STRONG REGULARITY

by

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1. Uniformly hyperbolic dynamical systems

The theory of *uniformly hyperbolic dynamical systems* was constructed in the 1960's under the dual leadership of Smale in the USA and Anosov and Sinai in the Soviet Union. It is nowadays almost complete. It encompasses various examples [**Sma67**]: expanding maps, horseshoes, solenoid maps, Plykin attractors, Anosov maps and DA, all of which are *basic pieces*.

We recall standard definitions. Let f be a C^1-diffeomorphism f of a finite dimensional manifold M. A compact f-invariant subset $\Lambda \subset M$ is *uniformly hyperbolic* if the restriction to Λ of the tangent bundle TM splits into two continuous invariant subbundles

$$TM|\Lambda = E^s \oplus E^u,$$

E^s being uniformly contracted and E^u being uniformly expanded.

Then for every $z \in \Lambda$, the sets

$$W^s(z) = \{z' \in M : \lim_{n\to+\infty} d(f^n(z), f^n(z')) = 0\},$$

$$W^u(z) = \{z' \in M : \lim_{n\to-\infty} d(f^n(z), f^n(z')) = 0\}$$

are called *the stable and unstable manifolds* of z. They are immersed manifolds tangent at z to respectively $E^s(z)$ and $E^u(z)$.

The ϵ-*local stable manifold* $W^s_\epsilon(z)$ of z is the connected component of z in the intersection of $W^s(z)$ with a ϵ-neighborhood of z. The ϵ-*local unstable manifold* $W^u_\epsilon(z)$ is defined likewise.

Definition 1.1. — A *basic set* is a compact, f-invariant, uniformly hyperbolic set Λ which is transitive and *locally maximal*: there exists a neighborhood N of Λ such that $\Lambda = \cap_{n\in\mathbb{Z}} f^n(N)$. A basic set is an *attractor* if the neighborhood N can be chosen in such a way that $\Lambda = \cap_{n\geq 0} f^n(N)$. Such a basic set contains the unstable manifolds of its points.

A diffeomorphism whose nonwandering set is a finite union of disjoint basic sets is called *uniformly hyperbolic* or *Axiom A*.

Such diffeomorphisms enjoy nice properties, which are proved in [**Sma67**] and the references therein.

1.0.0.1. SRB and physical measure. — Let $\alpha > 0$, and let Λ be an attracting basic set for a $C^{1+\alpha}$-diffeomorphism f. Then there exists a unique invariant, ergodic probability μ supported on Λ such that its conditional measures, with respect to any measurable partition of Λ into plaques of unstable manifolds, are absolutely continuous with respect to the Lebesgue measure class (on unstable manifolds). Such a probability is called *SRB* (for Sinai-Ruelle-Bowen). It turns out that a SRB -measure is *physical*: the Lebesgue measure of its basin $B(\mu)$

$$B(\mu) = \{z \in M : \frac{1}{n}\sum_{i<n} \delta_{f^i(x)} \rightharpoonup \mu\}, \tag{$\mathcal{B}$}$$

is positive. Actually, up to a set of Lebesgue measure 0, $B(\mu)$ is equal to the topological basin of Λ, i.e the set of points attracted by Λ.

1.0.0.2. Persistence. — A basic set Λ for a C^1-diffeomorphism f is *persistent*: every C^1-perturbation f' of f leaves invariant a basic set Λ' which is homeomorphic to Λ, via a homeomorphism which conjugates the dynamics $f|\Lambda$ and $f'|\Lambda'$.

1.0.0.3. Coding. — A basic set Λ for a C^1-diffeomorphism f admits a (finite) Markov partition. This implies that its dynamics is semi-conjugated with a subshift of finite type. The semi-conjugacy is 1-1 on a generic set. Its lack of injectivity is itself coded by subshifts of finite type of smaller topological entropy. This enables to study efficiently all the invariant measures of Λ, the distribution of its periodic points, the existence and uniqueness of the maximal entropy measure, and if f is $C^{1+\alpha}$, the Gibbs measures which are related to the geometry of Λ.

