Two-vector bundles

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(joint work with Bjørn Ian Dundas and John Rognes)

Two-vector bundles, as defined by Baas, Dundas and Rognes [6], are a 2-categorical analogue of ordinary complex vector bundles. A two-vector bundle of rank n over a space X can be thought of as a locally trivial bundle of categories with fibre $\mathcal{V}^n$, where $\mathcal{V}$ is the bimonoidal category of finite dimensional complex vector spaces and isomorphisms. It can be defined by means of an open cover of X by charts with specified trivialisations, gluing data which represents a weakly invertible matrix of ordinary vector bundles on the intersection of two charts, and coherence isomorphisms on the intersection of three charts [6, §2]. Equivalence classes of two-vector bundles of rank n over a finite CW-complex X are in bijective correspondence with homotopy classes of maps from X to $|B\,\mathrm{GL}_n(\mathcal{V})|$ [5]. By group-completing with respect to the direct sum of matrices we obtain the space

$$K(\mathcal{V}) = \Omega B\Big(\coprod_n |B\,\mathrm{GL}_n(\mathcal{V})|\Big)$$

that represents virtual two-vector bundles. Gerbes with band U(1) coincide with two-vector bundles of rank 1.

Very little is known about the geometry of two-vector bundles. However, by a theorem of Baas, Dundas, Richter and Rognes [7], there is a weak equivalence

$$K(\mathcal{V}) \simeq K(ku)\,,$$

where $K(ku)$ is the algebraic K-theory space of the connective complex K-theory spectrum ku (viewed as a ring in a suitable sense). This permits us to study the space $K(\mathcal{V})$ by means of invariants of algebraic K-theory, like the Bökstedt trace map to topological Hochschild homology, or the cyclotomic trace map to topological cyclic homology. In joint work with Rognes [3, 1] we applied trace methods to compute $K(ku)$ with suitable finite p-primary coefficients for $p \geq 5$. We prove that the spectrum $K(ku)$ is of chromatic complexity 2 in the sense of stable homotopy theory. This means that two-vector bundles define a cohomology theory that, from the view-point of stable homotopy theory, is a suitable candidate for elliptic cohomology. In particular, it is a strictly finer invariant than topological K-theory.

The rational information carried by a two-vector bundle is fairly well understood : it is contained in the associated dimension and determinant bundles. Let $\pi : ku \to \mathbb{Z}$ be the unique ring-map that is a π_0-isomorphism, and let

$$\pi : K(ku) \to K(\mathbb{Z})$$

be the induced map in algebraic K-theory. This map represents the forgetful map that associates to a two-vector-bundle its dimension bundle, or decategorification. There is also a rational determinant map [4]

$$\det_{\mathbb{Q}} : K(ku) \to B\,\mathrm{SL}_1(ku)_{\mathbb{Q}}\,.$$

Up to homotopy, this is a rational retraction of the “inclusion of units” map

$$w : B\,\mathrm{SL}_1(ku) \to K(ku)\,.$$

Thus, any virtual two-vector bundle has an associated (rational) determinant bundle. We proved in [4] that the maps π and $\det_{\mathbb{Q}}$ define a rational equivalence

$$K(ku)_{\mathbb{Q}} \simeq B\,\mathrm{SL}_1(ku)_{\mathbb{Q}} \times K(\mathbb{Z})_{\mathbb{Q}}\,.$$

The space of units $\mathrm{SL}_1(ku)$ is equivalent as an infinite loop-space to the space $B\mathrm{U}_{\otimes}$ representing virtual complex line bundles and their tensor product. By a result of Borel [8], the space $K(\mathbb{Z})$ is rationally equivalent to $\mathbb{Z} \times \mathrm{SU}/\mathrm{SO}$.

The map $\pi : K(ku) \to K(\mathbb{Z})$ is 3-connected, from which we deduce that $K_1(ku) \cong \mathbb{Z}/2$ and $K_2(ku) \cong \mathbb{Z}/2$. We expect that in higher degrees, the integral homotopy groups of $K(ku)$ will reflect the high complexity of $K(\mathbb{Z})$ and of some of the v_2-periodic families in the stable homotopy groups of spheres [3, §9]. This is illustrated in the following example. An obvious and meaningful invariant to detect higher dimensional classes in $K_*(ku)$ would be a determinant map $\det : K(ku) \to B\,\mathrm{SL}_1(ku)$ that is an (integral) homotopy retraction of w. However, as observed by Dundas and Rognes, such a map cannot possibly exist : a first obstruction to its existence is an intriguing virtual two-vector bundle ς on the sphere S^3. In effect, we show in [2] that there is an isomorphism of Abelian groups

$$K_3(ku) \cong \mathbb{Z} \oplus \mathbb{Z}/24\,,$$

where the torsion free summand is generated by ς, and the torsion subgroup is generated by a class named ν (the image of the class with the same name in the stable homotopy of S^3). We prove that the $\mathrm{U}(1)$-gerbe μ over S^3 representing the fundamental class in $H^3(S^3;\mathbb{Z}) \cong \mathbb{Z}$ (also known as Dirac's magnetic monopole) decomposes as

$$\mu = 2\varsigma - \nu \in K_3(ku)$$

when viewed as a virtual two-vector bundle of rank one. Therefore, the element ς classifies a virtual two-vector bundle over S^3 that, modulo torsion, is half the magnetic monopole. Its associated dimension bundle is a generator of $K_3(\mathbb{Z}) \cong \mathbb{Z}/48$.

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