FIXED POINT FREE ACTIONS ON $\mathbb{Z}$-ACYCLIC 2-COMPLEXES[1]

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In this paper, we give a complete description of the finite groups which can act on 2-dimensional $\mathbb{Z}$-acyclic complexes without fixed points. One example of such an action (by the group A_5) has been known for a long time, but as far as we know it is the only such action constructed earlier. In fact, we construct here actions of this type for many different finite simple groups.

More precisely, our main theorem is the following.

Theorem A. *For any finite group G, there is an essential fixed point free 2-dimensional (finite) $\mathbb{Z}$-acyclic G-complex if and only if G is isomorphic to one of the simple groups $PSL_2(2^k)$ for $k \geq 2$, $PSL_2(q)$ for $q \equiv \pm 3 \pmod 8$ and $q \geq 5$, or $\mathrm{Sz}(2^k)$ for odd $k \geq 3$. Furthermore, the isotropy subgroups of any such G-complex are all solvable.*

Here "G-complex" means a G-CW complex; but the same result holds if one instead uses simplicial complexes with admissible G-action in the sense of [S1] or [AS] (see Proposition A.4 in the appendix). The word "finite" is in parentheses because the theorem holds whether or not this condition is included. The condition that the action be essential was put in to insure that an action of a quotient group G/N does not automatically produce an action of G:

Definition. *A G-complex X is* essential *if there is no normal subgroup $1 \neq N \lhd G$ with the property that for each $H \subseteq G$, the inclusion $X^{HN} \to X^H$ induces an isomorphism on integral homology.*

In other words, if there is such a subgroup $N \lhd G$, then the G-action on X is "essentially" the same as the G-action on X^N, which factors through a G/N-action. In the case of actions on acyclic 2-complexes, the relation between essential actions and arbitrary actions is made precise in the next theorem.

Theorem B. *Let G be any finite group, and let X be any 2-dimensional $\mathbb{Z}$-acyclic G-complex. Let N be the subgroup generated by all normal subgroups $N' \lhd G$ such that $X^{N'} \neq \emptyset$. Then X^N is $\mathbb{Z}$-acyclic; X is essential if and only if $N = 1$; and if $N \neq 1$ then the action of G/N on X^N is essential.*

The proofs of Theorems A and B rely on the earlier works [O1], [O2], [S1], and [AS], as well as on the classification theorem for finite simple groups. In [S1], Y. Segev proved that if a finite group G acts on an acyclic 2-complex X, the fixed point set X^G is either $\mathbb{Z}$-acyclic or empty, and is $\mathbb{Z}$-acyclic if G is solvable or $G \cong A_n$ for $n \geq 6$. Later, in [AS],

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Aschbacher and Segev extended these results, and proved that $X^G \neq \emptyset$ if G is simple, except perhaps when G is of Lie type and Lie rank one, or the first Janko group J_1 (a sporadic group).

Techniques for constructing fixed point free actions of finite groups on finite acyclic or contractible complexes (without restrictions on dimension) were developed by B. Oliver in several earlier papers such as [O1] and [O2]. In particular, in [O2], actions for which the fixed point set of each subgroup is contractible or empty are studied.

The proof of Theorem A — both when constructing actions of G and when proving their nonexistence — is based on refinements of the techniques developed in these earlier papers of both authors. The main new input comes from a more detailed analysis of the subgroup lattice of G and its orbit space. In particular, necessary and sufficient conditions for the existence of actions are stated in terms of this lattice in Proposition 1.9. Afterwards, the proofs of nonexistence of actions of particular groups require identifying homology in certain "pieces" of the subgroup lattice of G.

In fact, relatively few solvable subgroups need occur as isotropy groups for the actions constructed when proving Theorem A, and those which do occur are listed explicitly. It is possible that these and similarly constructed G-complexes can give new information about decompositions of BG, and about the cohomology of G.

Theorem A leaves open the question as to whether or not it is possible for a finite group to act on a 2-dimensional *contractible* complex without fixed points. Understanding actions on acyclic 2-complexes is clearly a first step towards investigating this question, but the first author feels that any serious attempt to answer it will require some very different methods than those used here.

This paper is intended for both group theorists and topologists, and we have attempted to write it in a way which will be appealing and readable for both. In particular, more background material has been included than might normally be the case, although we have tried to put most of that in the appendix at the end of the paper.

The paper is organized as follows. In Section 1, conditions are established, in terms of homological properties of the subgroup lattice of G, which determine the minimal dimensions of certain "universal" G-complexes. In particular, this section includes the general machinery for constructing such actions. After proving some technical results in Section 2, the constructions of the G-complexes described in Theorem A are carried out in Section 3. In Section 4, we show that any finite group G which acts essentially on a 2-dimensional acyclic complex must be almost simple (i.e., there is a nonabelian simple group L such that $L \subseteq G \subseteq \mathrm{Aut}(L)$). In Section 5, we develop machinery to show the nonexistence of actions on acyclic 2-complexes; and this is applied in Section 6 to prove Theorem A for simple groups of Lie type and Lie rank one. The sporadic groups are dealt with in Theorem 7; except for the first Janko group J_1 this repeats results already shown in [AS]. Theorem B is proven in Section 4, and Theorem A in Section 8. All of this is preceded by a preliminary "Section 0" where we present some general results about G-posets and construction of G-complexes; and is followed by an appendix which includes background material about G-complexes, $\mathbb{Z}[G]$-modules, and simple groups of Lie type, as well as a sketch of the proofs in [S1] and [AS] of certain cases of Theorem A. References of the form A.x, B.x, etc. all refer to the appendix. After the appendix, we attach a list of the notation used throughout the paper.

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0. G-complexes and G-posets

Posets, and in particular families of subgroups considered as posets, will play an important role as "bookkeeping" devices for controlling dimensions of certain acyclic complexes. For any poset S, we let $\mathcal{N}(S)$ denote its nerve: the simplicial complex with one vertex for each element of S, and one n-simplex for each chain $\alpha_0 < \alpha_1 < \cdots < \alpha_n$ of elements of S. By a *G-poset* is meant a poset with G-action which preserves the ordering. A *terminal subposet* of a poset S is a subset $S' \subseteq S$ such that $\beta \geq \alpha \in S'$ implies $\beta \in S'$. For any element α in a poset S, we set $S_{\geq\alpha} = \{\beta \in S \,|\, \beta \geq \alpha\}$. The next lemma provides a general setting for comparing G-complexes with coverings to the nerves of the coverings.

Lemma 0.1. *Let X be a G-complex, let S be a finite G-poset, and let $\{X_\alpha\}_{\alpha\in S}$ be a covering of X by subcomplexes which satisfies the following conditions:*

(a) $\alpha \leq \beta$ implies $X_\alpha \supseteq X_\beta$.

(b) For all $x \in X$, the set $\{\alpha \in S \,|\, x \in X_\alpha\}$ has a largest element.

(c) $X_{g(\alpha)} = g(X_\alpha)$ for all $\alpha \in S$, $g \in G$.

Then there is a G-map $f_X\colon X \to \mathcal{N}(S)$ with the property that

$$f_X(X_\alpha) \subseteq \mathcal{N}(S_{\geq\alpha}) \quad \text{for all } \alpha \in S. \tag{1}$$

If, furthermore, X_α is acyclic (contractible) for each α, then for any map $f\colon X \to \mathcal{N}(S)$ which satisfies (1), and any terminal subposet $S' \subseteq S$, f restricts to a homology equivalence (homotopy equivalence) $f_{S'}\colon X_{S'} = \bigcup_{\alpha\in S'} X_\alpha \to \mathcal{N}(S')$.

Proof. For each $n \geq 0$, let J_n denote the G-set of n-cells of X, and let $\varphi_n\colon J_n \times D^n \to X$ denote the characteristic map for the n-cells. Let $\theta\colon J_n \to S$ be the map which sends $j \in J_n$ to the largest element in the set $\{\alpha \in S \,|\, \varphi_n(j,0) \in X_\alpha\}$; this is well defined by (b) and equivariant by (c). For each $\alpha \in S$, we let $[\alpha]$ denote the corresponding vertex in $\mathcal{N}(S)$.

First define $f_0\colon X^{(0)} \to \mathcal{N}(S)$ by setting $f_0(\varphi_0(j,0)) = [\theta(j)]$ for each $j \in J_0$. This clearly satisfies condition (1).

Now assume that $f_{n-1}\colon X^{(n-1)} \to \mathcal{N}(S)$ has been defined, satisfying (1). For any $j \in J_n$ and any $v \in S^{n-1}$, $\varphi_n(j,0) \in X_{\theta(j)}$ by construction, and so $\varphi_n(j,v) \in X_{\theta(j)}$ since $X_{\theta(j)}$ is a subcomplex of X. So $f_{n-1}(\varphi_n(j,v)) \in \mathcal{N}(S_{\geq\theta(j)})$ by (1), hence it is in some simplex which contains the vertex $[\theta(j)]$, and the segment from $f_{n-1}(\varphi_n(j,v))$ to $[\theta(j)]$ lies in $\mathcal{N}(S)$. So we can define

$$f_n\colon X^{(n)} \longrightarrow \mathcal{N}(S)$$

by setting $f_n(x) = f_{n-1}(x)$ for $x \in X^{(n-1)}$, and

$$f_n(\varphi_n(j, tv)) = t \cdot f_{n-1}(\varphi_n(j, v)) + (1-t) \cdot [\theta(j)] \qquad \text{for } j \in J_n,\ v \in S^{n-1},\ t \in [0,1].$$

This is well defined as a map of sets, since the two definitions agree on $\varphi_n(J_n \times S^{n-1}) \subseteq X^{(n-1)}$. So it is continuous by Lemma A.3 ($f_n|_{X^{(n-1)}}$ and $f_n \circ \varphi_n$ are both continuous). Condition (1) still holds for f_n, since for all $j \in J_n$ and $v \in \text{int}(D^n)$, and all $\alpha \in S$,

$$\begin{aligned} \varphi_n(j, v) \in X_\alpha \iff \varphi_n(j, 0) \in X_\alpha &\implies \alpha \le \theta(j) \\ &\implies f_n(\varphi_n(j, v)) \in \mathcal{N}(S_{\ge \theta(j)}) \subseteq \mathcal{N}(S_{\ge \alpha}). \end{aligned}$$

And f_n is equivariant since θ is equivariant, since f_{n-1} is equivariant (by induction), and since the G-action on $\mathcal{N}(S)$ is affine.

Finally, define $f_X \colon X \to \mathcal{N}(S)$ to be the union of the f_n; this is again continuous by Lemma A.3, and condition (1) holds since it holds for each f_n.

Now let f be any map which satisfies (1), and assume that X_α is acyclic (contractible) for each $\alpha \in S$. We want to show that f is a homology (homotopy) equivalence. The group action no longer plays a role here, so we can assume $G = 1$. We can assume inductively that for any properly contained terminal poset $S' \subsetneqq S$, f restricts to an equivalence $\bigcup_{\alpha \in S'} X_\alpha \to \mathcal{N}(S')$ (since the subspace and subposet still satisfy conditions (a) and (b) above). If S has a smallest element σ, then $X = X_\sigma$ is acyclic (contractible) and $\mathcal{N}(S)$ is contractible, so any map $f \colon X \to \mathcal{N}(S)$ is a homology (homotopy) equivalence, and we are done.

Assume now that S contains no smallest element. In this case, we can write $S = S_1 \cup S_2$, where S_1 and S_2 are proper terminal subposets of S. Set $S_0 = S_1 \cap S_2$; and set $X_i = \bigcup_{\alpha \in S_i} X_\alpha$ for each $i = 0, 1, 2$. Clearly, $\mathcal{N}(S_0) = \mathcal{N}(S_1) \cap \mathcal{N}(S_2)$, and condition (b) implies that $X_0 = X_1 \cap X_2$. By the inductive assumption, f restricts to homology (homotopy) equivalences $f_i \colon X_i \to \mathcal{N}(S_i)$, and so f is a homology (homotopy) equivalence by Proposition B.3. □

By a *family* of subgroups of G will here be meant any subset $\mathcal{F} \subseteq \mathcal{S}(G)$ which is closed under conjugation. We do *not* assume here that subgroups of elements of the family are also in the family.

For any family $\mathcal{F}$ of subgroups of G, a $(G, \mathcal{F})$-*complex* will mean a G-CW-complex all of whose isotropy subgroups lie in $\mathcal{F}$. A $(G, \mathcal{F})$-complex is *universal* if the fixed point set of each subgroup in $\mathcal{F}$ is contractible. (The "universality" property of such spaces is explained in Proposition A.6.) One can, in fact, construct universal $(G, \mathcal{F})$-complexes for any family $\mathcal{F}$ of subgroups of G, but in most cases any such complex must be infinite dimensional. For example, when $\mathcal{F} = \{1\}$ contains only the trivial subgroup, a universal $(G, \mathcal{F})$-complex is just a contractible complex upon which G acts freely; and so its orbit space is a classifying space for G. The results in Section 1 will make it clear what conditions are needed on $\mathcal{F}$ for there to be a finite (or finite dimensional) universal $(G, \mathcal{F})$-complex.

The following lemma is the starting point for the constructions of universal $(G, \mathcal{F})$-complexes, and of other G-complexes satisfying certain homological conditions. Roughly, it describes the effect on the homology of X of attaching cells of one orbit type G/H to X. By "attaching an orbit of cells of type $G/H \times D^i$" to a G-complex X, we mean replacing X by the complex $X\cup_\varphi(G/H\times D^n)$ for some G-map $\varphi\colon G/H\times S^{n-1}\to X^{(n-1)}$. We refer to Lemma A.2 for more detail.

Proposition 0.2. *Fix a finite G-complex X, and a subgroup $H\subseteq G$. Then the following hold.*

(a) For any $n\ge 1$, there is a finite G-complex $Y\supseteq X$, obtained by attaching to X orbits of cells of type $G/H\times D^i$ for $1\le i\le n$, such that Y^H is $(n-1)$-connected and $H_i(Y^H)\cong H_i(X^H)$ for all $i>n$. Also, $H_n(Y^H)$ is $\mathbb{Z}$-free if $H_n(X^H)$ is $\mathbb{Z}$-free.

(b) Assume $n\ge 1$, and that X^H is $(n-1)$-connected. For any homomorphism

$$\varphi\colon (\mathbb{Z}[N(H)/H])^k \longrightarrow H_n(X^H)$$

of $\mathbb{Z}[N(H)/H]$-modules, there is a finite G-complex $Y\supseteq X$, obtained by attaching k orbits of cells $G/H\times D^{n+1}$ to X, such that $H_i(Y^H)\cong H_i(X^H)$ for all $i\ne n,n+1$, such that

$$H_n(Y^H)\cong \operatorname{Coker}(\varphi), \tag{1}$$

and such that there is a short exact sequence

$$0\longrightarrow H_{n+1}(X^H)\longrightarrow H_{n+1}(Y^H)\longrightarrow \operatorname{Ker}(\varphi)\longrightarrow 0. \tag{2}$$

(c) Assume, for some $n\ge 1$, that $\widetilde{H}_*(X^H)=H_n(X^H)$ is a stably free $\mathbb{Z}[N(H)/H]$-module; more precisely that

$$H_n(X^H)\oplus(\mathbb{Z}[N(H)/H])^k\cong(\mathbb{Z}[N(H)/H])^m$$

(where $k,m\ge 0$). Then there exists a G-complex $Y\supseteq X$, obtained by attaching to X k orbits of cells of type $G/H\times D^n$ and m orbits of cells of type $G/H\times D^{n+1}$, such that Y^H is acyclic.

(d) Assume that all connected components of X^H are acyclic, and that one of the components of X^H is fixed by the action of $N(H)/H$ and the others are permuted freely. Then there exists a G-complex $Y\supseteq X$, obtained by attaching to X cells of orbit type $G/H\times D^1$, such that Y^H is acyclic.

Proof. (b) Since X^H is $(n-1)$-connected, the Hurewicz theorem applies to show that each element $h\in H_n(X^H)$ is represented by a map $\varphi\colon S^n\to X^H$, in the sense that $h=\varphi_*([S^n])$ for some fixed generator $[S^n]$ of $H_n(S^n)$. (See, e.g., [Hu, Theorem II.9.1] if $n>1$, or [Hu, Theorem II.6.1] if $n=1$). And we can assume that $\varphi(S^n)\subseteq(X^H)^{(n)}$ by the cellular approximation theorem [LW, Theorem II.8.5], which says that any map $S^n\to X^H$ is homotopic to a cellular map, and in particular a map with image in the n-skeleton.

Now let $E=\{e_1,\dots,e_k\}$ denote the canonical basis of $(\mathbb{Z}[N(H)/H])^k$, and fix maps $f_i\colon S^n\to(X^H)^{(n)}$ which represent $\varphi(e_i)\in H_n(X^H)$. Define

$$f\colon (E\times G/H)\times S^n\longrightarrow X^{(n)}$$

by setting $f(e_i,gH,x)=g\cdot f_i(x)$; and let f^H be the restriction of f to the H-fixed point sets. In particular, for each i and each $g\in N(H)$, $f|_{e_i\times gH\times S^n}$ (as a map $S^n\to X^H$) represents the class $g\cdot\varphi(e_i)\in H_n(X^H)$. In other words, $H_n(f^H)=\varphi$ under the identification

$$H_n((E\times G/H)^H\times S^n)=H_n((E\times N(H)/H)\times S^n)\cong(\mathbb{Z}[N(H)/H])^k.$$

Set

$$Y = X \cup_f \big((E{\times}G/H){\times}D^{n+1}\big)$$

(Lemma A.2). Then

$$Y^H = X^H \cup_{f^H} \big((E{\times}N(H)/H){\times}D^{n+1}\big);$$

and (1) and (2) now follow from Lemma B.1.

(a) We prove this inductively. Fix $n \geq 0$ such that X^H is $(n-1)$-connected. We will construct a finite G-complex $Y \supseteq X$, obtained by attaching orbits of cells of type $G/H{\times}D^{n+1}$ to X, such that Y^H is n-connected.

If $n = 0$ and X^H is not connected, then let v_{-1} and v_1 be two vertices in different connected components of X^H, define $f\colon G/H{\times}S^0 \to X$ by setting $f(gH,t) = gv_t$, and set $X' = X \cup_f (G/H{\times}D^1)$. By construction, $(X')^H$ has fewer connected components than X^H, and by continuing the procedure we obtain a finite G-complex Y such that Y^H is connected.

If $n = 1$ and $\pi_1(X^H) \neq 1$, then choose any element $1 \neq \phi \in \pi_1(X^H)$, represent it by a map $f_0\colon S^1 \to X^H$, and extend this to a G-map $f\colon G/H{\times}S^1 \to X$ by setting $f(gH,v) = g{\cdot}f_0(v)$. Set $X' = X \cup_f (G/H{\times}D^2)$. Then $\pi_1((X')^H) = \pi_1(X^H)/N$, where N is a normal subgroup of $\pi_1(X^N)$ which contains ϕ (in fact, the normal closure of $\langle\phi\rangle$). Since $\pi_1(X^H)$ is finitely generated, we can repeat this procedure and obtain a finite G-complex Y such that Y^H is 1-connected.

If $n > 1$, then the result follows from part (b), where we choose φ to be any surjection ($H_n(X^H)$ is finitely generated as an abelian group, hence as a $\mathbb{Z}[N(H)/H]$-module).

(c) Upon applying point (b) to the trivial homomorphism

$$\varphi_0\colon (\mathbb{Z}[N(H)/H])^k \to H_{n-1}(X^H) = 0,$$

we get a finite G-complex $Y_0 \supseteq X$, obtained by attaching k-orbits of cells $G/H{\times}D^n$ to X, such that $H_i((Y_0)^H) \cong H_i(X^H) = 0$ for all $i \neq n$ and

$$H_n((Y_0)^H) \cong H_n(X^H) \oplus (\mathbb{Z}[N(H)/H])^k \cong (\mathbb{Z}[N(H)/H])^m.$$

If we now apply (b) to any isomorphism $\varphi\colon (\mathbb{Z}[N(H)/H])^m \to H_n((Y_0)^H)$, we obtain a finite G-complex $Y \supseteq Y_0$, constructed by attaching m orbits of cells $G/H{\times}D^{n+1}$, such that Y^H is acyclic.

(d) Here, we assume that all connected components of X^H are acyclic, and that one is invariant under the action of $N(H)/H$ and the others are permuted freely. Let $X_0 \subseteq X^H$ denote the component which is $N(H)/H$-invariant, and let $X_1, X_2, \ldots, X_k$ be $N(H)/H$-orbit representatives for the other components. (If $N(H)/H = 1$, then let X_0 be any of the connected components.) Fix vertices $x_i \in X_i$ for $i = 0, \ldots, k$. Set $J = \{1, \ldots, k\}$, and define $\varphi\colon (G/H{\times}J){\times}S^0 \to X$ by setting

$$\varphi(gH, i, 1) = gx_i \qquad \text{and} \qquad \varphi(gH, i, -1) = gx_0.$$

Now set $Y = X \cup_\varphi ((G/H{\times}J){\times}D^1)$. Then

$$Y^H = X^H \cup_{\varphi|} ((N(H)/H{\times}J){\times}D^1),$$

and this is acyclic since X_0 has been connected (by a unique 1-cell) to each of the other connected components of X. □

We finish the section with two lemmas which involve elementary properties of nerves of posets. We first recall the following results of Quillen.

Lemma 0.3 [Q2, 1.3–1.5]**.** *(a) Let $T \subseteq S$ be posets, and let $r\colon S \to T$ be any order preserving map such that $r|_T = \mathrm{Id}_T$, and such that $r(\alpha) \le \alpha$ for all $\alpha \in S$ (or $r(\alpha) \ge \alpha$ for all α). Then the inclusion of $\mathcal{N}(T)$ in $\mathcal{N}(S)$ is a homotopy equivalence.*

(b) Let G be a finite group, and let $\mathcal{H}$ be any set of subgroups of G. Assume there is some $H_0 \in \mathcal{H}$ such that either $H \cap H_0 \in \mathcal{H}$ for all $H \in \mathcal{H}$, or $\langle H, H_0\rangle \in \mathcal{H}$ for all $H \in \mathcal{H}$. Then $\mathcal{N}(\mathcal{H})$ is contractible.

Proof. Point (a) is shown in [Q2, 1.3]. In fact, $\mathcal{N}(T)$ is a strong deformation retract of $\mathcal{N}(S)$, where r induces the retraction $\mathcal{N}(S) \to \mathcal{N}(T)$, and where the homotopy with the identity comes from the assumption that $r(x)$ is always $\le x$ or always $\ge x$ [Q2, 1.3].

If $\mathcal{H}$ is as in (b), then its nerve is "conically contractible" in the sense of Quillen, and hence is contractible [Q2, 1.4–1.5]. □

The following lemma will also be useful, when showing that certain subgroups of G need not occur as isotropy subgroups in acyclic G-complexes.

Lemma 0.4. *Let S be any finite poset, and let $S' \subseteq S$ be any subposet with the property that $\mathcal{N}(S_{>\alpha}) \simeq *$ for all $\alpha \in S \smallsetminus S'$. Then $\mathcal{N}(S') \simeq \mathcal{N}(S)$ (the inclusion induces a homotopy equivalence).*

Proof. It suffices to show this when $S \smallsetminus S'$ contains just one element α. In this case, $\mathcal{N}(S)$ is the union of $\mathcal{N}(S')$ with the cone over the subcomplex $A \subseteq \mathcal{N}(S')$, where

$$A = \mathcal{N}\big(S_{<\alpha} \amalg S_{>\alpha}\big) = \mathcal{N}(S_{<\alpha}) * \mathcal{N}(S_{>\alpha}). \tag{1}$$

Note that the nerve of the disjoint union in (1) is identified with the join of the nerves, since every element in $S_{<\alpha}$ is less than every element in $S_{>\alpha}$. Then A is contractible, since $\mathcal{N}(S_{>\alpha}) \simeq *$ by assumption. □

Lemma 0.4 does, in fact, hold without the assumption that S is finite: it follows as a consequence of Quillen's Theorem A [Q1] (see also [Q2, Proposition 1.6]).

A central problem throughout this paper, especially in Sections 5 and 6, is to find ways to detect 2-dimensional homology in nerves of certain posets. Given a 2-cycle in $\mathcal{N}(S)$, the simplest way to show it is nonvanishing in $H_2(\mathcal{N}(S))$ is to show that some 2-simplex with nonzero coefficient is maximal in $\mathcal{N}(S)$; i.e., not in the boundary of any 3-simplex. The following lemma provides a refinement of this observation, and will be used in Section 5.

Lemma 0.5. *Let S be a finite poset, and let z be a 2-cycle in the nerve of S. Fix elements $m < M$ in S, where m is minimal and M is maximal. Set $Q = \{x{\in}S \,|\, m{<}x{<}M\}$, and let $Q' \subseteq Q$ be the set of all $x \in Q$ such that the simplex (m, x, M) occurs with nonzero coefficient in z. Assume that $Q' \ne \emptyset$, and that some element of Q' lies in a separate connected component of $\mathcal{N}(Q)$ from all of the other elements of Q'. Then $0 \ne [z] \in H_2(\mathcal{N}(S))$.*

Proof. Set $X = \mathcal{N}(S)$, for short, and let $Y \subseteq X$ be the subcomplex of all simplices which do not contain both vertices m, M. Let $C_*(X) \supseteq C_*(Y)$ be the simplicial chain complexes; and write

$$z = \sum_{x \in Q'} a_x(m, x, M) \pmod{C_2(Y)}$$

(where $0 \neq a_x \in \mathbb{Z}$ for each x).