1.1. End of Smale's program. — Smale wished to prove the density of Axiom A in the space of C^r-diffeomorphisms. In higher dimensions, obstructions were soon discovered by Shub [**Shu71**]. For surfaces Newhouse showed the non-density of Axiom A diffeomorphisms for $r \geq 2$: he constructed robust tangencies between stable and unstable manifolds of a thick horseshoe [**New74**]. Numerical studies by Lorenz [**Lor63**] and Hénon [**Hén76**] explored dynamical systems with hyperbolic features that did not fit in the uniformly hyperbolic theory. In order to include many examples such as the Hénon one, the *non-uniform hyperbolic theory* is still under construction.

2. Non-uniformly hyperbolic dynamical systems

2.1. Pesin theory. — The natural setting for non-uniform hyperbolicity is Pesin theory [**BP06, LY85**], from which we recall some basic concepts. We first consider the simpler settings of invertible dynamics.

Let f be a $C^{1+\alpha}$-diffeomorphism (for some $\alpha > 0$) of a compact manifold M and let μ be an ergodic f-invariant probability measure on M. The Oseledets multiplicative ergodic theorem produces Lyapunov exponents (w.r.t. μ) for the tangent cocycle of f, and an associated μ-a.e f-invariant splitting of the tangent bundle into characteristic subbundles.

Denote by $E^s(z)$ (resp. $E^u(z)$) the sum of the the characteristic subspaces associated to the negative (resp. positive) Lyapunov exponents.

The *stable and unstable Pesin manifolds* are defined respectively for μ-a.e. z by

$$W^s(z) = \{z' \in M : \limsup_{n\to+\infty} \frac{1}{n} \log d(f^n(z), f^n(z')) < 0\},$$

$$W^u(z) = \{z' \in M : \liminf_{n\to-\infty} \frac{1}{n} \log d(f^n(z), f^n(z')) > 0\}.$$

They are immersed manifolds through z tangent respectively at z to $E^s(z)$ and $E^u(z)$.

The measure μ is *hyperbolic* if 0 is not a Lyapunov exponent w.r.t. μ. Every invariant ergodic measure, which is supported on a uniformly hyperbolic compact invariant set, is hyperbolic.

2.1.0.4. SRB, physical measures. — An invariant ergodic measure μ is *SRB* if the largest Lyapunov exponent is positive and the conditional measures of μ w.r.t. a measurable partition into plaques of unstable manifolds are μ-a.s. absolutely continuous w.r.t. the Lebesgue class (on unstable manifolds). When μ is SRB and hyperbolic, it is also *physical*: its basin has positive Lebesgue measure.

The paper [**You98**] provides a general setting where appropriate hyperbolicity hypotheses allow to construct hyperbolic SRB measures with nice statistical properties.

2.1.0.5. Coding. — Let μ be a f-invariant ergodic hyperbolic SRB measure. Then there is a partition mod.0 of M into finitely many disjoint subsets $\Lambda_1, \dots, \Lambda_k$, which are cyclically permuted by f and such that the restriction $f^k_{|\Lambda_1}$ is metrically conjugated to a Bernoulli automorphism.

Of a rather different flavor is Sarig's recent work [**Sar13**]. For a $C^{1+\alpha}$-diffeomorphism of a compact surface of positive topological entropy and any $\chi > 0$, he constructs a countable Markov partition for an invariant set which has full measure w.r.t. any ergodic invariant measure with metric entropy $> \chi$. The semi-conjugacy associated to this Markov partition is finite-to-one.

2.1.0.6. Non-invertible dynamics. — One should distinguish between the non-uniformly expanding case and the case of general endomorphisms.

In the first setting, a SRB measure is simply an ergodic invariant measure whose all Lyapunov exponents are positive and which is absolutely continuous.

Defining appropriately unstable manifolds and SRB measures for general endomorphisms is more delicate. One has typically to introduce the inverse limit where the endomorphism becomes invertible.

