TOPOLOGICAL HOCHSCHILD HOMOLOGY OF CONNECTIVE COMPLEX K-THEORY

By Christian Ausoni

Abstract. Let ku be the connective complex K-theory spectrum, completed at an odd prime p. We present a computation of the mod (p, v_1) homotopy algebra of the topological Hochschild homology spectrum of ku.

1. Introduction. Since the discovery of categories of spectra with a symmetric monoidal smash product, as for instance the $\mathbb{S}$-modules of [EKMM], the topological Hochschild homology spectrum $THH(A)$ of a structured ring spectrum A can be defined by translating the definition of Hochschild homology of an algebra into topology, using a now standard "Algebra – Brave New Algebra" dictionary. The algebraic origin of this definition sheds light on many features of topological Hochschild homology, and has also led to more conceptual proofs of results that were based on Bökstedt's original definition [Bö1] of topological Hochschild homology for functors with smash products, see for instance [SVW2]. As can be expected by analogy with the algebraic situation, this definition also highlights the role that topological Hochschild (co-)homology plays in the classification of $\mathbb{S}$-algebra extensions. See for example [SVW1], [La] or [BJ] for applications to extensions.

The aim of this paper is to exploit the advantages of such an algebraic definition to compute the mod (p, v_1) homotopy groups of $THH(ku)$ as an algebra, which we denote by $V(1)_*THH(ku)$. Here ku is the connective complex K-theory spectrum completed at an odd prime p, with a suitable $\mathbb{S}$-algebra structure. Let us give a succinct description of the graded-commutative $\mathbb{F}_p$-algebra $V(1)_*THH(ku)$, referring to Theorem 9.15 for the complete structure.

Theorem 1.1. *Let p be an odd prime. Then $V(1)_*THH(ku)$ contains a class μ_2 of degree $2p^2$ that generates a polynomial subalgebra $\mathbb{F}_p[\mu_2]$, and $V(1)_*THH(ku)$ is a free module of rank $4(p-1)^2$ over $\mathbb{F}_p[\mu_2]$. Its localization*

$$A_* = \mu_2^{-1}V(1)_*THH(ku)$$

Manuscript received November 27, 2003; revised July 20, 2004.
Research supported in part by the SFB 478 – Geometrische Strukturen in der Mathematik.
American Journal of Mathematics 127 (2005), 1261–1313.

away from μ_2 is a Frobenius algebra over the graded field $k_* = \mathbb{F}_p[\mu_2^{\pm 1}]$, in the sense that there is an isomorphism

$$A_* \cong \mathrm{Hom}_{k_*}(A_*, \Sigma^{2p^2+2p-2} k_*)$$

of graded A_*-modules.

The justification for performing this computation in $V(1)$-homotopy is that $V(1)_*THH(ku)$ is a complicated but finitely presented $\mathbb{F}_p$-algebra. On the other hand, a presentation of the mod p homotopy algebra $V(0)_*THH(ku)$ requires infinitely many generators and relations. We nevertheless evaluate the additive structure of $V(0)_*THH(ku)$ in Theorem 7.9.

A first motivation for these computations is to approach the algebraic K-theory spectrum of ku. By work of Baas, Dundas and Rognes [BDR], the spectrum $K(ku)$ is conjectured to represent a cohomology theory whose zeroth group classifies equivalence classes of virtual two-vector bundles. It is also expected that the spectrum $K(ku)$ is of chromatic complexity two, which essentially means that it is suitable for studying v_2-periodic and v_2-torsion phenomena in stable homotopy. Thus $K(ku)$ should represent a form of elliptic cohomology with a genuine geometric content, something which has long been wished for.

Topological Hochschild homology is the target of a trace map from algebraic K-theory, which refines over the cyclotomic trace map trc: $K(ku) \to TC(ku;p)$. The topological cyclic homology spectrum $TC(ku;p)$ is a very close approximation of $K(ku)_p$, since by Dundas [Du] and Hesselholt-Madsen [HM1] it sits in a cofibre sequence

$$K(ku)_p \xrightarrow{\text{trc}} TC(ku;p) \to \Sigma^{-1} H\mathbb{Z}_p.$$

The spectrum $TC(ku;p)$ is built by taking the homotopy limit of a diagram whose vertices are the fixed points of $THH(ku)$ under the action of the cyclic p-subgroups of the circle. Thus computing $THH(ku)$ is a first step in the study of $K(ku)$ by trace maps.

A second motivation for the computations presented in this paper is to pursue the exploration of the "brave new world" of ring spectra and their arithmetic. In the classical case, arithmetic properties of a ring or of a ring extension are to a large extent reflected in algebraic K-theory or its approximations, as topological cyclic homology, topological Hochschild homology or even Hochschild homology. An important example is descent in its various forms. Etale descent has been conjectured in algebraic K-theory by Lichtenbaum and Quillen, and has been proven for various classes of rings [RW], [HM2]. For Hochschild homology, Geller and Weibel [GW] proved étale descent by showing that for an étale extension $A \hookrightarrow B$ there is an isomorphism $\mathbb{H}_*(B) \cong B \otimes_A \mathbb{H}_*(A)$. A form of tamely ramified descent for topological Hochschild homology, topological cyclic homology and algebraic K-theory of discrete valuation rings has been proven by Hesselholt and Madsen [HM2], see also [Ts].

Laying the foundations of a theory of extensions for $\mathbb{S}$-algebras is work in progress, see for instance [Ro]. It is not yet known to what extent such descent results can be generalized, and so far only very few computations are available. This paper provides an example of what we expect to be tamely ramified descent.

The $\mathbb{S}$-algebra ku has a subalgebra ℓ, called the Adams summand. The spectrum ku splits as an $\mathbb{S}$-module into a sum of $p-1$ shifted copies of ℓ, namely

$$ku \simeq \bigvee_{i=0}^{p-2} \Sigma^{2i}\ell. \tag{1.2}$$

McClure and Staffeldt computed the mod p homotopy groups of $THH(\ell)$ in [MS]. This computation was was then used by Rognes and the author [AR] to further evaluate the mod (p, v_1) homotopy groups of $TC(\ell; p)$ and $K(\ell)$. In view of the above splitting, one could expect that similar computations should follow quite easily for ku. However, we found out that a computation of $THH(ku)$ involves some surprising new features. For example the Bökstedt spectral sequence

$$E^2_{*,*}(ku) = \mathbb{H}^{\mathbb{F}_p}_{*,*}\big(H_*(ku; \mathbb{F}_p)\big) \Longrightarrow H_*(THH(ku); \mathbb{F}_p)$$

has higher differentials than that for ℓ (Lemma 9.6), and in computing the algebra structure of $H_*(THH(ku); \mathbb{F}_p)$ we have to deal with extra multiplicative extensions (Proposition 9.10). It turns out that the multiplicative structure of $V(1)_*THH(ku)$, given in Theorem 9.15, is highly nontrivial. All this reflects the well known fact that the splitting (1.2) is not multiplicative. In fact, much of the added complexity of $THH(ku)$, as compared to $THH(\ell)$, can be accounted for by speculating on the extension $\ell \to ku$. We would like to think of it as the extension defined by the relation

$$ku = \ell[u]/(u^{p-1} = v_1) \tag{1.3}$$

in commutative $\mathbb{S}$-algebras. This formula holds on coefficients, since the homomorphism $\ell_* \to ku_*$ is the inclusion $\mathbb{Z}_p[v_1] \hookrightarrow \mathbb{Z}_p[u]$ with $v_1 = u^{p-1}$. First we notice that the prime (v_1) in ℓ ramifies, and hence the extension $\ell \to ku$ should not qualify as étale. This is confirmed by the computations of $V(1)_*THH(\ell)$ and $V(1)_*THH(ku)$, which show that

$$THH(ku) \not\simeq ku \wedge_\ell THH(\ell)$$

(compare with the Geller-Weibel Theorem). But if we invert v_1 in ℓ and ku we obtain the periodic Adams summand L and the periodic K-theory spectrum KU. Here the ramification has vanished so the extension $L \to KU$ should be étale.

And indeed we have an equivalence

$$THH(KU)_p \simeq KU \wedge_L THH(L)_p.$$

This is a consequence of McClure and Staffeldt's computation of $THH(L)_p$, which we adapt to compute $THH(KU)_p$ in Proposition 7.13.

Returning to $\ell \to ku$ and formula (1.3), we notice that the ramification index is $(p-1)$ and that this extension ought to be tamely ramified. The behavior of topological Hochschild homology with respect to tamely ramified extensions of discrete valuation rings was studied by Hesselholt and Madsen in [HM2]. Let us assume that their results hold also in the generality of commutative $\mathbb{S}$-algebra. The ring ku has a prime ideal (u), with residue ring $H\mathbb{Z}_p$ and quotient ring KU. Following [HM2, Theorem 1.5.6] we expect to have a localization cofibre sequence in topological Hochschild homology

$$THH(H\mathbb{Z}_p) \xrightarrow{i^!} THH(ku) \xrightarrow{j} THH(ku \mid KU). \tag{1.4}$$

This requires that we can identify by dévissage $THH(H\mathbb{Z}_p)$ with the topological Hochschild homology spectrum of a suitable category of finite u-torsion ku-modules. The tame ramification of $\ell \to ku$ should be reflected by an equivalence

$$ku \wedge_\ell THH(\ell \mid L) \simeq THH(ku \mid KU).$$

Now $THH(\ell \mid L)$ can be computed using the localization cofibre sequence

$$THH(H\mathbb{Z}_p) \xrightarrow{i^!} THH(\ell) \xrightarrow{j} THH(\ell \mid L)$$

and McClure-Staffeldt's computation of $THH(\ell)$. Thus $THH(ku \mid KU)$ is also known, and $THH(ku)$ can be evaluated from the cofibre sequence (1.4). We elaborate more on this in Paragraph 10.4. At this point we do not know if this conceptual line of argument can be made rigorous. This would of course require a generalization of the results in [HM2] for $\mathbb{S}$-algebras. But promisingly, the description of $V(1)_*THH(ku)$ it provides is perfectly compatible with our computation of it given in Theorem 9.15.

The units $\mathbb{Z}_p^\times$ act as p-adic Adams operations on ku. Let Δ be the cyclic subgroup of order $p-1$ of $\mathbb{Z}_p^\times$. Then we have a weak equivalence

$$\ell \simeq ku^{h\Delta}$$

where $(-)^{h\Delta}$ denotes taking the homotopy fixed points. We prove the following result as Theorem 10.2.

THEOREM 1.5. *Let p be an odd prime. There are weak equivalences of p-completed spectra*

$$\begin{aligned} THH(ku)^{h\Delta} &\simeq THH(\ell), \\ TC(ku;p)^{h\Delta} &\simeq TC(\ell;p), \text{ and} \\ K(ku)^{h\Delta} &\simeq K(\ell). \end{aligned}$$

We would like to interpret this theorem as an example of tamely ramified descent for topological Hochschild homology, and of étale descent for topological cyclic homology and algebraic K-theory.

Let us briefly review the content of the present paper. Our strategy for computing $THH(ku)$ can be summarized as follows. Taking Postnikov sections we obtain a sequence of $\mathbb{S}$-algebra maps $ku \to M \to H\mathbb{Z}_p$, where M is the section $ku[0, 2p-6]$. Using naturality of topological Hochschild homology we construct a sequence

$$THH(ku) \to THH(ku, M) \to THH(ku, H\mathbb{Z}_p) \to THH(H\mathbb{Z}_p).$$

We then use this sequence to interpolate from $THH(H\mathbb{Z}_p)$ to $THH(ku)$, the point being that at each step the added complexity can be handled by essentially algebraic means. In §2 we discuss some properties of ku and compute its homology. We present in §3 the computations in Hochschild homology that will be needed later on as input for various Bökstedt spectral sequences. In §4 we review the definition of topological Hochschild homology, following [MS], [EKMM] and [SVW2], and we set up the Bökstedt spectral sequence. We present in §5 a simplified computation of the mod p homotopy groups of $THH(H\mathbb{Z}_p)$, for odd primes p. We also briefly review a computation of $V(1)_*THH(\ell)$. In §6 we determine the homotopy type of the spectrum $THH(ku, H\mathbb{Z}_p)$, which is given in Corollary 6.9 by an equivalence of p-completed spectra

$$THH(ku, H\mathbb{Z}_p) \simeq S^3_+ \wedge THH(H\mathbb{Z}_p).$$

Its mod p homotopy groups $V(0)_*THH(ku, H\mathbb{Z}_p)$ are the input for a Bockstein spectral sequence which is computed in §7. It yields a description of $V(0)_*THH(ku)$ as a module over $V(0)_*ku$, given in Theorem 7.9. Note that §7 is not used in the later sections. In §8 we compute the mod p homology of $THH(ku, M)$. The core of this paper is §9. Here we compute the mod p homology Bökstedt spectral sequence for $THH(ku)$, and evaluate $V(1)_*THH(ku)$ as an algebra over $V(1)_*ku$ in Theorem 9.15. Finally, in §10 we compare $THH(\ell)$ and $THH(ku)$ and elaborate on the properties of the extension $\ell \to ku$ mentioned above.

Acknowledgments. It is a pleasure to acknowledge the help I received from John Rognes. He suggested many improvements over an earlier draft. I especially thank him and Vigleik Angeltveit for making a preliminary version of [AnR] available to me. I also thank Lars Hesselholt for explaining to me how [HM2] could be used to make a guess about $V(1)_*THH(ku)$. I am grateful to the referee for valuable suggestions.

Notations and conventions. Throughout the paper p will be a fixed odd prime, and $\mathbb{Z}_p$ will denote the p-adic integers. For an $\mathbb{F}_p$-vector space V, let $E(V)$, $P(V)$ and $\Gamma(V)$ be the exterior algebra, polynomial algebra and divided power algebra on V, respectively. If V has a basis $\{x_1, \ldots, x_n\}$, we write $V = \mathbb{F}_p\{x_1, \ldots, x_n\}$ and $E(x_1, \ldots, x_n)$, $P(x_1, \ldots, x_n)$ and $\Gamma(x_1, \ldots, x_n)$ for these algebras. By definition, $\Gamma(x)$ is the $\mathbb{F}_p$-vector space $\mathbb{F}_p\{\gamma_k x \mid k \geqslant 0\}$ with product given by $\gamma_i x \cdot \gamma_j x = \binom{i+j}{i}\gamma_{i+j}x$, where $\gamma_0 x = 1$ and $\gamma_1 x = x$. Let $P_h(x) = P(x)/(x^h = 0)$ be the truncated polynomial algebra of height h. For an algebra A, we denote by $A\{x_1, \ldots, x_n\}$ the free A-module generated by $x_1, \ldots, x_n$.

An infinite cycle in a spectral sequence is a class x such that $d^r(x) = 0$ for all r. A permanent cycle is an infinite cycle that is not in the image of d^r for any r. In the description of spectral sequences, we will often determine differentials or multiplicative extensions only up to multiplication by a unit. We therefore introduce the following notation. In an $\mathbb{F}_p$-vector space we write $v \doteq w$ if $v = \alpha w$ holds for some unit α in $\mathbb{F}_p$.

We denote the mod p Moore spectrum by $V(0)$. It has a periodic v_1-multiplication $\Sigma^{2p-2}V(0) \to V(0)$ whose cofibre is called $V(1)$. We define the mod p homotopy groups of a spectrum X by $V(0)_*X = \pi_*(V(0) \wedge X)$, and its mod (p, v_1) homotopy groups by $V(1)_*X = \pi_*(V(1) \wedge X)$. By the symbol $X \simeq_p Y$ we mean that X and Y are weakly equivalent after p-completion. We denote by β the primary mod p homology Bockstein, by $\beta_{0,r}$ the rth mod p homotopy Bockstein, and by $\beta_{1,r}$ the rth mod v_1 homotopy Bockstein.

For a ring R, let HR be the Eilenberg-MacLane spectrum associated to R. If A is a (-1)-connected ring spectrum, we call the ring map $A \to H\pi_0 A$ that induces the identity on π_0 the linearization map.

2. Connective complex K-theory. Let ku be the p-completed connective complex K-theory spectrum, having coefficients $ku_* = \mathbb{Z}_p[u]$ with $|u| = 2$.

An E_∞ model. Since we would like to take ku as input for topological Hochschild homology, we need to specify a structured ring spectrum structure on ku. Following [MS, Section 9], we will take as model for ku the p-completion of the algebraic K-theory spectrum of a suitable field. Let q be a prime that generates $(\mathbb{Z}/p^2)^\times$, and let μ_{p^∞} be the set of all pth-power roots of 1 in $\bar{\mathbb{F}}_q$. We define k to be the field extension obtained by adjoining the elements of μ_{p^∞} to $\mathbb{F}_q$. Hence $k = \mathbb{F}_q[\mu_{p^\infty}] = \bigcup_{i \geqslant 0} \mathbb{F}_{q^{p^i(p-1)}}$. Quillen [Qu] proved that the Brauer lift induces a weak equivalence

$$K(k)_p \xrightarrow{\simeq} ku.$$

Notice that the inclusion $k \subset \bar{\mathbb{F}}_q$ induces an equivalence $K(k)_p \simeq K(\bar{\mathbb{F}}_q)_p$, so that we do not need to go all the way up to the algebraic closure of $\mathbb{F}_q$ to get a model for ku. The algebraic K-theory spectrum of a commutative ring comes equipped with a natural structure of commutative $\mathbb{S}$-algebra in the sense of [EKMM], and p-completion preserves this structure. In particular the Galois group

$$\mathrm{Gal}(k/\mathbb{F}_q) \cong \mathbb{Z}_p \times \mathbb{Z}/(p-1)$$

acts on $K(k)_p$ by $\mathbb{S}$-algebra maps. From now on ku will stand for $K(k)_p$.

The Bott element. The mod p homotopy groups of ku are given by $V(0)_*ku = P(u)$, where u is the mod p reduction of a generator of ku_2. We call such a class u a Bott element.

The algebraic K-theory groups of k were computed by Quillen [Qu], and are given by

$$K_n(k) = \begin{cases} \mathbb{Z} & \text{for } n = 0, \\ k^\times & \text{for } n \text{ odd } \geqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

We have $k^\times \cong \bigoplus_{l\,\mathrm{prime} \neq q} \mathbb{Z}/l^\infty$. The universal coefficient formula for mod p homotopy implies that we have an isomorphism

$$V(0)_2 K(k) \xrightarrow{\cong} \mathrm{Tor}_1^{\mathbb{Z}}(k^\times, \mathbb{F}_p) = \mathbb{F}_p.$$

Identifying $V(0)_*ku$ with $V(0)_*K(k)$, a Bott element $u \in V(0)_2ku$ corresponds under this isomorphism to a primitive pth-root of 1 in k.

2.1. The Adams summand. Let δ be a chosen generator of Δ, the cyclic subgroup of order $p-1$ of $\mathrm{Gal}(k/\mathbb{F}_q)$. Then δ permutes the primitive p-th roots of 1 in k via a cyclic permutation of order $p-1$. In particular, if $u \in V(0)_2ku$ is a chosen Bott element, then

$$\delta_*\colon\ V(0)_*ku \to V(0)_*ku$$

maps u to αu for some generator α of $\mathbb{F}_p^\times$. Let k' be the subfield of k fixed under the action of Δ. Then the homotopy fixed point spectrum $ku^{h\Delta} = K(k')_p$ is a commutative $\mathbb{S}$-algebra model for the p-completed Adams summand ℓ, with coefficients $\ell_* = \mathbb{Z}_p[v_1]$. The spectrum ku is then a commutative ℓ-algebra. It splits as an $\mathbb{S}$-module into

$$ku \simeq \bigvee_{i=0}^{p-2} \Sigma^{2i}\ell.$$

In $V(0)_*ku$ we have the relation $u^{p-1} = v_1$. We would like to think of ku as the extension $\ell[u]/(u^{p-1} = v_1)$ of ℓ in commutative $\mathbb{S}$-algebras.

The dual Steenrod algebra. Let A_* be the dual Steenrod algebra

$$A_* = P(\xi_1, \xi_2, \dots) \otimes E(\tau_0, \tau_1, \dots)$$

where ξ_i and τ_j are the generators defined by Milnor [Mi], of degree $2p^i - 2$ and $2p^j - 1$, respectively. We denote by $\bar{\xi}_i$ and $\bar{\tau}_j$ the images of ξ_i and τ_j under the canonical involution of A_*. The coproduct ψ on A_* is given by

$$\psi(\bar{\xi}_k) = \sum_{i+j=k} \bar{\xi}_j \otimes \bar{\xi}_i^{p^j} \quad \text{and} \quad \psi(\bar{\tau}_k) = 1 \otimes \bar{\tau}_k + \sum_{i+j=k} \bar{\tau}_j \otimes \bar{\xi}_i^{p^j}$$

where by convention $\bar{\xi}_0 = 1$.