For any 3-simplex σ in X, either σ is in Y (and so $\partial(\sigma) \in C_2(Y)$), or $\sigma = (m, x, y, M)$ for some $x, y \in Q$ in the same connected component of $\mathcal{N}(Q)$ and

$$\partial(\sigma) = (m, x, M) - (m, y, M) \pmod{C_2(Y)}.$$

Thus, if z is a boundary, then the sum of the coefficients a_x in the above expression for z, taken over all $x \in Q'$ which lie in any given connected component of $\mathcal{N}(Q)$, is zero. And this contradicts the assumption that some element of Q' is in a component by itself. □

1. Minimal dimensions of universal G-spaces

We will now establish necessary and sufficient conditions for the existence of universal complexes satisfying certain dimensional restrictions. These conditions will be expressed in terms of the homology of the nerves of certain posets.

Throughout this section, G will be a finite group. A nonempty family $\mathcal{F} \subseteq \mathcal{S}(G)$ will be called *separating* if it has the following three properties: (a) $G \notin \mathcal{F}$; (b) any subgroup of an element of $\mathcal{F}$ is in $\mathcal{F}$; and (c) for any $H \lhd K \subseteq G$ with K/H solvable, $K \in \mathcal{F}$ if $H \in \mathcal{F}$. The following property of separating families is immediate.

Lemma 1.1. *Each maximal subgroup in a separating family of subgroups of G is self-normalizing.* □

If G is solvable, then it has no separating family of subgroups. If G is not solvable, then we let $\mathcal{SLV}$ denote the family of solvable subgroups: the minimal separating family for G. We also let $\mathcal{MAX}$ denote the maximal separating family for G, which can be described as follows. Let L be the maximal normal perfect subgroup of G; i.e., the last term in the derived series of G. Then $\mathcal{MAX}$ is the family of all subgroups of G which do not contain L. In particular, if G is perfect, then $\mathcal{MAX}$ is the family of all proper subgroups of G.

A $(G, \mathcal{F})$-complex will be called *H-universal* if the fixed point set of each $H \in \mathcal{F}$ is acyclic. The importance of universal, and H-universal, $(G, \mathcal{F})$-complexes when studying 2-dimensional actions comes from the following lemma.

Lemma 1.2. *Let X be any 2-dimensional acyclic G-complex without fixed points. Let $\mathcal{F}$ be the set of subgroups $H \subseteq G$ such that $X^H \neq \emptyset$. Then $\mathcal{F}$ is a separating family of subgroups of G, and X is an H-universal $(G, \mathcal{F})$-complex.*

Proof. By [S1, Theorem 3.4], X^H is acyclic for each $H \in \mathcal{F}$; i.e., for each H such that $X^H \neq \emptyset$. (Another proof of this, which does not depend on the odd order theorem, is given in Theorem 4.1 here.) So by definition, X is an H-universal $(G, \mathcal{F})$-complex. Also, if $H \lhd K \subseteq G$ are subgroups such that $H \in \mathcal{F}$ and K/H is solvable, then X^H is acyclic, and so $X^K = (X^H)^{K/H}$ is acyclic by [S1, Theorem 3.1] (see also Theorem 4.1). Thus, $\mathcal{F}$ is a separating family. □

For any family $\mathcal{F}$ of subgroups of G, we consider $\mathcal{N}(\mathcal{F})$ as a G-complex via the conjugation action. Note, however, that $\mathcal{N}(\mathcal{F})$ is not itself a $(G,\mathcal{F})$-complex in general. For example, when $\mathcal{F} = \{1\}$, then $\mathcal{N}(\mathcal{F})$ is a point, while a $(G,\mathcal{F})$-complex must have a free G-action.

Recall that for any family $\mathcal{F}$ of subgroups of G and any set $\mathcal{H}$ of subgroups, $\mathcal{F}_{\geq\mathcal{H}}$ denotes the poset of those subgroups in $\mathcal{F}$ which contain some element of $\mathcal{H}$. Also, for any $H \subseteq G$, $\mathcal{F}_{\geq H}$ and $\mathcal{F}_{>H}$ denote the posets of subgroups in $\mathcal{F}$ which contain H, or strictly contain H, respectively. The following proposition follows immediately from Lemma 0.1.

Proposition 1.3. *Fix any family $\mathcal{F}$ of subgroups of G. Let $\mathcal{N}(\mathcal{F})$ be the nerve of the poset $\mathcal{F}$, regarded as a G-complex via the action by conjugation. Then for any $(G,\mathcal{F})$-complex X, there is a G-map $f\colon X \to \mathcal{N}(\mathcal{F})$ with the property that $f(X^H) \subseteq \mathcal{N}(\mathcal{F}_{\geq H})$ for all $H \subseteq G$. And if X is universal (H-universal), then for any set $\mathcal{H}$ of subgroups of G, any such map f restricts to a homotopy equivalence (homology equivalence) $X^{\mathcal{H}} \to \mathcal{N}(\mathcal{F}_{\geq\mathcal{H}})$.*

Proof. We apply Lemma 0.1, with $S = \mathcal{F}$ (regarded as a poset via inclusion), and $X_H = X^H$ for $H \in \mathcal{F}$. Since X is a $(G,\mathcal{F})$-complex, every element of X is fixed by some $H \in \mathcal{F}$, and so $\{X^H\}_{X\in\mathcal{F}}$ is a covering of X. Condition (a) of Lemma 0.1 clearly holds, and condition (b) holds since the largest element of $\{H \in \mathcal{F} \,|\, x \in X^H\}$ is the isotropy group G_x. And condition (c) holds since $X^{gHg^{-1}} = g(X^H)$. □

The following lemma, which helps to limit the number of orbit types needed when constructing "minimal" universal $(G,\mathcal{F})$-complexes, is an easy consequence of Lemma 0.4.

Lemma 1.4. *Let $\mathcal{F}$ be any family of subgroups of G, and let $\mathcal{F}_0 \subseteq \mathcal{F}$ be any subfamily such that $\mathcal{N}(\mathcal{F}_{>H}) \simeq *$ for all $H \in \mathcal{F}\smallsetminus\mathcal{F}_0$. Then any (H-)universal $(G,\mathcal{F}_0)$-complex is also an (H-)universal $(G,\mathcal{F})$-complex; and*

$$\mathcal{N}((\mathcal{F}_0)_{\geq\mathcal{H}}) \simeq \mathcal{N}(\mathcal{F}_{\geq\mathcal{H}}) \tag{1}$$

for any set $\mathcal{H}$ of subgroups of G.

Proof. For any set $\mathcal{H}$ of subgroups of G, point (1) follows from Lemma 0.4, applied to the posets $S \overset{\text{def}}{=} \mathcal{F}_{\geq\mathcal{H}}$ and $S' \overset{\text{def}}{=} (\mathcal{F}_0)_{\geq\mathcal{H}}$.

Let X be an (H-) universal $(G,\mathcal{F}_0)$-complex. All isotropy subgroups of X lie in $\mathcal{F}_0 \subseteq \mathcal{F}$, so X is also a $(G,\mathcal{F})$-complex. For each $K \in \mathcal{F}$, X^K is homotopy (homology) equivalent to $\mathcal{N}((\mathcal{F}_0)_{\geq K})$ by Proposition 1.3 (applied with $\mathcal{H} = \{K\}$); this in turn is homotopy (homology) equivalent to $\mathcal{N}(\mathcal{F}_{\geq K})$ by (1); and this last complex is contractible (acyclic). So X is also (H-) universal as a $(G,\mathcal{F})$-complex. □

We are now ready to deal directly with the problem of controlling the dimensions of universal or H-universal $(G,\mathcal{F})$-complexes. This will be done by attaching cells, one orbit type at a time, at each stage arranging for the appropriate fixed point set to be contractible or acyclic. The key problem is how to do this with cells in free orbits.

This will be described in the following three lemmas. The first will be needed when constructing contractible 1-complexes.

Lemma 1.5. *Let X be any finite G-set with the property that $|X^H| = 1$ for each subgroup $1 \neq H \subseteq G$ of prime power order. Then X has one fixed point and is otherwise free.*

Proof. We may assume that $X^G = \emptyset$; otherwise the result is clear. We may also assume that X has no free orbits (otherwise just remove them). By assumption, each Sylow subgroup of G acts freely on X away from one fixed point; and so $|X| \equiv 1 \pmod{|G|}$.

Write $X = G/H_1 \amalg G/H_2 \amalg \cdots \amalg G/H_k$, where $1 \neq H_i \subsetneqq G$ for all i. In particular,

$$\sum_{i=1}^{k} [G{:}H_i] = |X| = r{\cdot}|G| + 1 \tag{1}$$

for some r. Furthermore, for each pair of distinct elements $x, y \in X$, the isotropy subgroups G_x and G_y have trivial intersection, since otherwise $G_x \cap G_y$ contains a nontrivial p-subgroup (some p) which fixes two points of X. It follows that

$$|G| - 1 \geq \sum_{x \in X} (|G_x| - 1) = \sum_{i=1}^{k} [G{:}H_i]{\cdot}(|H_i| - 1) = k{\cdot}|G| - \sum_{i=1}^{k} [G{:}H_i]. \tag{2}$$

Upon adding (1) and (2), we see that (2) is an equality, and that $r = k - 1$. But then after dividing (1) by $|G|$, we get that

$$\sum_{i=1}^{k} \frac{1}{|H_i|} > k - 1.$$

Since $|H_i| \geq 2$ for all i, we must have $k = 1$, and hence $|X| = 1$. □

A complex X will be called *homologically m-dimensional* if $H_n(X) = 0$ for all $n > m$, and $H_m(X)$ is $\mathbb{Z}$-free. (Technically, this should be called homologically $\leq m$-dimensional, since it only provides an upper bound on the degrees of homology of X.) We note first the following properties of subcomplexes of acyclic complexes.

Lemma 1.6. *Let X be any m-dimensional acyclic CW complex ($m \geq 1$). Then any subcomplex of X is homologically $(m-1)$-dimensional. And if $A_1, \ldots, A_n \subseteq X$ are homologically $(m-2)$-dimensional subcomplexes, then their intersection is also homologically $(m-2)$-dimensional.*

Proof. For any subcomplex $A \subseteq X$, $\widetilde{H}_i(A) \cong H_{i+1}(X, A)$ must be zero for $i \geq m$ and $\mathbb{Z}$-free for $i = m - 1$. Hence A is homologically $(m-1)$-dimensional.

It suffices to prove the last statement when $n = 2$. For each $i \geq m - 2$, there is a Mayer-Vietoris exact sequence

$$0 \longrightarrow H_{i+1}(A_1 \cup A_2) \longrightarrow H_i(A_1 \cap A_2) \longrightarrow H_i(A_1) \oplus H_i(A_2).$$

If $i \geq m - 1$, then the first and last groups are zero, and so $H_i(A_1 \cap A_2) = 0$. And if $i = m - 2$, then the first and last groups are $\mathbb{Z}$-free, and so $H_{m-2}(A_1 \cap A_2)$ is also $\mathbb{Z}$-free. □

The next lemma is essentially included in the proof of [O2, Proposition 6].

Proposition 1.7. *Let X be a finite G-complex with the following two properties.*

(a) For each $1 \neq H \subseteq G$, X^H is acyclic or empty, and is acyclic if H has prime power order.

(b) For some $n > 0$, $\widetilde{H}_*(X) = H_n(X)$, and is $\mathbb{Z}$-free.

Then $H_n(X)$ is stably free as a $\mathbb{Z}[G]$-module.

Proof. For each prime p and each Sylow p-subgroup $S \subseteq G$, consider the subcomplex

$$X' = \bigcup_{1 \neq H \subseteq S} X^H = \{x \in X \mid S_x \neq 1\}.$$

By Proposition 1.3, applied with $\mathcal{H} = \{1 \neq H \subseteq S\}$, X' is acyclic ($\mathcal{N}(\mathcal{H}) \simeq *$ since $\mathcal{H}$ has maximal element S). Hence $H_*(X, X') \cong \widetilde{H}_*(X)$ also vanishes in degrees different from n. Furthermore, since all cells in $X \smallsetminus X'$ are permuted freely by S, $C_*(X, X')$ is a chain complex of finitely generated free $\mathbb{Z}[S]$-modules (Lemma C.1). So by Proposition C.2, the unique nonvanishing homology group $H_n(X, X') \cong H_n(X)$ is $\mathbb{Z}[S]$-stably free. (Since all but one summand in (1) of Proposition C.2 is stably free, so is the remaining summand, by definition.) In particular, $H_n(X)$ is a $\mathbb{Z}[G]$-module which is projective after restriction to each Sylow subgroup, and is hence $\mathbb{Z}[G]$-projective by Rim's theorem [Rim, Proposition 4.9].

Now set $Y = X \times \Sigma X$, where ΣX is the unreduced suspension of X (see Lemma A.5). We identify X with the subcomplex $X \times \{x_0\}$ of Y, where $x_0 \in \Sigma X$ is one of the suspension vertices. Then $H_*(\Sigma X, x_0) = H_{n+1}(\Sigma X, x_0) \cong H_n(X)$; and so by the Künneth formula

$$H_i(Y, X) \cong H_{i-n-1}(X) \otimes H_{n+1}(\Sigma X, x_0) \cong \begin{cases} H_n(X) \otimes_{\mathbb{Z}} H_n(X) & \text{if } i = 2n+1 \\ H_n(X) & \text{if } i = n+1 \\ 0 & \text{otherwise.} \end{cases}$$

Consider the subcomplexes

$$X_s = \bigcup_{1 \neq H \subseteq G} X^H \qquad \text{and} \qquad Y_s = \bigcup_{1 \neq H \subseteq G} Y^H.$$

We claim that the inclusion map $X_s \hookrightarrow Y_s$ is a homology equivalence. To see this, set $\mathcal{F} = \{1 \neq H \subseteq G \mid X^H \neq \emptyset\}$. By Proposition 1.3, there is a map $f\colon Y_s \to \mathcal{N}(\mathcal{F})$ such that $f((Y_s)^H) \subseteq \mathcal{N}((\mathcal{F})_{\geq H})$ for all $H \subseteq G$; and $f|_{X_s}$ has the same property. Since X_s and Y_s are both H-universal $(G, \mathcal{F})$-complexes ($Y^H = X^H \times \Sigma X^H$ is acyclic if X^H is), Proposition 1.3 implies that f restricts to homology equivalences $Y_s \to \mathcal{N}(\mathcal{F})$ and $X_s \to \mathcal{N}(\mathcal{F})$; and thus that the inclusion $X_s \subseteq Y_s$ is a homology equivalence.

In particular, this shows that $H_*(Y, X) \cong H_*(Y, X \cup Y_s)$ (see Lemma B.2). Thus, $C_*(Y, X \cup Y_s)$ is a chain complex of free $\mathbb{Z}[G]$-modules (by Lemma C.1, since G acts freely on $Y \smallsetminus (X \cup Y_s)$) with only two nonzero homology groups. Since $H_n(X) \otimes_{\mathbb{Z}} H_n(X)$ is stably free by Proposition C.3, the other homology group $H_n(X)$ must also be stably free by Proposition C.2. □

For any G-space X and any $H \subseteq G$, we write

$$X^{>H} = \{x \in X \mid G_x \supsetneq H\}:$$

i.e., the union of fixed point sets of subgroups which strictly contain H. Also, for any family $\mathcal{F} \subseteq \mathcal{S}(G)$, $\mathcal{F}_{>H}$ denotes the set of elements of $\mathcal{F}$ which strictly contain H.

Proposition 1.8. *Let G be any finite group, and let $\mathcal{F}$ be a separating family for G. Let $\mathcal{F}_0 \subseteq \mathcal{F}$ be any subfamily with the property that $\mathcal{N}(\mathcal{F}_{>H})$ is contractible (and nonempty) for all $H \in \mathcal{F} \smallsetminus \mathcal{F}_0$. Let $d\colon \mathcal{F}_0 \to \mathbb{N}$ be any function which is constant on conjugacy classes of subgroups, such that $d(H) = 0$ for H maximal in $\mathcal{F}$, such that $\mathcal{N}((\mathcal{F}_0)_{>H})$ is homologically $(d(H)-1)$-dimensional for each non-maximal subgroup $H \in \mathcal{F}_0$, and such that $d(H) \geq d(H')$ whenever $H \subseteq H'$. Then there is a finite H-universal $(G, \mathcal{F}_0)$-complex X with the property that $\dim(X^H) \leq d(H)$ for each $H \in \mathcal{F}_0$. Furthermore, X can be taken to be universal if $d(H) \neq 2$ for each $H \in \mathcal{F}_0$. Also, X can be chosen such that every vertex of X is fixed by some maximal subgroup in $\mathcal{F}$.*

Proof. Let $\mathcal{F}_{\max}$ be the set of maximal subgroups in $\mathcal{F}$. Set $X_0 = \mathcal{F}_{\max}$, regarded as a zero-dimensional G-complex. Since the elements of $\mathcal{F}_{\max}$ are all self-normalizing (Lemma 1.1), this is a 0-dimensional $(G, \mathcal{F})$-complex, and $(X_0)^H$ contains exactly one point for each $H \in \mathcal{F}_{\max}$. Let $H_1, \ldots, H_k = 1$ be conjugacy class representatives for the elements of $\mathcal{F} \smallsetminus \mathcal{F}_{\max}$, ordered such that $d(H_1) \leq d(H_2) \leq \cdots d(H_k)$, and such that $i \leq j$ if H_i contains a subgroup conjugate to H_j. For each $i = 0, \ldots, k$, let $\mathcal{H}_i$ be the set of all maximal subgroups in $\mathcal{F}$, together with all subgroups conjugate to H_j for any $j \leq i$. In particular, $\mathcal{H}_0 = \mathcal{F}_{\max}$ and $\mathcal{H}_k = \mathcal{F}$. We construct a sequence of G-complexes $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_k$, such that for each $i \geq 1$,

(a) $\dim(X_i) \leq d(H_i)$ and $X_i^{(0)} = X_0^{(0)}$,

(b) $X_i \smallsetminus X_{i-1}$ has only orbit types G/H_i,

(c) $X_i = X_{i-1}$ if $H_i \notin \mathcal{F}_0$, and

(d) $(X_i)^{H_i}$ is acyclic, and is contractible if $H_i \in \mathcal{F}_0$ and $d(H_i) \neq 2$.

Note that for each $H \in \mathcal{F}_{\max}$, $(X_0)^H = \{H\}$ is contractible, and hence $(X_i)^H$ will be contractible for all $i > 0$. Once the X_i have been constructed, we set $X = X_k$. This is a $(G, \mathcal{F}_0)$-complex; and for all $H \in \mathcal{F}_0$, $\dim(X^H) \leq d(H)$, and X^H is acyclic, and contractible if $d(H) \neq 2$. And by (a), each vertex of X is in X_0, and hence fixed by a maximal subgroup of $\mathcal{F}$.

It remains to construct the X_i. Assume that X_{i-1} has been constructed ($i \geq 1$). Then X_{i-1} is an H-universal $(G, \mathcal{H}_{i-1})$-complex. By Proposition 1.3 (and by definition of the $\mathcal{H}_j$),

$$H_*\big((X_{i-1})^{H_i}\big) = H_*\big((X_{i-1})^{>H_i}\big) \cong H_*\big(\mathcal{N}((\mathcal{H}_{i-1})_{>H_i})\big) = H_*\big(\mathcal{N}(\mathcal{F}_{>H_i})\big).$$

In particular, by Lemma 1.4, $(X_{i-1})^{H_i}$ is homologically $(d(H_i)-1)$-dimensional, and is acyclic if $H_i \notin \mathcal{F}_0$. Also, $\dim({X_{i-1}}^{H_i}) \leq d(H_i)$: this is clear if $i = 0$ ($\dim(X_0) = 0$), and holds for $i \geq 1$ by (a) since $d(H_j) \leq d(H_i)$ for $j < i$ by assumption. Thus, if $H_i \notin \mathcal{F}_0$, we can set $X_i = X_{i-1}$.

Assume now that $H_i \in \mathcal{F}_0$. Write $H = H_i$ and d=d(H) for short. If $d = 1$, then $(X_{i-1})^H$ is 1-dimensional, and its connected components are all acyclic. By Lemma 1.5, applied to the $N(H)/H$-set $\pi_0((X_{i-1})^H)$ (the set of connected components of $(X_{i-1})^H$), $(X_{i-1})^H$ has one connected component which is fixed by the action of $N(H)/H$, and the other components are permuted freely by $N(H)/H$. So Proposition 0.2(d) applies to show that there is a finite G-complex X_i, obtained by attaching orbits of cells $G/H\times D^1$ to X_{i-1}, such that $(X_i)^H$ is acyclic.

If $d > 1$, then by Proposition 0.2(a), there is a G-complex $Y \supseteq X_{i-1}$, constructed by attaching cells $G/H\times D^k$ for $1 \le k \le d-1$, such that Y^H is $(d-2)$-connected and $H_{d-1}(Y^H)$ is $\mathbb{Z}$-free. In particular, Y^H is still homologically $(d-1)$-dimensional, and $\dim(Y^H) \le d$. For any subgroup $1 \ne K/H \subseteq N(H)/H$ of prime power order, $(Y^H)^{K/H} = Y^K = (X_{i-1})^K$ is acyclic by (d): $K \in \mathcal{F}$ by definition of a separating family, and so $K \in \mathcal{H}_{i-1}$. Proposition 1.7 now applies to show that $H_{d-1}(Y^H)$ is stably free as a $\mathbb{Z}[N(H)/H]$-module. So by Proposition 0.2(c), we can attach orbits of cells of type $G/H\times D^k$ for $k = d-1, d$ to Y, to obtain a finite G-complex $X_i \supseteq Y$ such that $(X_i)^H$ is acyclic. □

In fact, one can show for any family $\mathcal{F}$ of subgroups of G that there is a universal $(G,\mathcal{F})$-complex. But such a complex must be infinite dimensional if $\mathcal{F}$ is not a separating family.

We can now state necessary and sufficient conditions for the existence of universal or H-universal $(G,\mathcal{F})$-complexes of a given dimension.

Proposition 1.9. *Let G be any finite group, and let $\mathcal{F}$ be a separating family for G. Let $\mathcal{F}_0 \subseteq \mathcal{F}$ be any subfamily with the property that $\mathcal{N}(\mathcal{F}_{>H})$ is contractible (and nonempty) for all $H \in \mathcal{F}\smallsetminus\mathcal{F}_0$. Then there is a finite universal $(G,\mathcal{F}_0)$-complex. Furthermore, the following four conditions are equivalent for any $m \ge 2$:*

(a) There exists an m-dimensional universal $(G,\mathcal{F})$-complex (H-universal if $m = 2$).

(b) There exists a finite m-dimensional universal $(G,\mathcal{F}_0)$-complex (H-universal if $m = 2$).

(c) $\mathcal{N}(\mathcal{F}_{>H})$ is homologically $(m-1)$-dimensional for each subgroup $H \in \mathcal{F}_0$.

(d) $\mathcal{N}((\mathcal{F}_0)_{\ge\mathcal{H}})$ is homologically $(m-1)$-dimensional for each set $\mathcal{H}$ of subgroups of G.

Proof. Since the nerve $\mathcal{N}(\mathcal{F})$ is finite dimensional, the existence of a finite universal $(G,\mathcal{F}_0)$-complex follows from Proposition 1.8.

(a ⇒ d) If X is an m-dimensional H-universal $(G,\mathcal{F})$-complex, then for any set of subgroups $\mathcal{H}$, $X^{\mathcal{H}}$ is homologically $(m-1)$-dimensional by Lemma 1.6. Since

$$H_*(X^{\mathcal{H}}) \cong H_*(\mathcal{N}(\mathcal{F}_{\ge\mathcal{H}})) \cong H_*(\mathcal{N}((\mathcal{F}_0)_{\ge\mathcal{H}}))$$

by Proposition 1.3 and Lemma 1.4, $\mathcal{N}((\mathcal{F}_0)_{\ge\mathcal{H}})$ is also homologically $(m-1)$-dimensional.

(d ⇒ c) Follows immediately from Lemma 1.4.

(c ⇒ b) Follows immediately from Proposition 1.8.

(b ⇒ a) Follows immediately from Lemma 1.4. □

As an immediate corollary of Proposition 1.9, we get:

Corollary 1.10. *Let G be any finite group, and let $\mathcal{F}$ be a separating family for G. Then there is a (finite) 2-dimensional H-universal $(G,\mathcal{F})$-complex if and only if $\mathcal{N}(\mathcal{F}_{>H})$ is homologically 1-dimensional for each subgroup $H \in \mathcal{F}$, if and only if $\mathcal{N}(\mathcal{F}_{\ge\mathcal{H}})$ is homologically 1-dimensional for each set $\mathcal{H}$ of subgroups of G.* □

2. Numbers of cells

Again, G will always be a finite group throughout this section. We prove here some results which will be useful for keeping track of Euler characteristics of (unions of) fixed point sets in H-universal G-complexes. The notation used for doing this is defined as follows:

Definition 2.1. *For any family $\mathcal{F}$ of subgroups of G, define*

$$i_{\mathcal{F}}(H) = i_{(G,\mathcal{F})}(H) = \frac{1}{[N(H){:}H]}\cdot\bigl(1 - \chi(\mathcal{N}(\mathcal{F}_{>H}))\bigr).$$

for each $H \in \mathcal{F}$. Set $I(G,\mathcal{F}) = i_{(G,\mathcal{F})}(1)$.

We first note the following elementary relation between Euler characteristics of G-complexes and of their orbit spaces.