2.2. Case studies. — The paradigmatic examples in low dimension can be summarized by the following table:

Uniformly hyperbolic	Non-uniformly hyperbolic
Expanding maps of the circle	Jakobson's Theorem
Conformal expanding maps of complex tori	Rees' Theorem
Attractors (Solenoid, DA, Plykin...)	Benedicks-Carleson's Theorem
Horseshoes	Non-uniformly hyperbolic horseshoes
Anosov diffeomorphisms	Standard map ?

Let us recall what are these theorems, and the correspondence given by the lines of the table.

Expanding maps of the circle may be considered as the simplest case of uniformly hyperbolic dynamics. The Chebychev quadratic polynomial $P_{-2}(x) := x^2 - 2$ on the invariant interval $[-2, 2]$ has a critical point at 0, but it is still semi-conjugated to the doubling map $\theta \mapsto 2\theta$ on the circle (through $x = 2\cos 2\pi\theta$). For $a \in [-2, -1]$, the quadratic polynomial $P_a(x) := x^2 + a$ leaves invariant the interval $[P_a(0), P_a^2(0)]$ which contains the critical point 0.

Theorem 2.1 **(Jakobson [Jak81]).** — *There exists a set $\Lambda \subset [-2, -1]$ of positive Lebesgue measure such that for every $a \in \Lambda$ the map $P(x) = x^2 + a$ leaves invariant an ergodic, hyperbolic measure which is equivalent to the Lebesgue measure on $[P_a(0), P_a^2(0)]$.*

Actually the set Λ is nowhere dense. Indeed the set of $a \in \mathbb{R}$ such that P_a is axiom A is open and dense [**GŚ97, Lyu97**].

Let L be a lattice in $\mathbb{C}$ and let c be a complex number such that $|c| > 1$ and $cL \subset L$. Then the homothety $z \mapsto cz$ induces an expanding map of the complex torus $\mathbb{C}/L$. The Weierstrass function associated to the lattice L defines a ramified covering of degree 2 from $\mathbb{C}/L$ onto the Riemann sphere which is a semi-conjugacy from this expanding map to a rational map of degree $|c|^2$ called a *Lattes map.* For any $d \geq 2$, the set Rat_d of rational maps of degree d is naturally parametrized by an open subset of $\mathbb{P}(\mathbb{C}^{2d+2})$.

Theorem 2.2 **(Rees [Ree86]).** — *For every $d \geq 2$, there exists a subset $\Lambda \subset \mathrm{Rat}_d$ of positive Lebesgue measure such that every map $R \in \Lambda$ leaves invariant an ergodic hyperbolic probability measure which is equivalent to the Lebesgue measure on the Riemann sphere.*

For rational maps in Λ, the Julia set is equal to the Riemann sphere. On the other hand, a conjecture of Fatou [**Mil06**] claims that the set of rational maps which satisfy Axiom A is open and dense in Rat_d. The restriction of such maps to their Julia set is uniformly expanding. For such maps, the Hausdorff dimension of the Julia set is smaller than 2.

The (real) Hénon family is the 2-parameter family of polynomial diffeomorphisms of the plane defined for $a, b \in \mathbb{R}$, $b \neq 0$ by

$$h_{a\,b}(x, y) = (x^2 + a + y, -bx)$$

Observe that $h_{a\,b}$ has constant Jacobian equal to b. For small $|b|$, there exists an interval $J(b)$ close to $[-2, -1]$ such that, for $a \in J(b)$, the Hénon map $h_{a\,b}$ has the following properties

- $h_{a\,b}$ has two fixed points; both are hyperbolic saddle points, one, called β with positive unstable eigenvalue, the other , called α, with negative unstable eigenvalue;
- there is a trapping open region B satisfying $h_{a\,b}(B) \Subset B$ which contains α (and therefore also its unstable manifold).

Hénon [**Hén76**] investigated numerically the behavior of orbits starting in B for $b = -0.3$, $a = -1.4$. Such orbits apparently converged to a "strange attractor ".