We view the mod p homology of a spectrum X as a left A_*-comodule, i.e., $H_*(X; \mathbb{F}_p) = \pi_*(H\mathbb{F}_p \wedge X)$. In particular we write

$$H_*(H\mathbb{F}_p; \mathbb{F}_p) = P(\bar{\xi}_1, \bar{\xi}_2, \dots) \otimes E(\bar{\tau}_0, \bar{\tau}_1, \dots).$$

We will denote by ν_* the coaction $H_*(X; \mathbb{F}_p) \to A_* \otimes H_*(X; \mathbb{F}_p)$. The mod p reduction map ρ: $H\mathbb{Z}_p \to H\mathbb{F}_p$ induces an injection in mod p homology, and we identify $H_*(H\mathbb{Z}_p; \mathbb{F}_p)$ with its image in $H_*(H\mathbb{F}_p; \mathbb{F}_p)$, namely

$$H_*(H\mathbb{Z}_p; \mathbb{F}_p) = P(\bar{\xi}_1, \bar{\xi}_2, \dots) \otimes E(\bar{\tau}_1, \bar{\tau}_2 \dots). \tag{2.2}$$

The homology of *ku*. The linearization map j: $ku \to H\mathbb{Z}_p$ has the 1-connected cover $ku[2, \infty]$ of ku as fiber. By Bott periodicity, we can identify this cover with $\Sigma^2 ku$. We assemble the iterated suspensions of the cofibre sequence

$$\Sigma^2 ku \xrightarrow{u} ku \xrightarrow{j} H\mathbb{Z}_p$$

into a diagram

$$\begin{tikzcd} \cdots \arrow[r, "u"] & \Sigma^4 ku \arrow[r, "u"] \arrow[d, "j"] & \Sigma^2 ku \arrow[r, "u"] \arrow[d, "j"] & ku \arrow[d, "j"] \\ & \Sigma^4 H\mathbb{Z}_p & \Sigma^2 H\mathbb{Z}_p & H\mathbb{Z}_p. \end{tikzcd} \tag{2.3}$$

Applying $H_*(-;\mathbb{F}_p)$ we obtain an unrolled exact couple in the sense of Boardman [Bo]. Placing $\Sigma^{2s}H\mathbb{Z}_p$ in filtration degree $-2s$, it yields a spectral sequence

$$E^2_{s,t} = \big(H_*(H\mathbb{Z}_p;\mathbb{F}_p)\otimes P(x)\big)_{(s,t)} \Longrightarrow H_{s+t}(ku;\mathbb{F}_p). \tag{2.4}$$

This is a second quadrant spectral sequence where $a \in H_*(H\mathbb{Z}_p;\mathbb{F}_p)$ has bidegree $(0,|a|)$ and x has bidegree $(-2,4)$ and represents the image of $u \in V(0)_2ku$ under the Hurewicz homomorphism $V(0)_*ku \to H_*(ku;\mathbb{F}_p)$.

THEOREM 2.5. (Adams) *There is an isomorphism of A_*-comodule algebras*

$$H_*(ku;\mathbb{F}_p) \cong H_*(\ell;\mathbb{F}_p)\otimes P_{p-1}(x)$$

where $H_*(\ell;\mathbb{F}_p) = P(\bar\xi_1,\bar\xi_2,\dots)\otimes E(\bar\tau_2,\bar\tau_3,\dots)\subset A_*$ is a sub-A_*-comodule algebra of A_* and $P_{p-1}(x)$ is spherical (hence primitive).

Proof. The unrolled exact couple given by applying $H_*(-;\mathbb{F}_p)$ to (2.3) is part of a multiplicative Cartan-Eilenberg system. Thus the spectral sequence (2.4) is a spectral sequence of A_*-comodule algebras. It is also strongly convergent. The E^2-term of this spectral sequence is

$$E^2_{*,*} = P(\bar\xi_1,\bar\xi_2,\dots)\otimes E(\bar\tau_1,\bar\tau_2,\dots)\otimes P(x).$$

There is a differential $d^{2p-2}(\bar\tau_1)\doteq x^{p-1}$ (Adams [Ad, Lemma 4]), after which the spectral sequence collapses for bidegree reasons, leaving

$$E^\infty_{*,*} = E^{2p-1}_{*,*} = P(\bar\xi_1,\bar\xi_2,\dots)\otimes E(\bar\tau_2,\bar\tau_2,\dots)\otimes P_{p-1}(x).$$

There are no nontrivial multiplicative extensions. For instance $x^{p-1}=0$ because the only other possibility would be $x^{p-1}\doteq\bar\xi_1$, which would contradict the fact that x is primitive. For degree reasons there cannot be any nontrivial A_*-comodule extensions. □

This formula for $H_*(ku;\mathbb{F}_p)$ reflects the splitting $ku\simeq\bigvee_{i=0}^{p-2}\Sigma^{2i}\ell$. The class $v_1 = u^{p-1}\in V(0)_*ku$ is of Adams filtration 1 and is in the kernel of the Hurewicz homomorphism, which accounts for the relation $x^{p-1}=0$ in $H_*(ku;\mathbb{F}_p)$.

LEMMA 2.6. *Let $\delta\colon ku\to ku$ be the map given in 2.1. The algebra endomorphism δ_* of $H_*(ku;\mathbb{F}_p)$ is the identity on the tensor factor $H_*(\ell;\mathbb{F}_p)$, and maps x to αx for some generator α of $\mathbb{F}_p^\times$.*

Proof. By definition ℓ is fixed under the action of δ, and x is the Hurewicz image of a Bott element. This implies the lemma. □

We will also need some knowledge of the integral homology of ku in low degrees. In the mod p homology of $H\mathbb{Z}_p$, the primary Bockstein homomorphism

β is given by

$$\beta\colon\ \bar{\tau}_i \mapsto \bar{\xi}_i \quad \text{for all} \quad i \geqslant 1.$$

In particular $H_0(H\mathbb{Z}_p;\mathbb{Z}) = \mathbb{Z}_p$ and $pH_n(H\mathbb{Z}_p;\mathbb{Z}) = 0$ for all $n \geqslant 1$.

PROPOSITION 2.7. *Consider the graded-commutative algebra Λ_* over $\mathbb{Z}_p$ defined as*

$$\Lambda_* = \mathbb{Z}_p[\tilde{x}, \tilde{\xi}_1]/(\tilde{x}^{p-1} = p\tilde{\xi}_1)$$

where $\tilde{x}$ has degree 2 and $\tilde{\xi}_1$ has degree $2p-2$. There exists a homomorphism of $\mathbb{Z}_p$-algebras

$$\lambda\colon\ \Lambda_* \to H_*(ku;\mathbb{Z})$$

such that λ is an isomorphism in degrees $\leqslant 2p^2-3$, and such that the composition

$$\Lambda_* \xrightarrow{\ \lambda\ } H_*(ku;\mathbb{Z}) \xrightarrow{\ \rho\ } H_*(ku;\mathbb{F}_p),$$

where ρ is the mod p reduction, maps $\tilde{x}$ to x and $\tilde{\xi}_1$ to $\bar{\xi}_1$.

Proof. By comparison with the case of $H\mathbb{Z}_p$, we have primary mod p Bocksteins $\beta(\bar{\tau}_i) = \bar{\xi}_i$ in $H_*(ku;\mathbb{F}_p)$, for all $i \geqslant 2$. We also have $\beta(x) = 0$ for degree reasons. Hence the Bockstein spectral sequence

$$E^1_* = H_*(ku;\mathbb{F}_p) \Longrightarrow (H_*(ku;\mathbb{Z})/\text{torsion}) \otimes \mathbb{F}_p,$$

whose first differential is β, collapses at the E^2-term, leaving

$$E^\infty_* = E^2_* = P_{p-1}(x) \otimes P(\bar{\xi}_1).$$

Let $\tilde{x} \in H_2(ku;\mathbb{Z})$ be a lift of x and $\tilde{\xi}_1 \in H_{2p-2}(ku;\mathbb{Z})$ be a lift of $\bar{\xi}_1$. Since $H_*(ku;\mathbb{Q})$ is polynomial over $\mathbb{Q}_p$ on one generator in degree 2, there is a multiplicative relation $\tilde{x}^{p-1} = a\tilde{\xi}_1$ in $H_*(ku;\mathbb{Z})$, for some non-zero $a \in p\mathbb{Z}_p$. The Postnikov invariant $ku[0, 2p-4] \to \Sigma^{2p-1}H\mathbb{Z}_p$ of ku is of order p, so we can choose $\tilde{\xi}_1$ such that $a = p$. We hence obtain the ring homomorphism λ. There is no torsion class in $H_*(ku;\mathbb{Z})$ of degree $\leqslant 2p^2-3$, and λ is an isomorphism in these degrees. □

3. Hochschild homology. In this section we recall some properties of Hochschild homology and present elementary computations that will be needed in the later sections.

Suppose R is a graded-commutative and unital ring, A a graded unital R-algebra, and M a graded A-bimodule. Let us simply write $\otimes$ for $\otimes_R$. The enveloping algebra A^{e} of A is the graded and unital R-algebra $A \otimes A^{\mathrm{op}}$, and it acts on A on the left and on M on the right in the usual way.

If A is flat over R, the Hochschild homology of A with coefficients in M is defined as the bigraded R-module

$$\mathbb{H}^R_{s,t}(A, M) = \mathrm{Tor}^{A^{\mathrm{e}}}_{s,t}(M, A).$$

Here s is the homological degree and t is the internal degree. The (two-sided) bar complex $C^{\mathrm{bar}}_*(A)$, with $C^{\mathrm{bar}}_n(A) = A^{\otimes(n+2)}$, is a standard resolution of A as left A^{e}-module, having the product $\mu\colon\ C^{\mathrm{bar}}_0(A) = A \otimes A \to A$ of A as augmentation. The complex $M \otimes_{A^{\mathrm{e}}} C^{\mathrm{bar}}_*(A)$ is isomorphic to the Hochschild complex $C_*(A, M)$, with $C_n(A, M) = M \otimes A^{\otimes n}$, see [Lo, Chapter 1].

Suppose now that A is graded-commutative and that M is a graded-commutative and unital A-algebra with A-bimodule structure given by forgetting part of the A-algebra structure. The standard product and coproduct on the bar resolution make the Hochschild complex into a graded differential M-bialgebra with unit and augmentation. In particular $\mathbb{H}^R_{*,*}(A, M)$ is a bigraded unital M-algebra, and if $\mathbb{H}^R_{*,*}(A, M)$ is flat over M, then $\mathbb{H}^R_{*,*}(A, M)$ is a bigraded unital and augmented M-bialgebra. The unit

$$\iota\colon\ M \xrightarrow{\ \cong\ } \mathbb{H}^R_{0,*}(A, M)$$

is given by the inclusion of the 0-cycles $M = C_0(A, M)$, and the augmentation is the projection $\mathbb{H}^R_{*,*}(A, M) \to \mathbb{H}^R_{0,*}(A, M) \cong M$. There is also an R-linear homomorphism

$$\sigma\colon\ A \to \mathbb{H}^R_{1,*}(A, M) \tag{3.1}$$

induced by $A \to C_1(A, M) = M \otimes A$, $a \mapsto 1 \otimes a$. It satisfies the derivation rule

$$\sigma(ab) = e(a)\sigma(b) + (-1)^{|a||b|} e(b)\sigma(a),$$

where $e\colon\ A \to M$ is the unit of M.

As usual, we write $\mathbb{H}^R_{*,*}(A)$ for $\mathbb{H}^R_{*,*}(A, A)$ and $C_*(A)$ for $C_*(A, A)$.

In the next three propositions we take $R = \mathbb{F}_p$ (considered as a graded ring concentrated in degree 0).

Proposition 3.2. (a) *Let $P(x)$ be the polynomial $\mathbb{F}_p$-algebra generated by x of even degree d. Then there is an isomorphism of $P(x)$-bialgebras*

$$\mathbb{H}^{\mathbb{F}_p}_{*,*}\big(P(x)\big) \cong P(x) \otimes E(\sigma x)$$

with σx primitive of bidegree $(1, d)$.

(b) *Let $E(x)$ be the exterior $\mathbb{F}_p$-algebra generated by x of odd degree d. Then there is an isomorphism of $E(x)$-bialgebras*

$$\mathbb{H}^{\mathbb{F}_p}_{*,*}\big(E(x)\big) \cong E(x) \otimes \Gamma(\sigma x)$$

with σx of bidegree $(1, d)$ and with coproduct given by

$$\Delta(\gamma_k \sigma x) = \sum_{i+j=k} \gamma_i \sigma x \otimes_{E(x)} \gamma_j \sigma x.$$

Proof. This is standard, see for instance [MS, Proposition 2.1]. □

PROPOSITION 3.3. *Let $P_h(x)$ be the truncated polynomial $\mathbb{F}_p$-algebra of height h generated by x of even degree d, with $(p, h) = 1$. Then there is an isomorphism of $P_h(x)$-algebras*

$$\mathbb{H}^{\mathbb{F}_p}_{*,*}\big(P_h(x)\big) \cong P_h(x)[z_i, y_j \mid i \geqslant 0, j \geqslant 1]/\sim$$

where z_i has bidegree $(2i+1, ihd+d)$ and y_j has bidegree $(2j, jhd+d)$. The relation $\sim$ is generated by

$$\begin{cases} x^{h-1} z_i \ = \ x^{h-1} y_j = z_i z_k = 0 \\ \quad z_i y_j \ = \ \binom{i+j}{i} x z_{i+j} \\ \quad y_j y_g \ = \ \binom{j+g}{j} x y_{j+g} \end{cases}$$

for all $i, k \geqslant 0$ and all $j, g \geqslant 1$. Moreover $z_0 = \sigma x$, the generator z_i is represented in the Hochschild complex $C_*\big(P_h(x)\big)$ by

$$\sum_{\substack{k_1,\ldots,k_{i+1}\geqslant 0 \\ k_1+\ldots+k_{i+1}=i(h-1)}} (x^{k_1} \otimes x \otimes x^{k_2} \otimes x \otimes x^{k_3} \otimes x \otimes \ldots \otimes x \otimes x^{k_{i+1}} \otimes x)$$

for all $i \geqslant 1$, and the generator y_j is represented by

$$\sum_{\substack{k_1,\ldots,k_{j+1}\geqslant 0 \\ k_1+\ldots+k_{j+1}=j(h-1)}} (x^{k_1+1} \otimes x^{k_2} \otimes x \otimes x^{k_3} \otimes x \otimes \ldots \otimes x \otimes x^{k_{j+1}} \otimes x)$$

for all $j \geqslant 1$.

Proof. Let $A = P_h(x)$. This proposition is proven by choosing a small differential bigraded algebra over A^{e} that is a projective resolution of A as left A^{e}-module. For example, one can take

$$X_{*,*} = A^{\mathrm{e}} \otimes E(\sigma x) \otimes \Gamma(\tau),$$

where $a \in A^{\mathrm{e}}$ has bidegree $(0,|a|)$, σx has bidegree $(1,d)$ and τ has bidegree $(2,dh)$. The differential d of X, of bidegree $(-1,0)$, is given on the generators by

$$d(\sigma x) = T \quad \text{and} \quad d(\tau) = N\sigma x,$$

where $T = x \otimes 1 - 1 \otimes x \in A^{\mathrm{e}}$ and $N = (x^h \otimes 1 - 1 \otimes x^h)/(x \otimes 1 - 1 \otimes x) \in A^{\mathrm{e}}$. The product of A gives an augmentation $X_0 = A^{\mathrm{e}} \to A$. Now $\mathbb{H}^{\mathbb{F}_p}_{*,*}\big(P_h(x)\big)$ is isomorphic to the homology of the differential graded algebra

$$A \otimes_{A^{\mathrm{e}}} X_{*,*} \cong A \otimes E(\sigma x) \otimes \Gamma(\tau)$$

with differential given by $d(\sigma x) = 0$ and $d(\tau) = hx^{h-1}\sigma x$. The class z_i is represented by the cycle $\sigma x \cdot \gamma_i \tau$ and the class y_j is represented by the cycle $x \cdot \gamma_j \tau$, for any $i \geqslant 0$ and $j \geqslant 1$. Representatives for the generators are obtained by choosing a homotopy equivalence $X_* \to C^{\mathrm{bar}}_*(A)$, see for instance [BAC, page 55]. □

Remark 3.4. Notice that $\mathbb{H}^{\mathbb{F}_p}_{*,*}\big(P_h(x)\big)$ is not flat over $P_h(x)$, so that there is no coproduct in this case. Moreover there are infinitely many algebra generators. However, the set $\{1, z_i, y_j \mid i \geqslant 0, j \geqslant 1\}$ of given algebra generators is also a set of $P_h(x)$-module generators for $\mathbb{H}^{\mathbb{F}_p}_{*,*}\big(P_h(x)\big)$. More precisely, $\mathbb{H}^{\mathbb{F}_p}_{*,*}\big(P_h(x)\big)$ has one $P_h(x)$-module generator in each non-negative homological degree, and is given by

$$\mathbb{H}^{\mathbb{F}_p}_{n,*}\big(P_h(x)\big) = \begin{cases} P_h(x) & n = 0, \\ P_{h-1}(x)\{z_{\frac{n-1}{2}}\} & n \geqslant 1 \ \text{ odd}, \\ P_{h-1}(x)\{y_{\frac{n}{2}}\} & n \geqslant 2 \text{ even}. \end{cases}$$

Let $B_n\colon \mathbb{H}^{\mathbb{F}_p}_{n,*}\big(P_h(x)\big) \to \mathbb{H}^{\mathbb{F}_p}_{n+1,*}\big(P_h(x)\big)$ be Connes' operator (B_0 coincides with the operator σ given above). Then we have

$$B_{2n}(y_n) \doteq (-1-nh)z_n$$

for all $n \geqslant 0$, where by convention we set $y_0 = x$ (see [BAC, Proposition 2.1]).

The next proposition shows that if we take Hochschild homology of $P_h(x)$ with coefficients having a lower truncation, we have both flatness and finite generation as an algebra.

PROPOSITION 3.5. *Let $P_h(x)$ be as above, and for $1 \leqslant g < h$ let $P_h(x) \to P_g(x)$ be the quotient by (x^g). We view $P_g(x)$ as a $P_h(x)$-algebra. Then there is an isomorphism of $P_g(x)$-bialgebras*

$$\mathbb{H}^{\mathbb{F}_p}_{*,*}\big(P_h(x), P_g(x)\big) \cong P_g(x) \otimes E(\sigma x) \otimes \Gamma(y)$$

where σx has bidegree $(1, d)$ and y has bidegree $(2, dh)$ and is represented in the Hochschild complex $C_*\big(P_h(x), P_g(x)\big)$ by

$$\sum_{i=0}^{g-1} x^i \otimes x^{h-i-1} \otimes x.$$

The class σx is primitive and

$$\Delta(\gamma_k y) = \sum_{i+j=k} \gamma_i y \otimes_{P_g(x)} \gamma_j y.$$

Proof. The proof is similar to that of Proposition 3.3, using the same resolution $X_{*,*}$ of $P_h(x)$. The differential algebra $X_{*,*}$ admits a coproduct, defined as follows. The class σx is primitive and the coproduct on $\Gamma(\tau)$ is given by

$$\Delta(\gamma_k \tau) = \sum_{i+j=k} \gamma_i \tau \otimes_{A^{\mathrm{e}}} \gamma_j \tau.$$

By inspection this coproduct on $X_{*,*}$ is compatible with that on $C_*^{\mathrm{bar}} P_h(x)$ under a suitable homotopy equivalence. □

PROPOSITION 3.6. *Consider $P(u, u^{-1})$ as an algebra over $P(v, v^{-1})$ with $v = u^h$ for some h prime to p. Then the unit*

$$P(u, u^{-1}) \to \mathbb{H}_{*,*}^{P(v,v^{-1})}\big(P(u, u^{-1})\big)$$

is an isomorphism.

Proof. Let $A = P(u, u^{-1})$. We have that $A = P(v, v^{-1})[u]/(u^h = v)$ is flat as $P(v, v^{-1})$-module. The enveloping algebra is

$$A^{\mathrm{e}} = P(v, v^{-1})[1 \otimes u, u \otimes 1]/\big((1 \otimes u)^h = (u \otimes 1)^h = v\big).$$

There is a two-periodic resolution of A as A^{e}-module

$$\cdots \xrightarrow{T} A^{\mathrm{e}} \xrightarrow{N} A^{\mathrm{e}} \xrightarrow{T} A^{\mathrm{e}},$$

with augmentation $A^{\mathrm{e}} \to A$ given by the product of A. Here T is multiplication by $1 \otimes u - u \otimes 1$ and N is multiplication by $\big((1 \otimes u)^h - (u \otimes 1)^h\big)/(1 \otimes u - u \otimes 1)$. Applying $A \otimes_{A^{\mathrm{e}}} -$ we obtain a two-periodic chain complex

$$\cdots \xrightarrow{0} A \xrightarrow{hu^{h-1}} A \xrightarrow{0} A$$

quasi-isomorphic to the Hochschild complex. Since hu^{h-1} is invertible in A the proposition follows. □

Remark 3.7. Notice that the requirement that $hv^{\frac{h-1}{h}}$ be invertible in $P(v, v^{-1})[v^{\frac{1}{h}}]$ is equivalent to the requirement that the extension $P(v, v^{-1}) \hookrightarrow P(v, v^{-1})[v^{\frac{1}{h}}]$ be étale.

4. Topological Hochschild homology. The category of $\mathbb{S}$-modules (in the sense of [EKMM]) has a symmetric monoidal smash product. Suppose that R is a unital and commutative $\mathbb{S}$-algebra, that A is a unital R-algebra, and that M is an A-bimodule. We denote the symmetric monoidal smash-product in the category of R-modules by $\wedge_R$, or simply by $\wedge$ if $R = \mathbb{S}$. We implicitly assume in the sequel that the necessary cofibrancy conditions are satisfied. Following [MS], [EKMM], [SVW2] we define the topological Hochschild homology spectrum of the R-algebra A with coefficients in M as the realization of the simplicial spectrum $THH^R_\bullet(A, M)$ whose spectrum of q-simplices is

$$THH^R_q(A, M) = M \wedge_R A^{\wedge_R q},$$

with the usual Hochschild-type face and degeneracy maps. If $R = \mathbb{S}$ we just write THH for $THH^{\mathbb{S}}$.