Lemma 2.2. *Let $X' \subseteq X$ be any pair of finite G-complexes, and assume that all orbits in $X \smallsetminus X'$ are of type G/H for some fixed subgroup $H \subseteq G$. Then*

$$\chi(X) - \chi(X') = |G/H|\cdot\bigl(\chi(X/G) - \chi(X'/G)\bigr).$$

Proof. For each $n \geq 0$, let c_n denote the number of n-cells in X not in X'. Then $\chi(X) - \chi(X') = \sum_{n\geq 0}(-1)^n c_n$. By assumption, each G-orbit of cells has order exactly $|G/H|$. So the number of n-cells in X/G not in X'/G is $\frac{1}{|G/H|}\cdot c_n$ for each n, and thus

$$\chi(X/G) - \chi(X'/G) = \sum_{n\geq 0}\frac{c_n}{|G/H|} = \frac{1}{|G/H|}\bigl(\chi(X) - \chi(X')\bigr). \qquad \square$$

The relation between these indices and Euler characteristics of universal complexes is given in the following two lemmas.

Lemma 2.3. *Fix a separating family $\mathcal{F}$, a finite H-universal $(G,\mathcal{F})$-complex X, and a subgroup $H \subseteq G$. For each n, let $c_n(H)$ denote the number of orbits of n-cells of type G/H. Then $i(H) = \sum_{n\geq 0}(-1)^n c_n(H)$.*

Proof. By Proposition 1.3, there is a G-map $f\colon X \to \mathcal{N}(\mathcal{F})$, which restricts to homology equivalences $X^H \to \mathcal{N}(\mathcal{F}_{\geq H})$ and $X^{>H} \to \mathcal{N}(\mathcal{F}_{>H})$. Thus, by Definition 2.1, and by Lemma 2.2 applied to the action of $N(H)$ on the complexes $X^{>H} \subseteq X^H$,

$$\begin{aligned} i_{\mathcal{F}}(H) = \frac{1}{[N(H){:}H]}\cdot\bigl(1 - \chi(\mathcal{N}(\mathcal{F}_{>H}))\bigr) &= \frac{1}{[N(H){:}H]}\cdot\bigl(\chi(X^H) - \chi(X^{>H})\bigr) \\ &= \chi(X^H/N(H)) - \chi(X^{>H}/N(H)). \end{aligned}$$

Each orbit of cells of type $G/H\times D^n$ in X restricts to one of type $(N(H)/H)\times D^n$ in X^H, and hence to exactly one n-cell in the orbit space $X^H/N(H)$. These are precisely the cells in $X^H/N(H)$ which are not in $X^{>H}/N(H)$, and hence

$$\chi(X^H/N(H)) - \chi(X^{>H}/N(H)) = \sum_{n\geq 0}(-1)^n c_n(H). \qquad \square$$

Lemma 2.4. *Let $\mathcal{F}$ be any separating family of subgroups of G, and let X be any finite H-universal $(G,\mathcal{F})$-complex. Let $\mathcal{H} \subseteq \mathcal{F}$ be any subset with the property that $K \supseteq H \in \mathcal{H}$ and $K \in \mathcal{F}$ implies $K \in \mathcal{H}$. Then*

$$\chi(\mathcal{N}(\mathcal{H})) = \chi\big(X^{\mathcal{H}}\big) = \sum_{H\in\mathcal{H}} [N(H){:}H]\cdot i_{\mathcal{F}}(H). \tag{1}$$

If, furthermore, $\mathcal{H}$ is a family (i.e., a union of G-conjugacy classes), then

$$\chi\big(X^{\mathcal{H}}/G\big) = \sum_{H\in\mathcal{H}/\mathrm{conj}} i_{\mathcal{F}}(H). \tag{2}$$

Proof. We prove these formulas by induction on $|\mathcal{H}|$; they clearly (vacuously) hold when $\mathcal{H} = \emptyset$. Let H be a minimal subgroup of $\mathcal{H}$, and set $\mathcal{H}' = \mathcal{H} \smallsetminus \{H\}$. Then $\mathcal{N}(\mathcal{H}) = \mathcal{N}(\mathcal{H}') \cup_{\mathcal{N}(\mathcal{F}_{>H})} C(\mathcal{N}(\mathcal{F}_{>H}))$; in other words, the union of $\mathcal{N}(\mathcal{H}')$ and $C(\mathcal{N}(\mathcal{F}_{>H}))$ (the cone over $\mathcal{N}(\mathcal{F}_{>H})$) with intersection $\mathcal{N}(\mathcal{F}_{>H})$. So by the Mayer-Vietoris sequence for the union,

$$\chi(\mathcal{N}(\mathcal{H})) = \chi(\mathcal{N}(\mathcal{H}')) + 1 - \chi(\mathcal{N}(\mathcal{F}_{>H})) = \chi(\mathcal{N}(\mathcal{H}')) + [N(H){:}H]\cdot i_{\mathcal{F}}(H);$$

and so $\chi(\mathcal{N}(\mathcal{H})) = \sum_{H\in\mathcal{H}}[N(H){:}H]\cdot i_{\mathcal{F}}(H)$ by induction. Since $\chi(\mathcal{N}(\mathcal{H})) = \chi(X^{\mathcal{H}})$ by Proposition 1.3, this proves (1).

Now assume that $\mathcal{H}$ is a family. For each $n \geq 0$ and each $H \in \mathcal{H}$, let $c_n(H)$ be the number of orbits of n-cells of type G/H. Let $c_n(\mathcal{H})$ be the sum, taken over conjugacy class representatives for all $H \in \mathcal{H}$, of the $c_n(H)$. Then $c_n(\mathcal{H})$ is precisely the number of n-cells in $X^{\mathcal{H}}/G$; and so

$$\chi\big(X^{\mathcal{H}}/G\big) = \sum_{n=0}^{\infty}(-1)^n c_n(\mathcal{H}) = \sum_{H\in\mathcal{H}/\mathrm{conj}} \sum_{n=0}^{\infty}(-1)^n c_n(H) = \sum_{H\in\mathcal{H}/\mathrm{conj}} i_{\mathcal{F}}(H)$$

by Lemma 2.3. □

Corollary 2.5. *For any separating family $\mathcal{F}$ of subgroups of G,*

$$\sum_{H\in\mathcal{F}/\mathrm{conj}} i_{\mathcal{F}}(H) = 1.$$

Proof. If X is any finite H-universal $(G,\mathcal{F})$-complex, then in particular X is acyclic, and so X/G is acyclic (cf. [Br, Theorem III.7.12]). Thus $\chi(X/G) = 1$, and so the result follows from Lemma 2.4 (applied with $\mathcal{H} = \mathcal{F}$). □

The following relations will be useful later, when manipulating nerves of subgroups of G.

Lemma 2.6. *Fix a separating family $\mathcal{F}$ of subgroups of G. Let $\mathcal{F}_c \subseteq \mathcal{F}$ be the subfamily of those subgroups $H \in \mathcal{F}$ such that $\mathcal{N}(\mathcal{F}_{>H})$ is not contractible. Fix a subgroup $H \in \mathcal{F}_c$ such that $H \subsetneqq N(H) \in \mathcal{F}$, and let $K_1, \ldots, K_r$ be G-conjugacy class representatives for the subgroups $K \in \mathcal{F}_c$ such that $K \supsetneqq H$ and $N_K(H) = H$. For each j, let a_j be the number of K_j-conjugacy classes of subgroups in K_j which are G-conjugate to H and self-normalizing in K_j. Then*

$$i_{(G,\mathcal{F})}(H) = -\sum_{j=1}^{r} a_j \cdot i_{(G,\mathcal{F})}(K_j). \tag{1}$$

Proof. For any subgroup $H \in \mathcal{F}\smallsetminus\mathcal{F}_c$, $\mathcal{N}(\mathcal{F}_{>H})$ is contractible, and so $i_{\mathcal{F}}(H) = 0$ by Definition 2.1. So we can assume that the $K_1, \dots, K_r$ contain G-conjugacy class representatives for *all* subgroups $K \in \mathcal{F}$ such that $K \supsetneqq H$ and $N_K(H) = H$ (not just those in $\mathcal{F}_c$), without changing the right-hand side in (1).

Let X be any finite H-universal $(G,\mathcal{F})$-complex. Set $\mathcal{H} = \mathcal{F}_{\geq H}$, and set $\mathcal{H}_0 = \{K \in \mathcal{F} \,|\, K \supseteq H,\ N_K(H) \supsetneqq H\}$. Then $\mathcal{N}(\mathcal{H})$ and $\mathcal{N}(\mathcal{H}_0)$ are both contractible by Lemma 0.3(b): the first since $\mathcal{H}$ has smallest element H; and the second since $N(H) \in \mathcal{H}_0$, and $N(H) \cap K \in \mathcal{H}_0$ for all $K \in \mathcal{H}_0$.

By Lemma 2.4,

$$\sum_{K \in \mathcal{H}\smallsetminus\mathcal{H}_0} [N(K){:}K]{\cdot} i_{\mathcal{F}}(K) = \chi(\mathcal{N}(\mathcal{H})) - \chi(\mathcal{N}(\mathcal{H}_0)) = 1 - 1 = 0. \tag{2}$$

Set $R = \{K \in \mathcal{F} \,|\, K \supsetneqq H,\ N_K(H) = H\}$; the subgroups $K_1, \dots, K_r$ are thus G-conjugacy class representatives for the elements of R. For each j, set

$$\begin{aligned} S_j &= \{g \in G \,|\, gK_jg^{-1} \supseteq H,\ N_{gK_jg^{-1}}(H) = H\} \\ &= \{g \in G \,|\, gK_jg^{-1} \supseteq H,\ N_{K_j}(g^{-1}Hg) = g^{-1}Hg\}. \end{aligned}$$

Then by (2),

$$\begin{aligned} i_{\mathcal{F}}(H) &= -\sum_{K \in R} \frac{|N(K)|{\cdot}|H|}{|N(H)|{\cdot}|K|} i_{\mathcal{F}}(K) = -\sum_{K \in R} \frac{|N(K)|}{|N(H){\cdot}K|} i_{\mathcal{F}}(K) \\ &= -\sum_{j=1}^{r} \left(\sum_{g \in S_j} \frac{1}{|N(g^{-1}Hg){\cdot}K_j|} \right) i_{\mathcal{F}}(K_j); \end{aligned}$$

and it remains only to show that the sum in parentheses is equal to a_j: the number of K_j-conjugacy classes of subgroups $g^{-1}Hg$ for $g \in S_j$. And this follows since for any $g, g_0 \in S_j$, $g^{-1}Hg$ and $g_0^{-1}Hg_0$ are K_j-conjugate if and only if there exists $a \in K_j$ such that $a^{-1}g_0^{-1}Hg_0a = g^{-1}Hg$, if and only if $g^{-1}g_0a \in N(g^{-1}Hg)$ for some $a \in K_j$, if and only if $g^{-1}g_0 \in N(g^{-1}Hg){\cdot}K_j$. □

3. Construction of 2-dimensional actions

Again, in this section, G always denotes a finite group. To simplify the statements of results here and later, for any separating family $\mathcal{F}$ of subgroups of G, we write $(G,\mathcal{F}) \in \mathcal{U}_2$ whenever there exists a 2-dimensional H-universal $(G,\mathcal{F})$-complex (and $(G,\mathcal{F}) \notin \mathcal{U}_2$ otherwise).

We are now ready to construct the 2-dimensional acyclic actions of the groups G listed in Theorem A. But we first must look more closely at the question of which subgroups of G need not appear as isotropy subgroups in a universal $(G,\mathcal{F})$-complex.

For any G and any separating family $\mathcal{F}$ of subgroups of G, we say that $H \in \mathcal{F}$ is a *critical* subgroup in $\mathcal{F}$ if $\mathcal{N}(\mathcal{F}_{>H})$ is not contractible. As seen in Proposition 1.9, subgroups which are not critical need not occur as isotropy subgroups in (H-) universal $(G,\mathcal{F})$-complexes. When notation is needed, we will denote by $\mathcal{F}_c$ the subfamily of critical subgroups in $\mathcal{F}$. In the following lemma, we note some conditions which allow us to show that certain subgroups in $\mathcal{F}$ are not critical.

Lemma 3.1. *Let $\mathcal{F}$ be any family of subgroups of G which has the property that $H \subseteq H' \subseteq H''$ and $H, H'' \in \mathcal{F}$ imply $H' \in \mathcal{F}$. Fix a subgroup $H \in \mathcal{F}$. Then $\mathcal{N}(\mathcal{F}_{>H}) \simeq *$ if any of the following conditions hold:*

(a) H is not an intersection of maximal subgroups in $\mathcal{F}$.

(b) There is a subgroup $\widehat{H} \supsetneqq H$, $\widehat{H} \in \mathcal{F}$, such that $K \cap \widehat{H} \supsetneqq H$ for all $H \subsetneqq K \in \mathcal{F}_c$.

Proof. (a) Let $\mathcal{F}' \subseteq \mathcal{F}$ be the subfamily of all intersections of maximal subgroups in $\mathcal{F}$, and let $\alpha\colon \mathcal{F} \to \mathcal{F}'$ be the function which sends a subgroup to the intersection of the members of $\mathcal{F}_{\max}$ which contain it. Then α induces a deformation retraction $\mathcal{N}(\mathcal{F}_{>H}) \to \mathcal{N}(\mathcal{F}'_{>H})$ (Lemma 0.3(a)); and $\mathcal{N}(\mathcal{F}'_{>H})$ is contractible since it contains the minimal element $\alpha(H)$.

(b) Set $\mathcal{H} = \{K \in \mathcal{F} \,|\, K \cap \widehat{H} \supsetneqq H\}$. Then $\widehat{H} \in \mathcal{H}$, and $K \cap \widehat{H} \in \mathcal{H}$ for all $K \in \mathcal{H}$. So $\mathcal{N}(\mathcal{H})$ is contractible by Lemma 0.3(b).

Now $(\mathcal{F}_c)_{>H} = (\mathcal{F}_c)_{\geq \mathcal{H}}$ by assumption, and so

$$\mathcal{N}(\mathcal{F}_{>H}) \simeq \mathcal{N}((\mathcal{F}_c)_{>H}) = \mathcal{N}((\mathcal{F}_c)_{\geq \mathcal{H}}) \simeq \mathcal{N}(\mathcal{F}_{\geq \mathcal{H}}) = \mathcal{N}(\mathcal{H}) \simeq *;$$

where the homotopy equivalences follow from Lemma 1.4. □

The following lemma provides a simple sufficient condition for the existence of a 2-dimensional H-universal $(G, \mathcal{F})$-complex.

Lemma 3.2. *Let $\mathcal{F}$ be any separating family of subgroups of G. Assume, for every nonmaximal critical subgroup $1 \neq H \in \mathcal{F}$, that $N(H) \in \mathcal{F}$, and that $K \cap N(H) \supsetneqq H$ for all nonmaximal critical subgroups $K \supsetneqq H$ in $\mathcal{F}$. Then $(G, \mathcal{F}) \in \mathcal{U}_2$.*

More precisely, let $M_1, \ldots, M_n$ be conjugacy class representatives for the maximal subgroups of $\mathcal{F}$, and let $H_1, \ldots, H_k$ be conjugacy class representatives for all nonmaximal critical subgroups of $\mathcal{F}$. Then there is a 2-dimensional H-universal $(G, \mathcal{F})$-complex X which consists of one orbit of vertices of type G/M_i for each $1 \leq i \leq n$, $(-i_{\mathcal{F}}(H_j))$-orbits of 1-cells of type G/H_j for each $1 \leq j \leq k$, and free orbits of 1- and 2-cells. If, furthermore, G is simple, then X can be constructed to contain exactly $i_{\mathcal{F}}(1)$ free orbits of 2-cells (and no free orbits of 1-cells).

Proof. Fix a nonmaximal critical subgroup $H = H_j \in \mathcal{F}$. If $(\mathcal{F}_c)_{>H} \subseteq \mathcal{F}_{\max}$, then $\mathcal{N}(\mathcal{F}_{>H}) \simeq \mathcal{N}((\mathcal{F}_c)_{>H})$ is homologically 0-dimensional by Lemma 1.4. Otherwise, let $\mathcal{H}$ be the set of all $K \in \mathcal{F}_{>H}$ such that $K \cap N(H) \supsetneqq H$, and set $\mathcal{H}_c = \mathcal{H} \cap \mathcal{F}_c$. Then $N(H) \in \mathcal{H}$, and $K \cap N(H) \in \mathcal{H}$ for all $K \in \mathcal{H}$, so $\mathcal{N}(\mathcal{H})$ is contractible (Lemma 0.3(b)). Since $\mathcal{H} \subseteq \mathcal{F}$ and $\mathcal{H}_c \subseteq \mathcal{F}_c$ are terminal subposets, Lemma 0.4 now applies to show that $\mathcal{N}(\mathcal{H}_c) \simeq *$. Thus, $\mathcal{N}((\mathcal{F}_c)_{>H})$ consists of one contractible component $\mathcal{N}(\mathcal{H}_c)$, together with some isolated vertices for those maximal subgroups $M \in \mathcal{F}_{>H}$ such that $M \cap N(H) = H$. In particular, $\mathcal{N}(\mathcal{F}_{>H})$ is homologically 0-dimensional.

Hence, by Proposition 1.8, there is a finite H-universal $(G, \mathcal{F}_c)$-complex X such that $\dim(X^M) = 0$ for each maximal subgroup $M \in \mathcal{F}$, such that $\dim(X^H) = 1$ for each nonmaximal subgroup $1 \neq H \in \mathcal{F}_c$, and such that each vertex of X is fixed by a maximal subgroup in $\mathcal{F}$. But by Proposition 1.3 and Lemma 1.4, $H_*(X_s) \cong H_*(\mathcal{N}(\mathcal{F}_{>1}))$, so

$\mathcal{N}(\mathcal{F}_{>1})$ is homologically 1-dimensional since X_s is; and by Proposition 1.8 again, X can be taken to be 2-dimensional.

By the above description of X, we see that all orbits of vertices in X are of type G/M for maximal M; that all orbits of edges are of type G/H_i for $1 \le i \le k$ or (possibly) free (of type $G/1$); and that all orbits of 2-cells are free. Hence the numbers of orbits of cells of type G/M_i or G/H_j follows from the formula in Lemma 2.3. (Note that $i_{\mathcal{F}}(M) = 1$ whenever M is maximal.) Also, by Proposition 1.7, $H_1(X_s)$ is stably free as a $\mathbb{Z}[G]$-module, and hence is free by Proposition C.4 if G is simple. So by Proposition 0.2(c), X can be constructed by attaching only free orbits of 2-cells to X_s; and the number of orbits of cells is again given by Lemma 2.3. □

Lemma 3.2 will be applied to construct 2-dimensional actions of the simple groups $L_2(q)$ $(= PSL_2(q))$ for certain q, and of the Suzuki groups. We first list some of the properties of subgroups of the $L_2(q)$ which will be needed here, and also later in Section 6.

Proposition 3.3. *Fix $q = p^k \ge 4$, where p is prime. Then the maximal solvable subgroups $H \subseteq L_2(q) = PSL_2(q)$ and $\overline{H} \subseteq PGL_2(q)$ are as described in the following table. (Note that $L_2(q) = PGL_2(q)$ when q is a power of 2.)*

$H \subseteq L_2(q)$ (q odd)		$\overline{H} \subseteq PGL_2(q)$		
H	Nr. classes	$\overline{H}$	Nr. classes	conditions
$\mathbb{F}\rtimes C_{(q-1)/2}$	1	$\mathbb{F}\rtimes C_{q-1}$	1	——
D_{q-1}	1	$D_{2(q-1)}$	1	——
D_{q+1}	1	$D_{2(q+1)}$	1	——
A_4	1	Σ_4	1	$q \equiv \pm 3 \pmod 8$
Σ_4	2	Σ_4	1	$q \equiv \pm 1 \pmod 8$

Here, in all cases (when q is odd), $\overline{H} = N_{PGL_2(q)}(H)$. Furthermore, each nonsolvable subgroup of $L_2(q)$ is conjugate in $PGL_2(q)$ to one of the groups $L_2(q_0)$ for $q_0 = p^{k_0}$ and $k_0|k$; or to $PGL_2(q_0)$ for $q_0 = p^{k_0}$ and $2k_0|k$; or (if q is odd and $q \equiv \pm 1 \pmod 5$) is isomorphic to A_5.

Proof. See [Sz2, §3.6]. The subgroups of $L_2(q)$ are described in [Sz2, Theorems 3.6.25–26], and in [H1, 8.27]. The uniqueness up to conjugacy of the dihedral groups follows from [Sz2, 3.6.23]; and the uniqueness of the $\mathbb{F}_q\rtimes C_{q-1}$ or $\mathbb{F}_q\rtimes C_{(q-1)/2}$ follows since they are normalizers of Sylow p-subgroups. The maximal subgroups A_4 or Σ_4 are normalizers of elementary abelian subgroups $(C_2)^2 \subseteq L_2(q)$, of which there is one or two conjugacy classes depending on $q \pmod 8$ (see also [H1, 8.16]). The fact that any subgroup isomorphic to $L_2(q_0)$ or $PGL_2(q_0)$ is conjugate (in $PGL_2(q)$) to the standard one follows from [Sz2, 3.6.20 and Ex. 3.6.1+3].

Note in particular that B ($\cong \mathbb{F}_q\rtimes C_{q-1}$ or $\cong \mathbb{F}_q\rtimes C_{(q-1)/2}$) is represented by the group of upper triangular matrices, and that $D_{2(q-1)}$ is the subgroup of monomial matrices. The other dihedral group $D_{2(q+1)}$ or D_{q+1} is the subgroup of $GL(\mathbb{F}_{q^2})$ (here $\mathbb{F}_{q^2}$ is viewed as a 2-dimensional vector space over $\mathbb{F}_q$) of all transformations of determinant one generated by multiplying by an element of $\mathbb{F}_{q^2}$ or by applying the Frobenius automorphism ($x \mapsto x^q$).

Finally, the results about maximal subgroups of $PGL_2(q)$ follow from the information about subgroups of $L_2(q^2) \supseteq PGL_2(q)$. □

We first construct actions of the groups $L_2(2^k)$.

Example 3.4. *Set $G = L_2(q)$, where $q = 2^k$ and $k \geq 2$. Then there is a a 2-dimensional acyclic fixed point free G-complex X, all of whose isotropy subgroups are solvable. More precisely, X can be constructed to have three orbits of vertices with isotropy subgroups isomorphic to $\mathbb{F}_q \rtimes C_{q-1}$, $D_{2(q-1)}$, and $D_{2(q+1)}$; three orbits of edges with isotropy subgroups isomorphic to C_{q-1}, C_2, and C_2; and one free orbit of 2-cells.*

Proof. Let $\mathcal{SLV}$ be the separating family of solvable subgroups of G, and let $\mathcal{SLV}_c \subseteq \mathcal{SLV}$ be the subfamily of all critical subgroups in $\mathcal{SLV}$. By Proposition 3.3, the maximal solvable subgroups of G are the groups $B = \mathbb{F}_q \rtimes C_{q-1}$, $D_{2(q-1)}$, and $D_{2(q+1)}$, where each occurs with exactly one conjugacy class.

The *Borel subgroups* of G are those conjugate to B; or equivalently those subgroups of G which fix a line (a 1-dimensional subspace of $(\mathbb{F}_q)^2$). Every subgroup of G of even order is contained in at most one Borel subgroup, since the subgroup of elements fixing any two distinct lines is cyclic of order $q-1$. Also, any subgroup contained in both a Borel subgroup and a dihedral subgroup must have order 2. Thus, C_2 is the only subgroup of even order contained in more than one maximal subgroup in $\mathcal{SLV}$. Any nontrivial odd order subgroup is contained in a unique maximal dihedral subgroup (its normalizer); and a subgroup C_r for $1 \neq r|(q-1)$ is contained in exactly two Borel subgroups corresponding to the two lines (eigenspaces) it leaves invariant. Thus, since each critical subgroup must be an intersection of maximal subgroups in $\mathcal{SLV}$ (Lemma 3.1), the only possible critical subgroups are the maximal subgroups, together with C_{q-1}, C_2, and 1 (one conjugacy class each).

Computations using Lemma 2.6 (and Corollary 2.5 to determine $i_{\mathcal{SLV}}(1)$) now yield the following table.

$H \in \mathcal{SLV}_c$	$K \cap N(H) = H$	$i(H)$
$B = \mathbb{F}_q \rtimes C_{q-1}$	—	1
$D_{2(q-1)}$	—	1
$D_{2(q+1)}$	—	1
C_{q-1}	B	-1
C_2	$D_{2(q\pm1)}$	-2
1	—	1

Table 1

From this, it is clear that the hypotheses of Lemma 3.2 are satisfied, and hence that $(L_2(q), \mathcal{SLV}) \in \mathcal{U}_2$. More precisely, the lemma and table show that there is an H-universal $(G, \mathcal{SLV}_c)$-complex, with three orbits G/B, $G/D_{2(q-1)}$, and $G/D_{2(q+1)}$ of vertices; with three orbits G/C_2, G/C_2, and G/C_{q-1} of 1-cells; and with one free orbit of 2-cells. □

Before continuing with the construction of the actions of other groups, we want to discuss the classical example of an A_5-action, and its relationship with the construction (when $G = L_2(4) \cong A_5$) in Example 3.4. We first establish our notation. We write $SO(3) = SO(3, \mathbb{R})$, and write $S^3 = SL_1(\mathbb{H}) \cong SU(2, \mathbb{C})$ for the group of unit quaternions (elements of norm one in the quaternion algebra $\mathbb{H}$ over $\mathbb{R}$). There is a homomorphism $S^3 \to SO(3)$, surjective with kernel $\{\pm 1\}$, which is defined by sending $a \in S^3 \subseteq \mathbb{H}$ to the matrix of the conjugation map $(x \mapsto axa^{-1})$ on the subspace $\langle i, j, k \rangle \subseteq \mathbb{H}$. Thus, we regard S^3 as a two fold cover of $SO(3)$.

