AN INTRODUCTION TO ALGEBRAIC K-THEORY

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Abstract. These are the notes of an introductory lecture given at The 20th Winter School for Geometry and Physics, at Srni. It was meant as a leisurely exposition of classical aspects of algebraic K-theory, with some of its applications to geometry and topology.

Introduction

Classically, algebraic K-theory of rings is the study of the family of K-theory functors

$$K_n : \mathit{Rings} \longrightarrow \mathit{Abelian\ Groups} \quad (n = 0, 1, 2).$$

For a given ring R, the groups K_0R, K_1R and K_2R were defined, around the 60's, in purely algebraic terms, and are closely related to classical invariants of rings. It soon became apparent that these functors were part of a kind of homology theory for rings, but no algebraic definition of higher K-groups K_3, $K_4, \dots$ was found. In the early 70's, D. Quillen came up with a definition that requires the use of *homotopy theory*. He defined the group K_nR as the n-th homotopy group of a certain *algebraic K-theory space* KR :

$$K_nR = \pi_n(KR) \quad (n = 0, 1, 2, \dots).$$

Although its construction is quite obscure, the space KR has very nice properties, and Quillen's definition of K_nR, which agrees with the classical one if $n = 0, 1, 2$, was immediately recognized as the proper extension of the classical K-theory functors. Many efforts have been made not only to compute the homotopy groups K_nR but also, and perhaps more fundamental, to better understand the structure of space KR itself. This is the task of *higher algebraic K-theory*. Later on, the definition of algebraic K-theory was extended by F. Waldhausen to a certain type of rings up to homotopy, called *brave new rings*. Algebraic K-theory of brave new rings provides a very interesting link between algebraic K-theory of rings and geometry. This note is divided into four parts : Classical K-theory, Quillen's higher K-theory, K-theory of brave new rings, and a short appendix collecting the few notions of homotopy theory used in the paper.

Classical K-theory

Let us first review the algebraic definitions of K_0R, K_1R and K_2R, and discuss some examples. The main sources used for this section are [12], [13] and [22], where many further references are provided.

K_0 of a ring.
Suppose $(S,*)$ is a commutative semi-group, i.e. a set S with a commutative and associative composition law $*$. The Grothendieck group $Gr(S) = Gr(S,*)$ of $(S,*)$ is the quotient of the free abelian group $F(S)$ generated by S modulo relations generated by $(s*t) - s - t$ for all $s,t \in S$. There is a canonical homomorphism of semi-groups $\iota : S \longrightarrow Gr(S)$ induced by the inclusion of S in $F(S)$, and it is universal in the obvious sense. Notice that ι is injective if and only if cancellation holds in $(S,*)$.

Let R be an associative and unital ring. Recall that a R-module P is called projective if any surjection $M \twoheadrightarrow P$ of R-modules admits a section, or equivalently if P is a direct summand of a free R-module. A R-module M is of finite type if it admits a finite number of generators over R.

Consider the semi-group $(\mathrm{Proj}(R),\oplus)$ of isomorphism classes of projective left R-modules of finite type, with the direct sum. Define the zeroth algebraic K-theory group of R as

$$K_0R = Gr(\mathrm{Proj}(R),\oplus).$$

There is a homomorphism of groups $\mathbb{Z} \longrightarrow K_0R$ given by $n \longmapsto n[R]$. It is not always injective. The reduced K-theory group of R is the quotient $\widetilde{K}_0R = (K_0R)/i(\mathbb{Z})$. In other words, we have killed in $\widetilde{K}_0R$ the "obvious" elements, which are the free modules.

Examples.
a) If R is a commutative ring, then $K_0R = \mathbb{Z} \oplus \widetilde{K}_0R$.
b) If k is a field, V a k-vector space of infinite countable dimension, and $R = \mathrm{End}_k(V)$ is the ring of endomorphisms of V, then $K_0R = 0$.
c) If R is a principal ideal domain, or a local ring, then any projective R-module of finite type is free and is characterized up to isomorphism by its rank. Therefore $K_0R = \mathbb{Z}$.

Projective modules occur naturally in different areas of mathematics. One of the oldest example is as ideals in Dedekind domains. Recall that a Dedekind domain is a commutative ring with no zero divisors, such that for any pair of ideals $\mathfrak{a} \subset \mathfrak{b}$, there exists an ideal $\mathfrak{c}$ with $\mathfrak{a} = \mathfrak{b}\mathfrak{c}$. Two ideals $\mathfrak{a},\mathfrak{b}$ of a Dedekind domain belong to the same *ideal class* if $x\mathfrak{a} = y\mathfrak{b}$ for some non zero elements x, y in R. The ideal classes form an abelian group under the multiplication of ideals, by the defining property of Dedekind domains, and the unit is the class of principal ideals. This group is called the *ideal class group* of R and is denoted by $\mathrm{Cl}(R)$.

Theorem [12]. *If R is a Dedekind domain, any ideal of R is a projective module of finite type. Conversely, every finitely generated projective R-module is isomorphic to a direct sum $\mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_\ell$ of ideals. There is an isomorphism*

$$K_0(R) \xrightarrow{\cong} \mathbb{Z} \oplus \mathrm{Cl}(R),$$
$$[\mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_\ell] \longmapsto (\ell, [\mathfrak{a}_1 \ldots \mathfrak{a}_\ell]).$$

A fundamental example of Dedekind domain is the ring of algebraic integers $\mathcal{O}_F$ in a number field F (i.e. a finite extension of $\mathbb{Q}$). It is known that the class group of such a ring is always finite, and measures for instance how far from a unique factorization domain the ring is.

Example. The ring of algebraic integers of $\mathbb{Q}[\sqrt{-5}]$ is $\mathbb{Z}[\sqrt{-5}]$. Here $\mathrm{Cl}(\mathbb{Z}[\sqrt{-5}]) = \mathbb{Z}/2$, admitting the ideal $(3, 2+\sqrt{-5})$ as generator.

The class group of the ring of integers $\mathbb{Z}[\xi_p]$ of the cyclotomic number field $\mathbb{Q}[\xi_p]$ (here p is an odd prime and ξ_p is a primitive p-th root of 1) was already studied by Kummer in his work on Fermat's Last Theorem. For instance, if p is a regular prime, that is if p does not divide the order of $\mathrm{Cl}(\mathbb{Z}[\xi_p])$, the first case of Fermat's Theorem can be proven by an elementary arithmetic argument (see [31]).

