mathbb{Z})$ (see Remark 3.7.a).

b) If p is any odd prime and ℓ an odd prime that generates the units of $\mathbb{Z}/p^2$, the space $\mathrm{F}\Psi^\ell_\mathbb{C}$ splits off $K\mathbb{Z}$ after being localized at p. The element $\omega_{p,n}$ generates the factor $\pi_n\big((\mathrm{F}\Psi^\ell_\mathbb{C})_{(p)}\big)$ of $K_n\mathbb{Z}$, and is in the image of the homomorphism $\partial_* : \pi_n(SU) \longrightarrow \pi_n\big((\mathrm{F}\Psi^\ell_\mathbb{C})_{(p)}\big)$. It then follows from the rule (3.2) and the Remark 3.7.b that $h(\omega_{p,n}) = \big(\frac{n-1}{2}\big)! z_n$, where z_n is the generator of a direct summand of $H_n(K\mathbb{Z};\mathbb{Z})$ of order the p-primary part of $\ell^{\frac{n+1}{2}} - 1$ (see (2.2) for a description of it). The assertions then follow from the arithmetic behavior of $\big(\frac{n-1}{2}\big)!$ in $\mathbb{Z}/(\ell^{\frac{n+1}{2}} - 1)$ at p.

c) If X is a simple space of finite type, p a prime and $\eta : X \longrightarrow X^\wedge_p$ the p-adic completion of X, the homomorphism $\eta_* : H_*(X;\mathbb{Z}) \longrightarrow H_*(X^\wedge_p;\mathbb{Z})$ restricts to an isomorphism from the p-torsion subgroup of $H_*(X;\mathbb{Z})$ onto the p-torsion subgroup of $H_*(X_p^\wedge;\mathbb{Z})$. The same is also true for homotopy groups. For us, this means that using the equivalence $\phi : K\mathbb{Z}_2^\wedge \longrightarrow JK\mathbb{Z}_2^\wedge$, we can just read off the Hurewicz map of $K\mathbb{Z}$ on 2-torsion elements from the Hurewicz map h' of $JK\mathbb{Z}(3)$.

The Eilenberg-Mac Lane space $K(\mathbb{Z}/2,1)$ splits off $JK\mathbb{Z}(3)$, so by the Hurewicz Theorem, h' must be an isomorphism in dimensions 1 and 2, and surjective in dimension 3.

The classes $\omega_{2,n}$ with $n \equiv 3 \bmod (4)$ correspond to classes of $\pi_n\big(JK\mathbb{Z}(3)\big)$ coming from $\pi_n(SU)$. Their image under the Hurewicz homomorphism can therefore be calculated as for the odd-p-torsion classes $\omega_{p,n}$ in part b) of this proof.

Choose $n \geq 9$ satisfying $n \equiv 1, 2 \bmod (8)$, and let us show that the class $\omega'_{2,n} \in \pi_n\big(JK\mathbb{Z}(3)\big)$ corresponding to $\omega_{2,n}$ is in the kernel of h'. Consider the following diagram

$$\begin{array}{ccc} \pi_{n+1}(JK\mathbb{Z}(3);\mathbb{Z}/2) & \xrightarrow{d_*} & \pi_n\big(JK\mathbb{Z}(3)\big) \\ \bar{h}' \downarrow & & h' \downarrow \\ H_{n+1}(JK\mathbb{Z}(3);\mathbb{Z}/2) & \xrightarrow{d_*} & H_n(JK\mathbb{Z}(3);\mathbb{Z}) \end{array}$$

where d_* denotes the connecting homomorphism associated to the coefficient exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2 \to 0$. It is commutative (see [13], Lemma 3.2). The class $\omega'_{2,n} \in \pi_n\big(JK\mathbb{Z}(3)\big)$ is of order 2 and is in the image of d_*, so it suffices to show that the mod 2 Hurewicz homomorphism $\bar{h}'$ is trivial in dimension $n+1$.

Consider the mod 2 Moore space $P^{n+1}(2) = S^n/2$. By definition, an element α in $\pi_{n+1}(JK\mathbb{Z}(3);\mathbb{Z}/2)$ is the homotopy class of a map $\alpha : P^{n+1}(2) \longrightarrow JK\mathbb{Z}(3)$, and $\bar{h}(\alpha)$ is defined as $\alpha_*(e)$, where α_* is the homomorphism induced by α in mod 2 homology, and where e is the generator of $H_{n+1}(P^{n+1}(2);\mathbb{Z}/2) \cong \mathbb{Z}/2$. We claim that any such induced homomorphism α_* is zero. By duality, it is equivalent to prove the corresponding statement in mod 2 cohomology. There exists an isomorphism of Hopf algebras and of modules over the Steenrod algebra

$$H^*(JK\mathbb{Z}(3);\mathbb{Z}/2) \cong H^*(BO;\mathbb{Z}/2) \otimes H^*(SU;\mathbb{Z}/2)$$