We now identify $A_5 \cong L_2(5)$ as the icosahedral subgroup of $SO(3)$, and let $A_5^* \cong SL_2(5)$ (the binary icosahedral group) denote its inverse image in S^3. Consider the action of A_5 via left multiplication on the space $\Sigma^3 = SO(3)/A_5 \cong S^3/A_5^*$ of left cosets. This space is the Poincaré sphere, a 3-manifold which has the homology of the 3-sphere, and whose fundamental group is isomorphic to the perfect group A_5^*. Then A_5 acts with fixed point set $(SO(3)/A_5)^{A_5} = N(A_5)/A_5 = \text{pt}$. Upon removing an open invariant ball around the fixed point, we obtain a compact acyclic 3-manifold M (with boundary) upon which A_5 acts without fixed points. This was the starting point for the construction by Floyd and Richardson [FR] of an action of A_5 on a disk without fixed points (see also [Br, §I.8] for more details). Since $\partial M \neq \emptyset$, M can now be collapsed to a 2-dimensional subcomplex $X \simeq M$, upon which A_5 still acts without fixed point.

This last step can be made more explicit. Let P denote the regular polytope with 120 dodecahedral faces, and let Γ be its symmetry group. Clearly, $\Gamma \subseteq SO(4) \cong S^3 \times_{C_2} S^3$, and Γ contains A_5 (the group of symmetries leaving one face invariant) with index 120. This implies that $\Gamma \cong A_5^* \times_{C_2} A_5^*$; and hence that Γ contains a binary icosahedral subgroup A_5^* which permutes freely the faces of P. So $\Sigma^3 \cong S^3/A_5^*$ can be identified with the space $D/\sim$, obtained by identifying opposite faces of the solid dodecahedron D in an appropriate way. This is in fact Poincaré's original construction of the Poincaré sphere. For more details on the identification, and another way of showing that these two constructions are equivalent, we refer to [KS, pp. 124–128].

Under this identification of Σ^3 with $D/\sim$, the A_5 action on Σ^3 is induced by the usual action on the dodecahedron. The fixed point is thus the center of D; and the operation of removing the fixed point and collapsing the remaining space to a 2-dimensional subcomplex corresponds to removing the center of D and then collapsing to its boundary. The result is an explicit 2-dimensional complex $X = \partial D/\sim$ with fixed point free action of A_5, which has 6 pentagonal 2-cells, 10 edges, and 5 vertices.

Here's another, quicker way to construct this last complex. Let X_0 be the 1-skeleton of the 4-simplex, with the obvious action of A_5 permuting the five vertices. Any 5-cycle in A_5 (in the vertices of X_0) tells us how to attach a pentagon to X_0; and two such pentagons will be in the same orbit of A_5 if and only if the corresponding 5-cycles are conjugate. So by attaching to X_0 six pentagons corresponding to one conjugacy class of 5-cycles in A_5, we obtain a 2-complex X with A_5-action. One can check directly that

X is acyclic (and with a bit more work show that $\pi_1(X) \cong A_5^*$); but one also sees easily that it is identical with the previous construction based on the dodecahedron.

If we now subdivide each pentagon in (either of) these spaces, as a union of ten 2-simplices (by adding extra vertices at the midpoints of edges and centers of faces), we have constructed an A_5-complex of the type constructed in Example 3.4 — except that the 2-cells have been attached explicitly. This is also identical to the A_5-simplicial complex constructed in [S1, §3] and in [AS, §9]. We also note here that for $k \geq 3$, the $L_2(2^k)$-complexes constructed in Example 3.4 have the same 1-skeleton as the complexes constructed in [AS, §9] (which were not acyclic); they differ only in the way the 2-cells are attached.

We now consider $G = L_2(q)$, when $q \equiv \pm 3 \pmod 8$ is an odd prime power.

Example 3.5. *Assume that $G = L_2(q)$, where $q = p^k \geq 5$ and $q \equiv \pm 3 \pmod 8$. Then there is a 2-dimensional acyclic fixed point free G-complex X, all of whose isotropy subgroups are solvable. More precisely, X can be constructed to have four orbits of vertices with isotropy subgroups isomorphic to $\mathbb{F}_q \rtimes C_{(q-1)/2}$, D_{q-1}, D_{q+1}, and A_4; four orbits of edges with isotropy subgroups isomorphic to $C_{(q-1)/2}$, C_2^2, C_3, and C_2; and one free orbit of 2-cells.*

Proof. Since $L_2(5) \cong L_2(4)$ has already been dealt with in Example 3.4, we assume for simplicity that $q > 5$. Let $\mathcal{SLV}$ be the separating family of solvable subgroups of G, and let $\mathcal{SLV}_c \subseteq \mathcal{SLV}$ be the subfamily of all critical subgroups in $\mathcal{SLV}$. By Proposition 3.3, the maximal solvable subgroups of G are the groups

$$D_{q-1}, \quad D_{q+1}, \quad A_4, \quad \text{and} \quad B = \mathbb{F}_q \rtimes C_{(q-1)/2},$$

where each occurs with exactly one conjugacy class.

Any subgroup $H \in \mathcal{SLV}$ of order a multiple of p is contained in a unique subgroup conjugate to B (it fixes a unique line in $(\mathbb{F}_q)^2$); and is contained in one of the other maximal subgroups only if $p = 3$ and $H \cong C_3$. If $1 \neq H \in \mathcal{SLV}$ has order prime to p, is not maximal, and is not isomorphic to C_2, then either it is cyclic of order dividing $(q-1)/2$ and contained in one dihedral group and two Borel subgroups (corresponding to the two lines in $(\mathbb{F}_q)^2$ fixed by H), or it is cyclic of order dividing $(q+1)/2$ and is contained in a unique D_{q+1} (its normalizer), or H is dihedral and contained in a unique maximal dihedral subgroup $D_{q\pm 1}$ (the normalizer of its index 2 subgroup). Since each critical subgroup must be an intersection of maximal subgroups in $\mathcal{SLV}$ (Lemma 3.1), we have now shown that the only possible critical subgroups are the maximal subgroups, together with one conjugacy class each of subgroups $C_{(q-1)/2}$, C_3, C_2^2, C_2, and 1.

In the following table, D_+ denotes the maximal dihedral subgroup of order $q \pm 1 \equiv 0 \pmod 4$, and D_- the other (conjugacy class of) maximal dihedral subgroup (note that $D_+ = N(C_2)$). Recall that we are assuming that $q > 5$ (otherwise $D_{q-1} = C_2^2$).

$H \in \mathcal{SLV}_c$	$K \cap N(H) = H$	$i(H)$
$B = \mathbb{F}_q \rtimes C_{(q-1)/2}$	—	1
D_{q-1}	—	1
D_{q+1}	—	1
A_4	—	1
$C_{(q-1)/2}$	B	-1
C_2^2	D_+	-1
C_3	A_4	-1
C_2	D_-	-1
1	—	1

Table 2

As before, the computations of $i_{\mathcal{SLV}}(H)$ for nonmaximal $1 \neq H \subseteq G$ all follow from Lemma 2.6, and the computation of $i_{\mathcal{SLV}}(1)$ then follows from Corollary 2.5.

Lemma 3.2 now applies to show that $(L_2(q), \mathcal{SLV}) \in \mathcal{U}_2$. More precisely, together with Table 2, it shows that a 2-dimensional H-universal $(L_2(q), \mathcal{SLV})$-complex X can be constructed with four orbits of vertices of types G/B, G/D_{q-1}, G/D_{q+1}, and G/A_4; four orbits of 1-cells of types G/C_2^2, $G/C_{(q-1)/2}$, G/C_3, and G/C_2; and one free orbit of 2-cells. Note that $G/C_{(q-1)/2}{\times}D^1$ always connects the orbits G/B and G/D_{q-1}, and $G/C_2{\times}D^1$ always connects the orbits G/D_{q-1} and G/D_{q+1}. The orbit of cells $G/C_2^2{\times}D^1$ connects G/A_4 to G/D_{q-1} or G/D_{q+1}, depending on q modulo 8. And the orbit of cells $G/C_3{\times}D^1$ connects G/A_4 to one of G/B (if $q = 3^k$), or to $G/D_{q\pm1}$ (whichever has order a multiple of 3). □

The third family of groups with 2-dimensional actions consists of the Suzuki groups $\mathrm{Sz}(q)$, for all $q = 2^{2k+1} \geq 8$. In order to specify more precisely subgroups of $\mathrm{Sz}(q)$, we regard it as a subgroup of $GL_4(\mathbb{F}_q)$ as described in [HB3, §XI.3]. The following properties of $\mathrm{Sz}(q)$ and its subgroups will be needed here, as well as in Section 6.

Proposition 3.6. *Fix $q = 2^{2k+1}$, and let $\theta \in \mathrm{Aut}(\mathbb{F}_q)$ be the automorphism $x^\theta = x^{2^{k+1}} = x^{\sqrt{2q}}$ (thus $(x^\theta)^\theta = x^2$). For $a, b \in \mathbb{F}_q$ and $\lambda \in (\mathbb{F}_q)^*$, define elements*

$$S(a,b) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & a^\theta & 1 & 0 \\ a^{2+\theta} + ab + b^\theta & a^{1+\theta} + b & a & 1 \end{pmatrix},$$

and

$$M(\lambda) = \begin{pmatrix} \lambda^{1+2^k} & 0 & 0 & 0 \\ 0 & \lambda^{2^k} & 0 & 0 \\ 0 & 0 & \lambda^{-2^k} & 0 \\ 0 & 0 & 0 & \lambda^{-1-2^k} \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Set $S(q,\theta) = \langle S(a,b) \mid a, b \in \mathbb{F}_q \rangle$, $T = \langle M(\lambda) \mid \lambda \in (\mathbb{F}_q)^* \rangle \cong C_{q-1}$, and

$$B = M(q,\theta) = S(q,\theta) {\rtimes} T \quad \text{and} \quad N = \langle T, \tau \rangle \cong D_{2(q-1)}.$$

Then $\mathrm{Sz}(q) \cong \langle M(q,\theta), \tau\rangle$*, and under this identification the following hold:*

(a) $S(q,\theta)$ *is a Sylow 2-subgroup of* $\mathrm{Sz}(q)$*.*

(b) There are four conjugacy classes of maximal subgroups in $\mathrm{Sz}(q)$ *which are solvable:* (B)*,* (N)*,* (M_+)*, and* (M_-)*, where*

$$M_+ \cong C_{q+\sqrt{2q}+1}\rtimes C_4 \qquad \text{and} \qquad M_- \cong C_{q-\sqrt{2q}+1}\rtimes C_4.$$

These are the only maximal solvable subgroups in $\mathrm{Sz}(q)$*.*

(c) Each nonsolvable subgroup of $\mathrm{Sz}(q)$ *is conjugate to* $\mathrm{Sz}(q_0)$*, for some* $q_0 = 2^{2m+1}$ *where* $(2m+1)|(2k+1)$*.*

(d) Sz(q) is contained in the 4-dimensional symplectic group over $\mathbb{F}_q$*:*

$$Sz(q) \subseteq Sp_4(q) \stackrel{\text{def}}{=} \{g \in GL_4(q) \mid g\tau g^t = \tau\},$$

where g^t *is the transpose of g and* τ *is as above.*

(e) All of the subgroups $B, N, T, S(q,\theta), \mathrm{Sz}(q)$ *are invariant under the automorphisms of* $GL_4(q)$ *induced by automorphisms of the field* $\mathbb{F}_q$*.*

(f) $|\mathrm{Sz}(q)| = q^2(q-1)(q^2+1) = q^2\cdot(q-1)\cdot(q+\sqrt{2q}+1)\cdot(q-\sqrt{2q}+1)$*, where the four factors in the second expression are pairwise relatively prime.*

Proof. See [HB3, §XI.3]. Note in particular the relations

$$S(a,b)\cdot S(c,d) = S(a+c, b+d+a^\theta c) \quad \text{and} \quad M(\lambda)^{-1}S(a,b)M(\lambda) = S(\lambda a, \lambda^{1+\theta}b).$$

The list of maximal subgroups of $\mathrm{Sz}(q)$ (points (b) and (c)) is shown in [Sz1, Theorem 9].

Note that if $q_0 = 2^{2m+1}$, where $(2m+1)|(2k+1)$, then $\mathrm{Sz}(q)\cap GL_4(q_0) = \mathrm{Sz}(q_0)$ (and similarly for the other subgroups). The inclusion $\mathrm{Sz}(q_0) \subseteq \mathrm{Sz}(q)$ follows since $2^k \equiv 2^m \pmod{2^{2m+1}-1}$, and hence $x^{2^k} = x^{2^m}$ for all $x \in \mathbb{F}_{q_0}$. The inclusion $\mathrm{Sz}(q)\cap GL_4(q_0) \subseteq \mathrm{Sz}(q_0)$ then follows from (c). □

We are now ready to construct actions of $\mathrm{Sz}(q)$ on acyclic 2-complexes.

Example 3.7. *Set* $q = 2^{2k+1}$*, for any* $k \geq 1$*. Then there is a 2-dimensional acyclic fixed point free* $\mathrm{Sz}(q)$*-complex* X*, all of whose isotropy subgroups are solvable. More precisely,* X *can be constructed to have four orbits of vertices with isotropy subgroups isomorphic to* $M(q,\theta)$*,* $D_{2(q-1)}$*,* $C_{q+\sqrt{2q}+1}\rtimes C_4$*, and* $C_{q-\sqrt{2q}+1}\rtimes C_4$*; four orbits of edges with isotropy subgroups isomorphic to* C_{q-1}*,* C_4*,* C_4*, and* C_2*; and one free orbit of 2-cells.*

Proof. Set $G = \mathrm{Sz}(q)$. By Proposition 3.6, G contains the following maximal solvable subgroups:

$$M(q,\theta), \quad D_{2(q-1)}, \quad C_{q+\sqrt{2q}+1}\rtimes C_4, \quad \text{and} \quad C_{q-\sqrt{2q}+1}\rtimes C_4;$$

with one conjugacy class for each isomorphism type. If $1 \neq H \in \mathcal{SLV}$ and $(|H|, q^2+1) \neq 1$, then H is contained in a unique maximal subgroup $C_{q^2\pm\sqrt{2q}+1}\rtimes C_4$: the normalizer of its unique maximal odd order subgroup. Likewise, if H is dihedral of order dividing $2(q-1)$ (and $|H| \neq 2$), then H is contained in a unique maximal subgroup $D_{2(q-1)}$; while if $|H|\big|(q-1)$ then H is contained in the same maximal subgroups as its centralizer of order $q-1$. Any subgroup of even order which is not dihedral is contained in at most one maximal subgroup, conjugate to $M(q,\theta)$. (The centralizer of any involution in G is a 2-group by [Sz1, Proposition 1], and each involution in the Sylow subgroup $S(q,\theta)$ is central. So an involution cannot be in two Sylow subgroups.) Thus, any subgroup which is an intersection of two or more maximal subgroups is isomorphic to one of the groups C_{q-1}, C_4, C_2, or 1; and these are the only possible critical subgroups by Lemma 3.1(a). There is just one conjugacy class each of subgroups C_{q-1} or C_2 (note, for example, that all subgroups of order 2 in $S(q,\theta)$ are conjugate in $M(q,\theta)$). By [Sz1, Proposition 18], G contains two conjugacy classes of elements of order 4, and it is easy to check by direct calculations that an element of order 4 in G is not conjugate to its inverse. Hence G contains just one conjugacy class of C_4's.

Now let $\mathcal{SLV}_c$ be the subfamily of critical subgroups in $\mathcal{SLV}$. Consider the following table of values for $i_{\mathcal{SLV}}(H)$ for $H \in \mathcal{SLV}_c$:

$H \in \mathcal{SLV}_c$	$K \cap N(H) = H$	$i(H)$
$B = M(q,\theta)$	—	1
$D_{2(q-1)}$	—	1
$C_{q+\sqrt{2q}+1} \rtimes C_4$	—	1
$C_{q-\sqrt{2q}+1} \rtimes C_4$	—	1
C_{q-1}	$M(q,\theta)$	-1
C_4	$C_{q\pm\sqrt{2q}+1} \rtimes C_4$	-2
C_2	$D_{2(q-1)}$	-1
1	—	1

Table 3

When $H \cong C_{q-1}$, C_4, or C_2, then $i_{\mathcal{SLV}}(H)$ is computed using Lemma 2.6. (Note that C_2 can never be self-normalizing in any group of order a multiple of 4.) The value of $i_{\mathcal{SLV}}(1)$ then follows from Corollary 2.5.

Lemma 3.2 now applies to show that $(\mathrm{Sz}(q), \mathcal{SLV}) \in \mathcal{U}_2$. More precisely, there is a 2-dimensional H-universal $(\mathrm{Sz}(q), \mathcal{SLV})$-complex which has four orbits of vertices and four orbits of edges (with isotropy subgroups as given in Table 3), and one free orbit of 2-cells. □

4. Reduction to simple groups

Throughout this section, G will be a finite group. Recall that a G-complex X is called *essential* if there is no normal subgroup $1 \neq N \lhd G$, with the property that the inclusion $X^N \subseteq X$ is a G-$\mathbb{Z}$-equivalence; i.e., such that $X^{NH} \to X^H$ is a homology equivalence for all $H \subseteq G$. We would like to be able to show directly that all groups which have essential fixed point free actions on acyclic 2-complexes are simple. Instead, in this section, we prove a slightly weaker result (Proposition 4.4), where we show that any group with such an action is an extension of a simple group by outer automorphisms.

The proof of this uses the result in [S1] that the fixed point set of any group acting on a 2-dimensional acyclic complex must be acyclic or empty. Since the proof in [S1] requires the odd order theorem, we give here a different one, which is more elementary.

Theorem 4.1 [S1, Theorem 3.4]**.** *Let X be any 2-dimensional acyclic G-complex (*not *necessarily finite). Then X^G is acyclic or empty, and is acyclic if G is solvable.*

Proof. The first half of the following proof is essentially the same as that in [S1], but is included here for the sake of completeness.

If G is a p-group for some prime p, then X^G is $\mathbb{Z}/p$-acyclic by Smith theory (cf. [Br, Theorem III.7.12]), and homologically 1-dimensional by Lemma 1.6. It follows that X^G is $\mathbb{Z}$-acyclic in this case.

Now assume that G is a minimal group for which there is a counterexample. Then G must be simple and nonabelian — since if $N \lhd G$ were a proper normal subgroup, then X^N would be acyclic, and hence $X^G = (X^N)^{G/N}$ would be acyclic or empty (acyclic if G is solvable) by the minimality of G. Also, X^H is acyclic for all $H \subsetneqq G$, and $X^G = \bigcap_{H \subsetneqq G} X^H$ is homologically 0-dimensional by Lemma 1.6 again. In other words, each connected component of X^G is acyclic, and it remains to show that there is at most one component.

Assume otherwise: let $k \geq 2$ be the number of connected components of X^G. Let $\mathcal{F}$ be the (separating) family of proper subgroups $H \subsetneqq G$. Very roughly, we will show that X "looks like" the join of an H-universal $(G, \mathcal{F})$-complex Y with a set of k points. But for X to be 2-dimensional, Y would have to be 1-dimensional, i.e., a tree; and this is impossible.

To make this precise, let $\mathcal{F}_+$ denote the poset which consists of $\mathcal{F}$, together with k elements (G, i) for $i = 1, \ldots, k$. Extend the ordering on $\mathcal{F}$ by setting $(G, i) \supseteq H$ for all $H \in \mathcal{F}$, and with no inclusion relations between the (G, i). Write $X^G = F_1 \amalg \cdots \amalg F_k$, where the F_i are the connected components. We now apply Lemma 0.1, with the covering of X given by $X_H = X^H$ for $H \in \mathcal{F}$, and $X_{(G,i)} = F_i$. Thus, X_α is acyclic for each $\alpha \in \mathcal{F}_+$. So by Lemma 0.1, for each $H \in \mathcal{F}$, $H_*(X^{>H}) \cong H_*(\mathcal{N}((\mathcal{F}_+)_{>H}))$, and thus $\mathcal{N}((\mathcal{F}_+)_{>H})$ is homologically 1-dimensional (Lemma 1.6). But the poset $(\mathcal{F}_+)_{>H}$ consists of $\mathcal{F}_{>H}$ together with the elements (G, i), and so its nerve is the union of k cones over $\mathcal{N}(\mathcal{F}_{>H})$. This complex contains the suspension of $\mathcal{N}(\mathcal{F}_{>H})$ as a retract (i.e., the case $k = 2$); and hence $\mathcal{N}(\mathcal{F}_{>H})$ is homologically 0-dimensional. Since this holds for all $H \in \mathcal{F}$, Proposition 1.8 now applies to show that there is a finite 1-dimensional universal $(G, \mathcal{F})$-complex Y. But then Y is a tree upon which G acts without fixed points, and this is impossible (cf. [Se, §I.6]). □

The following easy consequence of Theorem 4.1 turns out to be very useful. Its proof involves collapsing out certain subcomplexes of a CW complex to create new fixed points, and get a contradiction to Theorem 4.1. In general, if X is a G-complex and $A \subseteq X$ is a G-invariant subcomplex, then X/A is defined to be the quotient space $X/\sim$, where $x \sim y$ if $x = y$ or $x, y \in A$. This quotient space has an obvious structure as a G-complex: where $(X/A)^{(n)} = X^{(n)}/\sim$, and where X/A has one vertex for the identification point A/A and otherwise one cell for each cell in X not in A (see [LW, Theorem II.5.11], taking $Y = \text{pt}$). The homology groups of X, A, and X/A are linked by exact sequences (coming from the fact that $C_n(X/A)/C_n(\text{pt}) \cong C_n(X)/C_n(A)$). In particular, if A is acyclic, then $H_*(X/A) \cong H_*(X)$.

Corollary 4.2. *Let X be any 2-dimensional acyclic G-complex. Assume that $A, B \subseteq X$ are G-invariant acyclic subcomplexes such that $A \cup B \supseteq X^G$. Then $A \cap B \neq \emptyset$.*

Proof. Assume otherwise: that $A \cap B = \emptyset$. Let Y be the G-complex obtained by identifying the subcomplexes A and B each to a point. Then Y is still acyclic, since A and B are, and Y^G consists of the two identification points. And this contradicts Theorem 4.1, which says that Y^G must be acyclic or empty. □

As immediate consequences of Corollary 4.2 we get:

Lemma 4.3. *Let X be a 2-dimensional acyclic G-complex. Then the following hold.*

(a) [AS, 4.5] *If $H, N \subseteq G$ are such that $H \subseteq N_G(K)$ and X^H and X^K are nonempty, then $X^{HK} \neq \emptyset$.*

(b) If $H \subseteq G$ is such that $X^H = \emptyset$, then $X^{C_G(H)} \neq \emptyset$.

Proof. If $X^G \neq \emptyset$, then (a) and (b) are obvious. So assume $X^G = \emptyset$.

(a) Since H normalizes K, both X^H and X^K are H-invariant acyclic subcomplexes of X. So by Corollary 4.2, if X^H and X^K are nonempty, then $X^H \cap X^K = X^{HK} \neq \emptyset$.

(b) It suffices to prove this when H is minimal among subgroups without fixed points. Fix a pair $M, M' \subseteq H$ of distinct maximal subgroups (H is nonsolvable). Then X^M and $X^{M'}$ are nonempty, but $X^M \cap X^{M'} = X^{\langle M,M'\rangle} = X^H = \emptyset$. Thus X^M and $X^{M'}$ are disjoint $C_G(H)$-invariant acyclic subcomplexes of X, and so $C_G(H)$ must have fixed points by Corollary 4.2. □

As a first consequence of Lemma 4.3, we can now prove:

Theorem B. *Let G be any finite group, and let X be any 2-dimensional acyclic G-complex. Let N be the subgroup generated by all normal subgroups $N' \lhd G$ such that $X^{N'} \neq \emptyset$. Then X^N is acyclic; X is essential if and only if $N = 1$; and if $N \neq 1$ then the action of G/N on X^N is essential.*

Proof. If $X^{N_1} \neq \emptyset$ and $X^{N_2} \neq \emptyset$, where $N_1, N_2 \lhd G$, then $X^{\langle N_1,N_2\rangle} \neq \emptyset$ by Lemma 4.3(a). Thus X^N is nonempty, and is acyclic by Theorem 4.1. The action of G/N on X^N is always essential, since any nontrivial normal subgroup of G/N has empty fixed point set.

Now assume that $N \neq 1$. For all $H \subseteq G$, X^H and X^{NH} are acyclic or empty by Theorem 4.1; and X^{NH} is nonempty if X^H is by Lemma 4.3(a). So the inclusion $X^{NH} \to X^H$ is always an equivalence of integral homology, and hence X is not essential. □

We are now ready to prove:

Proposition 4.4. *If G is a nontrivial finite group for which there exists an essential 2-dimensional acyclic G-complex X, then G is almost simple. More precisely, there is a normal subgroup $L \lhd G$, such that L is simple, such that $X^L = \emptyset$, and such that $C_G(L) = 1$ (i.e., $G \subseteq \text{Aut}(L)$).*

Proof. By Theorem B, $X^N = \emptyset$ for all proper normal subgroups $1 \neq N \lhd G$. In particular, $X^G = \emptyset$.

Fix a minimal normal subgroup $1 \neq L \lhd G$. Then L is nonsolvable, since $X^L \neq \emptyset$. Hence L is a direct product of isomorphic nonabelian simple groups (cf. [Go, Theorem 2.1.5]).

Assume first that L is not simple. By Lemma 4.3(b), $X^H \neq \emptyset$ for some simple factor $H \lhd L$; and $L = \langle gHg^{-1} \,|\, g \in G \rangle$ since it is a minimal normal subgroup. Since $X^{gHg^{-1}} = g(X^H) \neq \emptyset$ for all g, $X^L \neq \emptyset$ by Lemma 4.3(a) (applied to the action of L on X). And this is a contradiction.