From now on, we assume furthermore that A is commutative, that M is a commutative and unital A-algebra, whose A-bimodule structure is induced by the unit $e\colon A \to M$. Then $THH^R(A, M)$ is a unital and commutative M-algebra. The unit

$$\iota\colon M \to THH^R(A, M)$$

is given by inclusion of the 0-simplices $M = THH^R_0(A, M)$. The level-wise products $THH^R_q(A, M) \to M$ assemble into an augmentation

$$THH^R(A, M) \to M.$$

In particular M splits off from $THH^R(A, M)$.

This construction of $THH^R(A, M)$ is functorial in M. Moreover, if $L \to M \to N$ is a cofibration of A-bimodules, then there is a cofibre sequence of R-modules

$$THH^R(A, L) \to THH^R(A, M) \to THH^R(A, N).$$

We also have functoriality in A: if $A \to B \to M$ are maps of commutative R-algebras inducing the A- and B-bimodule structures on M, then there is a map of

M-algebras

$$THH^R(A, M) \to THH^R(B, M).$$

The M-algebra $THH^R(A, M)$ has, in the homotopy category, the structure of an M-bialgebra. To construct the coproduct

$$THH^R(A, M) \to THH^R(A, M) \wedge_M THH^R(A, M)$$

in the homotopy category, one can use the weak equivalence $THH^R(A, M) \simeq M \wedge_{A^e} B(A)$, where $B(A)$ is the two-sided bar construction, and take advantage of the coproduct $B(A) \to B(A) \wedge_{A^e} B(A)$ defined in the homotopy category (see [MSV], [AnR]).

The Bökstedt spectral sequence. Let E be a commutative $\mathbb{S}$-algebra, for instance $E = H\mathbb{F}_p$ or $H\mathbb{Z}_p$. Alternatively, one can take E such that E_*A is flat over E_*R. The skeletal filtration of $THH^R(A, M)$ induces in E_*-homology a conditionally convergent spectral sequence [EKMM, Th. 6.2 and 6.4]

$$E^2_{p,q}(A, M) = \mathrm{Tor}^{E_*(A^e)}_{p,q}(E_*A, E_*M) \Longrightarrow E_{p+q}THH^R(A, M).$$

We call this spectral sequence the Bökstedt spectral sequence, and denote it by $E^*_{*,*}(A, M)$, or just by $E^*_{*,*}(A)$ if $A = M$. If E_*A is flat over E_*R, we can identify the E^2-term of this spectral sequence with

$$E^2_{p,q}(A, M) = \mathbb{H}^{E_*R}_{p,q}(E_*A, E_*M).$$

In good cases, the rich structure of $THH^R(A, M)$ carries over to the Bökstedt spectral sequence. Indeed, the unit, the augmentation and the product of $THH^R(A, M)$ are compatible with the skeletal filtration, and the spectral sequence is one of unital and augmented E_*M-algebras. In particular the 0-th column $E^2_{0,*}(A, M)$ consists of permanent cycles and the edge homomorphism

$$E_*M \cong E^2_{0,*}(A, M) = E^{\infty}_{0,*}(A, M) \to E_*THH(A, M) \tag{4.1}$$

is a split injection. If $E^r_{*,*}(A, M)$ is flat over E_*M for all r, it is also a spectral sequence of E_*M-bialgebras. Finally, if moreover E_*E is flat over E_*, then E_*A and E_*M are E_*E-comodule algebras and the Bökstedt spectral sequence is one of E_*E-comodules E_*M-bialgebra.

On the E^2-term of the Bökstedt spectral sequence, these structures coincide with the corresponding structures for Hochschild homology described in the previous section. See [AnR] for a detailed discussion of these structures on topological Hochschild homology and on the Bökstedt spectral sequence.

The map σ. There is a map ω: $S^1_+ \wedge A \to THH^R(A)$ defined in [MS, Proposition 3.2], and which is induced by the S^1-action on the 0-simplices. The choice of a base point in S^1 provides a cofibre sequence

$$S^0 \xrightarrow{\eta} S^1_+ \xrightarrow{j} S^1,$$

and smashing it with A we obtain the cofibre sequence

$$A = S^0 \wedge A \xrightarrow{\eta\wedge 1} S^1_+ \wedge A \xrightarrow{j\wedge 1} S^1 \wedge A = \Sigma A.$$

The retraction $S^1_+ \to S^0$ induces a canonical retraction of $\eta \wedge 1$. Thus in the stable homotopy category the map $j \wedge 1$ has a canonical section κ: $\Sigma A \to S^1_+ \wedge A$. Composing it with ω we get a map denoted by σ: $\Sigma A \to THH(A)$. We will also denote by

$$\sigma\colon\ \Sigma A \to THH^R(A, M) \tag{4.2}$$

the composition of this map with the map $THH(A) \to THH(A, M)$ induced by the unit e: $A \to M$.

Let us assume that E_*A is flat over E_*R. The interplay between the homomorphism σ_*: $E_*A \to E_{*+1}THH^R(A, M)$ induced by (4.2) and the homomorphism σ: $E_*A \to \mathbb{H}^{E_*R}_{1,*}(E_*A, E_*M)$ given by (3.1) is described in the following proposition. We first specify a notation.

Notation 4.3. If w is an infinite cycle in $E^2_{*,*}(A, M)$, we denote by $[w]$ a class in $E_*THH^R(A, M)$ that represents w. We will only use this notation when the infinite cycle w uniquely determines a representative $[w]$, for a reason internal to the spectral sequence. Typically, such a reason is that w lies in filtration degree 1 (and the part of filtration degree 0 splits off from $E_*THH(A, M)$ naturally), or that there are no non-zero permanent cycles of lower filtration and same total degree than w.

Proposition 4.4. (McClure-Staffeldt) *For any $a \in E_*A$ we have*

$$\sigma_*(a) = [\sigma a]$$

in $E_{*+1}THH^R(A, M)$.

Proof. This is Proposition 3.2 of [MS]. □

In the case $R = \mathbb{S}$, the map σ has another useful feature: it commutes with the Dyer-Lashof operations. Let us denote by Q^i the Dyer-Lashof operation of degree $2i(p-1)$ on the mod p homology of a commutative $\mathbb{S}$-algebra.

PROPOSITION 4.5. (Bökstedt) *For any* $a \in H_*(A;\mathbb{F}_p)$ *we have*

$$Q^i\sigma_*(a) = \sigma_*(Q^i a)$$

in $H_{*+2i(p-1)+1}(THH(A);\mathbb{F}_p)$.

Proof. Bökstedt gives a proof of this proposition in [Bö2, Lemma 2.9]. His approach is to analyze the pth-reduced power of the map $S^1_+ \wedge A \to THH(A)$. Another elegant proof is presented by Angeltveit and Rognes in [AnR]. □

5. Topological Hochschild Homology of $\mathbb{Z}_p$. In a very influential but unpublished paper, Bökstedt [Bö2] computed the homotopy type of the $H\mathbb{Z}$-module $THH(H\mathbb{Z})$. In this section we present a simplified computation of $V(0)_*THH(H\mathbb{Z}_p)$ for $p \geqslant 3$, since we will need this result in the sequel.

We start by computing the Bökstedt spectral sequence

$$E^2_{s,t}(H\mathbb{Z}_p) = \mathbb{H}^{\mathbb{F}_p}_{s,t}\big(H_*(H\mathbb{Z}_p;\mathbb{F}_p)\big) \Longrightarrow H_{s+t}(THH(H\mathbb{Z}_p);\mathbb{F}_p). \tag{5.1}$$

The description of $H_*(H\mathbb{Z}_p;\mathbb{F}_p)$ given in (2.2) and Proposition 3.2 imply that the E^2-term of this spectral sequence is

$$H_*(H\mathbb{Z}_p;\mathbb{F}_p) \otimes E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2, \ldots) \otimes \Gamma(\sigma\bar{\tau}_1, \sigma\bar{\tau}_2, \ldots)$$

where $a \in H_*(H\mathbb{Z}_p;\mathbb{F}_p)$ has bidegree $(0,|a|)$, the class $\sigma\bar{\xi}_i$ has bidegree $(1, 2p^i - 2)$ and $\sigma\bar{\tau}_j$ has bidegree $(1, 2p^j - 1)$, for $i, j \geqslant 1$.

LEMMA 5.2. *There are multiplicative relations*

$$[\sigma\bar{\tau}_i]^p = [\sigma\bar{\tau}_{i+1}]$$

in $H_*(THH(H\mathbb{Z}_p);\mathbb{F}_p)$, *for all* $i \geqslant 1$.

Proof. This follows from the Dyer-Lashof operations $Q^{p^i}\bar{\tau}_i = \bar{\tau}_{i+1}$. By Propositions 4.4 and 4.5 we have

$$[\sigma\bar{\tau}_i]^p = \sigma_*(\bar{\tau}_i)^p = Q^{p^i}\sigma_*(\bar{\tau}_i) = \sigma_*(Q^{p^i}\bar{\tau}_i) = \sigma_*(\bar{\tau}_{i+1}) = [\sigma\bar{\tau}_{i+1}]. \qquad \square$$

LEMMA 5.3. *In the spectral sequence (5.1) we have* $d^r = 0$ *for* $2 \leqslant r \leqslant p-2$, *and there are differentials*

$$d^{p-1}(\gamma_{p+k}\sigma\bar{\tau}_i) \doteq \sigma\bar{\xi}_{i+1} \cdot \gamma_k\sigma\bar{\tau}_i$$

for all $i \geqslant 1$ and $k \geqslant 0$. Taking into account the algebra structure, this leaves

$$E^p_{*,*}(H\mathbb{Z}_p) = H_*(H\mathbb{Z}_p; \mathbb{F}_p) \otimes E(\sigma\bar{\xi}_1) \otimes P_p(\sigma\bar{\tau}_1, \sigma\bar{\tau}_2, \ldots).$$

At this stage the spectral sequence collapses.

Proof. The E^2-term of the spectral sequence (5.1) is flat over $H_*(H\mathbb{Z}_p; \mathbb{F}_p)$, so this is a spectral sequence of unital augmented A_*-comodule $H_*(H\mathbb{Z}_p; \mathbb{F}_p)$-bialgebras, at least until a differential puts an end to flatness. The mod p primary Bockstein β in the mod p homology of a ring spectrum is a derivation. If $i \geqslant 1$, we deduce from the relation

$$[\sigma\bar{\xi}_{i+1}] = \sigma_*(\bar{\xi}_{i+1}) = \sigma_*\big(\beta(\bar{\tau}_{i+1})\big) = \beta\big(\sigma_*(\bar{\tau}_{i+1})\big) = \beta([\sigma\bar{\tau}_{i+1}]) = \beta([\sigma\bar{\tau}_i]^p) = 0$$

in $H_*(THH(H\mathbb{Z}_p); \mathbb{F}_p)$ that $\sigma\bar{\xi}_{i+1}$ is not a permanent cycle in $E^r_{*,*}(H\mathbb{Z}_p)$. The rich algebra structure of this spectral sequence now only leaves enough freedom for the claimed pattern of differentials.

Let $r \geqslant 2$ be minimal with $d^r \neq 0$, and let $w \in E^r_{*,*}(H\mathbb{Z}_p) = E^2_{*,*}(H\mathbb{Z}_p)$ be a homogeneous algebra generator of minimal total degree $|w|$ with $d^r(w) \neq 0$. This requires w to lie in a filtration degree $\geqslant p$. By minimality of $|w|$ the class $d^r(w)$ is both an $H_*(H\mathbb{Z}_p; \mathbb{F}_p)$-coalgebra and an A_*-comodule primitive. The coalgebra primitives lie in filtration degree 0 or 1, which forces $r \geqslant p - 1$. This implies that $d^r = 0$ for $2 \leqslant r \leqslant p-2$. We prove the existence of the claimed differentials by induction on $i \geqslant 1$. Suppose given $i \geqslant 1$ and proven the claim that for any $1 \leqslant j \leqslant i-1$ and any $k \geqslant 0$ there is a differential

$$d^{p-1}(\gamma_{p+k}\sigma\bar{\tau}_j) \doteq \sigma\bar{\xi}_{j+1} \cdot \gamma_k\sigma\bar{\tau}_j$$

(for $i = 1$ the claim is empty). By induction hypothesis $E^p_{*,*}(H\mathbb{Z}_p)$ is a sub-quotient of the algebra

$$H_*(H\mathbb{Z}_p; \mathbb{F}_p) \otimes E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_{i+1}, \sigma\bar{\xi}_{i+2}, \ldots) \otimes P_p(\sigma\bar{\tau}_1, \ldots, \sigma\bar{\tau}_{i-1}) \otimes \Gamma(\sigma\bar{\tau}_i, \sigma\bar{\tau}_{i+1}, \cdots),$$

which we denote by $B_{*,*}$. Recall from the splitting (4.1) that classes in filtration degree 0 are permanent cycles. Hence, if $r \geqslant p$, classes x with $d^r(x) \neq 0$ lie in filtration degree $\geqslant p+1$. But algebra generators in $B_{*,*}$ of filtration degree $\geqslant p+1$ have total degree $\geqslant 2p^{i+2}$. Therefore, either all classes in $B_{*,*}$ of total degree $\leqslant 2p^{i+2}-2$ are permanent cycles for $E^*_{*,*}(H\mathbb{Z}_p)$, or the class $\gamma_p\sigma\bar{\tau}_i$ supports a nontrivial d^{p-1} differential. Since the class $\sigma\bar{\xi}_{i+1}$ of total degree $2p^{i+1} - 1$ is not a permanent cycle, the later is true. Knowing that $d^{p-1}(\gamma_p\sigma\bar{\tau}_i) \neq 0$, we can use the A_*-comodule structure to determine its value. The A_*-coaction on $\gamma_p\sigma\bar{\tau}_i$ is given by

$$\nu_*(\gamma_p\sigma\bar{\tau}_i) = \bar{\tau}_0 \otimes \sigma\bar{\xi}_i \cdot \gamma_{p-1}\sigma\bar{\tau}_i + 1 \otimes \gamma_p\sigma\bar{\tau}_i.$$

This can be computed using the A_*-comodule bialgebra structure and the formula

$$\nu_*(\sigma\bar{\tau}_i) = \bar{\tau}_0 \otimes \sigma\bar{\xi}_i + 1 \otimes \sigma\bar{\tau}_i$$

(see (5.6), or recall that $\sigma\bar{\tau}_i$ is represented by $1\otimes\bar{\tau}_i$ in $E^1_{1,*}(H\mathbb{Z}_p)$). Since $d^{p-1}(\sigma\bar{\xi}_i\cdot\gamma_{p-1}\sigma\bar{\tau}_i) = 0$, we deduce that $d^{p-1}(\gamma_p\sigma\bar{\tau}_i)$ is an A_*-comodule primitive. But in this bidegree the comodule primitives belong to $\mathbb{F}_p\{\sigma\bar{\xi}_{i+1}\}$, which implies that $d^{p-1}(\gamma_p\sigma\bar{\tau}_i) \doteq \sigma\bar{\xi}_{i+1}$. Finally, the differential on $\gamma_{p+k}\sigma\bar{\tau}_i$ for $k \geqslant 1$ is detected using the $H_*(H\mathbb{Z}_p;\mathbb{F}_p)$-coproduct on $E^{p-1}_{*,*}(H\mathbb{Z}_p)$. We deduce from the coproduct formulas of Proposition 3.2 that $d^{p-1}(\gamma_{p+k}\sigma\bar{\tau}_i) - \gamma_k\sigma\bar{\tau}_i \cdot \sigma\bar{\xi}_{i+1}$ is a coalgebra primitive element in filtration degree $k+1$. But all non-zero coalgebra primitives of $E^{p-1}_{*,*}(H\mathbb{Z}_p)$ lie in filtration degree 0 or 1, thus $d^{p-1}(\gamma_{p+k}\sigma\bar{\tau}_i) = \gamma_k\sigma\bar{\tau}_i \cdot \sigma\bar{\xi}_{i+1}$. This completes the induction step. There are no further possible differentials for bidegree reasons, and the spectral sequence collapses at the E^p-term. □

Remark 5.4. The idea to use the coproduct to show that $d^r = 0$ for $2 \leqslant r \leqslant p-2$ and to detect the differential on $\gamma_{p+k}\sigma\bar{\tau}_i$ for $k \geqslant 1$ is borrowed from [AnR]. Bökstedt's original argument to prove this lemma relies on a Kudo-type formula for differentials in the spectral sequence

$$\mathbb{H}^{\mathbb{F}_p}_{s,t}\big(H_*(A;\mathbb{F}_p)\big) \Longrightarrow H_*(THH(A);\mathbb{F}_p),$$

namely

$$d^{p-1}(\gamma_{p+k}\sigma x) = \sigma(\beta Q^{\frac{n+1}{2}} x) \cdot \gamma_k\sigma x$$

whenever $x \in H_*(A;\mathbb{F}_p)$ is a class of odd degree n. See [Bö2] or [Hu].

Proposition 5.5. *There is an isomorphism of A_*-comodule algebras*

$$H_*(THH(H\mathbb{Z}_p);\mathbb{F}_p) \cong H_*(H\mathbb{Z}_p;\mathbb{F}_p) \otimes E([\sigma\bar{\xi}_1]) \otimes P([\sigma\bar{\tau}_1]).$$

The A_*-coaction ν_* is given on the tensor factor $H_*(H\mathbb{Z}_p;\mathbb{F}_p)$ by the inclusion in the coalgebra A_*. The class $[\sigma\bar{\xi}_1]$ is primitive and

$$\nu_*([\sigma\bar{\tau}_1]) = 1 \otimes [\sigma\bar{\tau}_1] + \bar{\tau}_0 \otimes [\sigma\bar{\xi}_1].$$

Proof. By Lemma 5.3 the E^∞-term of the spectral sequence (5.1) is

$$E^\infty_{*,*}(H\mathbb{Z}_p) = H_*(H\mathbb{Z}_p;\mathbb{F}_p) \otimes E(\sigma\bar{\xi}_1) \otimes P_p(\sigma\bar{\tau}_1, \sigma\bar{\tau}_2, \ldots).$$

Lemma 5.2 implies that the subalgebra $P_p(\sigma\bar{\tau}_1, \sigma\bar{\tau}_2, \ldots)$ of $E^\infty_{*,*}(H\mathbb{Z}_p)$ lifts as a subalgebra $P([\sigma\bar{\tau}_1])$ of $H_*(THH(H\mathbb{Z}_p);\mathbb{F}_p)$. There are no further possible multiplicative extensions. The A_*-coaction on the tensor factor $H_*(H\mathbb{Z}_p;\mathbb{F}_p)$ is determined by naturality with respect to the unit map $H\mathbb{Z}_p \to THH(H\mathbb{Z}_p)$. The values of $\nu_*(\sigma\bar{\xi}_1)$ and $\nu_*(\sigma\bar{\tau}_1)$ follow by naturality with respect to $\sigma\colon \Sigma H\mathbb{Z}_p \to THH(H\mathbb{Z}_p)$, which is expressed in the formula

$$\nu_*\sigma_* = (1 \otimes \sigma_*)\nu_*. \tag{5.6}$$

We use that in $A_* \otimes H_*(H\mathbb{Z}_p; \mathbb{F}_p)$ we have

$$\nu_*(\bar{\xi}_1) = \bar{\xi}_1 \otimes 1 + 1 \otimes \bar{\xi}_1 \quad \text{and} \quad \nu_*(\bar{\tau}_1) = \bar{\tau}_1 \otimes 1 + 1 \otimes \bar{\tau}_1 + \bar{\tau}_0 \otimes \bar{\xi}_1,$$

and that σ_* is a derivation. □

THEOREM 5.7. (Bőkstedt) *For any prime $p \geqslant 3$ there is an isomorphism of $\mathbb{F}_p$-algebras*

$$V(0)_*THH(H\mathbb{Z}_p) \cong E(\lambda_1) \otimes P(\mu_1),$$

where $|\lambda_1| = 2p - 1$ and $|\mu_1| = 2p$.

Proof. The proof we give here is adapted from the proof of [AR, Proposition 2.6]. Since $H\mathbb{F}_p \simeq V(0) \wedge H\mathbb{Z}_p$, the spectrum $V(0) \wedge THH(H\mathbb{Z}_p)$ is an $H\mathbb{F}_p$-module. In particular the Hurewicz homomorphism

$$V(0)_*THH(H\mathbb{Z}_p) \to H_*(V(0) \wedge THH(H\mathbb{Z}_p); \mathbb{F}_p)$$

is an injection with image the A_*-comodule primitives. Let λ_1 and μ_1 be classes that map respectively to $[\sigma\bar{\xi}_1]$ and $[\sigma\bar{\tau}_1] - \bar{\tau}_0[\sigma\bar{\xi}_1]$ under this homomorphism. By inspection these classes generate the subalgebra of A_*-comodule primitive elements in $H_*(V(0) \wedge THH(H\mathbb{Z}_p); \mathbb{F}_p)$. □

Remark 5.8. Bökstedt proved also that there are higher mod p homotopy Bocksteins

$$\beta_{0,r}(\mu_1^{p^{r-1}}) \doteq \mu_1^{p^{r-1}-1}\lambda_1$$

in $V(0)_*THH(H\mathbb{Z}_p)$, for all $r \geqslant 1$. This implies a weak equivalence of $H\mathbb{Z}_p$-modules

$$THH(H\mathbb{Z}_p) \simeq_p H\mathbb{Z}_p \vee \bigvee_{k \geqslant 1} \Sigma^{2k-1} H\mathbb{Z}/p^{\upsilon_p(k)},$$

where υ_p is the p-adic valuation.