Topological K-theory and Swan's Theorem.
Topological K-theory provides a very nice example of projective modules occurring in geometry. Let X be a compact Hausdorff space, and $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$. A $\mathbb{F}$-vector bundle of rank n over X is a triple $\xi = (E, p, X)$, where $p : E \longrightarrow X$ is a continuous surjection of topological spaces such that :
a) for all $x \in X$, $p^{-1}(\{x\})$ is a n-dimensional $\mathbb{F}$-vector space, and
b) there exists an open cover $\{U_\alpha\}$ of X and homeomorphisms $\phi_\alpha : U_\alpha \times \mathbb{F}^n \longrightarrow p^{-1}(U_\alpha)$ such that $p\phi_\alpha$ is the projection $U_\alpha \times \mathbb{F}^n \longrightarrow U_\alpha$, and such that $\phi_\alpha : \{x\} \times \mathbb{F}^n \longrightarrow p^{-1}(\{x\})$ is a linear isomorphism.

Typical examples of $\mathbb{F}$-vector bundles of rank n are the trivial fibre bundle $(X \times \mathbb{F}^n, p_1, X)$, or the tangent bundle (TM, π, M) to a differentiable manifold M of dimension n.

If $\xi = (E, p, X)$ and $\xi' = (E', p', X)$ are two $\mathbb{F}$-vector bundles over X, a morphism $f : \xi \longrightarrow \xi'$ is a map $f : E \longrightarrow E'$ such that $p'f = p$ and $f : p^{-1}(\{x\}) \longrightarrow p'^{-1}(\{x\})$ is linear. The Whitney sum $\xi \oplus \xi' = (E'', p'', X)$ of ξ and ξ' is a vector bundle over X having $E'' = \{(e, e') \in E \times E' | p(e) = p'(e')\}$ as total space, with the obvious projection p'' to X. Consider the semi-group $(\mathrm{Vect}_{\mathbb{F}}(X), \oplus)$ of isomorphism classes of $\mathbb{F}$-vector bundles of finite rank over X. Its Grothendieck group is denoted $K^0_{\mathbb{F}}X$ and is called the topological $K_{\mathbb{F}}$-theory of X. Swan's Theorem asserts that there is an isomorphism

$$\psi : K^0_{\mathbb{F}}X \xrightarrow{\cong} K_0C(X, \mathbb{F}),$$

where $C(X, \mathbb{F})$ is the ring of continuous functions $X \longrightarrow \mathbb{F}$. This isomorphism is induced by $\xi = (E, p, X) \longmapsto \Gamma(\xi)$, where $\Gamma(\xi)$ is the set of sections of p, i.e. of continuous maps $\gamma : X \longrightarrow E$ such that $p\gamma = \mathrm{id}_X$. There is an obvious $C(X, \mathbb{F})$-module structure on $\Gamma(\xi)$, making it a module of finite type (by compactness of X). The projectivity of $\Gamma(\xi)$ follows from the fact that there exists a vector bundle ξ' such that $\xi \oplus \xi'$ is a trivial bundle. For instance, if ξ is the tangent bundle to a manifold, ξ' can be chosen as its normal bundle. See [2] for more on topological K-theory.

K_1 of a ring.
The group K_0 deals with classifying projective modules of finite type, while the groups K_i for $i \geq 1$ are tools to study the automorphisms group of such modules. Let GL_nR be the group of invertible matrices of size n with coefficients in R, and GLR the union $\bigcup_{n\geq 1} GL_nR$, where the inclusion $GL_nR \hookrightarrow GL_{n+1}R$ is given by $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. The group GLR is usually called *the general linear group*. Define K_1R as the abelianisation of GLR :

$$K_1R = (GLR)_{\mathrm{ab}} = GLR/[GLR; GLR].$$

The commutator subgroup $[GLR; GLR]$ is equal to the subgroup $ER \subset GLR$ generated by the set of elementary matrices $\{e_{ij}^r | r \in R \text{ and } i, j \in \mathbb{N}, i \neq j\}$. Here e_{ij}^r denotes the matrix whose diagonal elements are 1's, whose (i,j)-entry is r and whose other entries are 0's. Notice that the group ER is *perfect*, that is to say it is equal to its commutator subgroup. The inclusion $R^\times = GL_1 R \hookrightarrow GLR$ induces a homomorphism $R^\times \longrightarrow K_1 R$ that factorises over a homomorphism $j : R^\times_{\text{ab}} \longrightarrow K_1 R$.

Examples.
a) If R is commutative, the determinant provides a retraction to j, and therefore $R^\times$ splits off $K_1 R$. The quotient $(K_1 R)/R^\times$ is usually denoted $SK_1 R$ as a remainder of the isomorphism $SK_1 R \cong SLR/ER$, were $SLR \subset GLR$ is the *special linear group* of matrices of determinant 1. There is an isomorphism $K_1 R = R^\times \oplus SK_1 R$.
b) If R is a local ring (not necessarily commutative), then j is an isomorphism, and the homomorphism $GLR \longrightarrow K_1 R \cong R^\times_{\text{ab}}$ constitutes a genuine generalization of the determinant in the commutative case. Notice that, in particular, if R is a commutative local ring (eg. a field), then $SK_1 R = 0$.
c) If R is an Euclidean domain, then $SL_n R = E_n R$ for all n. It follows that $K_1 R = R^\times$. The same is not true for principal ideal domains, but the known counterexamples are complicated (see for instance [5]).

If R is a Dedekind domain, we know more on $SK_1 R$: it is generated by the image of the composite homomorphism $SL_2 R \hookrightarrow SLR \twoheadrightarrow SK_1 R$. The image of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is called a Mennicke symbol and denoted by $[a, b]$, because it depends only on a and b. Playing with matrices allows to produce several general relations among the Mennicke symbols. It also follows that if in addition R/P is a finite field for any non-trivial prime ideal P in R, then $SK_1 R$ is a torsion group.

In the particular case of a ring of integers $\mathcal{O}_F$ in a number field F, the group $SK_1 \mathcal{O}_F$ vanishes (see [12], Cor. 16.3), and we get that $K_1 \mathcal{O}_F = \mathcal{O}_F^\times$. This was computed by Dirichlet :

Theorem (Dirichlet). *If F is a number field having r_1 distinct real embeddings and r_2 distinct pairs of conjugate complex embeddings, then*

$$\mathcal{O}_F^\times \cong \mu(F) \oplus \mathbb{Z}^{r_1 + r_2 - 1},$$

where $\mu(F)$ is the finite cyclic group of roots of unity in F.

K_2 of a ring.