Thus, L is simple. Set $H = C_G(L)$. Then $H \lhd G$, and $X^H \neq \emptyset$ by Lemma 4.3(b); and so $H = 1$ (again since the G-action on X is essential). □

Using Proposition 4.4, when determining which finite groups have essential fixed point free actions on 2-dimensional acyclic complexes, it suffices first to determine which simple groups have such actions, and then consider automorphism groups only of those simple groups which do have them.

5. Some conditions for nonexistence of 2-dimensional actions

Again, throughout this section, G is a finite group. We recall two definitions introduced in Section 3. If $\mathcal{F}$ is a separating family for G, then $\mathcal{F}_c$ denotes the subfamily of critical subgroups for $\mathcal{F}$: the set of all $H \in \mathcal{F}$ such that $\mathcal{N}(\mathcal{F}_{>H}) \not\simeq *$. And $\mathcal{U}_2$ denotes the class of pairs $(G, \mathcal{F})$ (where $\mathcal{F}$ is a separating family for G) for which there exists a 2-dimensional H-universal $(G, \mathcal{F})$-complex. We have already constructed some examples of pairs $(G, \mathcal{F})$ which do lie in $\mathcal{U}_2$, and next want to show that they are the only ones. In this section, we develop some general techniques for doing this.

For any G-complex X, and any $n > 1$, it will be convenient to write $X^{[n]}$ to denote the union of fixed point sets of subgroups of order a multiple of n; or equivalently the set of all $x \in X$ for which $n \big| |G_x|$. Also, for any family $\mathcal{F}$ of subgroups of G, we write $\mathcal{F}_{[n]}$ to denote the subfamily of those subgroups in $\mathcal{F}$ of order a multiple of n. We will see that if $(G, \mathcal{F}) \in \mathcal{U}_2$, then not only is $\mathcal{N}(\mathcal{F}_{[n]})$ homologically 1-dimensional for all n, but its orbit space $\mathcal{N}(\mathcal{F}_{[n]})/G$ is homologically 0-dimensional (i.e., its connected components are acyclic).

In Section 5a, conditions are established which allow us to directly detect elements in $H_2(\mathcal{N}(\mathcal{F}_{[n]}))$, for appropriate n, via Euler characteristic arguments. The properties of $\mathcal{N}(\mathcal{F}_{[n]})/G$ are shown in Section 5b, and then another set of criteria are found which detect elements in $H_1(\mathcal{N}(\mathcal{F}_{[n]})/G)$. Afterwards, conditions on G and $\mathcal{F}$ are set up in Section 5c which imply that for any 2-dimensional H-universal $(G, \mathcal{F})$-complex X, the singular set X_s is itself acyclic (and hence H-universal); and then Section 5d deals with the problem of showing that this is impossible.

5a. Detecting 2-cycles in nerves of posets of subgroups

Our main tool here for directly detecting elements in the second homology of nerves of posets of subgroups will be certain "coset complexes". We adopt the following notation:

Definition 5.1. *Fix any group G, and any triple K_1, K_2, K_3 of subgroups of G. Define*

$$\ll K_1, K_2, K_3 \gg \, = \, \ll K_1, K_2, K_3 \gg_G$$

to be the G-simplicial complex with vertex set $(G/K_1) \amalg (G/K_2) \amalg (G/K_3)$ (where G acts by left translation), and with a 1- or 2-simplex for every pair or triple of cosets with nonempty intersection.

Thus, each edge in $\ll K_1, K_2, K_3 \gg$ has the form $[aK_i, aK_j]$ for some $a \in G$ and some $1 \le i < j \le 3$, and each 2-simplex has the form $[aK_1, aK_2, aK_3]$ for some $a \in G$. In many cases, one can show that $H_2(\ll K_1, K_2, K_3 \gg) \neq 0$ via an easy counting argument:

Lemma 5.2. *Fix any group G, and any sequence K_1, K_2, K_3 of subgroups of G. Set $K_{ij} = K_i \cap K_j$, $K = K_1 \cap K_2 \cap K_3$, and $G' = \langle K_1, K_2, K_3 \rangle$. Assume that*

$$\frac{1}{[K_{12}{:}K]} + \frac{1}{[K_{13}{:}K]} + \frac{1}{[K_{23}{:}K]} \le 1; \tag{1}$$

or (more generally) that

$$\sum_{i<j} \frac{1}{[K_{ij}{:}K]} < 1 + \sum_{i=1}^{3} \frac{1}{[K_i{:}K]} - \frac{1}{[G'{:}K]}. \tag{2}$$

Then $H_2(\ll K_1, K_2, K_3 \gg_G) \neq 0$.

Proof. Set $X = \ll K_1, K_2, K_3 \gg_G$ for short. By construction, X is the union of its closed 2-simplices, each of which is of the form $a\Delta \overset{\text{def}}{=} [aK_1, aK_2, aK_3]$ for some $a \in G$. Two 2-simplices $a\Delta$ and $b\Delta$ intersect if and only if $aK_i = bK_i$ for some i. Upon making this relation transitive, we see that $a\Delta$ and $b\Delta$ are in the same connected component of X if and only if a and b are in the same left coset of $G' = \langle K_1, K_2, K_3 \rangle$; and so there are exactly $[G{:}G']$ connected components.

By definition, X has three orbits of vertices of type G/K_i, three orbits of edges of type G/K_{ij}, and one orbit of 2-simplices of type G/K. Hence

$$\begin{aligned} \chi(X) &= [G{:}K] - \sum_{i<j} [G{:}K_{ij}] + \sum_{i=1}^{3} [G{:}K_i] \\ &= [G{:}K] \cdot \left(1 - \sum_{i<j} \frac{1}{[K_{ij}{:}K]} + \sum_{i=1}^{3} \frac{1}{[K_i{:}K]} \right) > [G{:}G'] = \operatorname{rk}(H_0(X)); \end{aligned}$$

where the inequality follows from (1) or (2). And this implies that $H_2(X) \neq 0$. □

The following proposition is a first application of Lemma 5.2. Recall that $\mathcal{F}_c$ denotes the subfamily of critical subgroups in a separating family $\mathcal{F}$.

Proposition 5.3. *Fix a finite group G and a separating family $\mathcal{F}$ for G. Fix subgroups $K_0 \supsetneqq K_1 \supsetneqq K_2$ in $\mathcal{F}$, and set $N_i = N_G(K_i)$, $N_{ij} = N_i \cap N_j$, and $N = N_0 \cap N_1 \cap N_2$. Set $\mathcal{F}_0 = \mathcal{F}_c \cup (K_1) \cup (K_2)$. Assume that the following hold:*

(a) $\sum_{i<j} \frac{1}{[N_{ij}:N]} < 1 + \sum_{i=0}^{2} \frac{1}{[N_i:N]} - \frac{1}{[G':N]}$*, where* $G' = \langle N_0, N_1, N_2\rangle$*.*

(b) K_0 *is maximal in* $\mathcal{F}$*.*

(c) If $H, H' \in \mathcal{F}_0$ *are such that* $K_2 \subsetneqq H \subseteq K_1$ *and* $H \subsetneqq H' \subsetneqq K_0$*, then* $H' \subseteq K_1$*.*

Then $H_2(\mathcal{F}_{\geq(K_2)}) \neq 0$*; and so* $(G, \mathcal{F}) \notin \mathcal{U}_2$*.*

Proof. Set $\mathcal{H} = (\mathcal{F}_0)_{\geq(K_2)}$. Consider the 2-simplex $\sigma = \{K_2, K_1, K_0\}$ in $\mathcal{N}(\mathcal{H})$, and let $X \subseteq \mathcal{N}(\mathcal{H})$ be the subcomplex generated by the 2-simplices $g\sigma$ for all $g \in G$ (where G acts by conjugation). Then $X \cong \ll N_0, N_1, N_2 \gg_G$; and $H_2(X) \neq 0$ by (a) and Lemma 5.2.

Let z be any 2-cycle in X such that $0 \neq [z] \in H_2(X)$. After conjugating, if necessary, we can assume that the coefficient in z of σ is nonzero. Set $Q = \mathcal{H}^{<K_0}_{>K_2}$, and let Q' be the set of those $H \in Q$ such that the coefficient in z of $\{K_2, H, K_0\}$ is nonzero. By construction, every element of Q' is G-conjugate (in fact, N_{02}-conjugate) to K_1; and by condition (c), every element of Q in the same $\mathcal{N}(Q)$-connected component as K_1 is contained in K_1. Lemma 0.5 now implies that $0 \neq [z] \in H_2(\mathcal{N}(\mathcal{H}))$; and so $(G, \mathcal{F}) \notin \mathcal{U}_2$ by Proposition 1.9. □

Two n-tuples of subgroups $(H_1, \ldots, H_n)$ and $(H'_1, \ldots, H'_n)$ in G will be called *G-conjugate* if there is some $g \in G$ such that $H'_i = gH_ig^{-1}$ for all i. The normalizer $N_G(H_1, \ldots, H_n)$ of such an n-tuple is just the intersection of the normalizers $N_G(H_i)$.

The next proposition is a somewhat more complicated application of Lemma 5.2.

Proposition 5.4. *Fix a separating family* $\mathcal{F}$ *of* G*. Let* $K_1, K_2, K_3 \in \mathcal{F}$ *be three subgroups such that neither* K_2 *nor* K_3 *is conjugate to* K_1*. Set* $K_{ij} = K_i \cap K_j$ *and* $K = K_1 \cap K_2 \cap K_3$*. Let* $\mathcal{F}_0 \subseteq \mathcal{F}$ *denote the subfamily consisting of* $\mathcal{F}_c$*, together with all subgroups conjugate to any of the* K_i*,* K_{ij}*, or* K*. Assume the following conditions hold:*

(a1) $\frac{1}{[K_{12}:K]} + \frac{1}{[K_{13}:K]} + \frac{1}{[K_{23}:K]} \leq 1$*; or more generally*

(a2) $\frac{1}{[K_{12}:K]} + \frac{1}{[K_{13}:K]} + \frac{1}{[K_{23}:K]} < 1 + \frac{1}{[K_1:K]} + \frac{1}{[K_2:K]} + \frac{1}{[K_3:K]} - \frac{1}{[G':K]}$*, where* $G' = \langle K_1, K_2, K_3\rangle$*.*

(b) K_1 *is maximal in* $\mathcal{F}$*.*

(c) There is no $H \in \mathcal{F}_0$ *such that* $K \subsetneqq H \subsetneqq K_{12}$ *or* $K_{12} \subsetneqq H \subsetneqq K_1$*.*

(d) $N_G(K_1, K_{12}, K) = K$*.*

(e) The triples (K_1, K_{12}, K) *and* (K_1, K_{13}, K) *are not G-conjugate.*

Then $H_2(\mathcal{F}_{\geq(K)}) \neq 0$*; and so* $(G, \mathcal{F}) \notin \mathcal{U}_2$*.*

Proof. Consider the complex $X = \ll K_1, K_2, K_3 \gg$ of Definition 5.1, and let X^* denote its barycentric subdivision. To distinguish between simplices of X^* and of $\mathcal{N}(\mathcal{F})$, we put parentheses $(-)$ around the former and curly brackets $\{-\}$ around the latter. The vertices in X^* will be denoted (gK_i) (the vertices in X), (gK_{ij}) (the midpoint of the edge (gK_i, gK_j)), and (gK) (the barycenter of the 2-simplex (gK_1, gK_2, gK_3)).

We have $H_2(X)\neq 0$ by (a1) or (a2), together with Lemma 5.2. Fix a 2-cycle z in X such that $0\neq [z]\in H_2(X)$. We can assume that the coefficient in z of the simplex (K_1,K_2,K_3) is nonzero (otherwise compose with the action of some appropriate element of G). Let z^* be the corresponding 2-cycle in the barycentric subdivision X^* of X.

Let $f\colon X^*\to\mathcal{N}((\mathcal{F}_0)_{\ge(K)})$ be the G-equivariant simplicial map which sends each vertex in X^* to its isotropy subgroup. Thus $f(gK_i)=\{gK_ig^{-1}\}$, $f(gK_{ij})=\{gK_{ij}g^{-1}\}$, and $f(gK)=\{gKg^{-1}\}$. By conditions (d) and (e), and since neither K_2 nor K_3 is conjugate to K_1, the only simplex in X^* which is sent to $\{K_1,K_{12},K\}$ is (K_1,K_{12},K), and this simplex has nonzero coefficient in the 2-cycle z^*. Hence $\{K_1,K_{12},K\}$ has nonzero coefficient in the 2-cycle $f(z^*)$. By (b) and (c), $\{K_1,K_{12},K\}$ is maximal in $\mathcal{N}((\mathcal{F}_0)_{\ge(K)})$ (not in the boundary of any 3-simplex), and hence $[f(z^*)]\neq 0$ in $H_2(\mathcal{N}((\mathcal{F}_0)_{\ge(K)}))=H_2(\mathcal{N}(\mathcal{F}_{\ge(K)}))$ (Lemma 1.4). And thus $(G,\mathcal{F})\notin\mathcal{U}_2$ by Proposition 1.9(a$\Rightarrow$d). ☐

5b. Detecting nonzero elements in $H_1(X^{[n]}/G)$

Recall that for any n and $\mathcal{F}$, $\mathcal{F}_{[n]}\subseteq\mathcal{F}$ denotes the subfamily of all subgroups in $\mathcal{F}$ of order a multiple of n. We first show, for $(G,\mathcal{F})\in\mathcal{U}_2$, that the connected components of the orbit space of $\mathcal{N}(\mathcal{F}_{[n]})$ are all acyclic, and then set up some conditions which detect elements in their first homology groups. The starting point for all of this is the following result, a consequence of Smith theory.

Proposition 5.5. *If X is any finite dimensional acyclic G-complex, then X/G is also acyclic. If $f\colon X\to Y$ is any equivariant map between finite dimensional G-complexes which induces an isomophism $H_*(X;\mathbb{Z})\cong H_*(Y;\mathbb{Z})$, then f/G induces an isomorphism $H_*(X/G;\mathbb{Z})\cong H_*(Y/G;\mathbb{Z})$.*

Proof. The first statement is shown, for example, in [Br, Theorem III.7.12]. The second statement follows from the first, since f induces an isomorphism in integral homology if and only if its mapping cone C_f is acyclic, and similarly for f/G. (Note that $C_{f/G}\cong(C_f)/G$.) ☐

The following result is similar to one used in [O3], but formulated here for acyclic rather than $\mathbb{F}_p$-acyclic spaces.

Proposition 5.6. *Fix a prime p, and let X be a finite dimensional acyclic G-complex with the property that X^P is acyclic for all p-subgroups $P\subseteq G$. Then for any (nonempty) family $\mathcal{P}$ of p-subgroups of G, $X^{\mathcal{P}}/G$ is acyclic.*

Proof. We assume that any p-group which contains an element of $\mathcal{P}$ also lies in $\mathcal{P}$ (if not just add these groups to the family). For the purposes of this proof, we define, for any p-subgroup $P\subseteq G$,

$$X^P_s=\bigcup_{\substack{Q\supsetneqq P\\ Q\text{ a }p\text{-subgr.}}}X^Q\qquad\text{and}\qquad X^{(P)}_s=G{\cdot}X^P_s=\bigcup_{\substack{Q\supsetneqq P\\ Q\text{ a }p\text{-subgr.}}}X^{(Q)}.$$

We first claim that for any $P\in\mathcal{P}$, the inclusion of X^P into $X^{(P)}$ induces an isomorphism of homology groups

$$H_*(X^P/N(P),X^P_s/N(P))\xrightarrow{\ \cong\ }H_*(X^{(P)}/G,X^{(P)}_s/G).\tag{1}$$

In fact, the inclusion induces an isomorphism

$$C_*(\iota) \,:\, C_*(X^P/N(P), X_s^P/N(P)) \xrightarrow{\;\cong\;} C_*(X^{(P)}/G, X_s^{(P)}/G)$$

between the cellular chain complexes of these pairs. The surjectivity of $C_*(\iota)$ is clear, since any open cell $\sigma \subseteq X^{(P)} \smallsetminus X_s^{(P)}$ lies in the G-orbit of some $\sigma \subseteq X^P \smallsetminus X_s^P$. To see its injectivity, fix open cells $\sigma, a(\sigma) \subseteq X^P \smallsetminus X_s^P$ in the same G-orbit ($a \in G$). Then P is a Sylow p-subgroup of the isotropy subgroups G_σ and $G_{a(\sigma)} = aG_\sigma a^{-1}$, so P and $a^{-1}Pa$ are both Sylow p-subgroups of G_σ, and hence $a^{-1}Pa = gPg^{-1}$ for some $g \in G_\sigma$. It follows that $ag \in N_G(P)$, and thus that σ and $a(\sigma) = ag(\sigma)$ lie in the same $N(P)$-orbit. This proves the injectivity of $C_*(\iota)$; and finishes the proof that (1) is an isomorphism.

Now set

$$\alpha = \max\{a \geq 0 \,|\, p^a | [G{:}P], \text{ some } P \in \mathcal{P}\}.$$

The proposition will be proven by induction on α. If $\alpha = 0$, then for any Sylow p-subgroup P of G, $X^{(P)} = X^{\mathcal{P}}$ and $X_s^{(P)} = X_s^P = \emptyset$; and so

$$H_*(X^{\mathcal{P}}/G) \cong H_*(X^{(P)}/G) \cong H_*(X^P/N(P))$$

by (1). Also, $X^P/N(P)$ is acyclic by Proposition 5.5 (since X^P is acyclic by assumption); and thus $X^{\mathcal{P}}/G$ is acyclic.

Now assume that $\alpha > 0$. Let $\mathcal{P}_0 \subseteq \mathcal{P}$ be the subfamily of all P such that $p^\alpha \nmid [G{:}P]$. Then $X^{\mathcal{P}_0}/G$ is acyclic by the induction hypothesis, and it remains to show that $H_*(X^{\mathcal{P}}/G, X^{\mathcal{P}_0}/G) = 0$. Let $P_1, \ldots, P_k$ be conjugacy class representatives for the subgroups in $\mathcal{P} \smallsetminus \mathcal{P}_0$, and set $\mathcal{P}_i = \mathcal{P}_0 \cup (P_i)$. Then by excision,

$$H_*(X^{\mathcal{P}}/G, X^{\mathcal{P}_0}/G) \cong \bigoplus_{i=1}^k H_*(X^{\mathcal{P}_i}/G, X^{\mathcal{P}_0}/G) \cong \bigoplus_{i=1}^k H_*(X^{(P_i)}/G, X_s^{(P_i)}/G).$$

It thus remains to show that $H_*(X^{(P)}/G, X_s^{(P)}/G) = 0$ for each $P = P_i$. By (1), this means showing that $H_*(X^P/N(P), X_s^P/N(P)) = 0$. But $X_s^P/N(P)$ is acyclic by the induction hypothesis again, and $X^P/N(P)$ is acyclic by Proposition 5.5 (since X^P is acyclic by assumption). □

Proposition 5.6 will be applied in particular to get information about the spaces $X^{[n]}/G$ and $\mathcal{N}(\mathcal{F}_{[n]})/G$.

Corollary 5.7. *Let $\mathcal{F}$ be any separating family for G, and let $\mathcal{F}_0 \subseteq \mathcal{F}$ be a subfamily which contains $\mathcal{F}_c$. Let X be a finite dimensional H-universal $(G, \mathcal{F})$-complex. Then for any subfamily $\mathcal{H}$ of $\mathcal{F}$,*

$$H_*(X^{\mathcal{H}}/G) \cong H_*(\mathcal{F}_{\geq \mathcal{H}}/G) \cong H_*((\mathcal{F}_0)_{\geq \mathcal{H}}/G).$$

In particular, $H_*(X_s/G) \cong H_*(\mathcal{N}(\mathcal{F}_{>1})/G) \cong H_*(\mathcal{N}((\mathcal{F}_0)_{>1})/G)$; and $H_*(X^{[n]}/G) \cong H_*(\mathcal{N}(\mathcal{F}_{[n]})/G) \cong H_*(\mathcal{N}((\mathcal{F}_0)_{[n]})/G)$ for all $n > 1$. And for any prime power q, $\mathcal{N}(\mathcal{F}_{[q]})/G$ is acyclic.

Proof. By Proposition 1.3, for any $\mathcal{H} \subseteq \mathcal{F}$, there is a G-map $f\colon X \to \mathcal{N}(\mathcal{F})$ which restricts to a homology equivalence $f_{\geq \mathcal{H}}\colon X^{\mathcal{H}} \to \mathcal{N}(\mathcal{F}_{\geq \mathcal{H}})$. By Lemma 1.4, the inclusion $\mathcal{N}((\mathcal{F}_0)_{\geq \mathcal{H}}) \subseteq \mathcal{N}(\mathcal{F}_{\geq \mathcal{H}})$ is a homotopy equivalence. So by Proposition 5.5, these maps induce homology equivalences in the orbit spaces. The isomorphisms involving $H_*(X_s/G)$

and $H_*(X^{[n]}/G)$ now follow from the case where $\mathcal{H} = \mathcal{F}_{>1}$ or $\mathcal{H} = \mathcal{F}_{[n]}$. In particular, $X^{[q]}/G$ is acyclic by Proposition 5.6 (and since X exists by Proposition 1.8). □

The importance of the families $\mathcal{F}_{[n]}$ comes from the following lemma. Note that for a family $\mathcal{F}$ of subgroups of G and a group A of automorphisms of G, the orbit space $\mathcal{N}(\mathcal{F})/A$ need not be a simplicial complex: there could, for example, be two edges of $\mathcal{N}(\mathcal{F})$ not in the same A-orbit, but whose endpoints are identified pairwise. But $\mathcal{N}(\mathcal{F})/A$ does always have the structure of a CW complex in a natural way (cf. Lemma A.5).

Lemma 5.8. *Let $\mathcal{F}$ be a separating family of subgroups of G such that $(G,\mathcal{F}) \in \mathcal{U}_2$, and let $\mathcal{F}_0 \subseteq \mathcal{F}$ be any subfamily which contains $\mathcal{F}_c$. Then for all $n > 1$, $\mathcal{N}((\mathcal{F}_0)_{[n]})/G$ is homologically 0-dimensional. More generally, if $\overline{G} \subseteq \mathrm{Aut}(G)$ is any subgroup which contains $\mathrm{Inn}(G)$, and such that $\mathcal{F}$ and $\mathcal{F}_0$ are $\overline{G}$-invariant, then $\mathcal{N}((\mathcal{F}_0)_{[n]})/\overline{G}$ is homologically 0-dimensional for all $n > 1$.*

Proof. Let X be any 2-dimensional H-universal $(G,\mathcal{F}_0)$-complex (X exists by Proposition 1.8). Then X/G is $\mathbb{Z}$-acyclic by Proposition 5.5. If $n = p^k$ where p is prime, then $X^{[n]}/G$ is acyclic by Proposition 5.6. If n is not a prime power, write $n = q_1 \cdots q_k$, where the q_i are prime powers for distinct primes. Then $X^{[n]}/G = \bigcap_{i=1}^{k} X^{[q_i]}/G$ is an intersection of acyclic subspaces of X/G; and hence is homologically 0-dimensional by Lemma 1.6 again.

Thus, $\mathcal{N}((\mathcal{F}_0)_{[n]})/G$ is also homologically 0-dimensional by Corollary 5.7, and its connected components are all acyclic. The last statement now follows from Proposition 5.5, since $\mathcal{N}((\mathcal{F}_0)_{[n]})/\overline{G}$ is the orbit space of the $\overline{G}/\mathrm{Inn}(G)$-action on $\mathcal{N}((\mathcal{F}_0)_{[n]})/G$. □

We end this subsection with an application of Lemma 5.8: one situation in which we can show that $\mathcal{N}(\mathcal{F}_{[n]})/\overline{G}$ is not homologically 0-dimensional, and thus that $(G,\mathcal{F}) \notin \mathcal{U}_2$. The argument is based on the following observation: given a 1-cycle ϕ in a simplicial complex K which involves at least one “free” edge (an edge with no higher dimensional simplices attached), then $0 \neq [\phi] \in H_1(K)$. Here, “simplicial complex” is used in the more general sense, where there can be two or more n-simplices ($n \geq 1$) having the same set of vertices.

When working with the orbit space $\mathcal{N}(\mathcal{F})/G$, we will let $[H]$ denote the vertex corresponding to a conjugacy class $(H) \subseteq \mathcal{F}$. More generally, for any chain $H_0 \subsetneqq H_1 \subsetneqq \cdots \subsetneqq H_n$ of subgroups in $\mathcal{F}$, $[H_0, H_1, \ldots, H_n]$ will denote the corresponding n-simplex in $\mathcal{N}(\mathcal{F})/G$.

Proposition 5.9. *Let $\mathcal{F}$ be a separating family of subgroups of G. Assume that there is a maximal subgroup $M \in \mathcal{F}$, and a pair of maximal subgroups $K, K' \subseteq M$ which are not conjugate in M, but are conjugate in G. Then $(G,\mathcal{F}) \notin \mathcal{U}_2$. More generally, the same conclusion holds if there is a subgroup $\overline{G} \subseteq \mathrm{Aut}(G)$ containing $\mathrm{Inn}(G)$, such that $\mathcal{F}$ is $\overline{G}$-invariant, and such that K and K' are in the same orbit of $\overline{G}$, but not in the same orbit of the action of the stabilizer of M.*

Proof. Set $n = |K|$. Then $\mathcal{F}_{[n]}/\overline{G}$ contains (at least) two edges which connect the vertices $[K]$ and $[M]$. The maximality properties guarantee that the resulting loop is nonzero in $H_1(\mathcal{F}_{[n]}/\overline{G})$. So $(G,\mathcal{F}) \notin \mathcal{U}_2$ by Lemma 5.8. □

5c. Acyclicity of $\mathcal{N}(\mathcal{F}_{>1})$

We now find conditions for showing that $\mathcal{N}(\mathcal{F}_{>1})$ is acyclic, under the assumption that $(G,\mathcal{F}) \in \mathcal{U}_2$. This can then be combined with results in Section 5d to obtain contradictions. We first note the following equivalent conditions on $\mathcal{F}$.