Theorem 2.3 **(Benedicks-Carleson [BC91]).** — *For every $b < 0$ close enough to 0, there exists a set $\Lambda_b \subset J(b)$ of positive Lebesgue measure, such that for every $a \in \Lambda_b$, the maximal invariant set $\bigcap_{n\geq 0} h_{a\,b}^n(B)$ is equal to the closure of the unstable manifold $W^u(\alpha)$ and contains a dense orbit along which the derivatives of iterates grow exponentially fast.*

An easy topological argument insures that this maximal invariant set is never uniformly hyperbolic. Later Benedicks-Young [**BY93**] showed that for every such parameters $a \in \Lambda_b$ the Hénon map $h_{a\,b}$ leaves invariant an ergodic hyperbolic SRB measure. Such a measure is physical. Benedicks-Viana [**BV01**] actually proved that the basin of this measure has full Lebesgue measure in the trapping region B.

From [**Ure95**], every $a \in \Lambda_b$ is accumulated by parameter intervals exhibiting Newhouse phenomenon: for generic parameters in these intervals, $h_{a\,b}$ has infinitely many periodic sinks in B. In particular, the set Λ_b is nowhere dense.

The starting point in [**PY09**] is a smooth diffeomorphism of a surface M having a horseshoe (1) K. It is assumed that there exist distinct fixed points $p_s, p_u \in K$ and $q \in M$ such that $W^s(p_s)$ and $W^u(p_u)$ have at q a quadratic heteroclinic tangency which is an isolated point of $W^s(K) \cap W^u(K)$. The authors consider a one-parameter family (f_t) unfolding the tangency and study the maximal f_t-invariant set L_t in a neighborhood of the union of K with the orbit of q. Writing d_s, d_u for the transverse Hausdorff dimensions of $W^s(K)$, $W^u(K)$ respectively, it was shown previously [**PT93**] that L_t is a horseshoe for most t when $d_s + d_u < 1$. By [**MY10**] this is no longer true when $d_s + d_u > 1$. However, when $d_s + d_u$ is only slightly larger (2) than 1, some dynamical and geometric information on L_t is obtained in [**PY09**] for most values of t: in particular, both the stable and unstable sets for L_t have Lebesgue measure 0, and an ergodic hyperbolic f_t-invariant probability measure supported on L_t with geometric content is constructed.

The two papers in this volume are related to these case studies.

1. A horseshoe is an infinite basic set of saddle type.
2. The exact condition is $(d_s + d_u)^2 + (\max\{d_s, d_u\})^2 < d_s + d_u + \max\{d_s, d_u\}$

In [**Yoc**], a proof of Jakobson's theorem is given. The main ingredient is the concept of *strong regularity* (explained below).

In [**Ber**], a class of endomorphisms of the plane containing the Hénon family is considered. Given any map $B \in C^2(\mathbb{R}^3, \mathbb{R}^2)$ with small C^2-uniform norm, one studies the one-parameter family

$$f_{a,B}(x, y) = (x^2 + a + y, 0) + B(x, y, a).$$

It is shown that there exists a set $\Lambda_B \subset \mathbb{R}$ of positive Lebesgue measure such that, for any $a \in \Lambda_B$, $f_{a,B}$ has an invariant ergodic hyperbolic physical SRB measure. The proof is based on an appropriate generalization of strong regularity.

2.3. Open problems. — Linear Anosov diffeomorphisms of $\mathbb{T}^2$ are area-preserving and uniformly hyperbolic. In the conservative setting, a very natural case study to consider is the Chirikov-Taylor standard map family. This is a one-parameter family of area-preserving diffeomorphisms of $\mathbb{T}^2$ defined for $a \in \mathbb{R}$ by

$$S_a(x, y) = (2x - y + a \sin 2\pi x, x).$$

One form of a conjecture of Sinai ([**Sin94**] P.144) about this family is

Conjecture 2.4. — *There exists a set $\Lambda \subset \mathbb{R}$ of positive Lebesgue measure such that, for $a \in \Lambda$, the Lebesgue measure on $\mathbb{T}^2$ is ergodic and hyperbolic for S_a.*

For such parameters, the map S_a cannot have any of the invariant curves produced by KAM-theory. In particular, a cannot be too small.

This conjecture is still completely open despite intense efforts. A weak argument in favor of this conjecture is that, when a is large, the maximal invariant set in the complement of an appropriate neighborhood of the critical lines $\{x = \pm 1/4\}$ is a uniformly hyperbolic horseshoe of dimension close to 2 [**Dua94, BC14**].