In K_1, we study the group GLR modulo the group ER. With K_2 at hand, it is possible to give a presentation of ER by generators and relations. For any integer $n \geq 3$, define the Steinberg group $St_n R$ of a ring R as having generators

$$\{x_{ij}^r | r \in R,\ 1 \leq i \neq j \leq n\}$$

and relations generated by

$$\begin{cases} x_{ij}^r x_{ij}^s = x_{ij}^{r+s} \\ [x_{ij}^r, x_{j\ell}^s] = x_{i\ell}^{rs} & \text{if } i \neq \ell, \\ [x_{ij}^r, x_{k\ell}^s] = 1 & \text{if } i \neq \ell \text{ and } j \neq k. \end{cases}$$

Recall the definition of elementary matrices e_{ij}^r given above. The correspondence $x_{ij}^r \longmapsto e_{ij}^r$ defines a surjective group homomorphism $\phi : St_nR \longrightarrow E_nR$, because the relations for the x_{ij}^r given above are obviously satisfied by the e_{ij}^r in E_nR. This homomorphism stabilizes to produce a homomorphism $\phi : StR \longrightarrow ER$, and define K_2R as

$$K_2R = \ker(\phi : StR \longrightarrow ER).$$

Hence K_2R consists of the non-trivial relations between elementary matrices seen as generators of ER. It is abelian, because ϕ is a central extension of ER. In fact, it is the universal central extension of the perfect group ER.

The group K_2R is usually quite complicated to compute. An effective way of constructing elements of K_2R is to search for non trivial matrices A, B in ER that commute with each other. Then, if x, y are pre-images of A and B by ϕ, the commutator $[x, y]$ lies in K_2R. In fact, $[x, y]$ does not depend on the choices of x and y since the extension ϕ is central. Let us consider for instance the case of a commutative ring R. Choose $u, v \in R^\times$, and define *the Steinberg symbol* $\{u, v\} \in K_2R$ by

$$\{u,v\} = \left[\phi^{-1}\begin{pmatrix} u & 0 & 0 \\ 0 & u^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \phi^{-1}\begin{pmatrix} v & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & v^{-1} \end{pmatrix}\right]$$

Example. $K_2\mathbb{Z} \cong \mathbb{Z}/2$, generated by $\{-1, -1\}$.

In the case of a field F, it is possible to describe K_2F in terms of Steinberg symbols :

Theorem (Matsumoto). *If F is field, the abelian group K_2F has a presentation by generators and relations as follows. It is generated by the Steinberg symbols $\{u, v\}$ for $(u, v) \in F^\times \times F^\times$, with relations generated by*

$$\begin{cases} \{u, 1-u\} = 1, \\ \{u_1u_2, v\} = \{u_1, v\}\{u_2, v\}, \\ \{u, v_1v_2\} = \{u, v_1\}\{u, v_2\}. \end{cases}$$

Example. If F is a finite field, then $K_2F = 0$.

The group K_2F of a field F is related to a classical invariant, namely the Brauer group $Br(F)$, which classifies the simple central F-algebras. If F contains a n-th primitive root ξ_n of 1, and if $\mathrm{char}(F)$ does not divide n, then define for any pair $(a, b) \in F^\times \times F^\times$ the simple central F-algebra $A(a, b)$ as generated by two elements x, y with relations generated by $x^n = a,\ y^n = b$ and $yx = \xi_n xy$. Using Matsumoto's Theorem, one can show that the map $A : F^\times \times F^\times \longrightarrow Br(F)$ factorises through a homomorphism $(K_2F)/n \longrightarrow {}_nBr(F)$, called the *power norm residue symbol*. Here ${}_nBr(F)$ denotes the n-torsion subgroup of $Br(F)$.

Theorem (Mercurjev-Suslin). *Suppose F is a field containing a n-th primitive root ξ_n of 1, and* $\mathrm{char}(F)$ *does not divide n. Then the power norm residue symbol $(K_2F)/n \longrightarrow {}_nBr(F)$ is an isomorphism.*

This Theorem was first proven by Tate in the case of number fields, and then by Mercurjev and Suslin, using higher K-theory. See for instance [24].

Milnor proposed a definition of higher algebraic K-theory for a field F in the spirit of Matsumoto's Theorem. Consider the graded tensor algebra $T(F^\times)$ over $\mathbb{Z}$, where an element of $F^\times$ has degree 1, and the ideal I in $T(F^\times)$ generated by elements of type $u \otimes (1-u)$ and $u \otimes (-u)$. Define $K_i^M F$ as the group of elements of degree i in the quotient $T(F^\times)/I$. Then $K_i^M F = K_i F$ for $i = 0, 1, 2$, but differs in general from Quillen's higher K-theory.

The constructions K_0, K_1 and K_2 are functorial : if $f : R \longrightarrow S$ is a ring homomorphism, then there is an induced homomorphism $f_* : K_i R \longrightarrow K_i S$. Its construction for $i = 1, 2$ is obvious. For K_0, it is given by $P \longmapsto S \otimes_R P$, the ring S being seen as right R-module via f. The functoriality is clear.

Finally, let us mention three applications of lower algebraic K-theory to topology and geometry. They are examples of problems concerning a space or manifold X, which are solved by means of obstructions lying in a quotient of a K-theory group. The ring involved is always the group ring $\mathbb{Z}[\pi]$ of the fundamental group π of X. K-groups of group rings are usually hard to compute. For instance, if π is cyclic of odd prime order p, then the K-theory of $\mathbb{Z}[\pi]$ is closely related to that of the cyclotomic integers $\mathbb{Z}[\xi_p]$, in the computation of which classical arithmetic questions are involved.

The Wall finiteness obstruction.
Let X be a topological space having the homotopy type of a CW-complex and whose fundamental group $\pi = \pi_1(X)$ is finitely presented, and let $S_*(X)$ be the singular chain complex of X. Suppose the $\mathbb{Z}[\pi]$-chain complex $S_*(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\pi]$ is homotopy equivalent to a chain complex C_* such that almost all C_i's are zero, and all C_i are projective $\mathbb{Z}[\pi]$-modules of finite type. This is equivalent to requiring that X is finitely dominated, which means that there exists a CW-complex Y with a finite number of cells, and maps $X \to Y \to X$ whose composite is homotopic to the identity of X. The Wall finiteness obstruction of X is the "Euler characteristic" $\chi(X) = \sum_{-\infty}^{+\infty} (-1)^i [C_i] \in K_0\mathbb{Z}[\pi]$, which does not depend on the choice of C_*. Then X is homotopy equivalent to a finite complex if and only if $\chi(X) = 0$ in the reduced K-group $\widetilde{K}_0\mathbb{Z}[\pi]$ (see [29] and [30]).