Lemma 5.10. *Fix a separating family $\mathcal{F}$ of subgroups of G, and assume that $(G,\mathcal{F}) \in \mathcal{U}_2$. Then the following are equivalent:*

(a) $\mathcal{N}(\mathcal{F}_{>1})/G$ is connected and $H_1(\mathcal{N}(\mathcal{F}_{>1})/G) = 0$.

(b) $\mathcal{N}(\mathcal{F}_{>1})$ is acyclic.

(c) $\mathcal{N}(\mathcal{F}_{>1})/G$ is acyclic.

Proof. For any 2-dimensional H-universal $(G,\mathcal{F})$-complex X, $H_*(X_s) \cong H_*(\mathcal{N}(\mathcal{F}_{>1}))$ by Proposition 1.3, and $H_*(X_s/G) \cong H_*(\mathcal{N}(\mathcal{F}_{>1})/G)$ by Corollary 5.7. So it suffices to show the equivalence of the above three conditions after replacing $\mathcal{N}(\mathcal{F}_{>1})$ by X_s.

Since X/G is acyclic (Proposition 5.5), X_s and X_s/G are homologically 1-dimensional by Lemma 1.6. Thus, (a) is equivalent to (c). Also, (b) implies (c) by Proposition 5.5 again; and it remains to show that (c) implies (b).

If X_s/G is acyclic, then in particular it has Euler characteristic one. Hence by Lemma 2.2,

$$1-\chi(X_s) = \chi(X)-\chi(X_s) = |G|\cdot\big(\chi(X/G)-\chi(X_s/G)\big) = |G|(1-1) = 0;$$

and so $\chi(X_s) = 1$. Since G acts transitively on the connected components of X_s (X_s/G being connected), all components of X_s have the same Euler characteristic, and so X_s must be connected. And since X_s is also homologically 1-dimensional, this shows that X_s is acyclic. □

The next proposition provides a tool for showing that condition (a) in Lemma 5.10 holds.

Proposition 5.11. *Assume G has even order, and let $\mathcal{F}$ be a separating family for G. Assume, for each member $M \in \mathcal{F}_{\max}$ of even order and each element $x \in M$ of odd prime order, that either*

(1a) $|N_M(\langle x\rangle)|$ is even; or

(1b) there is an element $y \in M$ of odd prime order such that $|N_G(\langle x\rangle)\cap N_G(\langle y\rangle)|$ and $|N_M(\langle y\rangle)|$ are both even.

Let $(M_1),\dots,(M_k)$ be the conjugacy classes of odd order subgroups in $\mathcal{F}_{\max}$. For $1 \le i \le k$, let $\mathcal{F}'_i$ be the set of all subgroups of M_i which are contained in members of $\mathcal{F}_{\max}$ of even order or in subgroups conjugate to M_j for $j < i$; and assume that

(2) the image of $\mathcal{N}((\mathcal{F}'_i)_{>1})$ in $\mathcal{N}(\mathcal{F}_{>1})/G$ is connected and nonempty for each i.

Then $\mathcal{N}(\mathcal{F}_{>1})/G$ is connected and $H_1(\mathcal{N}(\mathcal{F}_{>1})/G) = 0$.

Proof. For any $x \in H \subseteq G$, we write $N_H(x) \overset{\text{def}}{=} N_H(\langle x\rangle)$, for short. For each $i = 0,\dots,k$, let $\mathcal{F}_i$ be the family of all subgroups in $\mathcal{F}$ contained in even order members of $\mathcal{F}_{\max}$, or in subgroups conjugate to M_j for $j \le i$; and set $X_i = \mathcal{N}((\mathcal{F}_i)_{>1})/G$ and $X = X_k$. In particular, $\mathcal{F}_k = \mathcal{F}$, and $\mathcal{F}_0$ is the set of all subgroups in $\mathcal{F}$ which are contained in members of $\mathcal{F}_{\max}$ of even order ($k = 0$ if all members of $\mathcal{F}_{\max}$ have even order). Set $Y = \mathcal{N}(\mathcal{F}_{[2]})/G \subseteq X_0$. By Corollary 5.7, Y is connected and $H_1(Y) = 0$. Then X_0 is connected, since each vertex of X_0 is joined by an edge to a vertex of Y. And for each $i \geq 1$, each vertex of X_i not in X_{i-1} is joined to $[M_i]$, which in turn is connected to X_{i-1} via a vertex in the nonempty set $\mathcal{F}'_i$. This shows that the X_i are all connected. In particular, X is connected, and it remains to show that $H_1(X) = 0$.

We first set up some notation for elements of $H_1(X)$. The homology class of a loop will be denoted $[H_0, H_1, \ldots, H_n]$, where $(H_0) = (H_n)$, and each H_i contains or is contained in H_{i+1}. Note that by specifying subgroups rather than just conjugacy classes, we eliminate all ambiguity as to which edge between two vertices is meant (recall that there can be more than one edge connecting a pair of vertices of X). Finally, to simplify the notation, we will sometimes replace a cyclic group $H_i = \langle x_i \rangle$ by x_i in this notation.

Step 1 We first show that $H_1(X_0)$ maps trivially to $H_1(X)$. Whenever $[H_0, H_1, \ldots, H_n]$ is a path in X with endpoints in Y, we write $[H_0, H_1, \ldots, H_n]_Y \in H_1(X)$ to denote the homology class of the 1-cycle $[H_0, \ldots, H_n] - \phi$ for any path ϕ from $[H_0]$ to $[H_n]$ in Y. This is well defined since Y is connected and $H_1(Y) = 0$.

Fix a loop in X_0; we can assume that it alternates "peaks" and "valleys" (vertices corresponding to larger or smaller subgroups); and furthermore that each peak is maximal in $\mathcal{F}$ (hence of even order) and each valley is minimal (i.e., of prime order). The loop thus splits into a sum of elements $[M, x, M']_Y$, where M and M' are maximal of even order, and where $|x|$ is prime. If $|x| = 2$, then $[M, x, M']_Y \in \mathrm{Im}(H_1(Y)) = 0$; so we can assume that x has odd prime order.

In either of cases (1a) or (1b) above, $N_G(x)$ has even order. Choose a maximal subgroup $M_x \in \mathcal{F}_{[2]}$ which contains the extension of $\langle x \rangle$ by a Sylow 2-subgroup of $N_G(x)/\langle x \rangle$ (this extension is solvable and hence in $\mathcal{F}_{[2]}$). Then $[M, x, M']_Y = [M, x, M_x]_Y + [M_x, x, M']_Y$, and we are reduced to showing that $[M, x, M_x]_Y = 0$ in $H_1(X)$.

If $|N_M(x)|$ is even, let $H \subseteq M$ be any subgroup which contains $\langle x \rangle$ with index 2. Then H is conjugate in $N_G(x)$ to some $H' \subseteq M_x$ (by choice of M_x); and so $[M, x, M_x]_Y = [M, H, x, H', M_x]_Y = [M, H]_Y + [H', M_x]_Y$ (the last equality holds because $[H, x] = [H', x]$). But these edges lie in $Y = \mathcal{N}(\mathcal{F}_{[2]})/G$, and so $[M, x, M_x]_Y = 0$. Thus, $[M, x, M_x]_Y = 0$ whenever $x \in M$ satisfies condition (1a).

Now assume that $x \in M$ satisfies condition (1b), and fix $y \in M$ as in (1b). Fix subgroups $M_y, M_{xy} \in \mathcal{F}_{\max}$ of even order, such that M_y contains the extension of $\langle y \rangle$ by a Sylow 2-subgroup of $N_G(y)/\langle y \rangle$, and M_{xy} contains the extension of $\langle x, y \rangle$ by a Sylow 2-subgroup of $N_G(x) \cap N_G(y)$ (this last extension must lie in $\mathcal{F}$ since $\langle x, y \rangle \subseteq M \in \mathcal{F}$ and $\mathcal{F}$ is separating). Consider the following diagram:

$$\begin{array}{ccccccccc} & & & & M & & & & \\ & & & & \uparrow & & & & \\ M_x & \longleftarrow & \langle x \rangle & \longrightarrow & \langle x, y \rangle & \longleftarrow & \langle y \rangle & \longrightarrow & M_y \\ & & & & \downarrow & & & & \\ & & & & M_{xy} & & & & \end{array}$$

By construction, condition (1a) is satisfied by each of the pairs $x \in M_{xy}$, $y \in M_{xy}$ and $y \in M$, and so

$$[M, \langle x,y\rangle, y, M_y]_Y = 0 = [M_{xy}, \langle x,y\rangle, y, M_y]_Y = [M_{xy}, \langle x,y\rangle, x, M_x]_Y.$$

And hence $[M, x, M_x]_Y = [M, \langle x,y\rangle, x, M_x]_Y = 0$.

Step 2 We now prove inductively, for $i \geq 1$, that $H_1(X_i)$ has finite image in $H_1(X)$ if $H_1(X_{i-1})$ does. Fix a loop in X_i. We can again assume that it alternates "peaks" and "valleys"; and that each peak is either equal $[M_i]$ or lies in X_{i-1}. If any of the valleys is a vertex $[H] \notin X_{i-1}$, then it must be connected on both sides to $[M_i]$ (but possibly by different edges). This forms a loop (two edges each connecting $[H]$ to $[M_i]$) whose homology class lies in the image of $H_1(\mathcal{F}_{[p]})/G$ for any prime $p\big||H|$, and this group vanishes by Corollary 5.7. We are thus reduced to looking at 1-cycles of the form $z = \phi - [H, M_i, H']$, where $H, H' \in \mathcal{F}'_i$ and ϕ is a path in X_{i-1} connecting $[H]$ and $[H']$. And since the image of $\mathcal{N}((\mathcal{F}'_i)_{>1})$ in X is connected by (2), the path $[H, M_i, H']$ is homotopic to a path in X_{i-1} (and hence $[z]$ is in the image of $H_1(X_{i-1})$), modulo loops of the form $[K, M_i, K']$ for G-conjugate subgroups $K, K' \in \mathcal{F}'_i$. □

The following proposition shows that in certain cases, one can replace $\mathcal{F}$ by a different separating family without changing the homology of $\mathcal{N}(\mathcal{F}_{>1})$ or of $\mathcal{N}(\mathcal{F}_{>1})/G$. Note, in its statement and proof, that any finite group G contains a (unique) maximal normal perfect subgroup $L \lhd G$: the last term in the derived series of G. This normal subgroup is also characterized by the properties that L is perfect and G/L is solvable.

Proposition 5.12. *Let $\mathcal{F}' \subsetneqq \mathcal{F}$ be two separating families in G, and let $\mathcal{H} \subseteq \mathcal{F}$ be any subfamily. Assume that one of the following two conditions holds: either*

(a) for each perfect subgroup $L \in \mathcal{F}\smallsetminus\mathcal{F}'$, there is a solvable subgroup $N \lhd C_G(L)$ with $N \in \mathcal{H}$; or

(b) the maximal normal perfect subgroup $L_{\max} \lhd G$ is simple, and $C_G(L) \in \mathcal{H}$ for each perfect subgroup $L \neq L_{\max}$ in $\mathcal{S}(G)\smallsetminus\mathcal{F}'$.

Then the inclusion of $\mathcal{N}(\mathcal{F}'_{\geq\mathcal{H}})$ into $\mathcal{N}(\mathcal{F}_{\geq\mathcal{H}})$ is a homotopy equivalence, and

$$H_*(\mathcal{N}(\mathcal{F}'_{\geq\mathcal{H}})/G) \cong H_*(\mathcal{N}(\mathcal{F}_{\geq\mathcal{H}})/G).$$

Proof. Note that the set of perfect subgroups in $\mathcal{F}\smallsetminus\mathcal{F}'$ is nonempty. Since for any $H \in \mathcal{F}\smallsetminus\mathcal{F}'$ with maximal normal perfect subgroup $L \lhd H$, $L \in \mathcal{F}\smallsetminus\mathcal{F}'$ since H/L is solvable.

We first check that condition (b) implies condition (a). Fix any perfect subgroup $L \in \mathcal{F}\smallsetminus\mathcal{F}'$, and let $L' \supseteq L$ be the maximal normal perfect subgroup of $L{\cdot}C_G(L)$. Then $C_G(L') \subseteq C_G(L)$, so $L'{\cdot}C_G(L') \subseteq L{\cdot}C_G(L)$, and $C_G(L')$ is solvable since $(L'{\cdot}C_G(L'))/L'$ is solvable and $L' \cap C_G(L') = Z(L')$ is abelian. Also, $C_G(L)$ normalizes L', and so $C_G(L') \lhd C_G(L)$. If (b) holds, then either $L = L'$ or L' is not simple (since $L \lhd L'$); and in either case $L' \neq L_{\max}$ and so $C_G(L') \in \mathcal{H}$. Condition (a) thus applies, with $N = C_G(L')$.

Now assume that condition (a) holds. Fix a conjugacy class $\mathcal{L}$ of maximal perfect subgroups in $\mathcal{F}\smallsetminus\mathcal{F}'$. Set $\mathcal{F}'' = \mathcal{F}\smallsetminus(\mathcal{F}_{\geq\mathcal{L}})$: the family of subgroups in $\mathcal{F}$ which do not contain any subgroup in $\mathcal{L}$. This is a separating family (if H/K is solvable and $H \supseteq L \in \mathcal{L}$ then $K \supseteq L$); and we can assume inductively that the inclusion of $\mathcal{N}(\mathcal{F}'_{>1})$

into $\mathcal{N}(\mathcal{F}''_{>1})$ is a homotopy equivalence. So upon setting $\mathcal{F}' = \mathcal{F}''$, we are reduced to the case where $\mathcal{F}\smallsetminus\mathcal{F}'$ contains a single conjugacy class $\mathcal{L}$ of perfect subgroups, and where $\mathcal{F}'$ is the set of subgroups in $\mathcal{F}$ which do not contain any subgroup in $\mathcal{L}$.

For each $L \in \mathcal{L}$, let $\mathcal{K}_L$ be the set of all subgroups $H \subseteq N(L)$ such that HL/L is solvable, and let $\mathcal{K}'_L$ be the set of all $H \in \mathcal{K}_L$ such that $L \not\subseteq H$. Then $\mathcal{K}_L \subseteq \mathcal{F}$ (HL/L solvable implies $HL \in \mathcal{F}$ and hence $H \in \mathcal{F}$), and $\mathcal{K}'_L = \mathcal{K}_L \cap \mathcal{F}'$. By assumption, there is a solvable normal subgroup $N \lhd C_G(L)$ with $N \in \mathcal{H}$. Upon replacing N by the subgroup generated by its conjugates in $N(L)$ (still solvable since it is generated by solvable *normal* subgroups of $C_G(L)$), we can assume that $N \lhd N(L)$ (and $N \in \mathcal{F}_{\geq\mathcal{H}}$). Then $HN \in \mathcal{K}_L$ for all $H \in \mathcal{K}_L$ (HNL/L is solvable if HL/L is since HL/L normalizes NL/L and NL/L is solvable). Also, $HN \in \mathcal{K}'_L$ for all $H \in \mathcal{K}'_L$: since for $H \in \mathcal{K}_L$, $H/(H\cap L) \cong HL/L$ is solvable, so $HN/(H\cap L)$ is solvable (since N is solvable and centralizes $H\cap L$), and thus HN contains L if and only if H does. The nerves $\mathcal{N}((\mathcal{K}_L)_{\geq\mathcal{H}})$ and $\mathcal{N}((\mathcal{K}'_L)_{\geq\mathcal{H}})$ are thus contractible by Lemma 0.4(b).

For each subgroup $H \in \mathcal{F}\smallsetminus\mathcal{F}'$, there is a unique $L \in \mathcal{L}$ contained in H: the subgroups in $\mathcal{L}$ are maximal among perfect subgroups in $\mathcal{F}\smallsetminus\mathcal{F}'$, and hence L must be the last term in the derived sequence for H. Thus, $L \lhd H$ and H/L is solvable; and L is the unique element of $\mathcal{L}$ for which $H \in \mathcal{K}_L \smallsetminus \mathcal{K}'_L$. In other words, $\mathcal{N}(\mathcal{F}_{\geq\mathcal{H}})$ is the union of $\mathcal{N}(\mathcal{F}'_{\geq\mathcal{H}})$ with the contractible complexes $\mathcal{N}((\mathcal{K}_L)_{\geq\mathcal{H}})$ for $L \in \mathcal{L}$, any two of the complexes $\mathcal{N}((\mathcal{K}_L)_{\geq\mathcal{H}})$ and $\mathcal{N}((\mathcal{K}_{L'})_{\geq\mathcal{H}})$ have intersection contained in $\mathcal{N}(\mathcal{F}'_{\geq\mathcal{H}})$, and $\mathcal{N}(\mathcal{F}'_{\geq\mathcal{H}}) \cap \mathcal{N}((\mathcal{K}_L)_{\geq\mathcal{H}}) = \mathcal{N}((\mathcal{K}'_L)_{\geq\mathcal{H}})$ is also contractible for each L. The inclusion of $\mathcal{N}(\mathcal{F}'_{\geq\mathcal{H}})$ into $\mathcal{N}(\mathcal{F}_{\geq\mathcal{H}})$ is thus a homotopy equivalence; and hence $H_*(\mathcal{N}(\mathcal{F}'_{\geq\mathcal{H}})/G) \cong H_*(\mathcal{N}(\mathcal{F}_{\geq\mathcal{H}})/G)$ by Proposition 5.5. □

5d. <u>Connectivity of links at vertices</u>

In Section 5c, conditions were found on a separating family $\mathcal{F}$ which imply that if $(G,\mathcal{F}) \in \mathcal{U}_2$, then $\mathcal{N}(\mathcal{F}_{>1})$ is acyclic, and hence there is a 2-dimensional H-universal $(G,\mathcal{F})$-complex with no free orbits. The results of this section amount to showing that if there is such an action, then the links at all of its vertices must be connected. This result, and its proof, are closely related to [S2, Theorem 2.8].

Proposition 5.13. *Let $\mathcal{F}$ be a nonempty family of subgroups of G, such that $G \notin \mathcal{F}$. Let $\mathcal{F}_{\max}$ be the set of maximal members of $\mathcal{F}$. Assume that*

(a) each member of $\mathcal{F}_{\max}$ is self-normalizing;

(b) each member of $\mathcal{F}\smallsetminus\mathcal{F}_{\max}$ is contained in at least two members of $\mathcal{F}_{\max}$; and

(c) $\mathcal{N}(\mathcal{F})$ is connected and $H_1(\mathcal{N}(\mathcal{F})) = 0$.

Then for each $M \in \mathcal{F}_{\max}$, $\mathcal{N}(\mathcal{F}_{<M})$ (i.e., the link of M) is connected.

Proof. Set $\mathcal{F}' = \mathcal{F}\smallsetminus\mathcal{F}_{\max}$, for short. Let $\mathcal{L}$ be the set of all pairs $(M,H) \in \mathcal{F}_{\max}\times\mathcal{F}'$ such that $M \supsetneqq H$; regarded as a poset via the relation $(M,H) \leq (M',H')$ if $M = M'$ and $H \subseteq H'$. In both $\mathcal{F}'$ and $\mathcal{L}$ we let $\sim$ denote the equivalence relation generated by the poset relation; so that $\mathcal{F}'/{\sim}$ and $\mathcal{L}/{\sim}$ are the sets of connected components of the nerves.

Let Γ be the graph with vertex set $\mathcal{F}_{\max} \amalg (\mathcal{F}'/{\sim})$, and whose set of edges is $\mathcal{L}/{\sim}$. The edge corresponding to an equivalence class $[M,H]$ connects the vertices $[M]$ and

$[H]$. There is an obvious map $\psi\colon \mathcal{N}(\mathcal{F}) \to \Gamma$, which sends each simplex in $\mathcal{N}(\mathcal{F}')$ to the vertex for its connected component, and which sends a simplex $\{M, H_1, \dots, H_k\}$ (for $M \supsetneqq H_1 \supsetneqq \cdots$) to the edge $[M, H_1]$.

We next construct a map $\varphi\colon \Gamma \to \mathcal{N}(\mathcal{F})$ in the other direction. For each vertex v in Γ, let $\varphi(v) \in \mathcal{F}$ be any subgroup in the equivalence class which v represents. And for each edge e in Γ, choose a representative $(M, H) \in \mathcal{L}$ for e, and send e to the path which follows the edge from M to H in $\mathcal{N}(\mathcal{F})$, and then follows any path in $\mathcal{N}(\mathcal{F}')$ from H to $\varphi([H])$.

The composite $\psi \circ \varphi\colon \Gamma \to \Gamma$ sends each vertex to itself, and sends each closed edge to itself (although not via the identity). In particular, $\psi \circ \varphi$ is homotopic to the identity, and so $H_*(\Gamma)$ is a direct summand of $H_*(\mathcal{N}(\mathcal{F}))$. Thus, Γ is connected and $H_1(\Gamma) = 0$; and so Γ is a tree.

Now, G acts on Γ via conjugation, and since Γ is a tree there must be a fixed point $x_0 \in \Gamma$. Since the members of $\mathcal{F}_{\max}$ are assumed to be self-normalizing, no element of $\mathcal{F}_{\max}$ is normal in G, and hence x_0 is not the vertex corresponding to any $M \in \mathcal{F}_{\max}$.

Assume there is some $M \in \mathcal{F}_{\max}$ for which $\mathcal{N}(\mathcal{F}_{<M})$ is not connected. Then there are two or more edges attached to $[M]$ in Γ, and so $\Gamma \smallsetminus [M]$ is disconnected. Let Γ_1 be the component of $\Gamma \smallsetminus [M]$ which contains x_0, and let Γ_2 be any other component. Let $[H]$ be a vertex in Γ_2. By assumption, either $H \in \mathcal{F}_{\max}$, or H is contained in at least two maximal subgroups of $\mathcal{F}$. In particular, H is contained in some maximal subgroup $M' \neq M$.

The action of M' on Γ fixes x_0 and $[M']$, and hence fixes the full minimal path which connects them. Since M lies on this path, this implies that M' normalizes M. But both are maximal in $\mathcal{F}$, and so this contradicts assumption (a) that M is self-normalizing. □

The following proposition combines the above result with those in earlier sections. For any family $\mathcal{F}$ of subgroups and any maximal element $M \in \mathcal{F}$, we set

$$\mathrm{Lk}_{\mathcal{F}_{>1}}(M) = \mathcal{N}(\mathcal{F}_{>1}^{<M}) = \mathcal{N}\big(\{H \in \mathcal{F} \mid 1 \neq H \subsetneqq M\}\big).$$

Proposition 5.14. *Fix a separating family $\mathcal{F}$ for G. Let $\mathcal{F}_0 \subseteq \mathcal{F}$ be any subfamily which contains $\mathcal{F}_c$, and such that each nonmaximal subgroup in $\mathcal{F}_0$ is contained in two or more maximal subgroups. Assume that $\mathcal{F}$ satisfies the following two conditions:*

(a) $\mathcal{N}(\mathcal{F}_{>1})/G$ is connected and $H_1(\mathcal{N}(\mathcal{F}_{>1})/G) = 0$.

(b) There is a maximal subgroup $M \in \mathcal{F}$ such that $\mathrm{Lk}_{(\mathcal{F}_0)_{>1}}(M)$ is not connected.

Then $(G, \mathcal{F}) \notin \mathcal{U}_2$.

Proof. Assume that $(G, \mathcal{F}) \in \mathcal{U}_2$. Then by (a) and Lemma 5.10, $\mathcal{N}(\mathcal{F}_{>1})$ is acyclic. So Proposition 5.13, applied to the family $(\mathcal{F}_0)_{>1}$, implies that $\mathrm{Lk}_{(\mathcal{F}_0)_{>1}}(M) = \mathcal{N}((\mathcal{F}_0)_{>1}^{<M})$ is connected for all maximal subgroups $M \in \mathcal{F}$, and this contradicts point (b). (Recall that all maximal subgroups in $\mathcal{F}$ are self-normalizing by Lemma 1.1.) □

6. Simple groups of Lie rank one

We now focus attention on the simple groups of Lie type and Lie rank one. There are four families of such groups: the two dimensional projective special linear groups $L_2(q)$, the three dimensional projective special unitary groups $U_3(q)$, the Suzuki groups $\mathrm{Sz}(2^{2k+1})$, and the Ree groups $\mathrm{Ree}(3^{2k+1}) = {}^2G_2(2^{2k+1})$. We refer to Appendix D for more detail on these groups, and on the classification of finite groups of Lie type in general.

We first show that the only 2-dimensional actions which involve the simple groups $L_2(q)$ or $Sz(q)$ are the ones constructed in Section 3. This will be done in a series of three lemmas, after which the results will be summarized in Proposition 6.4.

Lemma 6.1. *Assume that $G = L_2(q)$ or $PGL_2(q)$, where $q = p^k$ and p is an odd prime. Let $\mathcal{F}$ be a separating family for G which contains no nonsolvable subgroups $L_2(q_0)$ or $PGL_2(q_0)$ for q_0 a smaller power of p. Assume also that $\mathcal{F} \neq \mathcal{SLV}$ if $G = L_2(q)$ and $q \equiv \pm 3 \pmod 8$. Then $(G, \mathcal{F}) \notin \mathcal{U}_2$.*

Proof. We refer to the description of maximal subgroups of G in Proposition 3.3. Note that if $G = L_2(q)$ and $q \equiv \pm 3 \pmod 8$, then $\mathcal{F}$ must contain a subgroup isomorphic to A_5 — the only nonsolvable subgroups of G not isomorphic to $L_2(q_0)$ or $PGL_2(q_0)$ for q_0 a smaller power of p. In particular, $q \equiv \pm 1 \pmod 5$ in this case.