Actually, a large Hausdorff dimension of the invariant sets under consideration appears to be a major difficulty on the way to prove non-uniform hyperbolicity.

For the parameters considered in [**BC91**] and subsequent papers, the Hausdorff dimension of the Hénon attractor is *a priori* close to 1. On the other hand, numerical studies [**RHO80**] of the values $a = -1.4$, $b = -0.3$ considered by Hénon indicate an (eventual) attractor of Hausdorff dimension 1.261 ± 0.003.

Problem 2.5. — *For every $d < 2$, find an open set of smooth families $(f_t)_t$ of smooth diffeomorphisms of $\mathbb{R}^2$ such that, with positive probability on the parameter, f_t leaves invariant an ergodic hyperbolic SRB probability measure whose support has dimension at least d.*

One should also recall that Carleson conjectured [**Car91**] that proving non-uniform hyperbolicity (or only the weaker conclusion of [**BC91**]) for a particular parameter value is in some rigorous sense undecidable.

A similar problem, in the setting of non-uniformly hyperbolic horseshoes, is

Problem 2.6. — *Prove the conclusions of* [**PY09**] *for an initial horseshoe K of transverse Hausdorff dimensions d_s, d_u satisfying*

$$d_s + d_u > 3/2.$$

Even the non-uniformly expanding case is still incomplete, since it regards only the case of real or complex dimension 1. A positive answer to the following problem would be a 2-dimensional generalization of Jakobson's Theorem for perturbation of the product dynamics:

$$P_a \times P_a \colon (x, y) \mapsto (x^2 + a, y^2 + a).$$

Problem 2.7. — *Does there exist an open set of* 1*-parameter smooth families (f_a) of endomorphisms of the plane, accumulating on $(P_a \times P_a)_a$, with the following property: with positive probability on the parameter, f_a leaves invariant an ergodic absolutely continuous invariant measure with two positive Lyapunov exponents.*

3. Proving non uniform hyperbolicity

There are now many proofs of both Jakobson's theorem and Benedicks-Carleson's theorem. Broadly speaking, they rely either on a binding approach, pioneered by Benedicks-Carleson, or on a strong regularity approach, closer to Jakobson's original proof. Both papers in this volume follow the second approach.

In both approaches, the study of the 2-dimensional setting depends very much on the 1-dimensional case.

We now explain some of the differences between the two methods.

3.1. The binding approach for quadratic maps. — Benedicks-Carleson proved Jakobson's theorem by focusing on the expansion of the post-critical orbit. There are many proofs in this spirit [**CE80, BC85, Tsu93a, Tsu93b, Luz00**].

One actually proves the existence of a set $\Lambda \subset \mathbb{R}$ of positive Lebesgue measure such that, for $a \in \Lambda$, the quadratic map $P_a(x) = x^2 + a$ satisfies the Collet-Eckmann condition:

$$\liminf_{+\infty} \frac{1}{n} \log \|DP^n(a)\| > 0.$$

This property implies the existence of an absolutely continuous ergodic invariant measure with positive Lyapunov exponent[**CE83**].

One starts with a parameter a_0 such that the critical value a_0 of P_{a_0} belongs to a repulsive periodic cycle. Then, there exists $\lambda > 1$ so that

(*i*) $DP^n_{a_0}(a_0) > \lambda^n$ for every large n,

(*ii*) for every $\delta > 0$, the map P_{a_0} is λ-expanding on the complement of $[-\delta, \delta]$ (for an adapted metric).

Then for every large M, for every a close to a_0 the post-critical orbit $(P^n_a(a))_{n\le M}$ is close to $(P^n_{a_0}(a_0))_{n\le M}$ and so has a similar expansion. At the next iterations $N = M + 1$, there are three possibilities:

(*a*) either $P^N_a(a)$ is not in $(-\delta, \delta)$ and so the expansion will continue by (*i*),

(b) or $P_a^N(a)$ is in $(-\delta, \delta)$ but is not too close to 0; then there exists an integer $k < N$, called *the binding time*, such that the orbits $P_a^{N+i}(a)$ and $P_a^i(0)$ remain close for $i \le k$ and separate for $i = k+1$. The expansion of $(DP_a^i(a))_{i<k}$ is transferred to $(DP_a^i(P_a^{N+1}(a)))_{i<k}$. The logarithmic contraction at time N, equal to $\log |DP_a(P_a^N(a))|$, is only roughly half the logarithmic expansion during the binding period $\log |DP_a^{k-1}(P_a^{N+1}(a))|$.