The Whitehead torsion.
Suppose given a pair of connected spaces $X \subset Y$ such that the inclusion is a homotopy equivalence, and denote by π the fundamental group of X. Under some extra assumptions, Whitehead defined an obstruction $\tau(Y, X)$ to this equivalence being *simple*, called the *Whitehead torsion* (see [32]). This obstruction lives in the group $Wh_1(\pi) = (K_1\mathbb{Z}[\pi])/j(\pi)$, where $j : (\mathbb{Z}[\pi])^\times \longrightarrow K_1\mathbb{Z}[\pi]$ is the homomorphism given above, and π is considered included in $(\mathbb{Z}[\pi])^\times$. The Whitehead torsion has a very nice application to geometry : it classifies h-cobordisms for closed manifolds of dimension ≥ 5. A h-cobordism between two smooth closed manifolds M and N of dimension n is a smooth compact manifold W of dimension $n+1$ having $M \coprod N$ as boundary, such that the inclusions $M \hookrightarrow W$ and $N \hookrightarrow W$ are weak deformation retracts (an inclusion $i : X \hookrightarrow Y$ is a *weak deformation retract* if there exists a map $r : Y \longrightarrow X$ with $ri = 1_X$ and ir being homotopic to 1_Y). The Barden-Mazur-Stallings Theorem (see [10]) states that if $n \geq 5$, then W is diffeomorphic to $M \times [0, 1]$ if and only if $\tau(W, M) = 0$ in $Wh_1\big(\pi_1(M)\big)$. In particular, the vanishing of $\tau(W, M)$ implies that M and N are diffeomorphic. As said above, K-groups

of group rings are difficult to compute, but if π is finite, then $Wh_1(\pi)$ is known (see [15]).

The Hatcher-Wagoner Theorem.
Consider a differentiable manifold M and its group $\mathrm{Diff}(M)$ of diffeomorphisms. An element of $\pi_0\mathrm{Diff}(M)$ is called an *isotopy class* of diffeomorphisms of M. Two diffeomorphisms h_0 and h_1 are called pseudo-isotopic if there exists a diffeomorphism h of $M \times [0,1]$ restricting to h_i on $M \times \{i\}$, $i = 0, 1$. Let $P(M)$ be the space of pseudo-isotopies h of M restricting to the identity on $M \times \{0\}$. If $\pi_0 P(M) = 0$, then any pseudo-isotopic diffeomorphisms h_0 and h_1 of M are actually isotopic. Let W_π be the subgroup of $St\mathbb{Z}[\pi]$ generated by elements of type x_{ij}^g for $g \in \pi$, and define the second Whitehead group of π by $Wh_2\pi = (K_2\mathbb{Z}[\pi])/(K_2\mathbb{Z}[\pi] \cap W_\pi)$. The Hatcher-Wagoner Theorem [6] states that if M is a smooth closed manifold of dimension ≥ 5 and with π as fundamental group, then there exists a surjection $\pi_0 P(M) \longrightarrow Wh_2\pi$. There also exists a formula for the kernel of this surjection, the corrected version of which is given in [7].

Quillen's higher K-theory

What was the motivation for calling the various groups defined above K_0, K_1 and K_2, as if they were related, and why should one want to generalize them to higher K_n's ? In fact, several results suggested that the groups K_0, K_1 and K_2 were the first of a quite mysterious "homology theory" for rings. Among such results is the existence of products. If R is a commutative ring, there exist products

$$\star : K_iR \otimes K_jR \longrightarrow K_{i+j}R, \ (0 \leq i, j \leq 2, \ i + j \leq 2).$$

For instance, if $i = j = 0$, the product is induced by the tensor product over R of R-modules. In the case of a field F, these products coincide with the products induced by the concatenation product on $T(F^\times)$. Another motivating result was the existence of exact sequences (a sequence of groups and homomorphisms is exact if at every group, the image of the incoming homomorphism is equal to the kernel of the outgoing homomorphism). For instance, if R is a Dedekind domain with field of fractions F, then there exists an exact sequence

$$K_2R \to K_2F \to \bigoplus_{\mathfrak{m}} K_1(R/\mathfrak{m}) \to K_1R \to K_1F \to \bigoplus_{\mathfrak{m}} K_0(R/\mathfrak{m}) \to K_0R \to K_0F$$

where $\mathfrak{m}$ runs over the non-zero maximal ideals of R .

There were several attempts to define higher K-groups in order to extend the products or exact sequences, with the hope to discover new invariants. Negative K-groups were defined algebraically in [3], but are trivial for many interesting rings, such as for instance Dedekind domains. A construction of higher K-groups was achieved by D. Quillen [17] in the early 70's. It requires the use of homotopy theory, applied to the K-theory space KR. One of the striking features of the space KR is that it admits several very different but equivalent constructions (see [9] for an overview). This fact is essential for many results in K-theory. We are going to describe only one of them, called *the group completion*. However, all these constructions use *the classifying space* construction, which is a link between algebra and topology. We will first describe it.

Simplicial sets and classifying spaces.
Define the standard n-simplex Δ_n as the topological space

$$\Delta_n = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \;\middle|\; 0 \le t_0, \dots, t_n \le 1,\ \sum t_i = 1 \right\} \subset \mathbb{R}^{n+1}$$

with the induced topology. A simplicial set is some combinatorial data that allows to glue together simplices in a coherent way, in order to get a "nice" topological space. Consider the category Δ whose objects are ordered sets $[n] = \{0, 1, \dots n\}$, and where the morphisms from $[n]$ to $[m]$ are all non-decreasing maps. In fact, all morphisms factorise as composition of the following two kind of morphisms :

$$\delta_i : [n] \longrightarrow [n+1], (0 \le i \le n+1),\ \text{whose image is } [n+1] \setminus \{i\},$$
$$\sigma_i : [n] \longrightarrow [n-1], (0 \le i \le n-1),\ \text{surjective with } \sigma_i(i) = \sigma_i(i+1) = i.$$

These morphisms satisfy some obvious relations, and any morphism $\alpha : [m] \longrightarrow [n]$ of Δ can be written in a unique way as $\alpha = \delta_{i_1} \dots \delta_{i_p} \sigma_{j_1} \dots \sigma_{j_q}$, where $\{i_p < \dots < i_1\} = [n] \setminus \alpha([m])$ and $j_1 < \dots < j_q$ are the elements j of $[m]$ with $\alpha(j) = \alpha(j+1)$.

There is a functor $F : \Delta \longrightarrow Top$, the category of topological spaces, given by $[n] \longmapsto \Delta_n$, with $F(\delta_i)(t_0, \dots, t_n) = (\dots, t_{i-1}, 0, t_i, \dots)$ and $F(\sigma_i)(t_0, \dots, t_n) = (\dots, t_i + t_{i+1}, \dots)$.