Case 1: Assume first that $p = 3$. If k is odd, then $q \equiv 3 \pmod 8$ and $q \equiv \pm 2 \pmod 5$; and so $G \not\cong L_2(q)$ by the above remarks. Thus, either $G = L_2(3^k)$ for k even, or $G = PGL_2(3^k)$.

Set $K_1 = PGL_2(3) \cong \Sigma_4$ (the subgroup of matrices with entries in $\mathbb{F}_3$), let K_2 be the subgroup of upper triangular matrices ($K_2 \cong \mathbb{F}_q \rtimes C_{(q-1)/2}$ or $\mathbb{F}_q \rtimes C_{q-1}$), and let K_3 be the subgroup of monomial matrices ($K_3 \cong D_{q-1}$ or $D_{2(q-1)}$). Set $K_{ij} = K_i \cap K_j$ and $K = K_1 \cap K_2 \cap K_3$. Then $K_{12} \cong D_6$, $K_{13} \cong C_2^2$, $K_{23} \cong C_{(q-1)/2}$ or C_{q-1}, and $K \cong C_2$. Since K_1 is a maximal subgroup in $\mathcal{F}$ (see the list of subgroups in Proposition 3.3), Proposition 5.4 now applies (using condition (a1), or (a2) if $G = L_2(9)$) to show that $(G, \mathcal{F}) \notin \mathcal{U}_2$.

Case 2: Now assume that $p \geq 5$. By Proposition 3.3, A_4 is a maximal subgroup of G only if $G = L_2(q)$ and $q \equiv \pm 3 \pmod 8$, in which case (as noted above) $\mathcal{F}$ must contain subgroups isomorphic to A_5. And since there is only one conjugacy class of $A_4 \subseteq G$ (Proposition 3.3 again), each such subgroup must be contained in some $A_5 \in \mathcal{F}$.

Thus, no maximal subgroup of $\mathcal{F}$ is isomorphic to A_4. From the lists of maximal subgroups in Proposition 3.3, we now see that each maximal subgroup in $\mathcal{F}$ is isomorphic to one of the groups $\mathbb{F}_q \rtimes C_{(q-1)/2}$ or $\mathbb{F}_q \rtimes C_{q-1}$ (triangular matrices); D_{q-1} or $D_{2(q-1)}$; D_{q+1} or $D_{2(q+1)}$; or Σ_4 or A_5. Also, by hypothesis, if $p = 5$, then $A_5 \cong L_2(5)$ is not in $\mathcal{F}$.

Let $M_1 \subseteq G$ be the (maximal) subgroup of upper triangular matrices, and let $T \subseteq M_1$ be the subgroup of diagonal matrices. From the above list (and since $p > 3$) we see that M_1 and its conjugates are the only maximal subgroups in $\mathcal{F}$ of order a multiple of p. Furthermore, for any subgroup $H \in \mathcal{F}$ with $p \big| |H|$, H leaves invariant a unique line in $(\mathbb{F}_q)^2$, and hence is contained in a unique subgroup conjugate to M_1 (and thus a unique maximal subgroup in $\mathcal{F}$). Also, each nontrivial subgroup $H \subseteq M_1$ of order prime to p is contained in a unique subgroup conjugate to T (i.e., $C_{M_1}(H)$).

We first check that $\mathcal{N}(\mathcal{F}_{>1})/G$ is connected and $H_1(\mathcal{N}(\mathcal{F}_{>1})/G) = 0$, using Proposition 5.11. From the above list of maximal subgroups in $\mathcal{F}$ (and since A_4 is not among them), we see that for each maximal subgroup $M \in \mathcal{F}$ of even order, and each $x \in M$ of odd prime order, $N_M(\langle x \rangle)$ has even order. Thus, condition (1a) in Proposition 5.11 holds. Also, the only maximal subgroups in G of odd order are those conjugate to $M_1 \cong \mathbb{F}_q \rtimes C_{(q-1)/2}$, when $G = L_2(q)$ and $q \equiv 3 \pmod 4$. Let $\mathcal{F}'_1$ be the set of subgroups of M_1 which are contained in maximal subgroups in other conjugacy classes; by the above remarks each $H \in \mathcal{F}'_1$ is conjugate to a subgroup of T. The image of $\mathcal{N}((\mathcal{F}'_1)_{>1})$ in $\mathcal{N}(\mathcal{F}_{>1})/G$ is thus connected, and so condition (2) in Proposition 5.11 holds. This finishes the proof that $\mathcal{N}(\mathcal{F}_{>1})/G$ is connected and $H_1(\mathcal{N}(\mathcal{F}_{>1})/G) = 0$.

Now let $\mathcal{F}_0 \subseteq \mathcal{F}$ be the subfamily consisting of all maximal subgroups in $\mathcal{F}$, together with all subgroups in $\mathcal{F}$ contained in two or more maximal subgroups. We have seen that each proper subgroup of M_1 contained in $\mathcal{F}_0$ is contained in a unique subgroup conjugate to T. In other words, $\mathrm{Lk}_{(\mathcal{F}_0)_{>1}}(M) = \mathcal{N}((\mathcal{F}_0)_{>1}^{<M})$ is not connected: it has one connected component for each subgroup of M_1 conjugate to T. So Proposition 5.14 now applies to show that $(G,\mathcal{F}) \notin \mathcal{U}_2$. □

In each of the next two lemmas, we deal simultaneously with simple groups $L = L_2(q)$ and $\mathrm{Sz}(q)$, where $q = p^k$ and p is prime ($p = 2$ if $L = \mathrm{Sz}(q)$). It will be convenient to fix subgroups $S, T, B, N \subseteq L$ of each of these groups, according to the following table:

L	$L_2(q)$	$\mathrm{Sz}(q)$
S	$\{\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) \mid a \in \mathbb{F}_q\} \cong \mathbb{F}_q$	$S(q,\theta)$
T	$\{\mathrm{diag}(\lambda, \lambda^{-1}) \mid \lambda \in (\mathbb{F}_q)^*\}/\{\pm I\}$	$\{M(\lambda) \mid \lambda \in (\mathbb{F}_q)^*\}$
B	$S \rtimes T$	$M(q,\theta) = S(q,\theta) \rtimes T$
N	$N(T) = \langle T, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\rangle$	$N(T) = \langle M(\lambda), \tau \rangle$

Table 4

When $L = \mathrm{Sz}(q)$, we are using the notation in Proposition 3.6 (where $\mathrm{Sz}(q)$ is regarded as a subgroup of $GL_4(q)$). All of these subgroups are invariant under the action of $\mathrm{Aut}(\mathbb{F}_q)$. In both cases, S is a Sylow p-subgroup, $B = N(S)$ is a Borel subgroup, T is cyclic (of order $q-1$ or $(q-1)/2$), and N is dihedral.

Lemma 6.2. *Assume that $G = L$ is one of the simple groups $L_2(q)$ or $\mathrm{Sz}(q)$, where $q = p^k$ and p is prime ($p = 2$ in the second case). Let $\mathcal{F}$ be any separating family for G which contains a nonsolvable subgroup isomorphic to $L_2(q_0)$ or $\mathrm{Sz}(q_0)$, where $q_0 = p^{k_0}$ (and $k_0|k$). Then $(G,\mathcal{F}) \notin \mathcal{U}_2$.*

Proof. Assume that $q_0 = p^{k_0}$ is chosen so that $\mathcal{F}$ contains a maximal subgroup isomorphic to $G_0 = L_2(q_0)$, $PGL_2(q_0)$, or $\mathrm{Sz}(q_0)$. Thus, G_0 is the subgroup of all matrices in G with entries in $\mathbb{F}_{q_0}$. (More precisely, if $G = L_2(q) \subseteq PGL_2(q)$, then $G_0 = L_2(q) \cap PGL_2(q_0)$.) By Proposition 3.3 or 3.6, if $G_0 \cong M \in \mathcal{F}$, then there is an automorphism $\sigma \in \mathrm{Aut}(G)$ such that $\sigma(M) = G_0$. Thus, upon replacing $\mathcal{F}$ by $\sigma(\mathcal{F})$, we can assume that $G_0 \in \mathcal{F}$.

We now apply Proposition 5.4, with the subgroups $K_1 = G_0$, $K_2 = B$, and $K_3 = N$ (as in Table 4). Then $K_{12} = B_0$, $K_{13} = N_0$, $K_{23} = T$, and $K = K_1 \cap K_2 \cap K_3 = T_0$.

Condition (b) of 5.4 holds by assumption ($K_1 = G_0$ is maximal in $\mathcal{F}$). Conditions (d) and (e) are clear: $N_G(G_0, B_0, T_0) = T_0$, and the triples (K_1, B_0, T_0) and (K_1, N_0, T_0) are not G-conjugate.

We next consider condition (c). Clearly, $K_{12} = B_0$ is a maximal subgroup of $K_1 = G_0$. If $G = L_2(q)$, then $K = T_0$ is maximal in $K_{12} = B_0$. And if $G = \mathrm{Sz}(q)$, then T_0 is maximal among critical subgroups of B_0. (There *is* one subgroup $T_0 \subsetneqq R \subsetneqq B_0$, where $R = Z(S(q_0,\theta)){\cdot}T_0 \cong \mathbb{F}_{q_0} \rtimes C_{q_0-1}$. But using Proposition 3.6(b), it is easy to check that every maximal subgroup of G which contains R also contains B_0. So by Lemma 3.1(a), R is not critical.)

It remains to check that inequality (a1) or (a2) holds. From the above description of the groups, we see that

$$[K_{12}{:}K] = [B_0{:}T_0] = \begin{cases} q_0 & \text{if } L = L_2(q) \\ (q_0)^2 & \text{if } L = \mathrm{Sz}(q), \end{cases} \quad [K_{13}{:}K] = 2, \quad [K_{23}{:}K] = \epsilon{\cdot}\tfrac{q-1}{q_0-1},$$

where $\epsilon = \frac{1}{2}$ if $G = L_2(q)$, p is odd, and $2k_0|k$ (so $G_0 = PGL_2(q_0)$), and $\epsilon = 1$ otherwise. Inequality (a1) now holds ($\sum_{i<j} \frac{1}{[K_{ij}:K]} \le 1$) unless $G = L_2(25)$ and $G_0 = PGL_2(5)$. In this last case,

$$\sum_{i<j} \frac{1}{[K_{ij}{:}K]} = \tfrac{1}{5} + \tfrac{1}{2} + \tfrac{1}{3} < 1 + \tfrac{1}{6} = 1 + \tfrac{1}{[K_3:K]},$$

and inequality (a2) holds.

The conditions of Proposition 5.4 thus hold, and so $(G, \mathcal{F}) \notin \mathcal{U}_2$. □

It remains to handle the case of extensions of $L_2(q)$ or $\mathrm{Sz}(q)$ by field automorphisms.

Lemma 6.3. *Let L be one of the simple groups $L = L_2(q)$ or $\mathrm{Sz}(q)$, where $q = p^k$ and p is prime. Let $A \subseteq \mathrm{Aut}(\mathbb{F}_q)$ be a subgroup of prime order, regarded as a subgroup of $\mathrm{Aut}(L)$, and set $G = L \rtimes A$. Then $(G, \mathcal{SLV}) \notin \mathcal{U}_2$.*

Proof. Let $L_0 \subseteq L$ be the subgroup of elements fixed by A. Let $\mathbb{F}_{q_0}$ be the fixed subfield of $A \subseteq \mathrm{Aut}(\mathbb{F}_q)$. If $q \equiv \pm 1 \pmod 8$, then $(L, \mathcal{SLV}) \notin \mathcal{U}_2$ by Lemma 6.1, and so we also have $(G, \mathcal{SLV}) \notin \mathcal{U}_2$. Thus, we can assume that q is a power of 2 or that $q = p^k \equiv \pm 3 \pmod 8$. In the second case, k must be odd, and hence $|A|\big|k$ is odd. Thus, $L_0 = L_2(q_0)$ if $L = L_2(q)$, and $L_0 = \mathrm{Sz}(q_0)$ if $L = \mathrm{Sz}(q)$.

To simplify the argument, we assume that $L \not\cong L_2(4)$ (the case $L_2(4) \rtimes C_2 \cong PGL_2(5)$ was already handled in Lemma 6.1).

Fix subgroups $S, T, B, N \subseteq L$ as in Table 4. All of these are A-invariant. Set $B_0 = B \cap L_0$, $N_0 = N \cap L_0$, and $T_0 = T \cap L_0$.

We claim that conditions (a,b,c) in Proposition 5.3 hold for the subgroups $K_0 = B \rtimes A$, $K_1 = T \rtimes A$, and $K_2 = A$; this will then imply that $(G, \mathcal{SLV}) \notin \mathcal{U}_2$. Condition (b) is clear: K_0 is a maximal subgroup of G since B is a maximal subgroup of L.

We next check condition (c). Let $H, H' \in \mathcal{SLV}_c$ be critical subgroups such that $A \subsetneqq H \subsetneqq H' \subsetneqq K_0 = B \rtimes A$ and $H \subseteq K_1 = T \rtimes A$. We must show that $H' \subseteq K_1$. Assume otherwise. Write $H = H_0 \rtimes A$ and $H' = H'_0 \rtimes A$ (where $H_0 = H \cap B$ and $H'_0 = H' \cap B$). Thus, $1 \ne H_0 \subsetneqq H'_0 \subsetneqq B = S \rtimes T$, $H_0 \subseteq T$, but $H'_0 \not\subseteq T$. So H'_0 intersects nontrivially with T and S. Since the intersection of any two distinct Sylow

p-subgroups of L is trivial (see [H1, Theorem II.8.5(a)] or [HB3, Theorem XI.3.10(c)]), the lists of maximal subgroups of L in Propositions 3.3 and 3.6 show that B is the unique maximal subgroup of L which contains H_0', and hence that $K_0 = B\rtimes A$ is the unique maximal subgroup of $G = L\rtimes A$ which contains $H' = H_0'\rtimes A$. And by Lemma 3.1(a), this contradicts the assumption that $H' \subsetneqq K_0$ is critical in $\mathcal{SLV}$. This proves condition (c).

It remains to check condition (a). To avoid conflicting notation, we set $R_i = N_G(K_i)$, $R_{ij} = R_i \cap R_j$, and $R = R_0 \cap R_1 \cap R_2$. Then

$$R_0 = K_0 = B\rtimes A, \qquad R_1 = N\rtimes A, \qquad R_2 = L_0\times A;$$

$$R_{01} = T\rtimes A, \qquad R_{02} = B_0\times A, \qquad R_{12} = N_0\times A, \qquad R = T_0\times A;$$

and so

$$[R_{01}{:}R] = [T{:}T_0] = \tfrac{q-1}{q_0-1}, \quad [R_{12}{:}R] = 2, \quad [R_{02}{:}R] = [B_0{:}T_0] = \begin{cases} q_0 & \text{if } L = L_2(q) \\ (q_0)^2 & \text{if } L = \mathrm{Sz}(q) \end{cases}$$

It follows that $\sum \frac{1}{[R_{ij}:R]} < 1$ except when $q_0 = 2$ and $L = L_2(q)$. And in this last case, since $q \ge 8$, we have $[R_2{:}R] = [L_0\times A{:}A] = 6 < \frac{q-1}{q_0-1} = [R_{01}{:}R]$, and so inequality (a) in 5.3 still holds. (In fact, inequality (a) in 5.3 also holds when $q = 4$, but one has to calculate each term explicitly.) □

The above three lemmas can now be summarized as follows.

Proposition 6.4. *Assume that L is one of the simple groups $L \cong L_2(q)$ or $\mathrm{Sz}(q)$, where $q = p^k$ and p is prime ($p = 2$ in the second case). Let $G \subseteq \mathrm{Aut}(L)$ be any subgroup containing L, and let $\mathcal{F}$ be a separating family for G. Then $(G,\mathcal{F}) \in \mathcal{U}_2$ if and only if $G = L$, $\mathcal{F} = \mathcal{SLV}$, and q is a power of 2 or $q \equiv \pm 3$ (mod 8).*

Proof. That $(G,\mathcal{SLV}) \in \mathcal{U}_2$ in the given cases was shown in Section 3 (Examples 3.4, 3.5, and 3.7). It remains only to check that all of the other cases have been eliminated by one of the above three lemmas.

If $G = L_2(q)$, then we are assuming that $\mathcal{F} \neq \mathcal{SLV}$ or $q \equiv \pm 1$ (mod 8). So $(G,\mathcal{F}) \notin \mathcal{U}_2$ by Lemma 6.1 (if $\mathcal{F}$ contains no nonsolvable subgroups $L_2(q_0)$) or by Lemma 6.2 (if $\mathcal{F}$ does contain such subgroups). If $G = \mathrm{Sz}(q)$ and $\mathcal{F} \neq \mathcal{SLV}$, then $\mathcal{F}$ must contain some nonsolvable subgroup $\mathrm{Sz}(q_0)$ (these are the only nonsolvable subgroups of G by [Sz1, Theorem 9]); and hence $(G,\mathcal{F}) \notin \mathcal{U}_2$ by Lemma 6.2. So we are finished if $G = L$ is simple.

Now assume that G is not simple: that $L \subsetneqq G \subseteq \mathrm{Aut}(G)$. If $L = L_2(q)$, then $\mathrm{Aut}(L)$ is generated by inner automorphisms, by "diagonal" automorphisms (conjugation by a matrix of non-square determinant), and by field automorphisms (cf. [Ca, Theorem 12.5.1]). In other words,

$$\mathrm{Aut}(L_2(q)) \cong P\Gamma L_2(q) \overset{\text{def}}{=} PGL_2(q)\rtimes \mathrm{Aut}(\mathbb{F}_q).$$

Also, by [Sz1, Theorem 11], all outer automorphisms of $\mathrm{Sz}(q)$ are given by field automorphisms.

Since $\mathrm{Aut}(L)/L$ is solvable, if $(G,\mathcal{F}) \in \mathcal{U}_2$, then $(G',\mathcal{F}\cap\mathcal{S}(G')) \in \mathcal{U}_2$ for any $G' \subseteq G$ containing L. So it suffices to consider the case where G is minimal; i.e., when G/L has prime order and where $(L,\mathcal{F}\cap\mathcal{S}(L)) \in \mathcal{U}_2$. In particular, $\mathcal{F} = \mathcal{SLV}$. If $L = L_2(2^k)$ or

$\mathrm{Sz}(2^k)$, then we are done by Lemma 6.3, since the only outer automorphisms of L are field automorphisms ($PGL_2(q) = L_2(q)$ in this case). If $L = L_2(q)$ and $q = p^k \equiv \pm 3 \pmod 8$, then k must be odd, and so

$$\mathrm{Out}(L) = PGL_2(q)/L_2(q) \times \mathrm{Aut}(\mathbb{F}_q) \cong C_2 \times C_k.$$

Since G/L has prime order, either $G \cong PGL_2(q)$ (and $(G, \mathcal{SLV}) \notin \mathcal{U}_2$ by Lemma 6.1); or $G = L \rtimes A$ for some group A of field automorphisms and $(G, \mathcal{SLV}) \notin \mathcal{U}_2$ by Lemma 6.3. □

The next groups we consider are the unitary groups $U_3(q) = PSU_3(q)$. The cases q odd and q a power of two will be handled separately. In both cases, we regard $U_3(q)$ as a group of projective unitary transformations of a vector space $V \cong (\mathbb{F}_{q^2})^3$ with hermitian product denoted $(-,-)$. We fix two bases of V: an orthonormal basis $\{e_1, e_2, e_3\}$, and a basis $\{v_1, v_2, v_3\}$ with respect to which the hermitian product has matrix $\left(\begin{smallmatrix} 0&0&1 \\ 0&1&0 \\ 1&0&0 \end{smallmatrix}\right)$. Elements of $U_3(q)$ will be regarded as matrices with respect to one or the other basis, depending on the situation.

Proposition 6.5. *Set $G = U_3(q)$, where $q = p^k$ is any odd prime power. Then there is no 2-dimensional G-complex without fixed points.*

Proof. Assume otherwise: let $\mathcal{F}$ be a separating family of subgroups of G such that $(G, \mathcal{F}) \in \mathcal{U}_2$. Set $d = (3, q+1)$: the order of the center of $SU_3(q)$.

Case 1 Assume first that $q \geq 7$, and regard elements of G as matrices with respect to the basis $\{v_1, v_2, v_3\}$ described above. We apply Proposition 5.4 with the following subgroups.

$K_1 \cong SO_3(q) \cong PGL_2(q)$: the subgroup of matrices with entries in $\mathbb{F}_q$.

$K_2 = S \rtimes T$, where $T = \left\{\mathrm{diag}(\lambda, \lambda^{q-1}, \lambda^{-q}) \,\middle|\, \lambda \in (\mathbb{F}_{q^2})^*\right\}/C_d$ and

$$S = \left\{ \left(\begin{smallmatrix} 1&a&b \\ 0&1&-a^q \\ 0&0&1 \end{smallmatrix}\right) \,\middle|\, a, b \in \mathbb{F}_{q^2},\ b + b^q = -a^{q+1} \right\}.$$

This is the subgroup of upper triangular matrices in $U_3(q)$, a Borel subgroup, and of order $q^3(q^2-1)/d$.

$K_3 = T \rtimes \left\langle \left(\begin{smallmatrix} 0&0&1 \\ 0&-1&0 \\ 1&0&0 \end{smallmatrix}\right) \right\rangle$.

Note that $SO_3(q) \cong PGL_2(q)$ (cf. [H1, Satz 10.11] or [Art, Theorem 5.20]). So all three subgroups lie in $\mathcal{F}$: the first by Proposition 6.4, and the others because they are solvable.

Set $K_{ij} = K_i \cap K_j$ and $K = K_1 \cap K_2 \cap K_3$, as usual. Then

$$K_{12} = \left\{ \left(\begin{smallmatrix} 1&a&-a^2/2 \\ 0&1&-a \\ 0&0&1 \end{smallmatrix}\right) \,\middle|\, a \in \mathbb{F}_q \right\} \rtimes \left\{\mathrm{diag}(\lambda, 1, \lambda^{-1})\right\} \cong \mathbb{F}_q \rtimes C_{q-1};$$

and

$$K_{13} \cong D_{2(q-1)}, \qquad K_{23} \cong C_{(q^2-1)/d}, \qquad K \cong C_{q-1}.$$

Then $K_1 = SO_3(q)$ is a maximal subgroup of G since $q \geq 7$ (see [GLS, Theorem 6.5.3] and its proof, where $U_3(q)$ is denoted $PSL_3^-(q)$). The other conditions in Proposition 5.4 are clear. So $(G, \mathcal{F}) \notin \mathcal{U}_2$ in this case.

Case 2 Assume $q = 3$. In particular, $d = 1$. Regard elements of G as matrices with respect to the orthonormal basis $\{e_1, e_2, e_3\}$. Then each subgroup of G isomorphic to C_2^2 is conjugate to the subgroup K of diagonal matrices with entries ± 1; and $N(K) \cong (C_4)^2 \rtimes \Sigma_3$. Thus, all subgroups A_4 containing K are conjugate, and have normalizer isomorphic to Σ_4. So G contains a unique conjugacy class of subgroups isomorphic to Σ_4.

Also, $G = U_3(3)$ contains a maximal subgroup $L_2(7)$ [Atl], which must be in $\mathcal{F}$ by Lemma 6.1. Since $L_2(7)$ contains two conjugacy classes of Σ_4's, and G only contains one such conjugacy class, Proposition 5.9 applies to show that $(G, \mathcal{F}) \notin \mathcal{U}_2$.

Case 3 Finally, assume $q = 5$ (hence $d = 3$). Set $\overline{G} = PGU_3(5)$, regarded as a subgroup of $\mathrm{Aut}(G)$. Then $\overline{G}/G \cong C_3$ permutes the three conjugacy classes of maximal subgroups A_7 in G [Atl]. Thus, the stabilizer of the $\overline{G}$-action on each subgroup A_7 is the group A_7 itself. Each A_7 contains two conjugacy classes of subgroups $L_2(7)$ (permuted by the outer automorphism of A_7). Since $L_2(7)$ has order prime to 5, one sees [Bl, Theorem 1.1] via complex characters that it has a unique 3-dimensional representation over $\mathbb{F}_{q^2}$ which is irreducible (unique up to outer automorphism); and this has a unique unitary structure (since any two would differ by an automorphism). Thus, there is exactly one $\overline{G}$-orbit of $L_2(7)$'s in G. Proposition 5.9 again applies to show that $(G, \mathcal{F}) \notin \mathcal{U}_2$. □

We next consider the unitary groups $U_3(2^k)$.

Proposition 6.6. *Set $G = U_3(q)$, where $q = 2^k > 2$ is a power of 2. Then there is no 2-dimensional G-complex without fixed points.*

Proof. Assume otherwise: let q be such that $U_3(q)$ is the smallest counterexample, and let $\mathcal{F}$ be a separating family of subgroups of G such that $(G, \mathcal{F}) \in \mathcal{U}_2$.

Set $d = (3, q{+}1)$. Then

$$|G| = \tfrac{1}{d} q^3 (q^2 - 1)(q^3 + 1) = q^3 {\cdot} (q - 1) {\cdot} (q + 1)^2 {\cdot} (\tfrac{q^2 - q + 1}{d}) \tag{1}$$

(cf. [Ca, Theorem 14.3.2], who writes $U_3(q) = {}^2A_2(q^2)$). Here, the factors in the second formula are pairwise relatively prime. (Note that $3 | (q^2 - q + 1) = \frac{q^3+1}{q+1}$ if and only if $3 | (q{+}1)$, and that $\frac{q^3+1}{q+1}$ cannot be divisible by 3^2.)