(c) or $P_a^N(a)$ is so close to 0 that (b) does not hold.

Cases (a) and (b) are allowed. Case (c) is excluded in the parameter selection by removing the parameter a for which this occurs. Then we can redo the same alternative with $N \leftarrow N+1$ in case (a) and $N \leftarrow N+k$ in case (b).

In case (b), roughly half of the original transferred logarithmic expansion is lost in the binding process. Therefore the Collet-Eckmann condition will not be satisfied if too much time is spent in iterated binding periods. To avoid this, it is asked that:

(H_N) the total length of all the binding periods before N is small with respect to N.

Actually, when appropriately formulated, the condition (H_N) implies that case (c) above does not hold. Hence if (H_N) holds for every N, the map is Collet-Eckmann.

To perform the parameter selection, we look at maximal *critical curves* $\gamma = (P_a^N(a))_{a\in\mathcal{I}}$ so that:

(P_1) Condition (H_n) holds for every $a \in \mathcal{I}$ and for every $n \le N$;

(P_2) the binding periods in $[0, N]$ are the same for every $a \in \mathcal{I}$, and the integer N is not part of a binding period;

(P_3) the length of the curve γ is bounded from below by some uniform constant.

Such a curve is split into different pieces according to which scenario holds at time $N+1$. Pieces corresponding to scenario (a) are iterated once. Pieces corresponding to scenario (c) (or to scenario (b), with a binding time k too long to satisfy (H_{N+k})) are discarded. The other pieces are iterated untill the end of the corresponding binding period. These new critical curves satisfy (P_1) and (P_2). Property ($P3$) is also satisfied, except for some boundary effects that are easily taken care of.

A large deviation argument, relying on property ($P3$), shows that the Lebesgue measure of the remaining parameters is positive (actually, a large proportion of the length of the starting parameter interval).

3.2. The binding approach for Hénon family. — There are many proofs in this spirit [**BC91, MV93, WY01, WY08, Tak11**].

A major difficulty of the 2-dimensional setting is that critical points are not defined beforehand, and will only be well-defined for good parameters.

Call a curve *flat* if it is C^2-close to a segment of $\mathbb{R} \times \{0\}$. Roughly speaking, given a flat segment $\gamma \subset W^u(\alpha)$ going across the critical strip $\{|x| \le \delta\}$, a critical point on γ should be a point of γ such that the vertical tangent vector is exponentially dilated under positive iteration, while the tangent vector to γ is exponentially contracted.

In the inductive construction of good parameters, only N iterations of the Hénon map are considered at a given stage. Under the appropriate induction hypotheses, one defines an approximate critical set $\mathcal{C}_N$. This is a finite set of cardinality exponentially large with N. Each point of $\mathcal{C}_N$ lies on a flat segment contained in $h_{a\,b}^{\theta N}(W^u_{loc}(\alpha))$, with $\theta \sim |\log|b||^{-1}$.

The main problem of the induction step is to extend the exponential dilation along the finitely many critical orbits beyond time N. As in the 1-dimensional case, this is automatic when the critical orbit at time N lies outside of the critical strip. On the other hand, when the critical orbit at time N returns to a point z_N of the critical strip, one has to find, after excluding inadequate parameters, a *binding* critical point $\tilde{z}_0$ whose initial expansion will be transferred (at some cost) to the orbit of z_N. It is here important that z_N should be in *tangential position*, i.e much closer to the flat segment containing $\tilde{z}_0$ than to $\tilde{z}_0$ itself.