A *simplicial set* $X_\bullet$ is a contra-variant functor $X_\bullet : \Delta \longrightarrow Sets$. The set $X_n = X_\bullet([n])$ is usually called the set of n-simplices of X. It's *geometric realization* $|X_\bullet|$ is the topological space

$$|X_\bullet| = \left(\coprod_{n \ge 0} X_n \times \Delta_n \right) / \sim$$

where the equivalence relation $\sim$ is generated by $\big(X_\bullet(\alpha)(x), t\big) = \big(x, F(\alpha)(t)\big)$ for any morphism $\alpha : [m] \longrightarrow [n]$ of Δ and any pair $(x, t) \in X_n \times \Delta_m$. The topology is the quotient topology, each X_n having the discrete topology. The space $|X_\bullet|$ is a CW-complex having one cell of dimension n for each non-degenerate simplex $x \in X_n$ (a simplex $y \in X_n$ is *degenerated* if it lies in the image of some $X(\sigma_i)$). It is possible to have extra structure by considering simplicial objects in other categories then $Sets$. For instance, *a simplicial space* is a contra-variant functor $X_\bullet : \Delta \longrightarrow Top$. By forming the geometric realization of a simplicial space, you take into account the topology of each X_n.

If $X_\bullet$, $Y_\bullet$ are simplicial objects, a simplicial map $f : X_\bullet \longrightarrow Y_\bullet$ is just a natural transformation of functors, and induces a continuous map $f : |X_\bullet| \longrightarrow |Y_\bullet|$. It is easy to verify that the geometric realization is a functor from the category of simplicial sets to the category of compactly generated topological spaces.

If $\mathcal{C}$ is a small category (i.e. whose objects form a set), you can associate to it a simplicial set $N_\bullet\mathcal{C}$ called *the nerve* of $\mathcal{C}$. Define $N_n\mathcal{C}$ as the set of functors $x : [n] \longrightarrow \mathcal{C}$. Here, $[n]$ is seen as a category whose objects are elements of $[n]$ and with a unique morphism $a \longrightarrow b$ if and only if $a \le b$. If α is a morphism of Δ, define $N_\bullet\mathcal{C}(\alpha)(x) = x \circ \alpha$. The geometric realization

$$B\mathcal{C} = |N_\bullet\mathcal{C}|$$

is called *the classifying space* of $\mathcal{C}$.

Examples.
a) Consider the small category $[n]$. Then $B[n]$ is homeomorphic to Δ_n.
b) A discrete group G can be seen as a category with one object 0, and where the set of morphisms $0 \longrightarrow 0$ is G, composition being given by multiplication. Then BG is the classifying space of G in the classical sense, and is characterized by the fact that $\pi_1(BG) = G$ and the other homotopy groups of BG are trivial.

Group completions.
We refer to the appendix for basic notions of homotopy theory and for notations. Let M be a homotopy commutative topological monoid. Recall that a monoid is a set with an associative composition law, and a unit. The classifying space BM of M is the classifying space of M seen as a category with one object, in a similar way as example b) above. The topology of M produces a topology on the n-simplices $N_nM = M^n$, so the nerve $N_\bullet M$ is a simplicial space.

Since $N_1M = M$, the 1-skeleton $(BM)^1$ of BM is homeomorphic to the suspension ΣM. The *group completion* of M is the map

$$\iota : M \longrightarrow \Omega BM$$

adjoint to the composition $\Sigma M \cong (BM)^1 \hookrightarrow BM$. This construction is the topological analogue to the Grothendieck group completion Gr discussed above in the case of commutative semi-group : there is an isomorphism $Gr(\pi_0 M) \cong \pi_0(\Omega BM)$. If M is *group-like*, i.e. $\pi_0 M$ is a group, then ι is a homotopy equivalence. Consider the direct system $\{M_\alpha | \alpha \in \pi_0(M)\}$, where M_α is the arc-wise connected component of M corresponding to α, and with maps $M_\beta \longrightarrow M_{\alpha\beta}$ given by left translation by α. Since all arc-wise connected components of ΩBM are homotopy equivalent, there are maps $\iota_\alpha : M_\alpha \longrightarrow \Omega_0 BM$ compatible with the direct system.

Group Completion Theorem [1]. *If M is a homotopy commutative topological monoid, then the maps $(\iota_\alpha)_* : H_*(M_\alpha; \mathbb{Z}) \longrightarrow H_*(\Omega_0 BM; \mathbb{Z})$ induce an isomorphism*

$$\varinjlim_{\pi_0 M} H_*(M_\alpha; \mathbb{Z}) \xrightarrow{\cong} H_*(\Omega_0 BM; \mathbb{Z}).$$

The K-theory space KR.
Recall that K_0R was about classifying isomorphism classes of projective modules of finite type, while K_1R and K_2R were defined to study the automorphisms groups of such modules. The following construction of KR is inspired from those tasks. Let P be a projective R-module of finite type, and $B\mathrm{Aut}(P)$ the classifying space of its (discrete) group of automorphisms. Consider the topological monoid $M_R = \coprod_P B\mathrm{Aut}(P)$, where P runs over a set of representatives of isomorphism classes of projective R-modules of finite type. The product on M_R is induced by the injective group homomorphisms $\mathrm{Aut}(P) \times \mathrm{Aut}(Q) \hookrightarrow \mathrm{Aut}(P \oplus Q)$. The K-theory space KR is the group completion of this topological monoid:

$$KR = \Omega BM_R$$

With this achieved, Quillen defined

$$K_n^Q R = \pi_n(KR),\ (n = 0, 1, 2, \dots).$$

The first sign that this is a good definition is that $\pi_0(KR) = Gr(\pi_0 M_R) = Gr(\mathrm{Proj}(R), \oplus) = K_0R$. The Group Completion Theorem gives a nice description of the homology groups of $(KR)_0 = \Omega_0 BM_R$. Notice that in the direct system $\{\mathrm{Aut}(P)\}$ indexed by $\mathrm{Proj}(R)$, the groups $\mathrm{Aut}(R^n) \cong GL_nR$ for $n \geq 1$ form a cofinal system, i.e. each $\mathrm{Aut}(P)$ is mapped into a $\mathrm{Aut}(R^n)$. This implies that $\varinjlim \mathrm{Aut}(P) = \varinjlim GL_nR = GLR$. Now filtered direct limits commute with the classifying space construction and homology. Therefore the Group Completion Theorem gives an isomorphism

$$H_*\big((KR)_0; \mathbb{Z}\big) = H_*(BGLR; \mathbb{Z}) = H_*(GLR; \mathbb{Z})$$

Here $H_*(GLR; \mathbb{Z})$ is the group homology of GLR, and the last equality can be taken as a definition. The homology of groups is a rich example of interplay between algebra and topology. This isomorphism is very important in algebraic K-theory, for instance because of the existence of the Hurewicz homomorphism, interpreted in this context as a homomorphism

$$h_n : K_n^Q R \longrightarrow H_n(GLR; \mathbb{Z}),\ (n = 1, 2, \dots).$$

Since $(KR)_0$ is an H-space, h_1 is an isomorphism. The first homology group of a group is its abelianisation, and so we obtain the desired isomorphism $K_1^Q R = \pi_1\big((KR)_0\big) = H_1(GLR; \mathbb{Z}) = GLR_{\mathrm{ab}} = K_1R$. In a similar way, one can prove that $\pi_2(KR)_0 = K_2R$ by using h_2 and the identity $K_2R = H_2(ER; \mathbb{Z})$. Quillen's definition of K_n^Q is compatible with the classical definitions in dimensions 0,1 and 2. We will from now write K_n for K_n^Q.

