1. The Chromatic Red-Shift in algebraic K-theory

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The algebraic K-theory of the sphere spectrum $\mathbf{S}$ is of interest in geometric topology, by Waldhausen's stable parametrized h-cobordism theorem [WJR] (ca. 1979). We wish to understand $K\mathbf{S}$ like we understand $K\mathbf{Z}$, via Galois descent. As a building block, the algebraic K-theory of the Bousfield localization $L_{K(n)}\mathbf{S}$ of $\mathbf{S}$ with respect to the n-th Morava K-theory $K(n)$ might be more accessible. John has developed a theory of Galois extensions for $\mathbf{S}$-algebras, and in this framework he has stated extensions of the Lichtenbaum-Quillen conjectures. Their precise formulation is distilled from the clues provided by our computations of the algebraic K-theory of topological K-theory and related spectra, and it is to be expected that they will keep maturing in a cask of skepticism for a few years. Writing X^{hG} for the homotopy fixed-point spectrum of a finite group G acting on a spectrum X, we recall:

Definition 1.1 ([Ro]). A map $A \to B$ of commutative $\mathbf{S}$-algebras is a *$K(n)$-local G-Galois extension* if G acts on B through commutative A-algebra maps, and the canonical maps $A \to B^{hG}$ and $B \wedge_A B \to \prod_G B$ are $K(n)$-equivalences.

Let E_n be Morava's E-theory [GH] with coefficients given by $(E_n)_* = W(\mathbf{F}_{p^n})[[u_1, \dots, u_{n-1}]][u^{\pm 1}]$. Then $L_{K(n)}\mathbf{S} \to E_n$ is an example of a $K(n)$-local pro-Galois extension. Let V be a finite CW-spectrum of chromatic type $n+1$, and let $T = v_{n+1}^{-1}V$ be the mapping telescope of its essentially unique v_{n+1}-self-map. For $n = 0$ take $V = V(0) = \mathbf{S}/p$ (the Moore spectrum), and for $n = 1$, $p \geq 3$ take $V = V(1) = V(0)/v_1$.

Conjecture 1.2. *Let $A \to B$ be a $K(n)$-local G-Galois extension. Then there is a homotopy equivalence $T \wedge KA \to T \wedge (KB)^{hG}$.*

For $n = 0$, $A \to B$ is a G-Galois extension of commutative $\mathbf{Q}$-algebras, and conjecture 1.2 is the descent conjecture of Lichtenbaum-Quillen (1973). For $n = 1$, conjecture 1.2 holds by [Au], [AR1], [BM] for the $K(1)$-local $\mathbf{F}_p^\times$-Galois extension $L_p \to KU_p$, where KU_p is the p-complete periodic K-theory spectrum and L_p its Adams summand.

CONJECTURE 1.3. *Let B be a suitably finite $K(n)$-local commutative $\mathbf{S}$-algebra (for example $L_{K(n)}\mathbf{S} \to B$ could be a G-Galois extension). Then the map $V \wedge KB \to T \wedge KB$ induces an isomorphism on homotopy groups in sufficiently high degrees.*

If $n = 0$ and $B = HF$ for a reasonable field F, then $V \wedge KF = K(F;\mathbf{Z}/p) \to T \wedge KF \simeq K^{\text{ét}}(F;\mathbf{Z}/p)$ induces an isomorphism on homotopy groups in sufficiently high degrees by Thomason's theorem (1985). For $n = 1$, $p \geq 5$ and $B = L_p$, KU_p or their connective versions ℓ_p and ku_p, it is known ([AR1], [BM]) that $V(1)_*KB$ is a finitely generated free $\mathbf{F}_p[v_2]$-module in high degrees, hence conjecture 1.3 holds for these $\mathbf{S}$-algebras. This is evidence for the "red-shift conjecture", which, in a less precise formulation than conjecture 1.3, asserts that algebraic K-theory increases chromatic complexity by one.

The algebraic K-theory of a ring of integers $\mathcal{O}_F$ in a number field F can be computed from the K-theory of its residue fields and the fraction field F, by a localization sequence. To compute $K(F;\mathbf{Z}/p)$ one uses Suslin's theorem (1983) that $K(\bar{F};\mathbf{Z}/p) \simeq V(0) \wedge ku$, and descent with respect to the absolute Galois group G_F. To generalize this program we wish to make sense of the $K(n)$-local $\mathbf{S}$-algebraic fraction field $\mathcal{F}$ of $L_{K(n)}\mathbf{S}$ (or one of its pro-Galois extensions), construct a separably closed extension Ω_n, and evaluate its algebraic K-theory.

CONJECTURE 1.4. *If Ω_n is a separable closure of the fraction field of $L_{K(n)}\mathbf{S}$, then there is a homotopy equivalence $L_{K(n+1)}K(\Omega_n) \simeq E_{n+1}$.*

For $n = 0$ this reduces to $L_{K(1)}K(\bar{\mathbf{Q}}_p) \simeq E_1 \simeq KU_p$, a weaker formulation of Suslin's theorem. For $n = 1$ we did some computations [AR2] aimed at understanding what the fraction field $\mathcal{F}$ of KU_p might be. We *define* $K\mathcal{F}$ to sit in a hypothetical localization sequence $K(KU/p) \to K(KU_p) \to K\mathcal{F}$, as the cofiber of the transfer map for $KU_p \to KU/p$. The result is that $V(1)_*K\mathcal{F}$ is, in high enough degrees, a free $\mathbf{F}_p[v_2]$-module on $2(p^2+3)(p-1)$ generators. In particular $\mathcal{F}$ cannot be the $H\mathbf{Q}_p$-algebra $KU_p[1/p]$. We rather believe that $\mathcal{F}$ is an $\mathbf{S}$-algebraic analogue of a two-dimensional local field. For example, there appears to be a perfect arithmetic duality pairing in the Galois cohomology of $\mathcal{F}$, analogous to Tate-Poitou duality (1963) for local number fields.

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