Let ℓ be the Adams summand defined in 2.1. A very similar computation can be performed for $THH(\ell)$, yielding a description of the $\mathbb{F}_p$-algebra $V(1)_*THH(\ell)$.

THEOREM 5.9. (McClure-Staffeldt) *For any prime $p \geqslant 3$ there are isomorphisms of $\mathbb{F}_p$-algebras*

$$H_*(THH(\ell);\mathbb{F}_p) \cong H_*(\ell;\mathbb{F}_p) \otimes E([\sigma\bar{\xi}_1],[\sigma\bar{\xi}_2]) \otimes P([\sigma\bar{\tau}_2])$$

and

$$V(1)_*THH(\ell) \cong E(\lambda_1,\lambda_2) \otimes P(\mu_2),$$

where $|\lambda_1| = 2p-1$, $|\lambda_2| = 2p^2-1$, and $|\mu_2| = 2p^2$.

Proof. The description of $V(1)_*THH(\ell)$ given here was made explicit in [AR, Proposition 2.6]. We briefly review this computation. The linearization map $\ell \to H\mathbb{Z}_p$ is injective in mod p homology and induces an injection on the E^2-terms of the respective Bökstedt spectral sequences. By comparison this determines both the differentials and the multiplicative extensions in the Bökstedt spectral sequence for ℓ, and the description of $H_*(THH(\ell);\mathbb{F}_p)$ given follows. Now there is an equivalence $V(1) \wedge \ell \simeq \mathbb{F}_p$, so $V(1) \wedge THH(\ell)$ is an $H\mathbb{F}_p$-module and the Hurewicz homomorphism

$$V(1)_*THH(\ell) \to H_*(V(1) \wedge THH(\ell);\mathbb{F}_p)$$

is injective with image the A_*-comodule primitives. The homotopy classes λ_1, λ_2 and μ_2 have as image the primitive homology classes $[\sigma\bar{\xi}_1]$, $[\sigma\bar{\xi}_2]$ and $[\sigma\bar{\tau}_2] - \bar{\tau}_0[\sigma\bar{\xi}_2]$. □

Remark 5.10. McClure and Staffeldt [MS, Corollary 7.2] computed $V(0)_*THH(\ell)$. We can reformulate their result in terms of the v_1 homotopy Bocksteins. Let

$$r(n) = \begin{cases} p^n + p^{n-2} + \ldots + p & \text{if } n \geqslant 1 \text{ is odd,} \\ p^n + p^{n-2} + \ldots + p^2 & \text{if } n \geqslant 2 \text{ is even, and} \end{cases}$$

$$s(n) = \begin{cases} 0 & \text{if } n = 1,2, \\ p^{n-2} - p^{n-3} + \ldots + p - 1 & \text{if } n \geqslant 3 \text{ is odd,} \\ p^{n-2} - p^{n-3} + \ldots + p^2 - p & \text{if } n \geqslant 4 \text{ is even.} \end{cases}$$

Then in $V(1)_*THH(\ell)$ there are v_1 Bocksteins

$$\beta_{1,r(n)}(\mu_2^{p^{n-1}}) \doteq \begin{cases} \lambda_1\mu_2^{s(n)} & \text{if } n \geqslant 1 \text{ is odd,} \\ \lambda_2\mu_2^{s(n)} & \text{if } n \geqslant 2 \text{ is even.} \end{cases}$$

6. The homotopy type of $THH(ku, H\mathbb{Z}_p)$. The linearization map $j\colon ku \to H\mathbb{Z}_p$ makes $H\mathbb{Z}_p$ into a commutative and unital ku-algebra. Our aim in this section is to determine the homotopy type of the $H\mathbb{Z}_p$-algebra $THH(ku, H\mathbb{Z}_p)$.

We first compute its mod p homology, using the Bökstedt spectral sequence

$$\begin{aligned} E^2_{s,t}(ku, H\mathbb{Z}_p) &= \mathbb{H}^{\mathbb{F}_p}_{s,t}\big(H_*(ku;\mathbb{F}_p), H_*(H\mathbb{Z}_p;\mathbb{F}_p)\big) \\ &\Longrightarrow H_{s+t}(THH(ku, H\mathbb{Z}_p);\mathbb{F}_p). \end{aligned} \tag{6.1}$$

The algebra homomorphism $j_*\colon H_*(ku;\mathbb{F}_p) \to H_*(H\mathbb{Z}_p;\mathbb{F}_p)$ is the edge homomorphism

$$H_*(ku;\mathbb{F}_p) \twoheadrightarrow E^\infty_{0,*} \subset E^2_{0,*} = H_*(H\mathbb{Z}_p;\mathbb{F}_p)$$

of the spectral sequence (2.4) described in the proof of Theorem 2.5. It is therefore given by

$$\begin{aligned} P(\bar\xi_1, \bar\xi_2, \dots) \otimes E(\bar\tau_2, \bar\tau_3, \dots) \otimes P_{p-1}(x) &\to P(\bar\xi_1, \bar\xi_2, \dots) \otimes E(\bar\tau_1, \bar\tau_2, \dots), \\ \bar\xi_i &\mapsto \bar\xi_i \ \text{ if } \ i \geqslant 1, \\ \bar\tau_i &\mapsto \bar\tau_i \ \text{ if } \ i \geqslant 2, \\ x &\mapsto 0. \end{aligned}$$

By Propositions 3.2 and 3.5, the E^2-term of (6.1) is

$$H_*(H\mathbb{Z}_p;\mathbb{F}_p) \otimes E(\sigma x, \sigma\bar\xi_1, \sigma\bar\xi_2, \dots) \otimes \Gamma(y, \sigma\bar\tau_2, \sigma\bar\tau_3, \dots)$$

where $a \in H_*(H\mathbb{Z}_p;\mathbb{F}_p)$ has bidegree $(0, |a|)$, a class $\sigma\omega$ for $\omega \in H_*(ku;\mathbb{F}_p)$ has bidegree $(1, |\omega|)$, and y has bidegree $(2, 2p-2)$.

Recall that a class $\sigma\omega$ is represented in the Hochschild complex by $1 \otimes \omega$ and y is represented by $1 \otimes x^{p-2} \otimes x$.

Lemma 6.2. *The classes $\sigma\bar\xi_1$ and y in $E^2_{*,*}(ku, H\mathbb{Z}_p)$ are permanent cycles, and in $H_*(THH(ku, H\mathbb{Z}_p);\mathbb{F}_p)$ there is a primary mod p Bockstein*

$$\beta([y]) \doteq [\sigma\bar\xi_1].$$

Proof. To detect the mod p Bockstein claimed in this lemma we will need some knowledge of the integral homology of $THH(ku, H\mathbb{Z}_p)$. For integral computations it is more convenient to work with $THH^{\mathbb{S}_p}(ku, H\mathbb{Z}_p)$, because in this way the ground ring for the Bökstedt spectral sequence is $H_*(\mathbb{S}_p;\mathbb{Z}) = \mathbb{Z}_p$ instead of $H_*(\mathbb{S};\mathbb{Z}) = \mathbb{Z}$ (recall that $H_*(ku;\mathbb{Z})$ and $H_*(H\mathbb{Z}_p;\mathbb{Z})$ are $\mathbb{Z}_p$-algebras). The natural map $THH(ku, H\mathbb{Z}_p) \to THH^{\mathbb{S}_p}(ku, H\mathbb{Z}_p)$ is an equivalence after p-completion, and induces an isomorphism of the mod p homology Bökstedt spectral sequences. It follows that the mod p homology Bockstein spectral sequences for $THH(ku, H\mathbb{Z}_p)$ and $THH^{\mathbb{S}_p}(ku, H\mathbb{Z}_p)$ are also isomorphic.

The class $\sigma\bar{\xi}_1$ is a permanent cycle for bidegree reasons. On the other hand y generates the component of total degree $2p$ in the E^2-term of the Bökstedt spectral sequence (6.1). We claim that $[\sigma\bar{\xi}_1]$ is the mod p reduction of a class of order p in integral homology. Then $[\sigma\bar{\xi}_1]$ must be in the image of the primary mod p Bockstein, and this forces $\beta([y]) \doteq [\sigma\bar{\xi}_1]$. In particular y is also a permanent cycle.

It remains to prove the claim that $[\sigma\bar{\xi}_1]$ is the mod p reduction of a class of order p in integral homology. Consider the commutative diagram

$$\begin{array}{ccc} H_*(ku;\mathbb{Z}) & \xrightarrow{\sigma_*} & H_{*+1}(THH^{\mathbb{S}_p}(ku, H\mathbb{Z}_p);\mathbb{Z}) \\ \rho\downarrow & & \rho\downarrow \\ H_*(ku;\mathbb{F}_p) & \xrightarrow{\sigma_*} & H_{*+1}(THH^{\mathbb{S}_p}(ku, H\mathbb{Z}_p);\mathbb{F}_p) \end{array}$$

where ρ is the mod p reduction. If $\tilde{\xi}_1 \in H_{2p-2}(ku;\mathbb{Z})$ is the class defined in 2.7, then

$$\rho\sigma_*(\tilde{\xi}_1) = \sigma_*\rho(\tilde{\xi}_1) = \sigma_*(\bar{\xi}_1) = [\sigma\bar{\xi}_1]$$

in $H_{2p-1}(THH^{\mathbb{S}_p}(ku, H\mathbb{Z}_p);\mathbb{F}_p)$. In particular $[\sigma\bar{\xi}_1]$ is the reduction of an integral class. We now prove that $pH_{2p-1}(THH^{\mathbb{S}_p}(ku, H\mathbb{Z}_p);\mathbb{Z}) = 0$, which implies the claim.

The Bökstedt spectral sequence converging to $H_*(THH^{\mathbb{S}_p}(ku, H\mathbb{Z}_p);\mathbb{Z})$ has an E^2-term given by

$$\tilde{E}^2_{*,*}(ku, H\mathbb{Z}_p) = \mathrm{Tor}_{*,*}^{H_*(ku\wedge_{\mathbb{S}_p} ku;\mathbb{Z})}\big(H_*(ku;\mathbb{Z}), H_*(H\mathbb{Z}_p;\mathbb{Z})\big). \tag{6.3}$$

Recall the graded ring homomorphism $\lambda\colon \Lambda_* \to H_*(ku;\mathbb{Z})$ defined in Proposition 2.7. Since Λ_* is torsion free and λ is an isomorphism in degrees $\leqslant 2p^2-3$, the map

$$\Lambda^{\mathrm{e}}_* = \Lambda_* \otimes_{\mathbb{Z}_p} \Lambda_* \to H_*(ku \wedge_{\mathbb{S}_p} ku;\mathbb{Z})$$

is also an isomorphism in this range of degrees. In particular, the Tor group of (6.3) is isomorphic to

$$\mathbb{H}^{\mathbb{Z}_p}_{*,*}\big(\Lambda_*, H_*(H\mathbb{Z}_p;\mathbb{Z})\big)$$

in total degrees $\leqslant 2p^2-3$. Here the Λ_*-bimodule structure of $H_*(H\mathbb{Z}_p;\mathbb{Z})$ is given by the ring homomorphism $\Lambda_* \to H_*(H\mathbb{Z}_p;\mathbb{Z})$ that sends $\tilde{x}$ to 0 and $\tilde{\xi}_1$ to a lift of $\bar{\xi}_1$ in $H_*(H\mathbb{Z}_p;\mathbb{Z})$. There is a free resolution X_* of Λ_* as Λ^{e}_*-module

$$0 \longrightarrow \Lambda^{\mathrm{e}}_*\{w\} \xrightarrow{d_2} \Lambda^{\mathrm{e}}_*\{\sigma\tilde{x}, \sigma\tilde{\xi}_1\} \xrightarrow{d_1} \Lambda^{\mathrm{e}}_*$$

having as augmentation the product $\Lambda^e_* \to \Lambda_*$. The bidegree of the generators is $|\sigma\tilde{x}| = (1, 2)$, $|\sigma\tilde{\xi}_1| = (1, 2p-2)$ and $|w| = (2, 2p-2)$. The differential is given by

$$\begin{aligned} d_1(\sigma\tilde{x}) &= 1 \otimes \tilde{x} - \tilde{x} \otimes 1, \\ d_1(\sigma\tilde{\xi}_1) &= 1 \otimes \tilde{\xi}_1 - \tilde{\xi}_1 \otimes 1, \quad \text{and} \\ d_2(w) &= \big((1 \otimes \tilde{x}^{p-1} - \tilde{x}^{p-1} \otimes 1)/(1 \otimes \tilde{x} - \tilde{x} \otimes 1)\big)\sigma\tilde{x} - p\sigma\tilde{\xi}_1. \end{aligned}$$

In total degrees $\leqslant 2p^2 - 3$, the E^2-term (6.3) is isomorphic to

$$\mathbb{H}^{\mathbb{Z}_p}_{*,*}\big(\Lambda_*, H_*(H\mathbb{Z}_p; \mathbb{Z})\big) = H_*(H_*(H\mathbb{Z}_p; \mathbb{Z}) \otimes_{\Lambda^e_*} X_*).$$

By inspection we have $\tilde{E}^2_{1,2p-2}(ku, H\mathbb{Z}_p) = \mathbb{F}_p\{\sigma\tilde{\xi}_1\}$, and the remaining groups in $\tilde{E}^2_{*,*}(ku, H\mathbb{Z}_p)$ of total degree $2p-1$ are all trivial. For degree reasons there are no differentials affecting $\tilde{E}^2_{1,2p-2}(ku, H\mathbb{Z}_p)$. This proves that

$$H_{2p-1}(THH^{\mathbb{S}_p}(ku, H\mathbb{Z}_p); \mathbb{Z}) \cong \mathbb{F}_p\{[\sigma\tilde{\xi}_1]\}. \qquad \square$$

Convention 6.4. We assume in the sequel that the class $y \in E^2_{2,2p-2}(ku, H\mathbb{Z}_p)$ has been chosen so that the equality $\beta([y]) = [\sigma\bar{\xi}_1]$ in $H_*(THH(ku, H\mathbb{Z}_p); \mathbb{F}_p)$ holds strictly, not just up to a unit.

LEMMA 6.5. *There are multiplicative relations*

$$[y]^p = [\sigma\bar{\tau}_2] \quad \textit{and} \quad [\sigma\bar{\tau}_i]^p = [\sigma\bar{\tau}_{i+1}]$$

in $H_*(THH(ku, H\mathbb{Z}_p); \mathbb{F}_p)$*, for all* $i \geqslant 2$.

Proof. The linearization $j\colon ku \to H\mathbb{Z}_p$ induces a map of $H\mathbb{Z}_p$-algebras

$$j\colon THH(ku, H\mathbb{Z}_p) \to THH(H\mathbb{Z}_p).$$

By naturality of σ, we have $j_*([\sigma\bar{\xi}_1]) = [\sigma\bar{\xi}_1]$ and $j_*([\sigma\bar{\tau}_i]) = [\sigma\bar{\tau}_i]$ for all $i \geqslant 2$. The Bockstein $\beta\colon H_{2p}(THH(H\mathbb{Z}_p); \mathbb{F}_p) \to H_{2p-1}(THH(H\mathbb{Z}_p); \mathbb{F}_p)$ is injective and maps $[\sigma\bar{\tau}_1]$ to $[\sigma\bar{\xi}_1]$. The relation $\beta([y]) = [\sigma\bar{\xi}_1]$ implies that $j_*([y]) = [\sigma\bar{\tau}_1]$. We have seen in Lemma 5.2 that $[\sigma\bar{\tau}_1]^p = [\sigma\bar{\tau}_2]$, and therefore $j_*([y]^p) = [\sigma\bar{\tau}_2]$. From the structure of $E^2_{*,*}(ku, H\mathbb{Z}_p)$ we deduce that

$$[y]^p \in \mathbb{F}_p\{[\sigma\bar{\tau}_2], [\bar{\xi}_1^{p-1}\bar{\tau}_1 \cdot \sigma\bar{\xi}_1]\}.$$

The restriction of j_* to this vector space is injective, and this proves that $[y]^p = [\sigma\bar{\tau}_2]$. The remaining multiplicative relations follow from the Dyer-Lashof operations on $\bar{\tau}_i$ and are proven as in Lemma 5.2. $\square$

LEMMA 6.6. *The Bökstedt spectral sequence (6.1) behaves as follows.*
(a) *For $2 \leqslant r \leqslant p-2$ we have $d^r = 0$, and there is a differential*

$$d^{p-1}(\gamma_{p+k}\sigma\bar{\tau}_i) \doteq \sigma\bar{\xi}_{i+1} \cdot \gamma_k\sigma\bar{\tau}_i$$

for all $i \geqslant 2$ and $k \geqslant 0$. Taking into account the algebra structure, this leaves

$$E^p_{*,*}(ku, H\mathbb{Z}_p) = H_*(H\mathbb{Z}_p; \mathbb{F}_p) \otimes E(\sigma x, \sigma\bar{\xi}_1, \sigma\bar{\xi}_2) \otimes P_p(\sigma\bar{\tau}_2, \sigma\bar{\tau}_3, \ldots) \otimes \Gamma(y).$$

(b) *For $p \leqslant r \leqslant 2p-2$ we have $d^r = 0$, and there is a differential*

$$d^{2p-1}(\gamma_{p+k}y) \doteq \sigma\bar{\xi}_2 \cdot \gamma_k y$$

for all $k \geqslant 0$. Taking into account the algebra structure, this leaves

$$E^{2p}_{*,*}(ku, H\mathbb{Z}_p) = H_*(H\mathbb{Z}_p; \mathbb{F}_p) \otimes E(\sigma x, \sigma\bar{\xi}_1) \otimes P_p(y, \sigma\bar{\tau}_2, \sigma\bar{\tau}_3, \ldots).$$

At this stage the spectral sequence collapses.

Proof. The proof is similar to that of Lemma 5.3. The spectral sequence is one of unital and augmented A_*-comodule $H_*(H\mathbb{Z}_p; \mathbb{F}_p)$-bialgebras, at least as long as the flatness requirement is fulfilled. Here also the coproduct and the coaction can be used to prove that $d^r = 0$ for $2 \leqslant r \leqslant p-2$. The non-zero classes that are both coalgebra and comodule primitives lie in filtration degree 0 or 1, except for the classes in $\mathbb{F}_p\{y\}$, which are permanent cycles by Lemma 6.2. On the other hand our given algebra generators of $E^2_{*,*}(ku, H\mathbb{Z}_p)$ lie in filtration degree 0, p^k or $2p^k$ for $k \geqslant 0$. Thus $d^r = 0$ for $2 \leqslant r \leqslant p-2$.

The differentials given in (a) can be detected by naturality with respect to the map j: $THH(ku, H\mathbb{Z}_p) \to THH(H\mathbb{Z}_p)$. Indeed, the induced homomorphism

$$j^2\colon\ E^2_{*,*}(ku, H\mathbb{Z}) \to E^2_{*,*}(H\mathbb{Z})$$

is injective on the tensor factor $H_*(H\mathbb{Z}; \mathbb{F}_p) \otimes E(\sigma\bar{\xi}_1, \ldots) \otimes \Gamma(\sigma\bar{\tau}_2, \ldots)$, with image the classes of same name, and maps the generators σx and $\gamma_{p^k}(y)$ for $k \geqslant 0$ to zero. The coalgebra and comodule structures imply that $d^{p-1}(\gamma_{p^k}y) = 0$ for all $k \geqslant 1$, and this proves that $E^p_{*,*}(ku, H\mathbb{Z}_p)$ is as claimed. Notice that $E^p_{*,*}(ku, H\mathbb{Z}_p)$ is flat over $H_*(H\mathbb{Z}_p; \mathbb{F}_p)$, and has an induced $H_*(H\mathbb{Z}_p; \mathbb{F}_p)$-coproduct structure. This structure implies that $d^r = 0$ for $p \leqslant r \leqslant 2p-2$. Because of the relation

$$[\sigma\bar{\xi}_2] = \beta([\sigma\bar{\tau}_2]) = \beta([y]^p) = 0$$

in $H_*(THH(ku, H\mathbb{Z}_p); \mathbb{F}_p)$, the class $\sigma\bar{\xi}_2 \in E^{2p-1}_{1,2p^2-2}(ku, H\mathbb{Z}_p)$ is a boundary at some stage. The set of classes in $E^{2p-1}_{*,*}(ku, H\mathbb{Z}_p)$ of total degree $2p^2$ and filtration degree $\geqslant 2p$ is equal to $\mathbb{F}_p\{\gamma_p y\}$. This implies the existence of a differential

$$d^{2p-1}(\gamma_p y) \doteq \sigma\bar{\xi}_2.$$

The differential $d^{2p-1}(\gamma_{p+k} y) \doteq \sigma\bar{\xi}_2 \cdot \gamma_k y$ for $k \geqslant 1$ is then detected using the coproduct on $\gamma_{p+k} y$. This leaves the E^{2p}-term as given in (b), where all homogeneous generators lie in filtration degrees less than $2p$, and the spectral sequence collapses. □

PROPOSITION 6.7. *There is an isomorphism of A_*-comodule algebras*

$$H_*(THH(ku, H\mathbb{Z}_p); \mathbb{F}_p) \cong H_*(H\mathbb{Z}_p; \mathbb{F}_p) \otimes E([\sigma x], [\sigma\bar{\xi}_1]) \otimes P([y])$$

where $[\sigma x]$ and $[\sigma\bar{\xi}_1]$ are A_*-comodule primitives and the coaction on $[y]$ is

$$\nu_*([y]) = \bar{\tau}_0 \otimes [\sigma\bar{\xi}_1] + 1 \otimes [y].$$

Proof. The Bökstedt spectral sequence described in Lemma 6.6 is strongly convergent and has an E^∞-term given by

$$E^\infty_{*,*}(ku, H\mathbb{Z}_p) = H_*(H\mathbb{Z}_p; \mathbb{F}_p) \otimes E(\sigma x, \sigma\bar{\xi}_1) \otimes P_p(y, \sigma\bar{\tau}_2, \sigma\bar{\tau}_3, \dots).$$

We have the multiplicative extensions $[y]^p = [\sigma\bar{\tau}_2]$ and $[\sigma\bar{\tau}_i]^p = [\sigma\bar{\tau}_{i+1}]$ established in Lemma 6.5. There are no further possible multiplicative extensions. The classes $[\sigma x]$ and $[\sigma\bar{\xi}_1]$ are comodule primitives by (5.6). The homomorphism

$$j_*\colon\ H_*(THH(ku, H\mathbb{Z}_p); \mathbb{F}_p) \to H_*(THH(H\mathbb{Z}_p); \mathbb{F}_p)$$

maps the class $[y]$ to $[\sigma\bar{\tau}_1]$, which has coaction $\nu_*([\sigma\bar{\tau}_1]) = \bar{\tau}_0 \otimes [\sigma\bar{\xi}_1] + 1 \otimes [\sigma\bar{\tau}_1]$. The formula for the coaction on $[y]$ follows by naturality because j_* is injective in the relevant degrees. □

THEOREM 6.8. *For any prime $p \geqslant 3$ there is an isomorphism of $\mathbb{F}_p$-algebras*

$$V(0)_* THH(ku, H\mathbb{Z}_p) \cong E(z, \lambda_1) \otimes P(\mu_1)$$

with $|z| = 3$, $|\lambda_1| = 2p - 1$ and $|\mu_1| = 2p$.