This definition of higher K-theory has very nice properties, and was immediately accepted as the good generalization of classical K-theory. Most of these properties are proven at the space level. Consider for example the extension of the product mentioned above. As for K_0, in the case of a commutative ring R, the tensor product of modules yields a continuous map

$$\star : KR \wedge KR \longrightarrow KR$$

This map induces a product $\star : K_iR \otimes K_jR \longrightarrow K_{i+j}R$ in the following way : if $f : S^i \longrightarrow KR$ and $g : S^j \longrightarrow KR$ represent an element $([f], [g]) \in K_iR \times K_jR$, then $[f] \star [g]$ is represented by

$$S^{i+j} = S^i \wedge S^j \xrightarrow{f \wedge g} KR \wedge KR \xrightarrow{\star} KR.$$

Concerning exact sequences, the machinery of higher K-theory allows to construct many of them. They arise as the long exact sequence in homotopy associated to a fibration. These fibrations are usually constructed using another definition of K-theory, called the "Q-construction". It allows to define the K-theory space of any exact category with small skeleton $\mathcal{C}$ (see [17]). If $\mathcal{C}$ is the exact category of finitely generated projective R-modules, then $K\mathcal{C} = KR$. For instance, if $\mathcal{A}$ is an abelian category, $\mathcal{B}$ a suitable enough subcategory to form a suitable quotient category $\mathcal{A}/\mathcal{B}$, then there is a homotopy fibration

$$K\mathcal{B} \longrightarrow K\mathcal{A} \longrightarrow K\mathcal{A}/\mathcal{B}.$$

This is Quillen's Localization's Theorem [17]. The following Theorem is a particular case of it.

Theorem. *If R is a Dedekind domain, and F its field of fractions, then there exists a long exact sequence*

$$\cdots \to K_{i+1}F \to \bigoplus_{\mathfrak{m}} K_i(R/\mathfrak{m}) \xrightarrow{\text{tr}} K_iR \to K_iF \to \cdots \to K_0F.$$

where $\mathfrak{m}$ runs over the maximal ideals of R.

These examples show how the topological approach to algebraic K-theory allows to construct by a single map (for the product) or by a fibration (for the exact sequence) what at an algebraic level would require a case-to-case definition.

Example. Quillen [18] computed the higher K-theory of finite fields : if $\mathbb{F}_q$ is a finite field with q elements, then

$$K_n\mathbb{F}_q = \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ \mathbb{Z}/(q^{\frac{n+1}{2}} - 1) & \text{if } n \text{ is odd}, \\ 0 & \text{otherwise.} \end{cases}$$

Higher K-theory of number rings.
As one might expect from the examples K_0 and K_1, algebraic K-theory of number rings, i.e. of rings of algebraic integers in number fields, is very interesting but hard to compute. Even in the case of the rational integers $\mathbb{Z}$, none of the groups $K_n\mathbb{Z}$ is completely known for $n \geq 5$. Nevertheless, a lot is known about them.

Quillen proved that if $\mathcal{O}_F$ is a number ring, then $K_n\mathcal{O}_F$ is a finitely generated (abelian) group for all $n \geq 0$. Its rank was computed by Borel [4]:

$$(K_n\mathcal{O}_F) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } n = 0, \\ \mathbb{Q}^{r_1+r_2-1} & \text{if } n = 1, \\ \mathbb{Q}^{r_2} & \text{if } n \equiv 3 \bmod 4, \\ \mathbb{Q}^{r_1+r_2} & \text{if } n \geq 5 \text{ and } n \equiv 1 \bmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore it remains to compute the torsion subgroup of each $K_n\mathcal{O}_F$, which is a finite group. A famous conjecture, called the Quillen-Lichtenbaum Conjecture, predicts the order of these groups, and relates them to the Riemann zeta function. It is supported by partial computations. We will state it in the case of the ring $\mathbb{Z}$.

We recall the definition of the Bernouilli numbers. Consider the function

$$F(t) = \frac{te^t}{e^t - 1}$$

and develop it as a power series $F(t) = \sum_{n=0}^{\infty} b_n \frac{t^n}{n!}$. The numbers b_n are rational, and are called Bernouilli numbers:

$$b_0 = 1,\ b_1 = \frac{1}{2},\ b_2 = \frac{1}{6},\ b_4 = \frac{-1}{30},\ \ldots,\ b_{12} = \frac{-691}{2730},\ \ldots$$

The Bernouilli numbers with odd indices ≥ 3 are trivial. Kummer discovered a surprising relation between the Bernouilli numbers and regular primes : a prime p is regular if and only if it does not divide the numerator of any $b_2, b_4, b_6, \ldots, b_{p-3}$. Notice that Bernouilli numbers are related to the Riemann zeta function by the relation ([31])

$$\zeta(1-n) = -\frac{b_n}{n} \quad \text{for} \quad n \geq 1.$$

Conjecture (Quillen-Lichtenbaum). *If n is an even integer ≥ 2, the following equality holds:*

$$\frac{\# K_{2n-2}\mathbb{Z}}{\# K_{2n-1}\mathbb{Z}} = \left|\frac{b_n}{n}\right| = |\zeta(1-n)|\,.$$

There is quite a lot of evidence for the Quillen-Lichtenbaum Conjecture. If n is an even integer, it is known by [19] that the denominator of b_n/n divides $\# K_{2n-1}\mathbb{Z}$, while its numerator divides $\# K_{2n-2}\mathbb{Z}$ (see Soulé's lecture in [14]).

Example. The group $K_{22}\mathbb{Z}$ has an element of order 691, and $K_{23}\mathbb{Z}$ has a cyclic direct factor of order 65'520.

There are several forms of the Quillen-Lichtenbaum Conjecture. Some of them consist of a description of the structure of the space $K\mathcal{O}_F$ itself (see [13] for instance). They also propose an explanation of why this structure arise, and its connection with arithmetic. Let us end by mentioning that the groups $K_n\mathbb{Z}\otimes\mathbb{Z}_{(2)}$ have been completely computed by Rognes and Weibel ([20]), using the Milnor Conjecture at 2, which was proved by Vœvodsky ([26]). It is compatible with the Quillen-Lichtenbaum Conjecture up to a factor 2 : the number $|\zeta(1-n)|$ should be replaced by $\frac{1}{2}|\zeta(1-n)|$.