To prove that the set of non-excluded parameters (at the end of the induction process) has positive Lebesgue measure, one has to investigate carefully how the whole structure of approximate critical points, analytical estimates and binding relationships survives through parameter deformation. This is certainly the trickiest part of the method.

3.3. Puzzles and parapuzzles. — Puzzles and parapuzzles are combinatorial structures which were first introduced in 1-dimensional complex dynamics to study the local connectivity of Julia sets and the Mandelbrot set [**Hub93, Mil00**]. In real 1-dimensional dynamics, they were instrumental in the proof that almost every quadratic map satisfies either axiom A or the Collet-Eckmann condition [**Lyu02, AM03**].

For real Julia sets of real quadratic maps, puzzle pieces are defined as follows. Let a be a parameter in $[-2,-1]$. Then the quadratic polynomial P_a has two fixed points α, β, both repelling, denoted so that $-\beta < \alpha < -\alpha < \beta$. The real Julia set is equal to $[-\beta, \beta]$. For $n \geq 0$, the *puzzle pieces of order* n are the closures of the connected components of $[-\beta,\beta] \setminus P_a^{-n}(\{\alpha, -\alpha\})$.

Puzzle pieces of successive orders are related in two fundamental ways: a puzzle piece of order n is contained in a puzzle piece of order $n-1$, and its image is contained in a puzzle piece of order $n-1$. The combinatorics of the partition by puzzle pieces of a given order depend on the sequence of nested puzzle pieces containing the critical value. This leads to a sequence of partitions of parameter space into *parapuzzle* pieces. It is a general rule of thumb that, assuming a mild level of hyperbolicity, the combinatorics and geometry of parapuzzle pieces around a given parameter a are closely related to the combinatorics and geometry of puzzle pieces for P_a around the critical value.

3.4. The strong regularity approach for quadratic maps. — Let a be a parameter in $[-2,-1]$. A *regular interval* is a puzzle piece of some order $n > 0$ which is sent diffeomorphically onto $A := [\alpha, -\alpha]$ by P_a^n. One also asks that the corresponding inverse branch extends to a fixed neighborhood of A, which insures a control of the distortion. The parameter a is *regular* if the measure of the set of points in A which are not contained in a regular interval of order $\leq n$ is exponentially small with n. A classical argument shows that regular parameters satisfy the conclusions of Jakobson's theorem.

To prove that the set of regular parameters has positive Lebesgue measure, one considers a more restrictive condition called *strong regularity.* Assume that the parameter is close to the Chebychev value $a_0 := -2$. Then the return time M of the critical point to A is large. Moreover, the complement in A of a neighborhood of 0 of approximate size 2^{-M} is covered by finitely many regular intervals of order $< M$, which are called *simple.* The parameter a is called *strongly regular* if

($\star$) there exists a sequence of regular intervals $(I_j)_{j>0}$ of order $(n_j)_j$ such that $P_a^{M+n_1+\cdots+n_{j-1}}(a) \in I_j$ for all $j > 0$;

($\diamondsuit$) most I_j are simple in the sense that $\sum_{i\le j: I_i \text{ is not simple}} n_i \ll \sum_{i\le j} n_i$ for all $j > 0$.

The most delicate part of the proof is to establish, through a careful analysis of the puzzle structures, that strongly regular parameters are regular. Then one is able to transfer the exponential regularity estimate from puzzles in phase space to parapuzzles in parameter space. Finally, one concludes through a large deviation argument that the set of strongly regular parameters has positive Lebesgue measure.

3.5. The strong regularity approach for Hénon family. — The hyperbolic fixed point $(\alpha, 0)$ of $h_{a,0}$ persists as a fixed point P for $h = h_{a\,b}$, with b small. One denotes by $Q \approx (-\alpha, 0)$ the first (transverse) intersection of the stable and unstable manifolds of P. Let $\mathbb{S}$ be the segment of $W^u(P)$ bounded by P and Q. It is a flat curve. Given a segment I of $W^u(P)$ one denotes by $W^s_\theta(\partial I)$ the union of the θ-local stable manifolds of the endpoints of I, with $\theta = 1/|\log b|$.