Proof. The proof is the same as for Theorem 5.7. Here z, λ_1 and μ_1 map respectively to $[\sigma x]$, $[\sigma\bar{\xi}_1]$ and $[y] - \bar{\tau}_0[\sigma\bar{\xi}_1]$ under the Hurewicz homomorphism

$$V(0)_* THH(ku, H\mathbb{Z}_p) \to H_*(V(0) \wedge THH(ku, H\mathbb{Z}_p); \mathbb{F}_p). \qquad \square$$

Notice that as in Remark 5.8 we have higher mod p Bocksteins

$$\beta_{0,r}(\mu_1^{p^{r-1}}) \doteq \mu_1^{p^{r-1}-1}\lambda_1$$

in $V(0)_*THH(ku, H\mathbb{Z}_p)$, for all $r \geqslant 1$.

COROLLARY 6.9. *For any prime $p \geqslant 3$, there is a weak equivalence*

$$THH(ku, H\mathbb{Z}_p) \simeq_p S^3_+ \wedge THH(H\mathbb{Z}_p).$$

We do not have a preferred map for this equivalence. Thus this corollary merely states that these two $H\mathbb{Z}_p$-modules have abstractly isomorphic homotopy groups.

7. The mod p homotopy groups of $THH(ku)$. In this section we compute $V(0)_*THH(ku)$ as a module over $P(u) = V(0)_*ku$. The strategy we use is similar to that developed by McClure and Staffeldt [MS] for computing $V(0)_*THH(\ell)$ as a $P(v_1)$-module, except that we use the mod u Bockstein spectral sequence

$$E^1_* = V(0)_*THH(ku, H\mathbb{Z}_p) \Longrightarrow \big(V(0)_*THH(ku)/(u\text{-torsion})\big) \otimes_{P(u)} \mathbb{F}_p \tag{7.1}$$

instead of the Adams spectral sequence.

PROPOSITION 7.2. *Let X be a connective ku-module such that $V(0)_*X$ is finite in each dimension. There is a one-column, strongly convergent spectral sequence*

$$E^1_* = V(0)_*(H\mathbb{Z}_p \wedge_{ku} X) \Longrightarrow \big(V(0)_*X/(u\text{-}torsion)\big) \otimes_{P(u)} \mathbb{F}_p,$$

called the mod u Bockstein spectral sequence. Its rth differential is denoted $\beta_{u,r}$ and decreases degree by $2r+1$. There is an isomorphism of $P(u)$-modules

$$V(0)_*X \cong P(u) \otimes E^\infty_* \oplus \bigoplus_{r\geqslant 1} P_r(u) \otimes \mathrm{im}\,(\beta_{u,r}).$$

Moreover, if X is a ku-algebra, then this is a spectral sequence of algebras.

Remark 7.3. In the target group of this spectral sequence, the graded $P(u)$-module structure of $\mathbb{F}_p$ is given by the augmentation $P(u) \to \mathbb{F}_p$ viewed as a map of graded rings.

Proof. This is very similar to the mod p Bockstein spectral sequence, see for instance [Mc, Theorem 10.3]. We just sketch the proof. Consider the diagram (2.3) and prolong it to the right by desuspending. Applying $V(0)_*(-\wedge_{ku}X)$ we obtain an unrolled exact couple. Placing $V(0)_*(\Sigma^{2s}H\mathbb{Z}_p \wedge_{ku} X)$ in filtration degree $-2s$, it yields a spectral sequence

$$V(0)_*(H\mathbb{Z}_p \wedge_{ku} X) \otimes P(u, u^{-1}) \Longrightarrow V(0)_*X \otimes_{P(u)} P(u, u^{-1}).$$

Here the class u represents the Bott element and has bidegree $(-2, 4)$. Strong convergence follows from the assumptions on X. This spectral sequence is one of differential $P(u, u^{-1})$-modules. In particular all columns are isomorphic at each stage. Extracting the column of filtration 0 and taking $\beta_{u,r} = u^{-r}d^r$, we obtain the mod u Bockstein spectral sequence.

The ring $P(u)$ is a graded principal ideal domain. Since by assumption the graded module $V(0)_*X$ is finite in each positive degree and trivial in negative degrees, it splits as a sum of shifted copies of $P(u)$ and its truncations (namely the quotients by an ideal generated by a homogeneous element). If there is a differential $\beta_{u,r}(a) = b$, then by definition of $\beta_{u,r}$ the class b is the image under

$$V(0)_*X \to V(0)_*(H\mathbb{Z}_p \wedge_{ku} X)$$

of a class $\tilde{b}$ not divisible by u, such that $u^{r-1}\tilde{b} \neq 0$ and $u^r\tilde{b} = 0$. The description of the $P(u)$-module $V(0)_*X$ given follows. Finally, if X is a ku-algebra, then our unrolled exact couple is part of a multiplicative Cartan-Eilenberg system (see [CE], XV.7) with

$$H(p, q) = V(0)_*\big((\Sigma^p ku/\Sigma^q ku) \wedge_{ku} X)\big),$$

and hence this spectral sequence is one of differential algebras. □

Let $K(1)$ be the Morava K-theory, with coefficients $K(1)_* = P(v_1, v_1^{-1})$. We compute $K(1)_*THH(ku)$, which allows us to determine the E^∞-term of (7.1). It will then turn out that only one pattern of differentials is possible.

PROPOSITION 7.4. *There is an isomorphism of $K(1)_*$-algebras*

$$K(1)_*ku \cong P(u, u^{-1}) \otimes K(1)_0\ell.$$

Here u is the Hurewicz image of the Bott element and on the right-hand side the $K(1)_*$-module structure is given by the inclusion $K(1)_* \to P(u, u^{-1})$ with $v_1 = u^{p-1}$.

Proof. The isomorphism

$$K(1)_*\ell \cong K(1)_* \otimes K(1)_0\ell$$

is established in [MS, Proposition 5.3(a)]. The splitting $ku \simeq \bigvee_{i=0}^{p-2} \Sigma^{2i}\ell$ implies that the formula claimed for $K(1)_*ku$ holds additively. The multiplication-by-u map $\Sigma^2 ku \to ku$ induces an isomorphism $K(1)_{*-2}ku \cong K(1)_*ku$ since for its cofibre $H\mathbb{Z}_p$ we have $K(1)_*H\mathbb{Z}_p = 0$. Thus multiplication by u is invertible in $K(1)_*ku$. The relation $v_1 = u^{p-1}$ follows from the corresponding relation in $V(0)_*ku$. $\square$

THEOREM 7.5. *The unit map $ku \to THH(ku)$ induces isomorphisms*

$$K(1)_*ku \xrightarrow{\cong} K(1)_*THH(ku)$$

and

$$v_1^{-1}V(0)_*ku \xrightarrow{\cong} v_1^{-1}V(0)_*THH(ku).$$

Proof. McClure and Staffeldt [MS, Th. 5.1 and Cor. 5.2] prove the corresponding statements for ℓ. Their argument extends to this case, and we just outline it, referring to [MS] for further details. By Proposition 7.4 we have an isomorphism

$$\mathbb{H}_{*,*}^{K(1)_*}\big(P(u,u^{-1})\big) \otimes \mathbb{H}_{*,*}^{\mathbb{F}_p}(K(1)_0\ell) \xrightarrow{\cong} \mathbb{H}_{*,*}^{K(1)_*}(K(1)_*ku).$$

By Proposition 3.6 and [MS, Proposition 5.3(c)] the unit for each of the tensor factors on the left-hand side is an isomorphism. This implies that the unit

$$K(1)_*ku \to \mathbb{H}_{*,*}^{K(1)_*}(K(1)_*ku)$$

is an isomorphism. The isomorphism $K(1)_*ku \cong K(1)_*THH(ku)$ follows from the collapse of the Bökstedt spectral sequence

$$E_{s,t}^2(ku) = \mathbb{H}_{s,t}^{K(1)_*}(K(1)_*ku) \Longrightarrow K(1)_{s+t}THH(ku).$$

Finally, by Lemma 5.4 of [MS] the first isomorphism claimed implies the second one. $\square$

Definition 7.6. Let

$$a(n) = \begin{cases} 0 & \text{if } n \leqslant -1, \\ p-2 & \text{if } n = 0, \\ p^{n+1} - p^n + p^{n-1} - \ldots + p^2 - p & \text{if } n \geqslant 1 \text{ is odd}, \\ p^{n+1} - p^n + p^{n-1} - \ldots + p^3 - p^2 + p - 2 & \text{if } n \geqslant 2 \text{ is even, and} \end{cases}$$

$$b(n) = \begin{cases} 0 & \text{if } n \leqslant 1, \\ p^{n-1} - p^{n-2} + \ldots + p - 1 & \text{if } n \geqslant 2 \text{ is even}, \\ p^{n-1} - p^{n-2} + \ldots + p^2 - p & \text{if } n \geqslant 3 \text{ is odd}. \end{cases}$$

We are now ready to describe the differentials in the spectral sequence (7.1). By Theorem 6.8 its E^1-term is

$$E^1_* = V(0)_* THH(ku, H\mathbb{Z}_p) = E(z, \lambda_1) \otimes P(\mu_1) \tag{7.7}$$

with $|z| = 3$, $|\lambda_1| = 2p - 1$ and $|\mu_1| = 2p$.

LEMMA 7.8. *The mod u Bockstein spectral sequence (7.1) for THH(ku) has an $E^{a(n)}$-term given by*

$$E^{a(n)}_* = \begin{cases} P(\mu_1^{p^n}) \otimes E(z\mu_1^{b(n)}, \lambda_1\mu_1^{b(n+1)}) & \text{if } n \geqslant 0 \text{ even,} \\ P(\mu_1^{p^n}) \otimes E(z\mu_1^{b(n+1)}, \lambda_1\mu_1^{b(n)}) & \text{if } n \geqslant 1 \text{ odd,} \end{cases}$$

and has differentials

$$\beta_{u,a(n)}(\mu_1^{p^n}) \doteq \begin{cases} z\mu_1^{b(n)} & \text{if } n \geqslant 0 \text{ is even,} \\ \lambda_1\mu_1^{b(n)} & \text{if } n \geqslant 1 \text{ is odd.} \end{cases}$$

Proof. Since we have the relation $v_1 = u^{p-1}$ in $V(0)_*ku$, we deduce from Theorem 7.5 that $u^{-1}V(0)_*THH(ku) = P(u, u^{-1})$. In particular the E^∞-term consists solely of a copy of $\mathbb{F}_p$ in degree 0, and $V(0)_0 THH(ku, H\mathbb{Z}_p) = \mathbb{F}_p\{1\}$ is the subgroup of permanent cycles in E^1_*. We prove the theorem by induction on even values of n, checking the claims for n and $n+1$ in a single induction step. Assume we have proven that for some even $n \geqslant 0$ we have

$$E^{a(n-1)+1}_* = P(\mu_1^{p^n}) \otimes E(z\mu_1^{b(n)}, \lambda_1\mu_1^{b(n+1)}).$$

For $n = 0$ this is given by (7.7). The algebra generators of $E^{a(n-1)+1}_*$ have degree

$$\begin{aligned} |\mu_1^{p^n}| &= 2p^{n+1}, \\ |z\mu_1^{b(n)}| &= 2b(n+1) + 3, \text{ and} \\ |\lambda_1\mu_1^{b(n+1)}| &= 2b(n+2) + 1. \end{aligned}$$

For degree reasons $z\mu_1^{b(n)}$ and $\lambda_1\mu_1^{b(n+1)}$ are infinite cycles. Indeed, a $\beta_{u,r}$ differential lowers degree by $2r + 1$, so there cannot be any differential of the form $\beta_{u,r}(\lambda_1\mu_1^{b(n+1)}) \doteq z\mu_1^{b(n)}$ (the two classes involved lie in odd degrees). The only other possibilities would be of the form $\beta_{u,r}(\lambda_1\mu_1^{b(n+1)}) \doteq 1$ or $\beta_{u,r}(z\mu_1^{b(n)}) \doteq 1$, which we can exclude since 1 is a permanent cycle. On the other hand, 1 and its scalar multiples are the only permanent cycles. Thus $\mu_1^{p^n}$ supports a differential, hitting either a scalar multiple of $z\mu_1^{b(n)}$ or of $\lambda_1\mu_1^{b(n+1)}$ (the class $z\mu_1^{b(n)} \cdot \lambda_1\mu_1^{b(n+1)}$

has higher degree than $\mu_1^{p^n}$). We have

$$|\mu_1^{p^n}| - |z\mu_1^{b(n)}| = 2a(n) + 1, \text{ and}$$
$$|\mu_1^{p^n}| - |\lambda_1\mu_1^{b(n+1)}| = 2a(n-1) + 1.$$

The second equality excludes the possibility of a differential $\beta_{u,r}(\mu_1^{p^n}) \doteq \lambda_1\mu_1^{b(n+1)}$ at this stage since it requires $r = a(n-1)$. It follows that $\beta_{u,r} = 0$ for $a(n-1)+1 \leqslant r < a(n)$, that $E_*^{a(n)} = E_*^{a(n-1)+1}$, and that the differential on $\mu_1^{p^n}$ we are looking for is

$$\beta_{u,a(n)}(\mu_1^{p^n}) \doteq z\mu_1^{b(n)}.$$

This leaves

$$E_*^{a(n)+1} = P(\mu_1^{p^{n+1}}) \otimes E(z\mu_1^{b(n+2)}, \lambda_1\mu_1^{b(n+1)}).$$

We can now repeat this argument, using degree considerations and the fact that the permanent cycles all lie in degree 0, to prove that $E_*^{a(n+1)} = E_*^{a(n)+1}$ and that there is a differential

$$\beta_{u,a(n+1)}(\mu_1^{p^{n+1}}) \doteq \lambda_1\mu_1^{b(n+1)}$$

leaving

$$E_*^{a(n+1)+1} = P(\mu_1^{p^{n+2}}) \otimes E(z\mu_1^{b(n+2)}, \lambda_1\mu_1^{b(n+3)}).$$

This completes the induction step. □

THEOREM 7.9. *Let p be an odd prime. There is an isomorphism of $P(u)$-modules*

$$V(0)_*THH(ku) \cong P(u) \oplus \bigoplus_{n\geqslant 0} P_{a(n)}(u) \otimes I_n,$$

where I_n is the graded $\mathbb{F}_p$-module

$$I_n = \begin{cases} E(\lambda_1\mu_1^{b(n+1)}) \otimes P(\mu_1^{p^{n+1}}) \otimes \mathbb{F}_p\{z\mu_1^{b(n)+p^nj} \mid j = 0, \ldots, p-2\}, & n \geqslant 0 \text{ even,} \\ E(z\mu_1^{b(n+1)}) \otimes P(\mu_1^{p^{n+1}}) \otimes \mathbb{F}_p\{\lambda_1\mu_1^{b(n)+p^nj} \mid j = 0, \ldots, p-2\}, & n \geqslant 1 \text{ odd,} \end{cases}$$

with $|z| = 3$, $|\lambda_1| = 2p-1$ and $|\mu_1| = 2p$.

Proof. Using the description of $\beta_{u,a(n)}$ given above, one checks that $\mathrm{im}\,(\beta_{u,a(n)}) \cong I_n$ for all $n \geqslant 0$. This theorem then follows from Proposition 7.2. □

Remark 7.10. A computation of $V(0)_*THH(ku)$ at the prime 2 has been performed by Angeltveit and Rognes [AnR]. Their argument is similar to that of [MS] at odd primes, and involves the Adams spectral sequence.

COROLLARY 7.11. *In any presentation, the $P(u)$-algebra $V(0)_*THH(ku)$ has infinitely many generators and infinitely many relations.*

Proof. This follows from Theorem 7.9 by degree and u-torsion considerations. □

We view this corollary as a motivation for pursuing, in the next two sections, a description of the algebra structure of $V(1)_*THH(ku)$. By analogy with the case of $THH(\ell)$ one expects it to be nicer than the structure of $V(0)_*THH(ku)$. Indeed, it will turn out that $V(1)_*THH(ku)$ admits finitely many generators and relations.

The periodic case. Let KU and L denote the periodic complex K-theory spectrum and the periodic Adams summand, both completed at p. They inherit a commutative $\mathbb{S}$-algebra structure as the $E(1)$-localizations of ku and ℓ, respectively. The homotopy type of the spectrum $THH(L)_p$ was computed by McClure and Staffeldt [MS, Theorem 8.1], and is given by

$$THH(L) \simeq_p L \vee \Sigma L_{\mathbb{Q}}, \tag{7.12}$$

where $L_{\mathbb{Q}}$ denotes the rationalization of the spectrum L. Their argument can be applied to compute $THH(KU)_p$, the only new ingredient being the computation of $K(1)_*THH(ku)$ given in Theorem 7.5. We therefore formulate without proof the following proposition.

PROPOSITION 7.13. *There is an equivalence $THH(KU) \simeq_p KU \vee \Sigma KU_{\mathbb{Q}}$.*

8. Coefficients in a Postnikov section. In this section we will assume that $p \geqslant 5$. Let M be the Postnikov section $M = ku[0, 2p-6]$ of ku with coefficients

$$M_n = \begin{cases} ku_n & \text{if } n \leqslant 2p-6, \\ 0 & \text{otherwise.} \end{cases}$$

It is known [Ba, Theorem 8.1] that the Postnikov sections of a commutative $\mathbb{S}$-algebra can be constructed within the category of commutative $\mathbb{S}$-algebras. We can therefore assume that the natural map $\phi\colon ku \to M$ is a map of commutative $\mathbb{S}$-algebras. In this section we compute the mod p homology groups of $THH(ku, M)$ using the Bökstedt spectral sequence. This will be useful in performing the corresponding computations for $THH(ku)$.

The mod p homology of M is given by an isomorphism of A_*-comodule algebras

$$H_*(M; \mathbb{F}_p) \cong H_*(H\mathbb{Z}_p; \mathbb{F}_p) \otimes P_{p-2}(x),$$

where $x = \phi_*(x)$ under the map $\phi_*\colon H_*(ku;\mathbb{F}_p) \to H_*(M;\mathbb{F}_p)$. The proof of this statement is a variation of the proof of Theorem 2.5.

By Propositions 3.2 and 3.5 the Bökstedt spectral sequence

$$(8.1)\quad E^2_{s,t}(ku,M) = \mathbb{H}^{\mathbb{F}_p}_{s,t}\big(H_*(ku;\mathbb{F}_p), H_*(M;\mathbb{F}_p)\big) \Longrightarrow H_{s+t}(THH(ku,M);\mathbb{F}_p)$$

has an E^2-term given by

$$E^2_{*,*}(ku,M) \cong H_*(H\mathbb{Z}_p;\mathbb{F}_p) \otimes P_{p-2}(x) \otimes E(\sigma x, \sigma\bar{\xi}_1, \sigma\bar{\xi}_2, \ldots) \otimes \Gamma(y, \sigma\bar{\tau}_2, \sigma\bar{\tau}_3, \ldots).$$

Let $\delta\colon ku \to ku$ be the operation corresponding to a chosen generator of Δ, as described in 2.1. It restricts to a map of $\mathbb{S}$-algebras $\delta\colon M \to M$ by naturality of the Postnikov section. Since topological Hochschild homology is functorial in both variables we have $\mathbb{S}$-algebra maps $\delta\colon THH(ku) \to THH(ku)$ and $\delta\colon THH(ku,M) \to THH(ku,M)$, inducing morphisms of spectral sequences $\delta^*\colon E^*_{*,*}(ku) \to E^*_{*,*}(ku)$ and $\delta^*\colon E^*_{*,*}(ku,M) \to E^*_{*,*}(ku,M)$. Suppose chosen a Bott element $u \in V(0)_2 ku$, and let $\alpha \in \mathbb{F}_p^\times$ be such that $\delta_*(u) = \alpha u$.