Algebraic k-theory of brave new rings

Classical algebraic K-theory groups come along with several applications to topology and geometry. Higher algebraic K-theory also has promising applications to these fields, and provides, for instance, a link between number theory and the structure of differentiable manifolds. One of such applications, inspired by the Hatcher-Wagoner Theorem, uses an extension of algebraic K-theory of rings to the so-called "brave new rings", which was developed by F. Waldhausen. Brave new rings are topological spaces having a sum and a product, such that the axioms for a ring are satisfied up to given homotopies. For technical reasons, these homotopies are required to satisfy some coherence conditions. The resulting structure is called an A_∞-ring structure (see for instance [25] for an introduction to the algebra of brave new rings). Roughly said, Waldhausen [28] defined the H-space $\widehat{GL}_nR$ of "invertible matrices" with coefficient in a brave new ring R, and proved that it admits a classifying space $B\widehat{GL}_nR$, so that the same construction as for classical rings, using the group completion, can be performed to construct a K-theory space of R:

$$KR = \Omega B(\coprod_n B\widehat{GL}_nR).$$

There exists a construction that associates to any pointed space X a brave new ring $Q\Omega X_+$. Here ΩX_+ is the pointed space obtained by adjoining to the loop space ΩX a disjoint base point. We briefly describe what the letter Q stands for. Let Y be a pointed space, and define the map $i_n : \Omega^n\Sigma^n Y \longrightarrow \Omega^{n+1}\Sigma^{n+1}Y$ as $\Omega^n j_n$, where $j_n : \Sigma^n Y \longrightarrow \Omega\Sigma^{n+1}Y$ is the adjoint of the identity map of $\Sigma^{n+1}Y$. This defines a direct system, and its direct limit $\varinjlim_n \Omega^n\Sigma^n Y$ is denoted by QY. The kth-homotopy group of QY is $\varinjlim_n \pi_k\Omega^n\Sigma^n Y$. By the Freudenthal Suspension Theorem, this direct system of groups stabilizes, so that $\pi_k QY = \pi_k\Omega^n\Sigma^n Y$ if $n \geq k+3$. The group $\pi_k QY$ is called the *k-th stable homotopy group* of Y, and is often denoted by $\pi_k^{\mathrm{s}}Y$. For instance $\pi_*^{\mathrm{s}} = \pi_*^{\mathrm{s}}S^0$ are the famous stable homotopy groups of spheres.

The space QS^0 has a nice connection to K-theory. Indeed, by a result of Baratt and Priddy [16], the space QS^0 is homotopy equivalent to the group completion of the topological monoid $\coprod_n B\Sigma_n$, where Σ_n denotes the symmetric group with $n!$ elements. The inclusions $\Sigma_n \longrightarrow \mathrm{Aut}(\mathbb{Z}^n)$ induce a map $QS^0 \longrightarrow K\mathbb{Z}$. Quillen used this map to prove the existence of elements in $K\mathbb{Z}$ of order the denominators of Bernouilli numbers, as predicted by the Quillen-Lichtenbaum Conjecture. Indeed, Adams had constructed a family of elements in π_*^{s}, known as the *image of J*, whose order coincide with denominators of Bernouilli numbers, and Quillen proved that these elements inject in $K_*\mathbb{Z}$ via the map $QS^0 \longrightarrow K\mathbb{Z}$.

If X is a brave new ring, then $\pi_0 X$ is an ordinary ring, $X \longrightarrow \pi_0 X$ is a map of brave new rings, and there is an induced map $KX \longrightarrow K\pi_0 X$. For instance, if X is a pointed space, then $\pi_0 Q\Omega X_+$ is isomorphic to $\mathbb{Z}[\pi_1 X]$, and there is an induced map $KQ\Omega X_+ \longrightarrow K\mathbb{Z}[\pi_1 X]$. Of course, if M is a closed differentiable manifold, one expects that $KQ\Omega M_+$ contains more geometrical information on M than that detected in $K\mathbb{Z}[\pi_1 M]$ via for instance the Hatcher-Wagoner Theorem. This is indeed the case : define the *stable pseudo-isotopy* space of M as $\mathcal{P}(M) = \varinjlim_n P(M \times I^n)$, were the map $P(M) \longrightarrow P(M \times I)$ is, roughly, crossing with the unit interval I. The space $\mathcal{P}(M)$ admits a double delooping $B^2\mathcal{P}(M)$, known also as the *smooth Whitehead space* of M and often denoted by $W^{\mathrm{Diff}}(M)$. Then the K-theory space of $Q\Omega M_+$ admits the following splitting:

Theorem (Waldhausen [27]). *There exists a homotopy equivalence*

$$KQ\Omega M_+ \simeq QM_+ \times W^{\mathrm{Diff}}(M).$$

For instance, if $M = \{*\}$ is a point, then $\Omega M_+ = S^0$, and this Theorem allows to interpret Quillen's map $QS^0 \longrightarrow K\mathbb{Z}$ as a composition $QS^0 \hookrightarrow KQS^0 \longrightarrow K\mathbb{Z}[\pi_0(*)] = K\mathbb{Z}$, shedding some light on the interplay between the K-theory of $\mathbb{Z}$ and the stable homotopy groups of spheres. It is known also that the map $KQS^0 \longrightarrow K\mathbb{Z}$ induced by π_0 is a rational homotopy equivalence : $(K_*QS^0)\otimes\mathbb{Q} \cong (K_*\mathbb{Z}) \otimes \mathbb{Q}$.

The contribution of algebraic K-theory of rings to the determination of the homotopy type of $\mathcal{P}(M)$ comes from the fact that in favorable cases, computations of the K-theory of brave new rings can be reduced to that of ordinary rings. This can be in theory achieved by introducing Bökstedt's topological cyclic homology functor TC, which can be thought of as an adaptation of Connes' negative cyclic homology to the world of brave new rings, and then by using a Theorem of Dundas-McCarthy, which states that if $f : R \longrightarrow S$ is a morphism of brave new rings such that $\pi_0 f$ is surjective with nilpotent kernel, then there is a homotopy Cartesian square

$$\begin{array}{ccc} (KR)^\wedge & \xrightarrow{Kf} & (KS)^\wedge \\ \downarrow & & \downarrow \\ TC(R) & \xrightarrow{TC(f)} & TC(S) \end{array}$$

(here $X^\wedge$ stands for the *completion* of the space X, which is a topological analogue of the completion of abelian groups). One can then apply this to the map $f : R \longrightarrow \pi_0 R$, thus reducing the computation of KR to that $TC(R)$, $TC(\pi_0 R)$ and

$K(\pi_0 R)$ and the maps between them appearing in the square above. The main interest of this is that $TC(R)$, although its definition is complicated, seems to be more accessible to computations than KR. This is mainly because all the steps in the construction of $TC(R)$ are quite explicit, while we have little control on what happens in the group completion process which is used in the construction of KR. See [11] for precise statements and outlines of proofs.