(see [12], Remark 4.5). Recall the isomorphisms $H^*(BO;\mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, w_2, \dots]$ and $H^*(SU;\mathbb{Z}/2) \cong \bigwedge_{\mathbb{Z}/2}(e_2, e_3, \dots)$, where w_i is the Stiefel-Whitney class of degree i, and e_i is primitive of degree $2i-1$. The action of the Steenrod algebra on these cohomology classes is well known. For instance, $Sq^1(w_i) = w_{i+1} + w_1 w_i$ and $Sq^2(w_i) = w_{i+2} + w_2 w_i$ if i is even, and $Sq^{2k} e_i = \binom{i-1}{k} e_{i+k}$. These relations, as well as the fact that $H^*(P^{n+1}(2);\mathbb{Z}/2)$ is concentrated in dimensions 0, n and $n+1$, force any induced homomorphism $H^{n+1}(JK\mathbb{Z}(3);\mathbb{Z}/2) \longrightarrow H^{n+1}(P^{n+1}(2);\mathbb{Z}/2)$ to be zero for the above choices of n. □

4. The order of the Postnikov invariants of $K\mathbb{Z}$

Let X be a connected simple space, for instance a connected H-space. For any integer $n \geq 1$, let us denote by $X \longrightarrow X[n]$ the nth-Postnikov section of X, and by k_X^{n+1} the $(n+1)$th-Postnikov invariant of X. Recall that k_X^{n+1} is an element of the cohomology group $H^{n+1}\big(X[n-1];\pi_n(X)\big)$, which can be chosen canonically as the image of the fundamental class $u_X^{n+1} \in H^{n+1}\big(X[n-1],X;\pi_n(X)\big)$ under the homomorphism induced by the inclusion of pairs $(X[n-1],\emptyset) \hookrightarrow (X[n-1],X)$. The Postnikov invariant k_X^{n+1} corresponds to a map $X[n-1] \longrightarrow K(\pi_n X, n+1)$ whose homotopy fiber is the nth-Postnikov section $X[n]$ of X.

If X is an H-space of finite type, all its Postnikov invariants are cohomology classes of finite order : this is the Arkowicz-Curjel Theorem ([2]). In particular, the Postnikov invariant of $K\mathbb{Z}$ are of finite order. The orders ρ_n of the Postnikov invariants $k_{K\mathbb{Z}}^{n+1}$ of $K\mathbb{Z}$ have previously been studied by Arlettaz and Banaszak in [3]. See especially their Proposition 5, which states that if $n \geq 5$ is an integer with $n \equiv 1 \bmod (4)$ and if $K_n\mathbb{Z}$ has no p-torsion, where p is an odd prime, then $v_p(\rho_n) \leq v_p\left(\frac{n-1}{2}!\right)$.

Theorem 4.1. *For any integer $n \geq 2$, the order ρ_n of the Postnikov invariant $k_{K\mathbb{Z}}^{n+1}$ of $K\mathbb{Z}$ verifies :*
a)

$$v_2(\rho_n) = \begin{cases} 1 & \textit{if } n = 3, 7, \textit{ or if } n \geq 10 \textit{ and } n \equiv 2 \bmod (8), \\ v_2(\frac{n-1}{2}!) & \textit{if } n \equiv 1 \bmod (4), \\ 4 & \textit{if } n \geq 11 \textit{ and } n \equiv 3 \bmod (8), \textit{ or if } n = 15, \\ v_2(n+1)+1 & \textit{if } n \geq 23 \textit{ and } n \equiv 7 \bmod (8), \\ 0 & \textit{otherwise.} \end{cases}$$

b) *If p is a Vandiver prime, and if $n \geq 5$ is an integer with $n \equiv 1 \bmod (4)$, then*

$$v_p(\rho_n) \geq v_p\left(\tfrac{n-1}{2}!\right),$$

and equality holds if the order e of the torsion subgroup of $K_n\mathbb{Z}$ verifies $v_p(e) \leq v_p\left(\frac{n-1}{2}!\right)$.
c) *Let p be an odd prime. If $2(p-1)$ is a proper divisor of $n+1$, then*

$$v_p(\rho_n) \geq v_p(n+1)+1$$

(except if $p = 3$ and $n = 11$, where $v_3(\rho_{11}) \geq 1$ holds).

Proof. This is a consequence of Theorem 3.2, using the following general argument. If X is a connected simple space, n an integer ≥ 2, and ρ an integer ≥ 1, then the following statements are equivalent :
a) The Postnikov invariant k_X^{n+1} verifies $\rho k_X^{n+1} = 0$ in $H^{n+1}\big(X[n-1],\pi_n(X)\big)$.
b) There exists a homomorphism $g : H_n(X;\mathbb{Z}) \longrightarrow \pi_n(X)$ the composition gh : $\pi_n(X) \longrightarrow \pi_n(X)$ is multiplication by ρ, where $h : \pi_n(X) \longrightarrow H_n(X;\mathbb{Z})$ is the Hurewicz homomorphism. □

Remark 4.2. If p is a regular prime and if the p-adic Quillen-Lichtenbaum Conjecture for $\mathbb{Z}$ holds, then equality for $v_p(\rho_n)$ holds in the inequalities b) and c) of Theorem 4.1, and for other values of n, $v_p(\rho_n) = 0$.

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