Definition 8.2. A class w in $E^r_{*,*}(ku)$ or $E^r_{*,*}(ku,M)$ has δ-weight $n \in \mathbb{Z}/(p-1)$ if $\delta^r(w) = \alpha^n w$. Similarly, a class v in $H_*(THH(ku);\mathbb{F}_p)$, $V(0)_*THH(ku)$, $V(1)_*THH(ku)$, or $H_*(THH(ku,M);\mathbb{F}_p)$ has δ-weight n if $\delta_*(v) = \alpha^n v$.

LEMMA 8.3. *In $E^2_{*,*}(ku,M)$ the classes belonging to the tensor factor*

$$H_*(H\mathbb{Z}_p;\mathbb{F}_p) \otimes E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2, \ldots) \otimes \Gamma(y, \sigma\bar{\tau}_2, \sigma\bar{\tau}_3, \ldots)$$

have δ-weight 0, while x and σx have δ-weight 1.

Proof. This is proven by inspection of the action of δ_* on the Hochschild complex. □

LEMMA 8.4. *There is an isomorphism of spectral sequences*

$$E^*_{*,*}(ku,M) \cong P_{p-2}(x) \otimes E^*_{*,*}(ku,H\mathbb{Z}_p).$$

Proof. It suffices to prove by induction that for all $r \geqslant 2$, the following two assertions hold.

(1) There is an isomorphism $E^r_{*,*}(ku,M) \cong P_{p-2}(x) \otimes E^r_{*,*}(ku,H\mathbb{Z}_p)$,

(2) For each $0 \leqslant i \leqslant p-3$ the d^r-differential maps $\mathbb{F}_p\{x^i\} \otimes E^r_{*,*}(ku,H\mathbb{Z}_p)$ to itself.

For $r = 2$ assertion (1) holds. Each algebra generator of $E^2_{*,*}(ku,M)$ that can support a differential has δ-weight 0. Since differentials preserve the δ-weight, the first nontrivial differential maps $\mathbb{F}_p\{x^i\} \otimes E^2_{*,*}(ku,H\mathbb{Z}_p)$ to itself. In particular it is detected by the morphism of spectral sequences

$$\varphi^*_{*,*}\colon\ E^*_{*,*}(ku,M) \to E^*_{*,*}(ku,H\mathbb{Z}_p),$$

induced by the linearization $\varphi\colon\ M \to H\mathbb{Z}_p$, and is given by $d^{p-1}(\gamma_p\sigma\bar{\tau}_i) \doteq \sigma\bar{\xi}_{i+1}$ for $i \geqslant 2$. As in the case of $E^*_{*,*}(ku,H\mathbb{Z}_p)$, taking into account the bialgebra structure, this leaves

$$E^p_{*,*}(ku,M) \cong P_{p-2}(x)\otimes H_*(H\mathbb{Z}_p;\mathbb{F}_p)\otimes E(\sigma x,\sigma\bar{\xi}_1,\sigma\bar{\xi}_2)\otimes P_p(\sigma\bar{\tau}_2,\sigma\bar{\tau}_3,\ldots)\otimes\Gamma(y).$$

Again (1) holds and algebra generators that can support a differential are of δ-weight 0. We can repeat the argument until we reach $E^{2p}_{*,*}(ku,M)$, where the spectral sequence collapses for bidegree reasons. □

PROPOSITION 8.5. *There is an isomorphism of A_*-comodule algebras*

$$H_*(THH(ku,M);\mathbb{F}_p) \cong H_*(M;\mathbb{F}_p) \otimes E([\sigma x],[\sigma\bar{\xi}_1]) \otimes P([y])$$

where $[\sigma x]$ and $[\sigma\bar{\xi}_1]$ are A_*-comodule primitives and the coaction on $[y]$ is

$$\nu_*([y]) = \bar{\tau}_0 \otimes [\sigma\bar{\xi}_1] + 1 \otimes [y].$$

Proof. The Bökstedt spectral sequence described in Lemma 8.4 is strongly convergent and has an E^∞-term given by

$$E^\infty_{*,*}(ku,M) = H_*(M;\mathbb{F}_p) \otimes E(\sigma x,\sigma\bar{\xi}_1) \otimes P_p(y,\sigma\bar{\tau}_2,\sigma\bar{\tau}_3,\ldots). \tag{8.6}$$

The map φ^∞ is surjective and its kernel is the ideal generated by x. The multiplicative and comodule extensions are detected using the homomorphism

$$\varphi_*\colon\ H_*(THH(ku,M);\mathbb{F}_p) \to H_*(THH(ku,H\mathbb{Z}_p);\mathbb{F}_p).$$ □

9. The $V(1)$ homotopy groups of *THH*(ku). In this section we compute the Bökstedt spectral sequence

$$E^2_{*,*}(ku) = \mathbb{H}^{\mathbb{F}_p}_{*,*}\big(H_*(ku;\mathbb{F}_p)\big) \Longrightarrow H_*(THH(ku);\mathbb{F}_p) \tag{9.1}$$

and describe $V(1)_*THH(ku)$ as an algebra over $V(1)_*ku$, for primes p with $p \geqslant 5$. We treat the case $p = 3$ separately at the end of the section. Unless otherwise specified, we assume throughout this section that $p \geqslant 5$.

Recall the description of $H_*(ku;\mathbb{F}_p)$ given in Theorem 2.5, and let $P_{p-1}(x)$ be the subalgebra of $H_*(ku;\mathbb{F}_p)$ generated by $x \in H_2(ku;\mathbb{F}_p)$. Let us denote by

$\Omega^2_{*,*}$ the bigraded $P_{p-1}(x)$-algebra $\mathbb{H}^{\mathbb{F}_p}_{*,*}\big(P_{p-1}(x)\big)$. It has generators

$$\begin{cases} z_i & \text{of bidegree } (2i+1, (2p-2)i+2) \text{ for } i \geqslant 0, \\ y_j & \text{of bidegree } (2j, (2p-2)j+2) \text{ for } j \geqslant 1, \end{cases}$$

subject to the relations given in Proposition 3.3. By Propositions 3.2 and 3.3, the E^2-term of the Bökstedt spectral sequence (9.1) is given by

$$E^2_{*,*}(ku) = H_*(\ell; \mathbb{F}_p) \otimes E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2, \dots) \otimes \Gamma(\sigma\bar{\tau}_2, \sigma\bar{\tau}_3, \dots) \otimes \Omega^2_{*,*}. \tag{9.2}$$

Notice that $E^2_{*,*}(ku)$ is not flat over $H_*(ku; \mathbb{F}_p)$, so there is no coproduct structure on this spectral sequence.

LEMMA 9.3. *Any class in the tensor factor*

$$H_*(\ell; \mathbb{F}_p) \otimes E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2, \dots) \otimes \Gamma(\sigma\bar{\tau}_2, \sigma\bar{\tau}_3, \dots)$$

of $E^2_{*,*}(ku)$ has δ-weight 0. The generators x, z_i and y_j of $\Omega^2_{*,*}$, for $i \geqslant 0$ and $j \geqslant 1$, have δ-weight 1.

Proof. This is a consequence of the action of δ_* on $H_*(ku; \mathbb{F}_p)$ which was described in Lemma 2.6. For z_i and y_j it follows from the fact that a representative for z_i or y_j in the Hochschild complex of $P_{p-1}(x)$ consists of a sum of terms having a number of factors x that is congruent to 1 modulo $(p-1)$. □

The $\mathbb{S}$-algebra map ϕ: $ku \to M$ from the previous section induces a map ϕ: $THH(ku) \to THH(ku, M)$ and a morphism of spectral sequences ϕ^*: $E^*_{*,*}(ku) \to E^*_{*,*}(ku, M)$. The term $E^2_{*,*}(ku, M)$ was given in (8.1).

LEMMA 9.4. *The homomorphism ϕ^2: $E^2_{*,*}(ku) \to E^2_{*,*}(ku, M)$ is characterized as follows. On the tensor factor $H_*(\ell; \mathbb{F}_p)$ it is the inclusion into $H_*(H\mathbb{Z}_p; \mathbb{F}_p)$, on the factor $E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2, \dots) \otimes \Gamma(\sigma\bar{\tau}_2, \sigma\bar{\tau}_3, \dots)$ it is the identity, and on $\Omega^2_{*,*}$ it is given by*

$$\begin{aligned} \phi^2(x) &= x, \\ \phi^2(z_i) &= \sigma x \cdot \gamma_i y \textit{ for all } i \geqslant 0, \\ \phi^2(y_j) &= x \cdot \gamma_j y \textit{ for all } j \geqslant 1. \end{aligned}$$

Proof. The homomorphism ϕ_*: $H_*(ku; \mathbb{F}_p) \to H_*(M; \mathbb{F}_p)$ is given by the tensor product of the inclusion of $H_*(\ell; \mathbb{F}_p)$ into $H_*(H\mathbb{Z}_p; \mathbb{F}_p)$ and the projection of $P_{p-1}(x)$ onto $P_{p-2}(x)$. This lemma follows from a computation in Hochschild homology, using the resolution given in the proof of Proposition 3.3. □

Definition 9.5. Let $\Omega^{\infty}_{*,*}$ be the submodule of $\Omega^2_{*,*}$ generated by

$$\begin{cases} 1 & \text{(the multiplicative unit of } \Omega^2_{*,*}), \\ z_i & \text{for } 0 \leqslant i \leqslant p-1, \text{ and} \\ y_j & \text{for } 1 \leqslant j \leqslant p-1 \end{cases}$$

over $P_{p-1}(x)$. Then $\Omega^{\infty}_{*,*}$ is closed under multiplication, and hence is a subalgebra of $\Omega^2_{*,*}$.

LEMMA 9.6. *The Bökstedt spectral sequence (9.1) behaves as follows.*
(a) *For* $2 \leqslant r \leqslant p-2$ *we have* $d^r = 0$*, and there are differentials*

$$d^{p-1}(\gamma_{p+k}\sigma\bar{\tau}_i) \doteq \gamma_k\sigma\bar{\tau}_i \cdot \sigma\bar{\xi}_{i+1}$$

for all $k \geqslant 0$ *and all* $i \geqslant 2$*. Taking into account the algebra structure, this leaves*

$$E^p_{*,*}(ku) = H_*(\ell;\mathbb{F}_p) \otimes E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2) \otimes P_p(\sigma\bar{\tau}_2, \sigma\bar{\tau}_3, \ldots) \otimes \Omega^2_{*,*}.$$

(b) *For* $p \leqslant r \leqslant 2p-2$ *we have* $d^r = 0$*, and there are differentials*

$$\begin{aligned} d^{2p-1}(z_i) &\doteq z_{i-p} \cdot \sigma\bar{\xi}_2 \ \textit{for all} \ i \geqslant p, \\ d^{2p-1}(y_p) &\doteq x \cdot \sigma\bar{\xi}_2, \\ d^{2p-1}(y_j) &\doteq y_{j-p} \cdot \sigma\bar{\xi}_2 \ \textit{for all} \ j > p. \end{aligned}$$

Taking into account the algebra structure, this leaves

$$E^{2p}_{*,*}(ku) = H_*(\ell;\mathbb{F}_p) \otimes E(\sigma\bar{\xi}_1) \otimes P_p(\sigma\bar{\tau}_2, \sigma\bar{\tau}_3, \ldots) \otimes \big(\mathbb{F}_p\{\sigma\bar{\xi}_2\} \oplus \Omega^{\infty}_{*,*}\big).$$

Here $\sigma\bar{\xi}_2 \cdot \omega = 0$ *for any* $\omega \in \Omega^{\infty}_{*,*}$ *of positive total degree. At this stage the spectral sequence collapses.*

Proof. We use the morphism of spectral sequences

$$\phi^r\colon\ E^r_{*,*}(ku) \to E^r_{*,*}(ku, M)$$

whose description on the E^2-terms is given in Lemma 9.4. Let us denote by d^r the differential of $E^r_{*,*}(ku)$ and by $\bar{d}^r$ the differential of $E^r_{*,*}(ku, M)$, which was determined in Lemmas 6.6 and 8.4. Let $V^r \subset E^r_{*,*}(ku)$ be the $\mathbb{F}_p$-vector space generated by the elements of δ-weight 0 or 1. The kernel of ϕ^2 is the ideal generated by x^{p-2} and $x^{p-3}y_j$ for all $j \geqslant 1$. In particular any element of $\ker\phi^2$ is of δ-weight $p-2$, and $\phi^2|_{V^2}$ is injective.

We now prove claim (a). Recall that $\bar{d}^r = 0$ for $2 \leqslant r \leqslant p-2$. If w is one of the algebra generators of $E^2_{*,*}(ku)$ given in formula (9.2), then both w and $d^2(w)$

lie in V^2 (the differentials preserve the δ-weight). The relation

$$\phi^2 d^2(w) = \bar{d}^2\phi^2(w) = 0$$

implies that $d^2(w) = 0$. This proves that $d^2 = 0$. By the same argument we obtain $d^r = 0$ for $2 \leqslant r \leqslant p-2$. Next, we have

$$\begin{aligned}\phi^{p-1}d^{p-1}(\gamma_{p+k}\sigma\bar{\tau}_i) &= \bar{d}^{p-1}\phi^{p-1}(\gamma_{p+k}\sigma\bar{\tau}_i) = \bar{d}^{p-1}(\gamma_{p+k}\sigma\bar{\tau}_i)\\ &= \gamma_k\sigma\bar{\tau}_i \cdot \sigma\bar{\xi}_{i+1} = \phi^{p-1}(\gamma_k\sigma\bar{\tau}_i \cdot \sigma\bar{\xi}_{i+1})\end{aligned}$$

for any $i \geqslant 2$ and $k \geqslant 0$, and $\phi^{p-1}d^{p-1}(w) = \bar{d}^{p-1}\phi^{p-1}(w) = 0$ on any remaining given algebra generator w of $E^{p-1}_{*,*}(ku) = E^2_{*,*}(ku)$. The description of d^{p-1} given follows from the injectivity of ϕ^{p-1} on V^{p-1}. This leaves $E^p_{*,*}(ku)$ as claimed.

The proof of claim (b) is similar. All algebra generators of $E^p_{*,*}(ku)$ given in (a) lie in V^p and $\phi^p|_{V^p}$ is injective. For $p \leqslant r \leqslant 2p-2$ the relation $\bar{d}^r = 0$ implies that $d^r = 0$, and the d^{2p-1} differential is determined by $\bar{d}^{2p-1}$. For example we have

$$\begin{aligned}\phi^{2p-1}d^{2p-1}(z_i) &= \bar{d}^{2p-1}\phi^{2p-1}(z_i) = \bar{d}^{2p-1}(\sigma x \cdot \gamma_i y)\\ &= \begin{cases} 0 & \text{if } i < p,\\ \sigma\bar{\xi}_2 \cdot \sigma x \cdot \gamma_{p-i} y = \phi^{2p-1}(\sigma\bar{\xi}_2 \cdot z_{p-i}) & \text{if } i \geqslant p,\end{cases}\end{aligned}$$

and thus $d^{2p-1}(z_i) = \sigma\bar{\xi}_2 \cdot z_{p-i}$ for $i \geqslant p$. Similarly we can check that $d^{2p-1}(y_j)$ is as claimed and that d^{2p-1} maps the remaining given algebra generators to zero. The differential algebra $E^{2p-1}_{*,*}(ku)$ is the tensor product of $H_*(\ell; \mathbb{F}_p) \otimes E(\sigma\bar{\xi}_1) \otimes P_p(\sigma\bar{\tau}_2, \sigma\bar{\tau}_3, \ldots)$ with the trivial differential and $E(\sigma\bar{\xi}_2) \otimes \Omega^2_{*,*}$ with the differential given in (b). We know from Remark 3.4 that this second factor can be written additively as

$$E(\sigma\bar{\xi}_2) \otimes \Omega^2_{*,*} = E(\sigma\bar{\xi}_2) \otimes \big(P_{p-1}(x) \oplus \bigoplus_{i \geqslant 0} P_{p-2}(x)\{z_i\} \oplus \bigoplus_{j \geqslant 1} P_{p-2}(x)\{y_j\}\big).$$

The d^{2p-1} differential maps the summands

$$\begin{aligned} P_{p-2}(x)\{z_i\} &\xrightarrow{\cong} P_{p-2}(x)\{z_{i-p}\} \otimes \mathbb{F}_p\{\sigma\bar{\xi}_2\} \quad \text{if } i \geqslant p,\\ P_{p-2}(x)\{y_p\} &\xrightarrow{\cong} \mathbb{F}_p\{x, x^2, \ldots, x^{p-2}\} \otimes \mathbb{F}_p\{\sigma\bar{\xi}_2\}, \quad \text{and}\\ P_{p-2}(x)\{y_j\} &\xrightarrow{\cong} P_{p-2}(x)\{y_{j-p}\} \otimes \mathbb{F}_p\{\sigma\bar{\xi}_2\} \quad \text{if } j > p\end{aligned}$$

isomorphically, and maps the summands $P_{p-2}(x)\{z_i\}$ and $P_{p-2}(x)\{y_j\}$ for $i, j < p$

to zero. From the factor $E(\sigma\bar{\xi}_2) \otimes \Omega^2_{*,*}$ only

$$\mathbb{F}_p\{\sigma\bar{\xi}_2\} \oplus \Omega^\infty_{*,*} = \mathbb{F}_p\{\sigma\bar{\xi}_2\} \oplus \left(P_{p-1}(x) \oplus \bigoplus_{i=0}^{p-1} P_{p-2}(x)\{z_i\} \oplus \bigoplus_{j=1}^{p-1} P_{p-2}(x)\{y_j\} \right)$$

survives to $E^{2p}_{*,*}(ku)$. Thus

$$E^{2p}_{*,*}(ku) = H_*(\ell; \mathbb{F}_p) \otimes E(\sigma\bar{\xi}_1) \otimes P_p(\sigma\bar{\tau}_2, \sigma\bar{\tau}_3, \ldots) \otimes \left(\mathbb{F}_p\{\sigma\bar{\xi}_2\} \oplus \Omega^\infty_{*,*}\right)$$

as claimed. In $E^{2p}_{*,*}(ku)$ all homogeneous algebra generators lie in filtration degrees smaller than $2p$, so the spectral sequence collapses for bidegree reasons. □

LEMMA 9.7. *For $0 \leqslant i \leqslant p-2$ let W_i be the graded $\mathbb{F}_p$-vector space of elements of δ-weight i in $H_*(THH(ku); \mathbb{F}_p)$. Then multiplication by x: $W_1 \to W_2$ is an isomorphism.*

Proof. By Lemma 9.3, multiplication by x on $E^2_{*,*}(ku)$ induces an isomorphism from the vector space of elements of δ-weight 1 to the vector space of elements of δ-weight 2. The differentials preserve the δ-weight and x is a permanent cycle, so the corresponding statement is true also for $E^\infty_{*,*}(ku)$. The filtration of $H_*(THH(ku); \mathbb{F}_p)$ given by the spectral sequence induces a filtration of the graded vector space W_i, for $0 \leqslant i \leqslant p-2$. We just argued that multiplication by x: $W_1 \to W_2$ induces an isomorphism of the associated graded groups $\mathrm{gr}(W_1) \cong \mathrm{gr}(W_2)$. This implies that multiplication by x: $W_1 \to W_2$ is an isomorphism. □

LEMMA 9.8. *Let ϕ_*: $H_*(THH(ku); \mathbb{F}_p) \to H_*(THH(ku, M); \mathbb{F}_p)$ be the algebra homomorphism induced by ϕ: $ku \to M$. Then* $\ker(\phi_*|_{W_1 \oplus W_2}) = 0$.

Proof. The morphism ϕ^2: $E^2_{*,*}(ku) \to E^2_{*,*}(ku, M)$ is described in Lemma 9.4. It follows from the computation of $E^\infty_{*,*}(ku, M)$ given in (8.6) that

$$\phi^\infty\colon\ E^\infty_{*,*}(ku) \to E^\infty_{*,*}(ku, M)$$

is injective on the vector space of elements of δ-weight 1 or 2. This implies the corresponding statement for ϕ_*. □

The $H_*(ku, \mathbb{F}_p)$-algebra $H_*(ku; \mathbb{F}_p)$ differs from its associated graded $E^\infty_{*,*}(ku)$ by multiplicative extensions. More precisely, the subalgebra

$$P_p(\sigma\bar{\tau}_2, \sigma\bar{\tau}_3, \ldots) \otimes (\mathbb{F}_p\{\sigma\bar{\xi}_2\} \oplus \Omega^\infty_{*,*})$$

of $E^\infty_{*,*}(ku)$ lifts to the subalgebra Ξ_* of $H_*(THH(ku); \mathbb{F}_p)$ defined as follows.