A flat curve is *stretched* if its end points belong to $W^s_\theta(\partial\mathbb{S})$. A *puzzle piece* of a flat curve S is a pair (I, n_I) of a segment I of S sent by f^{n_I} *onto* a flat *stretched* curve. The puzzle piece is *hyperbolic* if $f^{n_I}|I$ satisfies some hyperbolicity conditions and regular if it satisfies moreover a distortion condition. A *puzzle pseudo-group* is the data of a pair $(\Sigma, \mathcal{Y})$ formed by a family Σ of flat stretched curves (formed in particular by $\mathbb{S}$), and by a set of hyperbolic puzzle pieces $\mathcal{Y}$ associated to the curves of Σ so that, $\forall (I, n_I) \in \mathcal{Y}$, $h^{n_I}(I)$ is a curve of Σ. The puzzle pseudo group is *regular* (and the map is *regular*) if for every curve $S \in \Sigma$, the measure of the set of points in S which are not contained in a regular piece of order $\le n$ given by $\mathcal{Y}$ is exponentially small with n, and if every puzzle piece in $\mathcal{Y}$ of $S \in \Sigma$ persists as a puzzle puzzle in $\mathcal{Y}$ for $S' \in \Sigma$ nearby S. A classical argument shows that regular maps leave invariant an ergodic, physical SRB measure.

The notion of strong regularity is also generalized to show the abundance regular maps. A Hénon map is *strongly regular* if it preserves a combinatorial and geometrical object called *puzzle algebra.* Such an object does not need the concept of critical point to be defined; it relies basically on the topology of the homoclinic tangle of P.

By hyperbolic continuity, the simple regular intervals persist as puzzle pieces of every flat stretched curve S, their complement in S is denoted by $S_\square$.

A *puzzle algera* is the data of: a puzzle pseudo-group $(\Sigma, \mathcal{Y})$, a family of "semi-artificial" flat stretched curves $\Sigma^\square$, and for every $S \in \Sigma \sqcup \Sigma^\square$ an admissible sequence of puzzle pieces $c(S) = (I_i, n_i)_i \in \mathcal{Y}^\mathbb{N}$ from $\mathbb{S}$ satisfying the condition ($\diamondsuit$). *Admissibility* means that the intersection $J_k(S) := \cap_{i=1}^k f^{-n_{i-1}-\cdots-n_1}(I_i)$ is a puzzle piece of $\mathbb{S}$ for every $k \ge 1$. One shows that ($\diamondsuit$) implies that the local stable manifolds $W^s_\theta(\partial J_k)$

have their end points in $\{y > \theta 2^{-M}\}$ and $\{y < -\theta 2^{-M}\}$. Hence one can ask $\forall k > 0$, $S \in \Sigma \sqcup \Sigma^{\square}$:

($\star$) the segment $S_{\square}$ is folded by f^M between both components of $W^s_\theta(\partial J_k(S))$.
This is the main ingredient of puzzle algebras definition. One notices that $f^M(S_{\square})$ is tangent to a local stable manifold of the singleton $\cap_k J_k(S)$. Conversely, from these topological conditions, some combinatorially defined puzzle pieces turn out to be necessarily regular. They form $\mathcal{Y}$. Also some combinatorially defined local unstable manifolds turn out to be necessarily flat. Those which are stretched form Σ, the other are artificially stretched to form $\Sigma^{\square}$. This combinatorial formalism is certainly the main novelty and difficulty of this proof: pure topological and combinatorial properties imply analytical properties. Then it is rather quick to prove the regularity of $(\Sigma, \mathcal{Y})$ and so the regularity of strongly regular maps.

To handle the parameter selection, by induction on k, for a C^2-open set of dynamics f, we can define combinatorially a finite family of flat stretched curves $\check{\Sigma}_k$. Similar conditions are asked on $\check{\Sigma}_k$. This implies the existence and regularity of many flat stretched curves and puzzle pieces. When the map is strong strongly regular, every curve in $\Sigma \sqcup \Sigma^{\square}$ can be approximated by a curve of $\check{\Sigma}_k$ for $k \geq 0$. These combinatorial definitions enable one to follow carefully how the whole structure survives by parameter deformation.

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