Applying such an argument to the map $f : QS^0 \longrightarrow \mathbb{Z}$, and using the fact that $K\mathbb{Z}$ is known at 2, J. Rognes computed in [21] the (spectrum) $\mathbb{Z}/2$-cohomology of KQS^0 and of its direct summand $W^{\mathrm{Diff}}(*)$. With an Adams' spectral sequence argument, he then determined the 2-primary part of the homotopy groups of $W^{\mathrm{Diff}}(*) \simeq B^2\mathcal{P}(*)$ up to dimension 18. This is particularly interesting because, by a result of Igusa [8], the map $P(M) \longrightarrow \mathcal{P}(M)$ induces an isomorphism on homotopy groups up to dimension roughly one third of the dimension of M. Rognes' computations therefore provide new and fundamental information on the homotopy type of the pseudo-isotopy space of highly connected differentiable manifolds. At the moment, the main obstruction to a similar computation at an odd prime p is the lack of a complete knowledge of the homotopy type $K\mathbb{Z}$ at p.

Appendix : one page of homotopy theory

We very briefly review in this appendix the few notions of homotopy theory that were used above. See for instance the book by Spanier [23] for more details. Let Top_* be the category of pointed compactly generated topological spaces. A space X is compactly generated if it is Hausdorff and if $A \subset X$ is closed if and only if $A \cap K$ is closed for all compacts subsets $K \subset X$. Each space has a base point chosen, denoted $*$, and maps are required to preserve base points. Define the space $Map_*(X, Y) \in Top_*$ of morphisms $X \longrightarrow Y$ in Top_*, endowed with the compact-open topology. It's base point is the constant map. If $W \in Top_*$, denote by $\pi_0(W)$ the set of arc-wise connected components of W. For instance, $[X, Y]_* = \pi_0\big(Map_*(X, Y)\big)$ is the set of *homotopy classes* of pointed maps from X to Y. The *homotopy category* $HTop_*$ has same objects as Top_*, but the set of morphisms from X to Y is $[X, Y]_*$. A map f in Top_* is a *homotopy equivalence* if it is an isomorphism in $HTop_*$.

Call $X \in Top_*$ an H-space if there exists a map $\mu : X \times X \longrightarrow X$ that makes X a monoid in the category $HTop_*$. Notice that if X is an H-space, then $\pi_0(X)$ is a monoid, and X is called *group-like* if $\pi_0(X)$ is actually a group. For instance, if $X \in Top_*$, then $\Omega X = \Omega^1 X = Map_*(S^1, X)$ is a group-like H-space, the product being composition of loops. All it's arc-wise connected components are homotopy equivalent, and define $\Omega_0 X$ as the component of the constant map. Notice that $\Omega_0 X$ is also a group-like H-space. Similarly, define $\Omega^n X = \Omega(\Omega^{n-1} X)$ for $n \geq 2$. Define the n-th *homotopy group* of X as

$$\pi_n(X) = \pi_0(\Omega^n X), \quad (n = 1, 2, \dots).$$

It is abelian if $n \geq 2$, or if $n = 1$ and X is an H-space. The homotopy groups are clearly functorial. If a surjective map $f : E \longrightarrow B$ has the homotopy lifting property (see [23]), we call the sequence $F = f^{-1}(\{*\}) \hookrightarrow E \longrightarrow B$ a *fibration*. It induces a long exact sequence :

$$\cdots \to \pi_{n+1}(B) \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \cdots \to \pi_0(B).$$

The functor Ω has a left adjoint in the category $HTop_*$, given by $X \longmapsto \Sigma X = S^1 \wedge X$, i.e. there is a natural isomorphism $[X, \Omega Y]_* \cong [\Sigma X, Y]_*$ for any $X, Y \in Top_*$. The smash product $X \wedge Y$ of two spaces is defined as the quotient of the product $X \times Y$ by its subspace $(\{*\} \times Y) \cup (X \times \{*\})$. Notice that $S^1 \wedge S^n$ is homeomorphic to S^{n+1}. From this and the obvious equality $\pi_0(X) = [S^0, X]_*$ we deduce that $\pi_n(X) = [S^n, X]_*$.

Another family of functors from Top_* to the category of abelian groups are the *homology functors* $H_n(-;\mathbb{Z})$, $n \geq 0$. The group $H_n(X;\mathbb{Z})$ is the n-th homology group of the singular chain complex $(S_*(X), \partial_*)$ defined as follows : $S_*(X)$ is the free $\mathbb{Z}$-module generated by all continuous maps $\sigma : \Delta_n \longrightarrow X$, and $\partial_n : S_n(X) \longrightarrow S_{n-1}(X)$ is given by $\partial_n(\sigma) = \sum_{i=0}^{n}(-1)^i \sigma \circ F(\delta_i)$, where the maps $F(\delta_i) : \Delta_{n-1} \longrightarrow \Delta_n$ are as defined above. The family of homology functors satisfies the famous *Eilenberg-Steenrod axioms*, which characterize it up to natural equivalence.

There is a natural transformation of functors $h_n : \pi_n \longrightarrow H_n$ called the *Hurewicz homomorphism*. It is defined as follows. The homology groups of spheres are well known, and are given by $H_m(S^n;\mathbb{Z}) = \mathbb{Z}$ if $m = n$ or $m = 0$, and 0 otherwise. Chose a generator i of $H_n(S^n;\mathbb{Z}) = \mathbb{Z}$. An element $\alpha \in \pi_n(X)$ is represented by a map $f : S^n \longrightarrow X$. This map induces, by functoriality of homology, a homomorphism $f_* : H_n(S^n;\mathbb{Z}) \longrightarrow H_n(X;\mathbb{Z})$, and define $h_n(\alpha) = f_*(i)$. The Hurewicz homomorphism is a useful tool to study homotopy groups. For instance, $H_1(X;\mathbb{Z})$ is the abelianisation of $\pi_1(X)$. Also, if $\pi_m(X) = 0$ for all $0 \leq m \leq n$, where $n \geq 1$, then the same is true for $H_m(X;\mathbb{Z})$, and moreover, h_{n+1} is an isomorphism. This is the Hurewicz Theorem.

Roughly speaking, a CW-complex is a space $Z \in Top_*$ that, as a set, is a union of cells (a *n-cell* is a closed disk of dimension n), such that the interiors of two distinct cells are disjoint, and such that the boundary of any cell is included in a finite union of cells of lower dimension. The topology of Z is required to be the weak topology with respect to the family of closed cells. The n-skeleton Z^n of Z is the subspace consisting of the closed cells of dimension $\leq n$. CW-complexes are particularly convenient to work with in homotopy theory. For instance, if f is a map of connected CW-complexes, then it is a homotopy equivalence if and only if $\pi_n(f)$ is an isomorphism for all $n \geq 1$.

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