Definition 9.9. Let Ξ_* be the graded-commutative unital $P_{p-1}(x)$-algebra with generators

$$\begin{cases} \bar{z}_i & 0 \leqslant i \leqslant p-1, \\ \bar{y}_j & 1 \leqslant j \leqslant p-1, \\ [\sigma\bar{\tau}_2], \end{cases}$$

and relations

$$\begin{cases} x^{p-2}\bar{z}_i = 0 & 0 \leqslant i \leqslant p-2, \\ x^{p-2}\bar{y}_j = 0 & 1 \leqslant j \leqslant p-1, \\ \bar{y}_i\bar{y}_j = x\bar{y}_{i+j} & i+j \leqslant p-1, \\ \bar{z}_i\bar{y}_j = x\bar{z}_{i+j} & i+j \leqslant p-1, \\ \bar{y}_i\bar{y}_j = x\bar{y}_{i+j-p}[\sigma\bar{\tau}_2] & i+j \geqslant p, \\ \bar{z}_i\bar{y}_j = x\bar{z}_{i+j-p}[\sigma\bar{\tau}_2] & i+j \geqslant p, \\ \bar{z}_i\bar{z}_j = 0 & 0 \leqslant i,j \leqslant p-1. \end{cases}$$

Here by convention $\bar{y}_0 = x$, and the degree of the generators is $|\bar{z}_i| = 2pi+3$, $|\bar{y}_j| = 2pj+2$ and $|[\sigma\bar{\tau}_2]| = 2p^2$.

Beware that in Ξ_* we have $x^{p-2}\bar{z}_{p-1} \neq 0$. In fact we prove below that there is a multiplicative extension $x^{p-2}\bar{z}_{p-1} \doteq [\sigma\bar{\xi}_2]$.

PROPOSITION 9.10. *There is an isomorphism of $H_*(\ell;\mathbb{F}_p)$-algebras*

$$H_*(THH(ku);\mathbb{F}_p) \cong H_*(\ell;\mathbb{F}_p) \otimes E([\sigma\bar{\xi}_1]) \otimes \Xi_*.$$

Proof. The Bökstedt spectral sequence (9.1) converges strongly and its E^∞-term is given in Lemma 9.6.b. For $1 \leqslant i \leqslant p-1$, we define $\bar{y}_i \in H_{2pi+2}(THH(ku);\mathbb{F}_p)$ by induction on i, with the following properties:

(1) $\bar{y}_i$ has δ-weight 1,

(2) $\bar{y}_i$ reduces to iy_i modulo lower filtration in $E^\infty_{*,*}(ku)$, and

(3) $\phi_*(\bar{y}_i) = x[y]^i + ic\bar{\tau}_1[z][y]^{i-1}$ in $H_{2pi+2}(THH(ku,M);\mathbb{F}_p)$ for some $c \in \mathbb{F}_p^\times$ independent of i.

Let $\bar{y}_1 = [y_1]$. Then (1) and (2) are obviously satisfied and by Lemma 9.4 $\phi_*(\bar{y}_1) \equiv x[y]$ modulo filtration less than or equal to 1. From the splitting (4.1) of the 0-th column in the Bökstedt spectral sequences, we deduce that $\phi_*(\bar{y}_1) \equiv x[y]$ modulo classes of filtration 1. The filtration-1 part of $H_{2p+2}(THH(ku,M);\mathbb{F}_p)$ is $\mathbb{F}_p\{\bar{\tau}_1[z]\}$. On the other hand the map

$$j_*\colon\ H_*(THH(ku);\mathbb{F}_p) \to H_*(THH(ku,H\mathbb{Z}_p);\mathbb{F}_p)$$

satisfies $j_*(\bar{y}_1) = c\bar{\tau}_1[z]$ for some unit $c \in \mathbb{F}_p$, because $j_*(\bar{y}_1) \neq 0$ since $\bar{y}_1$ is not divisible by x and $\bar{\tau}_1[z]$ generates $H_{2p+2}(THH(ku,H\mathbb{Z}_p);\mathbb{F}_p)$. The homomorphism

j_* factorizes as $\varphi_*\phi_*$, so this proves that

$$\phi_*(\bar{y}_1) = x[y] + c\bar{\tau}_1[z].$$

Assume that $\bar{y}_i$ satisfying conditions (1), (2) and (3) has been defined for some $1 \leqslant i \leqslant p-2$. The class $\bar{y}_1\bar{y}_i$ has δ-weight 2, so is divisible by x in a unique way by Lemma 9.7. Let $\bar{y}_{i+1} = x^{-1}\bar{y}_1\bar{y}_i$. Then $\bar{y}_{i+1}$ satisfies conditions (1) and (2). By inspection there is no non-zero class w of δ-weight 1 in $H_{2p(i+1)+2}(THH(ku, M); \mathbb{F}_p)$ with $xw = 0$, so we can write

$$\phi_*(\bar{y}_{i+1}) = x^{-1}\phi_*(\bar{y}_1)\phi_*(\bar{y}_i),$$

which proves that $\bar{y}_{i+1}$ satisfies (3).

Next, we define $\bar{z}_0 = c[z_0]$, where $c \in \mathbb{F}_p$ is the same as in condition (3) above. The class $\bar{z}_0\bar{y}_i$ has δ-weight 2, and we define $\bar{z}_i = x^{-1}\bar{z}_0\bar{y}_i$. Then $\bar{z}_i$ has δ-weight 1, and if $i \geqslant 1$ it reduces to icz_i modulo lower filtration in $E^{\infty}_{*,*}(ku)$. Moreover we have $\phi_*(\bar{z}_i) = c[z][y]^i$ for $0 \leqslant i \leqslant p-1$.

The relations $\bar{y}_i\bar{y}_j = x\bar{y}_{i+j}$ and $\bar{z}_i\bar{y}_j = x\bar{z}_{i+j}$ for $i + j \leqslant p - 1$ are satisfied by definition of $\bar{z}_i$ and $\bar{y}_j$. It remains to check the following relations:

$$\begin{cases} x^{p-2}\bar{z}_i = 0 & 0 \leqslant i \leqslant p-2, \\ x^{p-2}\bar{y}_j = 0 & 1 \leqslant j \leqslant p-1, \\ \bar{z}_i\bar{z}_j = 0 & 0 \leqslant i,j \leqslant p-1, \\ \bar{y}_i\bar{y}_j = x\bar{y}_{i+j-p}[\sigma\bar{\tau}_2] & i+j \geqslant p. \\ \bar{z}_i\bar{y}_j = x\bar{z}_{i+j-p}[\sigma\bar{\tau}_2] & i+j \geqslant p. \end{cases}$$

The class $x^{p-2}\bar{y}_j$ is of δ-weight 0 and is in the kernel of multiplication by x. It follows that it is in the ideal generated by $[\sigma\bar{\xi}_2]$, so for degree reasons it must be zero. Similarly we have $x^{p-2}\bar{z}_i = 0$ if $0 \leqslant i \leqslant p-2$. The product $\bar{z}_i\bar{z}_j$ is of δ-weight 2 and in the kernel of ϕ_*, so must be zero by Lemma 9.8. In $H_{2p^2}(THH(ku, M); \mathbb{F}_p)$ we have the relation $[y]^p = [\sigma\bar{\tau}_2]$ (Proposition 6.7), which implies that $\phi_*([\sigma\bar{\tau}_2]) = [y]^p$ holds. If $i + j \geqslant p$, we then have

$$\phi_*(\bar{y}_i\bar{y}_j) = x^2[y]^{i+j} + (i+j)c\bar{\tau}_1 x[z][y]^{i+j-1} = \phi_*(x\bar{y}_{i+j-p}[\sigma\bar{\tau}_2]).$$

Since both $\bar{y}_i\bar{y}_j$ and $x\bar{y}_{i+j-p}[\sigma\bar{\tau}_2]$ have δ-weight 2, they are equal by Lemma 9.8. The proof that $\bar{z}_i\bar{y}_j = x\bar{z}_{i+j-p}[\sigma\bar{\tau}_2]$ for $i + j \geqslant p$ is similar.

Finally, the class $[\sigma\bar{\xi}_2]$ maps to zero via

$$j_*\colon\ H_*(THH(ku); \mathbb{F}_p) \to H_*(THH(ku, H\mathbb{Z}_p); \mathbb{F}_p),$$

and hence must be divisible by x. The only possibility left is a multiplicative extension $x^{p-2}z_{p-1} \doteq [\sigma\bar{\xi}_2]$. We have now established all possible multiplicative relations involving the classes x, $\bar{z}_i$, $\bar{y}_j$ and $[\sigma\bar{\tau}_2]$. The associated graded of Ξ_* is isomorphic to

$$P_p(\sigma\bar{\tau}_2, \sigma\bar{\tau}_3, \dots) \otimes (\mathbb{F}_p\{\sigma\bar{\xi}_2\} \oplus \Omega^{\infty}_{*,*}).$$

This proves the proposition. $\square$

Proposition 9.11. *The A_*-coaction on $H_*(THH(ku); \mathbb{F}_p)$ is as follows:*

- *on $H_*(\ell; \mathbb{F}_p)$ it is induced by inclusion into the coalgebra A_* ;*
- *the classes x, $[\sigma\bar{\xi}_1]$ and $\bar{z}_0$ are primitive ;*
- *on the remaining algebra generators, we have*

$$\begin{aligned}
\nu_*(\bar{z}_i) &= 1 \otimes \bar{z}_i + i\bar{\tau}_0 \otimes [\sigma\bar{\xi}_1]\bar{z}_{i-1} \quad \textit{for } i \geqslant 1,\\
\nu_*(\bar{y}_1) &= 1 \otimes \bar{y}_1 + \bar{\tau}_0 \otimes (x[\sigma\bar{\xi}_1] + \bar{\xi}_1\bar{z}_0) + \bar{\tau}_1 \otimes \bar{z}_0,\\
\nu_*(\bar{y}_j) &= 1 \otimes \bar{y}_j + i\bar{\tau}_0 \otimes ([\sigma\bar{\xi}_1]\bar{y}_{j-1} + \bar{\xi}_1\bar{z}_{j-1}) + i\bar{\tau}_1 \otimes \bar{z}_{j-1}\\
&\quad + i\bar{\tau}_0\bar{\tau}_1 \otimes [\sigma\bar{\xi}_1]\bar{z}_{j-2} \quad \textit{for } j \geqslant 2,\\
\nu_*([\sigma\bar{\tau}_2]) &= 1 \otimes [\sigma\bar{\tau}_2] + \bar{\tau}_0 \otimes x^{p-2}\bar{z}_{p-1}.
\end{aligned}$$

Proof. The class x is known to be primitive. On classes in the image of σ_*, like $\bar{z}_0$, $[\sigma\bar{\xi}_1]$ and $[\sigma\bar{\tau}_2]$, the coaction is determined by (5.6).

The class $\bar{y}_1$ was defined such that $\phi_*(\bar{y}_1) = x[y] + c\bar{\tau}_1[z]$. By Proposition 6.7 we have

$$\nu_*\phi_*(\bar{y}_1) = 1 \otimes x[z] + \bar{\tau}_0 \otimes (x[\sigma\bar{\xi}_1] + c\bar{\xi}_1[z]) + \bar{\tau}_1 \otimes c[z]$$

in $A_* \otimes H_*(THH(ku, M); \mathbb{F}_p)$. Since ϕ_* is injective on classes of δ-weight 1 we have by naturality

$$\nu_*(\bar{y}_1) = 1 \otimes \bar{y}_1 + \bar{\tau}_0 \otimes (x[\sigma\bar{\xi}_1] + \bar{\xi}_1\bar{z}_0) + \bar{\tau}_1 \otimes \bar{z}_0.$$

The product formulas $x\bar{y}_j = \bar{y}_1\bar{y}_{j-1}$ and $x\bar{z}_i = \bar{z}_0\bar{y}_i$ allow us to compute inductively the coaction on $\bar{y}_j$ and $\bar{z}_i$ for $2 \leqslant j \leqslant p-1$ and $1 \leqslant i \leqslant p-1$. Again, by Lemma 9.7 there is no indeterminacy upon dividing $\nu_*(x\bar{z}_i)$ and $\nu_*(x\bar{y}_j)$ by $1 \otimes x$. $\square$

Our next aim is to describe $V(1)_*THH(ku)$ as an algebra over $V(1)_*ku$.

Remark 9.12. Recall from [Ok] that $V(1)$ is a commutative ring spectrum if and only if $p \geqslant 5$. The obstruction in [Ok, Example 4.5] for $V(1)$ to be a ring spectrum at $p = 3$ vanishes when $V(1)$ is smashed with $H\mathbb{Z}_p$ or ku. In particular $V(1)\wedge THH(ku)$ is a ring spectrum for all $p \geqslant 3$, because $THH(ku)$ is a ku-algebra. Notice, however, that $TC(ku; p)$ and $K(ku)$ are not ku-algebras.

We define a $P_{p-1}(u)$ algebra Θ_*. It is the counterpart of Ξ_* in $V(1)$-homotopy and is abstractly isomorphic to it.

Definition 9.13. Assume $p \geqslant 3$, and let Θ_* be the graded-commutative unital $P_{p-1}(u)$-algebra with generators

$$\begin{cases} a_i & 0 \leqslant i \leqslant p-1, \\ b_j & 1 \leqslant j \leqslant p-1, \\ \mu_2, & \end{cases}$$

and relations

$$\begin{cases} u^{p-2}a_i = 0 & 0 \leqslant i \leqslant p-2, \\ u^{p-2}b_j = 0 & 1 \leqslant j \leqslant p-1, \\ b_i b_j = u b_{i+j} & i+j \leqslant p-1, \\ a_i b_j = u a_{i+j} & i+j \leqslant p-1, \\ b_i b_j = u b_{i+j-p}\mu_2 & i+j \geqslant p, \\ a_i b_j = u a_{i+j-p}\mu_2 & i+j \geqslant p, \\ a_i a_j = 0 & 0 \leqslant i,j \leqslant p-1. \end{cases}$$

Here by convention $b_0 = u$, and the degree of the generators is $|a_i| = 2pi + 3$, $|b_j| = 2pj + 2$ and $|\mu_2| = 2p^2$.

Remark 9.14. Let us describe the $P_{p-1}(u)$-algebra Θ_* more explicitly. The class μ_2 generates a polynomial subalgebra $P(\mu_2) \subset \Theta_*$. There is an isomorphism of $P(\mu_2)$-modules $\Theta_* \cong P(\mu_2) \otimes Q_*$, where Q_* is the $\mathbb{F}_p$-module

$$Q_* = P_{p-1}(u) \oplus P_{p-2}(u)\{a_0, b_1, a_1, b_2, \dots, a_{p-2}, b_{p-1}\} \oplus P_{p-1}(u)\{a_{p-1}\}.$$

Thus Θ_* is a free $P(\mu_2)$-module of rank $2(p-1)^2$. If we invert μ_2, then $\mu_2^{-1}\Theta_*$ is a graded Frobenius algebra of degree $2p^2 - 1$ over the graded field $k_* = P(\mu_2^{\pm 1})$. By this we mean that there is an isomorphism of graded $\mu_2^{-1}\Theta_*$-modules

$$\mu_2^{-1}\Theta_* \cong \mathrm{Hom}_{k_*}(\mu_2^{-1}\Theta_*, \Sigma^{2p^2-1}k_*).$$

This isomorphism is induced from the perfect pairing

$$\mu_2^{-1}\Theta_* \otimes_{k_*} \mu_2^{-1}\Theta_* \to \mu_2^{-1}\Theta_* \xrightarrow{\varepsilon} \Sigma^{2p^2-1}k_*.$$

Here the first map is the product of $\mu_2^{-1}\Theta_*$, and the graded counit ε is obtained by tensoring the identity of k_* with the homomorphism $Q_* \to \Sigma^{2p^2-1}\mathbb{F}_p$ sending $u^{p-2}a_{p-1}$ to 1 (it is unique for degree reasons). This pairing is non-degenerate since by inspection any non-zero homogeneous class in Q_* is a divisor of $u^{p-2}a_{p-1}$ in Θ_*. For example $(u^k a_i)(u^\ell b_j) = u^{p-2}a_{p-1}$ if $k + \ell = p-3$ and $i + j = p-1$.

THEOREM 9.15. *Let p be an odd prime and let Θ_* be the $P_{p-1}(u)$-algebra defined above. There is an isomorphism of $P_{p-1}(u)$-algebras*

$$V(1)_*THH(ku) \cong E(\lambda_1) \otimes \Theta_*,$$

where λ_1 is of degree $2p-1$.

Remark 9.16. Arguing as in Remark 9.14, we deduce that $V(1)_*THH(ku)$ is a free $P(\mu_2)$-module of rank $4(p-1)^2$ isomorphic to $P(\mu_2) \otimes E(\lambda_1) \otimes Q_*$, and that $\mu_2^{-1}V(1)_*THH(ku)$ is a Frobenius algebra of degree $2p^2+2p-2$ over $k_* = P(\mu_2^{\pm 1})$. Here the graded counit

$$\varepsilon\colon\ \mu_2^{-1}V(1)_*THH(ku) \to \Sigma^{2p^2+2p-2}k_*$$

is induced from the unique homomorphism $E(\lambda_1) \otimes Q_* \to \Sigma^{2p^2+2p-2}\mathbb{F}_p$ sending $\lambda_1 u^{p-2} a_{p-1}$ to 1. Theorem 1.1 follows from Theorem 9.15 and this remark.

Proof of Theorem 9.15 *for $p \geqslant 5$.* Since $V(1) \wedge \ell \simeq H\mathbb{F}_p$, the spectrum $V(1) \wedge THH(ku)$ is an $H\mathbb{F}_p$-module and its homology is given by

$$H_*(V(1) \wedge THH(ku); \mathbb{F}_p) \cong A_* \otimes E([\sigma\bar{\xi}_1]) \otimes \Xi_*.$$

The Hurewicz map

$$V(1)_*THH(ku) \to H_*(V(1) \wedge THH(ku); \mathbb{F}_p)$$

is injective with image the A_*-comodule primitives. We identify $V(1)_*THH(ku)$ with its image (in particular $P_{p-1}(u)$ is identified with $P_{p-1}(x)$). Consider the following classes in $H_*(V(1) \wedge THH(ku); \mathbb{F}_p)$:

$$\begin{aligned} a_0 &= \bar{z}_0, \\ b_1 &= \bar{y}_1 - \bar{\tau}_0 x[\sigma\bar{\xi}_1] - \bar{\tau}_1\bar{z}_0, \\ \lambda_1 &= [\sigma\bar{\xi}_1], \\ \mu_2 &= [\sigma\bar{\tau}_2] - \bar{\tau}_0 x^{p-2}\bar{z}_{p-1}. \end{aligned}$$

By Proposition 9.11 these classes are comodule primitives. Lemma 9.7 also holds for $H_*(V(1) \wedge THH(ku); \mathbb{F}_p)$. We define inductively $b_{j+1} = u^{-1}b_1 b_j$, for $1 \leqslant j \leqslant p-2$, and $a_i = u^{-1}a_0 b_i$, for $1 \leqslant i \leqslant p-1$. These classes b_j and a_i are primitive by construction. By inspection, the classes a_i, b_j and μ_2 satisfy the relations over $P_{p-1}(u)$ given in Definition 9.13. There is an isomorphism

$$H_*(V(1) \wedge THH(ku); \mathbb{F}_p) \cong A_* \otimes E(\lambda_1) \otimes \Theta_*.$$

where the $P_{p-1}(u)$-algebra $E(\lambda_1) \otimes \Theta_*$ consists of A_*-comodule primitives. $\square$

Proof of Theorem 9.15 *for* $p = 3$. Applying $V(1)_* THH(ku, -)$ to the diagram (2.3) we obtain an unrolled exact couple and a strongly convergent spectral sequence of algebras

$$E^2_{*,*} = V(1)_* THH(ku, H\mathbb{Z}_p) \otimes P(u) \Longrightarrow V(1)_* THH(ku) \tag{9.17}$$

analogous to the mod u spectral sequence of Proposition 7.2. Here u is of bidegree $(-2, 4)$ and represents the mod v_1 reduction of the Bott element. By Theorem 6.8 we have

$$E^2_{0,*} = V(1)_* THH(ku, H\mathbb{Z}_p) \cong E(z, \lambda_1, \varepsilon) \otimes P(\mu_1)$$

where ε has degree $2p - 1$ with a primary v_1 Bockstein $\beta_{1,1}(\varepsilon) = 1$. We deduce from Lemma 7.8 that there is a differential

$$d^2(\mu_1) \doteq uz.$$

The ku-module structure of $THH(ku)$ implies a differential

$$d^4(\varepsilon) \doteq u^2.$$

At this point the spectral sequence collapses and this leaves

$$E^\infty_{*,*} = E^5_{*,*} = E(\lambda_1) \otimes P(\mu_1^3) \otimes \big[E(\varepsilon) \otimes \mathbb{F}_p\{z, z\mu_1\} \oplus E(z\mu_1^2) \otimes P_2(u)\big].$$

Defining $\mu_2 = \mu_1^3$, $a_i = z\mu_1^i$ and $b_j = z\varepsilon\mu_1^{j-1}$ we obtain the claimed $\mathbb{F}_p$-module structure of $V(1)_* THH(ku)$ for $p = 3$.

The permanent cycles are concentrated in filtration degrees 0 and -2, so there is not much room for multiplicative extensions. All multiplicative relations for Θ_* can be read off from the E^∞-term of this spectral sequence, except for $a_0 b_2 = a_1 b_1 = u a_2$ and $b_1 b_2 = 0$. These relations can be established by mapping to homology and using the map of $THH(ku)$-modules $THH(ku, H\mathbb{Z}_p) \to THH(ku, \Sigma^3 ku)$. This map is induced from the map of ku-modules $H\mathbb{Z}_p \to \Sigma^3 ku$ obtained by extending the cofibre sequence

$$\Sigma^2 ku \xrightarrow{u} ku \xrightarrow{j} H\mathbb{Z}_p$$

of (2.3) to the right. We omit the details. $\square$

Remark 9.18. The proof given for $p = 3$ is also valid for primes $p \geqslant 5$, and provides an alternative way of determining the additive structure of

$V(1)_*THH(ku)$. The differentials of (9.17) are given by

$$\begin{cases} d^{2(p-2)}(\mu_1) \doteq zu^{p-2}, \\ d^{2(p-1)}(\varepsilon) \doteq u^{p-1}. \end{cases}$$

The permanent cycles are now scattered through $p-1$ filtration degrees, and there are many nontrivial multiplicative extensions.

At this point it is of course also possible to study the mod v_1 Bockstein spectral sequence in order to recover $V(0)_*THH(ku)$ from $V(1)_*THH(ku)$. However, because of the relation $u^{p-1} = v_1$ in $V(0)_*ku$, the mod u Bockstein spectral sequence of §7 is more appropriate. Let us just describe the primary mod v_1 Bockstein, which involves some of the generators of Θ_*. The following proposition is a consequence of Theorem 7.9.

PROPOSITION 9.19. *Let $p \geqslant 3$. In $V(1)_*THH(ku)$ there are primary mod v_1 Bocksteins*

$$\beta_{1,1}(b_i) \doteq a_{i-1}$$

for $1 \leqslant i \leqslant p-1$.

10. On the extension $\ell \to ku$. In this final section we analyze the homomorphism

$$V(1)_*THH(\ell) \to V(1)_*THH(ku)$$

induced by the $\mathbb{S}$-algebra map $\ell \to ku$ defined in 2.1. We then interpret our computations above in terms of number-theoretic properties of the extension $\ell \to ku$.

The $\mathbb{F}_p$-algebras $V(1)_*THH(\ell)$ and $V(1)_*THH(ku)$ were described in Theorems 5.9 and 9.15, respectively. Let Δ be the group defined in 2.1, and recall the notion of δ-weight from Definition 8.2.

PROPOSITION 10.1. *The classes λ_1 and μ_2 have δ-weight 0 in $V(1)_*THH(ku)$, and the classes u, a_i and b_j have δ-weight 1. The homomorphism $V(1)_*THH(\ell) \to V(1)_*THH(ku)$ is given by $\lambda_1 \mapsto \lambda_1$, $\lambda_2 \mapsto u^{p-2}a_{p-1}$ and $\mu_2 \mapsto \mu_2$. In particular it is injective with image the classes of δ-weight 0, and induces a canonical isomorphism*

$$V(1)_*THH(\ell) = \big(V(1)_*THH(ku)\big)^{\Delta}.$$

Proof. These statements are proven in homology, where they follow directly from the definition of the various algebra generators. □

Let us denote by $TC(\ell;p)$ the topological cyclic homology spectrum of ℓ, and by $K(\ell)$ its algebraic K-theory. I thank John Rognes for pointing out to me the following consequence of Proposition 10.1.

THEOREM 10.2. *Let p be an odd prime. There are weak equivalences*

$$\begin{aligned} THH(ku)^{h\Delta} &\simeq_p THH(\ell), \\ TC(ku;p)^{h\Delta} &\simeq_p TC(\ell;p), \quad \textit{and} \\ K(ku)^{h\Delta} &\simeq_p K(\ell). \end{aligned}$$

Proof. Consider the homotopy fixed point spectral sequence

$$E^2_{s,t} = H^{-s}\big(\Delta; V(1)_t THH(ku)\big) \Longrightarrow V(1)_{t+s} THH(ku)^{h\Delta}.$$

By Proposition 10.1, and since the order of Δ is prime to p, its E^2-term is given by

$$E^2_{s,t} = \begin{cases} V(1)_t THH(\ell) & \text{if} \quad s = 0, \\ 0 & \text{if} \quad s \neq 0. \end{cases}$$

Thus the spectral sequence collapses and its edge homomorphism yields an isomorphism

$$V(1)_* THH(ku)^{h\Delta} \cong V(1)_* THH(\ell).$$

The spectra $THH(ku)^{h\Delta}$ and $THH(\ell)$ are both connective. In particular their $V(1)_*$-localization and their $V(0)_*$-localization (or p-completion) agree. Thus we have an equivalence of p-completed spectra $THH(ku)^{h\Delta} \simeq_p THH(\ell)$.

The spectrum $TC(ku;p)$ is defined as the homotopy limit

$$TC(ku;p) = \operatorname*{holim}_{F,R} THH(ku)^{C_{p^n}}$$

taken over the Frobenius and the restriction maps

$$F, R\colon\ THH(ku)^{C_{p^n}} \to THH(ku)^{C_{p^{n-1}}}$$

that are part of the cyclotomic structure of THH. In particular we have an equivalence

$$TC(ku;p)^{h\Delta} \simeq \operatorname*{holim}_{F^{h\Delta},R^{h\Delta}} (THH(ku)^{C_{p^n}})^{h\Delta}.$$

Thus the equivalence $TC(ku;p)^{h\Delta} \simeq_p TC(\ell;p)$ will follow from the claim that for each $n \geqslant 0$, there is an equivalence $(THH(ku)^{C_{p^n}})^{h\Delta} \simeq_p THH(\ell)^{C_{p^n}}$. We proceed by induction on n, the case $n = 0$ having been proven above. Let $n \geqslant 1$ and assume that the claim has been proven for $m < n$. Consider the homotopy commutative diagram

$$\begin{array}{ccccc}
THH(\ell)_{hC_{p^n}} & \xrightarrow{N} & THH(\ell)^{C_{p^n}} & \xrightarrow{R} & THH(\ell)^{C_{p^{n-1}}} \\
\downarrow & & \downarrow & & \downarrow \\
(THH(ku)_{hC_{p^n}})^{h\Delta} & \xrightarrow{N^{h\Delta}} & (THH(ku)^{C_{p^n}})^{h\Delta} & \xrightarrow{R^{h\Delta}} & (THH(ku)^{C_{p^{n-1}}})^{h\Delta}
\end{array}$$

where the top line is the norm-restriction fiber sequences for $THH(\ell)$ and the bottom line is obtained by taking the Δ homotopy-fixed points of the one for $THH(ku)$. By induction hypothesis the right-hand side vertical arrow is an equivalence. Since $THH(ku)_{hC_{p^n}}$ is p-complete and Δ is of order prime to p, the homotopy norm map $(THH(ku)_{hC_{p^n}})_{h\Delta} \to (THH(ku)_{hC_{p^n}})^{h\Delta}$ is an equivalence. This implies that

$$(THH(ku)_{hC_{p^n}})^{h\Delta} \simeq (THH(ku)^{h\Delta})_{hC_{p^n}}.$$

The left-hand side vertical arrow is therefore also an equivalence. Thus the middle vertical arrow is an equivalence, which completes the proof of the claim.

Finally, by [HM1] and [Du] we have natural cofibre sequences

$$K(\ell)_p \to TC(\ell; p) \to \Sigma^{-1}H\mathbb{Z}_p \quad \text{and} \quad K(ku)_p \to TC(ku; p) \to \Sigma^{-1}H\mathbb{Z}_p.$$

We take the Δ-homotopy fixed points of the latter one and assemble these cofibre sequences into a commutative diagram

$$\begin{array}{ccccc}
K(\ell)_p & \xrightarrow{\mathrm{trc}} & TC(\ell; p) & \longrightarrow & \Sigma^{-1}H\mathbb{Z}_p \\
\downarrow & & \downarrow & & \downarrow{=} \\
K(ku)_p^{h\Delta} & \xrightarrow{\mathrm{trc}^{h\Delta}} & TC(ku; p)^{h\Delta} & \longrightarrow & \Sigma^{-1}H\mathbb{Z}_p
\end{array}$$

where the middle and right-hand side vertical arrows are equivalences. Thus the map $K(\ell)_p \to K(ku)_p^{h\Delta}$ is also an equivalence. $\square$

The computations given in this paper provide evidence for interesting speculations on the properties of the extension $\ell \to ku$, and on how these properties are reflected in topological Hochschild homology.

Let us assume that we can make sense at a spectrum level of the formula

$$ku = \ell[u]/(u^{p-1} = v_1) \tag{10.3}$$

which holds for the coefficients rings. The prime (v_1) of ℓ ramifies as $(u)^{p-1}$ in ku, so the extension $\ell \to ku$ should not qualify as an étale extension. And indeed, the computations of $V(1)_*THH(\ell)$ and $V(1)_*THH(ku)$ given in Theorems 5.9 and 9.15 imply that

$$ku \wedge_\ell THH(\ell) \not\simeq_p THH(ku).$$

Compare with the algebraic situation, where the Geller-Weibel Theorem [GW] states that if an extension $A \to B$ of k-algebras is étale, this is reflected by an isomorphism

$$B \otimes_A \mathbb{H}_*^k(A) \cong \mathbb{H}_*^k(B)$$

in Hochschild homology.

Inverting v_1 in ℓ and ku, we obtain the periodic Adams summand L and the periodic K-theory spectrum KU (both p-completed). The map $L \to KU$ induces on coefficients the inclusion

$$L_* = \mathbb{Z}_p[v_1, v_1^{-1}] \hookrightarrow \mathbb{Z}_p[u, u^{-1}] = KU_*.$$

Now $(p-1)u^{p-2}$ is invertible in KU_* and we expect the extension $L \to KU$ to be étale (compare with Remark 3.7). Evidence for this is provided by the computations in topological Hochschild homology given in (7.12) and Proposition 7.13, which imply that we have an equivalence

$$KU \wedge_L THH(L) \simeq_p THH(KU).$$

10.4. Tame ramification. The extension $\ell \to ku$ is not unramified, but from formula (10.3) we nevertheless expect it to be tamely ramified. In particular we view Theorem 10.2 as an example of tamely ramified descent.

The behavior of topological Hochschild homology with respect to tamely ramified extensions of discrete valuation rings was studied by Hesselholt and Madsen in [HM2]. Their results can be used to provide an interesting, at this point very speculative explanation of the structure of $THH(ku)$. It is due to Lars Hesselholt, and I would like to thank him for sharing the ideas exposed in the remaining part of this paper.

Let us briefly recall the results of [HM2] that are relevant here. Let A be a discrete valuation ring, K its quotient field (of characteristic 0) and k its perfect residue field (of characteristic p). The localization cofibre sequence in algebraic K-theory maps via the trace map to a localization sequence in topological Hochschild homology. We have a commutative diagram

$$\begin{array}{ccccc}
K(k) & \xrightarrow{i^!} & K(A) & \xrightarrow{j} & K(K) \\
\downarrow & & \downarrow & & \downarrow \\
THH(k) & \xrightarrow{i^!} & THH(A) & \xrightarrow{j} & THH(A \mid K)
\end{array}$$

whose rows are cofibre sequences. Here the map $i^!$ is the transfer, and j is a map of ring spectra. The cofibre $THH(A \mid K)$ is defined in [HM2, Definition 1.5.5] as topological Hochschild homology of a suitable linear category.

Let $M = A \cap K^\times$, and consider the log ring (A, M) with pre-log structure given by the inclusion $\alpha\colon M \hookrightarrow A$. Then the homotopy groups $(\pi_* THH(A \mid K), M)$ form a log differential graded ring over (A, M). The universal example of such a log differential graded ring is the de Rham-Witt complex with log poles $\omega^*_{(A,M)}$, and there is a canonical map $\omega^*_{(A,M)} \to \pi_* THH(A \mid K)$. Hesselholt and Madsen define an element $\kappa \in V(0)_2 THH(A \mid K)$ and prove in [HM2, Theorem 2.4.1] that there is a natural isomorphism

$$\omega^*_{(A,M)} \otimes_{\mathbb{Z}} P(\kappa) \xrightarrow{\cong} V(0)_* THH(A \mid K). \tag{10.5}$$

Let L be a finite, tamely ramified extension of K, and B be the integral closure of A in L. If follows from [HM2, Lemma 2.2.4 and 2.2.6] that there is an isomorphism

$$B \otimes_A \omega^*_{(A,M_A)} \otimes_{\mathbb{Z}} \mathbb{F}_p \xrightarrow{\cong} \omega^*_{(B,M_B)} \otimes_{\mathbb{Z}} \mathbb{F}_p. \tag{10.6}$$

In fact this isomorphism is essentially the property that qualifies a map of log rings $(A, M_A) \to (B, M_B)$ for being log-étale.

Assembling (10.5) and (10.6) we obtain an isomorphism

$$B \otimes_A V(0)_* THH(A \mid K) \xrightarrow{\cong} V(0)_* THH(B \mid L).$$

In particular we have an equivalence

$$HB \wedge_{HA} THH(A \mid K) \simeq_p THH(B \mid L) \tag{10.7}$$

of p-completed spectra.

Let us now optimistically assume that these results hold also in the generality of commutative $\mathbb{S}$-algebras. The ring ℓ has a prime ideal (v_1), with residue ring $H\mathbb{Z}_p$ and quotient ring L. Similarly, ku has a prime ideal (u), with residue ring $H\mathbb{Z}_p$ and quotient ring KU. The localization cofibre sequences in topological

Hochschild homology fit into a commutative diagram

$$\begin{tikzcd} THH(H\mathbb{Z}_p) \arrow[r, "i^!"] \arrow[d, "\simeq"'] & THH(\ell) \arrow[r, "j"] \arrow[d] & THH(\ell \mid L) \arrow[d] \\ THH(H\mathbb{Z}_p) \arrow[r, "i^!"] & THH(ku) \arrow[r, "j"] & THH(ku \mid KU). \end{tikzcd} \tag{10.8}$$

This requires that we can identify by dévissage $THH(H\mathbb{Z}_p)$ with the topological Hochschild homology spectrum of a suitable category of finite v_1-torsion ℓ-modules or ku-modules. Since $\ell \to ku$ is tamely ramified, we expect that there is an equivalence

$$ku \wedge_\ell THH(\ell \mid L) \simeq THH(ku \mid KU) \tag{10.9}$$

analogous to (10.7). The top cofibration of (10.8) induces a long exact sequence

$$\cdots \xrightarrow{\partial_*} V(1)_n THH(H\mathbb{Z}_p) \xrightarrow{i^!_*} V(1)_n THH(\ell) \xrightarrow{j_*} V(1)_n THH(\ell \mid L) \xrightarrow{\partial_*} \cdots$$

in $V(1)$ homotopy. There are isomorphisms

$$V(1)_* THH(H\mathbb{Z}_p) \cong E(\lambda_1, \varepsilon) \otimes P(\mu_1),$$
$$V(1)_* THH(\ell) \cong E(\lambda_1, \lambda_2) \otimes P(\mu_2).$$

Here ε has degree $2p-1$ and supports a primary v_1-Bockstein $\beta_{1,1}(\varepsilon) = 1$. From the structure of the higher v_1-Bocksteins we know that $i^!_*\big(E(\lambda_1) \otimes P(\mu_1)\big) = 0$ and that $P(\mu_2)$ injects into $V(1)_* THH(\ell \mid L)$ via j_*. Thus we expect that

$$V(1)_* THH(\ell \mid L) = E(d, \lambda_1) \otimes P(\mu_1),$$

where $d \in V(1)_1 THH(\ell \mid L)$ satisfies $\partial_*(d) = 1$. We should have $\partial_*(d\lambda_1) = \lambda_1$, $\partial_*(d\mu_1^j) = \mu_1^j$, $\partial_*(\mu_1) = \varepsilon$, $j_*(\lambda_1) = \lambda_1$, $j_*(\mu_2^k) = \mu_1^{pk}$ and $i^!(\varepsilon\mu_1^{p-1}) = \lambda_2$. How the remaining classes map under ∂_*, $i^!_*$ or j_* is then forced by the grading and the exactness. We deduce from (10.9) that

$$V(1)_* THH(ku \mid KU) \cong P_{p-1}(u) \otimes E(\lambda_1, d) \otimes P(\mu_1).$$

Assembling these computations in diagram (10.8) and chasing, we obtain an (additive) isomorphism

$$V(1)_* THH(ku) \cong \big[E(\lambda_1, \lambda_2) \otimes P(\mu_2)\big] \oplus \big[E(\lambda_1, d) \otimes P(\mu_1) \otimes \mathbb{F}_p\{u, \dots, u^{p-2}\}\big].$$

Under the identifications $a_i = du\mu_1^i$ and $b_j = u\mu_1^j$, this is compatible with Theorem 9.15.

INSTITUTE OF MATHEMATICS, UNIVERSITY OF BONN, 53115 BONN, GERMANY
E-mail: ausoni@math.uni-bonn.de

REFERENCES

[Ad] J. F. Adams, On Chern characters and the structure of the unitary group, *Proc. Cambridge Philos. Soc.* **57** (1961), 189–199.

[AnR] V. Angeltveit and J. Rognes, Hopf algebra structure on the topological Hochschild homology of commutative S-algebras, preprint, 2003.

[AR] Ch. Ausoni and J. Rognes, Algebraic K-theory of topological K-theory, *Acta Math.* **188** (2002), 1–39.

[Ba] M. Basterra, André-Quillen cohomology of commutative S-algebras, *J. Pure Appl. Algebra* **144** (1999), 111–143.

[BDR] N. Baas, B. Dundas, and J. Rognes, Two-vector bundles and forms of elliptic cohomology, *London Math. Soc. Lecture Note Ser.*, vol. 308, Cambridge University Press, 2004, pp. 18–45.

[BJ] A. Baker and A. Jeanneret, Brave new Hopf algebroids and extensions of MU-algebras, *Homology Homotopy Appl.* **4** (2002), 163–173 (electronic).

[Bo] J. M. Boardman, Conditionally convergent spectral sequences, *Homotopy Invariant Algebraic Structures (Baltimore, MD, 1998)*, *Contemp. Math.*, vol. 239, Amer. Math. Soc., Providence, RI, 1999, pp. 49–84.

[Bö1] M. Bökstedt, Topological Hochschild homology, unpublished.

[Bö2] ———, The topological Hochschild homology of $\mathbb{Z}$ and $\mathbb{Z}/p$, unpublished.

[BAC] Buenos Aires Cyclic Homology Group, Cyclic homology of algebras with one generator, *K-Theory* **5** (1991), 51–69 (Jorge A. Guccione, Juan José Guccione, María Julia Redondo, Andrea Solotar and Orlando E. Villamayor participated in this research).

[CE] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton University Press, 1956.

[Du] B. I. Dundas, Relative K-theory and topological cyclic homology, *Acta Math.* **179** (1997), 223–242.

[EKMM] A. D. Elmendorff, I. Kriz, M. A. Mandell and J. P. May, *Rings, Modules, and Algebras in Stable Homotopy Theory*, *Math. Surveys Monogr.*, vol. 47, Amer. Math. Soc., Providence, RI, 1997.

[GW] S. Geller and C. Weibel, Etale descent for Hochschild and cyclic homology, *Comment. Math. Helv.* **66** (1991), 368–388.

[HM1] L. Hesselholt and I. Madsen, On the K-theory of finite algebras over Witt vectors of perfect fields, *Topology* **36** (1997), 29–101.

[HM2] ———, On the K-theory of local fields, *Ann. of Math. (2)* **158** (2003), 1–113.

[Hu] T. J. Hunter, On the homology spectral sequence for topological Hochschild homology, *Trans. Amer. Math. Soc.* **348** (1996), 3941–3953.

[La] A. Lazarev, Homotopy theory of A_∞ ring spectra and applications to MU-modules, *K-Theory* **24** (2001), 243–281.

[Lo] J.-L. Loday, *Cyclic Homology*, 2nd ed., *Grundlehren Math. Wiss.*, vol. 301, Springer-Verlag, 1998 (Appendix E by M. O. Ronco; Chapter 13 by the author in collaboration with T. Pirashvili).

[Mc] J. McCleary, *A User's Guide to Spectral Sequences*, 2nd ed., *Cambridge Stud. Adv. Math.*, vol. 58, Cambridge University Press, 2001.

[MS] J. E. McClure and R. E. Staffeldt, On the topological Hochschild homology of bu. I, *Amer. J. Math.* **115** (1993), 1–45.

[MSV] J. McClure, R. Schwänzl, and R. M. Vogt, $THH(R) \cong R \otimes S^1$ for E_∞ ring spectra, *J. Pure Appl. Algebra* **121**(2) (1997), 137–159.

[Mi] J. Milnor, The Steenrod algebra and its dual, *Ann. of Math. (2)* **67** (1958), 150–171.

[Ok] S. Oka, Ring spectra with few cells, *Japan J. Math. (N.S.)* **5** (1979), 81–100.

[Qu] D. Quillen, On the cohomology and K-theory of the general linear groups over a finite field, *Ann. of Math. (2)* **96** (1972), 552–586.

[Ro] J. Rognes, Galois extensions of structured ring spectra, preprint 2005.

[RW] J. Rognes and C. A. Weibel, Two-primary algebraic K-theory of rings of integers in number fields, *J. Amer. Math. Soc.* **13** (2000), 1–54 (Appendix A by Manfred Kolster).

[SVW1] R. Schwänzl, R. M. Vogt, and F. Waldhausen, Adjoining roots of unity to E_∞ ring spectra in good cases—a remark, *Homotopy Invariant Algebraic Structures (Baltimore, MD, 1998)*, *Contemp. Math.*, vol. 239, Amer. Math. Soc., Providence, RI, 1999, pp. 245–249.

[SVW2] ———, Topological Hochschild homology, *J. London Math. Soc. (2)* **62** (2000), 345–356.

[Ts] S. Tsalidis, On the étale descent problem for topological cyclic homology and algebraic *K*-theory, *K-Theory* **21** (2000), 151–199.