Let θ be the Frobenius automorphism of order 2 on $\mathbb{F}_{q^2}$; and write $x^\theta = \theta(x) = x^q$ for any x.

The following list of maximal subgroups of G can be found in [Ha, p. 158] or in [GLS, Theorem 6.5.3(a,b,c,g)]. Note also the thesis of Peter Kleidman [Kl1, §5], where maximal subgroups are listed for classical groups of low rank, and a general procedure for determining them is described.

(M_1): $M_1 \cong [q^3] \rtimes C_{(q^2-1)/d}$; the stabilizers of isotropic lines (generated by v with $(v, v) = 0$); the Borel subgroups of G. We choose M_1 to be the stabilizer of $\langle v_1 \rangle$, or equivalently the group of upper triangular matrices with respect to the basis $\{v_1, v_2, v_3\}$.

(M_2): $M_2 \cong GU_2(q)/C_d \cong C_{(q+1)/d} \times L_2(q)$; the stabilizers of anisotropic lines (generated by v with $(v, v) \neq 0$). We choose M_2 to be the subgroup of matrices (a_{ij}) (with respect to either of the above bases) for which a_{22} is the only nonzero entry in the second row or column.

(M_3): $M_3 \cong [(C_{q+1})^2 \rtimes \Sigma_3]/C_d$; the stabilizer of (the union of) three pairwise orthogonal lines. We choose M_3 to be the group of monomial matrices with respect to the orthonormal basis $\{e_1, e_2, e_3\}$.

$(M_4^{q_0})$, if $q = q_0^b$ and b is an odd prime: $M_4^{q_0} = N(U_3(q_0))$, isomorphic to $U_3(q_0)$ (if $(b,d) = 1$) or $PGU_3(q_0)$ (if $b = d = 3$). There are (b,d) conjugacy classes of such subgroups (all conjugate in $PGU_3(q)$).

(M_5): $M_5 \cong C_{(q^2-q+1)/d} \rtimes C_3$. Consider the hermitian form $\langle -,- \rangle$ on $\mathbb{F}_{q^6}$ (viewed as a vector space over $\mathbb{F}_{q^2}$) defined by $\langle x,y \rangle = \mathrm{Tr}(xy^{q^3})$, where $\mathrm{Tr}\colon \mathbb{F}_{q^6} \to \mathbb{F}_{q^2}$ is the trace map. Let $\sigma \in \mathrm{Aut}(\mathbb{F}_{q^6})$ be the automorphism $\sigma(x) = x^{q^2}$, let $H \subseteq (\mathbb{F}_{q^6})^*$ be the subgroup of order q^3+1, and set

$$M = H \rtimes \langle \sigma \rangle \subseteq (\mathbb{F}_{q^6})^* \rtimes \mathrm{Aut}(\mathbb{F}_{q^6}).$$

Then M preserves $\langle -,- \rangle$, and M_5 is the intersection of $U_3(q)$ with the image of M in $PGU_3(q)$. In particular, C_3 acts on $C_{(q^2-q+1)}$ via $x \to x^{q^2}$.

We can assume inductively that none of the groups $M_4^{q_0} = N(U_3(q_0))$, for $q_0 > 2$, can act on an acyclic 2-complex without fixed points. So they must all be contained in $\mathcal{F}$. Also, by Proposition 6.4, if $M_2 \notin \mathcal{F}$, then the only subgroups of M_2 (and its conjugates) which are in $\mathcal{F}$ are solvable subgroups. So either $\mathcal{F} = \mathcal{MAX}$, the family of all proper subgroups of G, or $\mathcal{F} = \mathcal{F}_0$, the family of all subgroups whose intersection with any subgroup in (M_2) is solvable — and this latter only when k is prime or a power of 2.

We first show that $\mathcal{N}(\mathcal{F}_{>1})/G$ is connected and $H_1(\mathcal{N}(\mathcal{F}_{>1})/G) = 0$, using Proposition 5.11. Since every perfect subgroup in $\mathcal{MAX} \smallsetminus \mathcal{F}_0$ is of the form $L_2(2^{k_0})$ where $1 < k_0 | k$ and has nontrivial centralizer, Proposition 5.12 applies, with $\mathcal{H} = \mathcal{MAX}_{>1}$ (and using condition (b)) to show that $H_*(\mathcal{N}((\mathcal{F}_0)_{>1})/G) \cong H_*(\mathcal{N}(\mathcal{MAX}_{>1})/G)$. So we can assume that $\mathcal{F} = \mathcal{MAX}$.

The even order maximal subgroups of G are those conjugate to M_1, M_2, M_3, or $M_4^{q_0}$. If $M = M_2$, M_3, or $M_4^{q_0}$ and $x \in M$ is of odd prime order, then one easily sees that $N_M(\langle x \rangle)$ has even order. Also, if $x \in M_1$ and $|x| \big| (q+1)/d$, then $C_M(x)$ has even order: if M_1 is the subgroup of upper triangular matrices with respect to the basis $\{v_1, v_2, v_3\}$, then x is conjugate to a diagonal matrix $\mathrm{diag}(\lambda, \lambda^{-2}, \lambda)$ and is centralized by the element $\left(\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. Thus, condition (1a) of Proposition 5.11 holds in all of these cases.

Now let $x \in M_1$ be of prime order dividing $p-1$. We check that condition (1b) of Proposition 5.11 holds. Let $y \in C_M(x) \cong C_{(q^2-1)/d}$ be any element of prime order dividing $(q+1)/d$. We have just seen that $N_M(\langle y \rangle)$ has even order, and $N_G(\langle x \rangle) \cap N_G(\langle y \rangle)$ also has even order since $N_M(\langle y \rangle) \cong M_2 \cong C_{(q+1)/d} \times L_2(q)$. Thus, condition (1b) of Proposition 5.11 holds in this case.

It remains to check condition (2) of 5.11. Let $\mathcal{F}_1$ be the set of all subgroups of $M_5 \cong C_{(q^2-q+1)/d} \rtimes C_3$ which are also contained in even order maximal subgroups. By inspection, $\mathcal{F}_1$ contains subgroups of order 3, and all maximal subgroups in $\mathcal{F}_1$ are of the form $C_a \rtimes C_3$ ($\subseteq M_4^{q_0}$) for some a. So the image of $\mathcal{N}((\mathcal{F}_1)_{>1})$ in $\mathcal{N}(\mathcal{F}_{>1})/G$ is nonempty and connected. Proposition 5.11 thus applies to show that $\mathcal{N}(\mathcal{F}_{>1})/G$ is connected and $H_1(\mathcal{N}(\mathcal{F}_{>1})/G) = 0$.

This shows that condition (a) in Proposition 5.14 holds, and it remains to check condition (b). Set $M = M_5 \cong C_{(q^2-q+1)/d} \rtimes C_3$. Let $\mathcal{F}_c \subseteq \mathcal{F}$ be the subfamily consisting of all critical subgroups in $\mathcal{F}$. Fix a prime $p|(q^2-q+1) = \frac{(q^6-1)(q-1)}{(q^3-1)(q^2-1)}$ such that $p\nmid(q_0^6-1)$ when q_0 is a smaller power of 2 (such a prime exists by Zsigmondy's theorem [HB2, Theorem IX.8.3]). Then for any proper subgroup $H \subseteq M$ with $p\big||H|$, M is the unique maximal subgroup of G which contains H, and so $H \notin \mathcal{F}_c$ (Lemma 3.1(a)). Let $T \lhd M$ be the subgroup of order $(q^2-q+1)/dp$; then $M/T \cong C_p \rtimes C_3$. And C_3 is not normal in M/T: since C_3 acts on C_p via $(x \mapsto x^{q^2})$, and $(q^2-1, (q^2-q+1)/d) = 1$.

By Proposition 5.14, we will be done upon showing that the nerve of $(\mathcal{F}_c)^{<M}_{>1}$ is not connected. For any $1 \neq H \subseteq T$, H is not critical by Lemma 3.1(b): $N(H) = M \in \mathcal{F}$, and $N_K(H) \supsetneqq K$ for all $K \supsetneqq H$ (note that K must be contained in M or in one of the subgroups $N(U_3(q_0))$). Thus, any critical subgroup properly contained in M must be of the form $H \rtimes C_3$ for $H \subseteq T$; and such subgroups do exist (any subgroup of M maximal among those contained in other maximal subgroups in $\mathcal{F}$ is critical). The image of the poset $(\mathcal{F}_c)^{<M}_{>1}$ in $\mathcal{S}(M/T)$ thus consists precisely of the subgroups of order 3. Since the continuous image of a connected space must be connected, this shows that $\mathcal{N}((\mathcal{F}_c)^{<M}_{>1})$ is not connected, and finishes the proof of the proposition. □

We note here that Proposition 6.6 can also be proven using Propositions 5.3 and 5.4; but this involves considering several different cases, and requires complicated arguments that certain subgroups are not critical.

We are now ready to consider the Ree groups ${}^2G_2(q)$.

Proposition 6.7. *When q is any odd power of 3, there is no 2-dimensional ${}^2G_2(q)$-complex without fixed points.*

Proof. Set $G = {}^2G_2(q)$, where $q = 3^k$ and k is odd; and assume that $\mathcal{F}$ is a separating family for G such that $(G, \mathcal{F}) \in \mathcal{U}_2$. We can assume inductively that q is the smallest power of 3 for which this happens. Since ${}^2G_2(3) \cong \mathrm{Aut}(L_2(8))$ [Jan], this subgroup has no fixed point free action on a $\mathbb{Z}$-acyclic 2-complex by Lemma 6.3. Thus, we must have ${}^2G_2(3) \in \mathcal{F}$.

The order of G is given by the formula

$$|G| = q^3(q-1)(q^3+1) = q^3{\cdot}2^3{\cdot}(\tfrac{q-1}{2}){\cdot}(\tfrac{q+1}{4}){\cdot}(q+\sqrt{3q}+1){\cdot}(q-\sqrt{3q}+1),$$

(cf. [Ca, Theorem 14.3.2]), where the factors in the last decomposition are pairwise relatively prime. The maximal solvable subgroups of G, as listed in [Kl2, Theorem C], all lie in the following conjugacy classes:

(M_1): the Borel subgroups $P \rtimes C_{q-1}$, where $|P| = q^3$ (a Sylow 3-subgroup of G). More precisely, $P = (\mathbb{F}_q)^3$ with multiplication given by

$$(x_1, y_1, z_1){\cdot}(x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2 + x_1{\cdot}x_2^\sigma, z_1 + z_2 - x_1{\cdot}y_2 + y_1{\cdot}x_2 - x_1{\cdot}x_1^\sigma{\cdot}x_2).$$

Here, $x^\sigma = x^{\sqrt{3q}}$ (so $x^{\sigma^2} = x^3$). The action of $(\mathbb{F}_q)^*$ on P is given by

$$\lambda(x, y, z)\lambda^{-1} = (\lambda x, \lambda\lambda^\sigma y, \lambda^2\lambda^\sigma z).$$

(See [HB3, Theorem XI.13.2].)

(M_2): $M_2 = C_G(C_2) \cong C_2 \times L_2(q)$ for any $C_2 \subseteq G$

(M_3): $M_3 = N(C_2^2) = (C_2^2 \times D(\frac{q+1}{2})) \rtimes C_3$ for any $C_2^2 \subseteq G$

(M_4^+) and (M_4^-): $M_4^\pm \cong C_{q\pm\sqrt{3q}+1} \rtimes C_6$, where C_6 acts via $(x \mapsto x^q)$. (The action of C_6 is determined by the fact that an element of order 2 or 3 has centralizer of order prime to q^2-q+1.)

$(M_5^{q_0})$: $M_5^{q_0} \cong {}^2G_2(q_0)$ whenever $q = q_0^p$ for some (odd) prime p.

By our inductive assumption, ${}^2G_2(q_0) \in \mathcal{F}$ for all $q_0 = 3^{k_0}$ where $k_0|k$. So all of the maximal subgroups must be included in $\mathcal{F}$, except possibly those in (M_2).

We first show that $\mathcal{N}(\mathcal{F}_{>1})$ is connected and that $H_1(\mathcal{N}(\mathcal{F}_{>1})/G) = 0$. By Proposition 5.12 (arguing as in the proof of Proposition 6.6), it suffices to do this when $\mathcal{F} = \mathcal{MAX}$: the family of all proper subgroups of G. We apply Proposition 5.11. From the above list, we see that all maximal subgroups of G have even order. If M is maximal and $x \in M$ has odd prime order, then $N_M(x)$ has even order, except possibly when M is conjugate to M_1 and $|x| = 3$. And under the above description of $P \lhd M_1$, any $x \in P$ of order 3 is of the form $x = (0, b, c)$ for $b, c \in \mathbb{F}_q$; x is normalized by $(-1) \in (\mathbb{F}_q)^*$ if $b = 0$ or $c = 0$; and if $b \neq 0$ then $x = (0, b, c)$ is conjugate to $(0, b, 0)$. Condition (1a) of Proposition 5.11 thus holds (and condition (2) is empty). It follows that $\mathcal{N}(\mathcal{F}_{>1})$ is connected and $H_1(\mathcal{N}(\mathcal{F}_{>1})/G) = 0$.

We have now shown that condition (a) in Proposition 5.14 holds. We claim that condition (b) holds for one of the maximal subgroups $M_4^\pm \cong C_{q\pm\sqrt{3q}+1} \rtimes C_6$; once this has been shown then we can conclude that $(G, \mathcal{F}) \notin \mathcal{U}_2$. By Zsigmondy's theorem [HB2, Theorem IX.8.3], there is a prime $p|(q^6 - 1) = (3^{6k} - 1)$ such that $p \nmid (3^m - 1)$ for any $m < 6k$. In particular, $p|\frac{q^3+1}{q+1} = (q + \sqrt{3q} + 1)(q - \sqrt{3q} + 1)$ — and thus divides the order of $M = M_4^+$ or M_4^- — but does not divide the order of ${}^2G_2(q_0)$ for any $q_0 < q$. We claim that the nerve of the poset of proper subgroups of M which are critical in $\mathcal{F}$ is not connected. Let $T \lhd M$ be the cyclic subgroup of index $6p$, and set

$$\mathcal{H} = \mathrm{Im}[(\mathcal{F}_c)_{>1}^{<M} \longrightarrow \mathcal{S}(M/T)].$$

From the above list of maximal subgroups, we see that for any proper subgroup $H \subsetneqq M$ of order a multiple of p, H is contained in no other maximal subgroup in $\mathcal{F}$, and hence H is not critical (Lemma 3.1(a)). Also, for any $1 \neq H \subseteq T$, Lemma 3.1(a) applies (with $\widehat{H} = N(H) = M$) to show that $H \notin \mathcal{F}_c$. Thus, $\mathcal{H}$ contains neither the trivial subgroup nor subgroups of order a multiple of p. Also, $\mathcal{H}$ contains the subgroups of order 6 in M/T, since any subgroup of the form $H \rtimes C_6 \subseteq M$ (for $H \subseteq T$) which is maximal among subgroups of M contained in other maximal subgroups of $\mathcal{F}$ must be critical. We have now shown that $\mathcal{H}$ consists of the subgroups of order 6 in $M/T \cong C_p \rtimes C_6$, as well as possibly the subgroups of order 2 and 3. Since none of these subgroups is normal (C_6 acts on C_p via $(x \mapsto x^q)$ and p is prime to $(q^2 - 1)$ and to $(q^3 - 1)$), this shows that the nerve of $\mathcal{H}$ is not connected. And since the continuous image of a connected space must be connected, this shows that $\mathcal{N}((\mathcal{F}_c)_{>1}^{<M})$ also fails to be connected. □

Proposition 6.7 can also be proven using Proposition 5.4 (when $\mathcal{F}$ contains centralizers of involutions), and Proposition 5.12 to reduce the general case to this case.

7. Sporadic simple groups

Aschbacher and Segev proved in [AS] that no sporadic simple group, with the possible exception of the first Janko group J_1, can act on a 2-dimensional acyclic complex without fixed points. In all cases, this was done by applying the four-subgroup criterion, presented here in Proposition E.1. Since the arguments in [AS] use a variety of structures and definitions unfamiliar to non-group-theorists, we now describe how these results — as well as the nonexistence of a J_1-action — can be proven using Proposition 5.4 instead. Note however that the arguments presented here, while fairly brief to present, are not really more elementary than those given in [AS]. They depend on information about maximal subgroups which has been collected together in [Atl] and [A2], but whose proofs (especially for the ten sporadic groups listed in Table 5) are scattered widely throughout the literature.

We first repeat some definitions in [A2, §28]. Fix a finite group G, a subgroup $A \subseteq \mathrm{Aut}(G)$, and an A-invariant subgroup $B \subseteq G$. A *regular* (A,B)*-basis* for G is a set $\{G_i \,|\, i \in I\}$ of subgroups containing B which satisfies the following two conditions:

(1) each subgroup $H \subseteq G$ containing B is in the A-orbit of $G_J \overset{\text{def}}{=} \bigcap_{j\in J} G_j$ for some unique $J \subseteq I$ (in particular, $B = G_I$); and

(2) for each $J, K \subseteq I$, if $a(G_K) \subseteq G_J$ for some $a \in A$, then $G_K \subseteq G_J$ and $a(G_K) = a'(G_K)$ for some $a' \in N_A(G_J)$.

If G has a regular (A,B)-basis of order at least four (for any A and B), then by [AS, 6.1] (and using the four-subgroup criterion described in Proposition E.1), $(G,\mathcal{F}) \notin \mathcal{U}_2$ for any separating family $\mathcal{F}$ which contains the basis. Using Proposition 5.4, this can be shown for bases of order three which satisfy certain additional conditions.

Lemma 7.1. *Fix a simple group G and a separating family $\mathcal{F}$ of subgroups of G. Assume, for some $A \subseteq \mathrm{Inn}(G)$ and some A-invariant subgroup $1 \neq K \subseteq G$, that there is a regular (A,K)-basis $\{K_i \,|\, i \in I\}$, and indices $r,s,t \in I$, such that $K_r, K_s, K_t \in \mathcal{F}$ and*

$$\frac{1}{[K_{rs}{:}K_{rst}]} + \frac{1}{[K_{rt}{:}K_{rst}]} + \frac{1}{[K_{st}{:}K_{rst}]} \le 1. \tag{1}$$

Then $(G,\mathcal{F}) \notin \mathcal{U}_2$. In particular, (1) holds if K contains a Sylow p-subgroup for any prime $p\big||G|$.

Proof. For simplicity, we write $I = \{1,2,\dots,k\}$, and assume that $\{r,s,t\} = \{1,2,3\}$. By [A2, 28.1], $\{K_1,K_2,K_3\}$ is a regular $(N_A(K_{123}),K_{123})$-basis; so we can assume $k=3$ and $K = K_{123}$. It is immediate from the definition of a regular (A,K)-basis that $K_{J\cup\{i\}}$ is a maximal subgroup of K_J for any $J \subsetneqq I$ and any $i \in I \smallsetminus J$.

We claim that the subgroups K_1, K_2, K_3 satisfy the hypotheses of Proposition 5.4; it then follows that $(G,\mathcal{F}) \notin \mathcal{U}_2$. We have just checked conditions (b) and (c) (maximality of subgroups). Condition (a1) holds by assumption, and condition (e) (the triples (K_1,K_{12},K) and (K_1,K_{13},K) are not G-conjugate) is immediate from the definition of a regular (A,K)-basis.

We next show that the K_i can be ordered so that $N_G(K_1,K_{12},K) = K$, thus proving condition (d). To see this, note first that $N_G(K)$ must be A-conjugate (hence equal) to one of the subgroups K_i, K_{ij}, or K. Also, the K_i are maximal in the simple group G and hence self-normalizing. If $N_G(K) = K$, then we are done. Otherwise, we can assume (after switching indices if necessary) that $N_G(K) = K_3$ or K_{23}. If $N_G(K) = K_{23}$, then $N_G(K_1, K_{12}, K) \subseteq K_1 \cap K_{23} = K$. So suppose $N_G(K) = K_3$. Since K_{12} is not normal in $G = \langle K_1, K_2\rangle$, K_{12} cannot be normal in both K_1 and K_2, and we can assume without loss of generality that K_{12} is not normal in K_1. Then $N_{K_1}(K_{12}) = K_{12}$, and so $N(K_1, K_{12}, K) \subseteq K_{12} \cap K_3 = K$. This finishes the proof that $(G, \mathcal{F}) \notin \mathcal{U}_2$.

It remains to show that (1) always holds if K contains a Sylow p-subgroup for some prime $p\big||G|$. By definition of a regular basis, $[K_{ij}{:}K] > 1$ for all i, j. If $[K_{ij}{:}K] = [K_{ik}{:}K] = 2$ for some i, then $K \lhd K_i = \langle K_{ij}, K_{ik}\rangle$ ([A2, 28.1(2)]); K_i/K is generated by two elements of order 2 and hence dihedral; and this is a contradiction since it means there are other overgroups of K not conjugate to any of the given ones.

Thus, $[K_{ij}{:}K] = 2$ for at most one pair of indices i, j. So if (1) does not hold, then the three indices $[K_{ij}{:}K]$ must be $(2, 3, 3)$, $(2, 3, 4)$, or $(2, 3, 5)$. Since each index is prime to p (K contains a Sylow p-subgroup), this shows that $p \geq 5$. If $[K_{ij}{:}K] = m$, then the permutation action of K_{ij} on the set K_{ij}/K restricts to a homomorphism $\varphi_{ij}{:}K \to \Sigma_{m-1} \subseteq \Sigma_4$ whose kernel R_{ij} is normal in K_{ij}. Set $H = O^{\{2,3\}}(K) \lhd K$: the smallest normal subgroup of index a product $2^r{\cdot}3^s$. Then H is characteristic in any subgroup of K which contains H, and in particular characteristic in each R_{ij}. So H is normal in each K_{ij}, and hence normal in $G = \langle K_{12}, K_{13}, K_{23}\rangle$. Since G is simple, $H = 1$, so 2 and 3 are the only primes dividing $|K|$. And this contradicts the assumption that K contains a Sylow p-subgroup for some $p \geq 5$ and $p\big||G|$. □

We are now ready to prove:

Proposition 7.2. *Let G be any of the sporadic simple groups, or the Tits group ${}^2F_4(2)'$. Then there is no 2-dimensional acyclic G-complex without fixed points.*

Proof. We first prove the proposition for ten of the sporadic groups as well as the Tits group, by direct application of Proposition 5.4. Since M_{22} is one of these groups, Proposition E.3 then applies to prove the proposition for the other four Mathieu groups. The last twelve sporadic groups are then handled using Lemma 7.1. Throughout the proof, whenever two names are given for one of the sporadic groups, the first is that used in [Atl], and the second the name used in [A1] or [A2].

Assume the proposition does not hold, and let G be the smallest such group which has a fixed-point free action on a 2-dimensional acyclic complex X. Let $\mathcal{F}$ be the separating family of subgroups $H \subseteq G$ such that $X^H \neq \emptyset$. Consider first the following table, which describes how Proposition 5.4 can be applied to these eleven simple groups:

G	K_1	K_2	K_3	K_{12}	K	K_{13}	K_{23}	$[K_{ij}{:}K]$
J_1	$2^3{:}7{:}3$	$7{:}6$ $= N(7)$	$C_3\times D_{10}$ $= C(3)$	$7{:}3$	3	6	6	$7,2,2$
M_{22}	$L_3(4)$ (point)	$2^4{:}S_5$ (duad)	$2^4{:}A_6$ (hexad)	$2^4{:}A_5$	$2^{2+4}{:}3$ $= 2^4{:}A_4$	$2^4{:}A_5$	$2^{2+4}{:}S_3$ $= 2^4{:}S_4$	$5,5,2$
J_2	$U_3(3)$	$3{\cdot}PGL_2(9)$ $= N(3A)$	$2_-^{1+4}{:}S_3$ $= N(4)$	$3_-^{1+2}{:}8$	$3{:}8$	$4{\cdot}S_4$	$3{:}D_{16}$	$9,4,2$
HS	$U_3(5){:}2$	$U_3(5){:}2$ $\supseteq N(\mathrm{Syl}_5)$	$C(2x)$	$5_+^{1+2}{:}8{:}2$	$5{:}8{:}2$	$2S_5.2$	$2S_5.2$	$25,6,6$
J_3	$L_2(16){:}2$	$L_2(17)$ $\supseteq N(17)$	$2_-^{1+4}{:}S_3$ $= N(4)$	$17{:}4$	4	$D_8\times 2$	D_{16}	$17,4,4$
He	$S_4\times L_3(2)$	$L_3(2)\times 7{:}3$ $= N(7)$	$N(3x)$	$S_4\times(7{:}3)$	$S_4\times 3$	$S_4\times D_6$	$L_3(2)\times 3$	$7,2,7$
Ru	${}^2F_4(2)$ (point)	$L_2(25){\cdot}2^2$ (edge)	$3{\cdot}A_6{\cdot}2^2$ $= N(3)$	$L_2(25){\cdot}2$	$D_{24}.2$	$(3_+^{1+2}{:}D_8).2$	$D_{24}.2^2$	$\bullet,9,2$
O'N	J_1	$L_3(7){:}2$ $\supseteq N(19)$	$(3^2\times A_6){\cdot}2$ $= N(3)$	$19{:}6$	6	$D_6\times D_{10}$	$S_3\times S_4$	$19,10,24$
HN $= F_5$	A_{12} (point)	$M_{12}{:}2$ (point pair)	$C(2x)$	M_{12}	$2\times S_5$ $= C_{M_{12}}(2A)$	$2^5{:}S_6$ $= C_{A_{12}}(2B)$	$(2^2\times A_5){:}2$	$\bullet,96,2$
Th $= F_3$	${}^3D_4(2){:}3$	$(3\times 13){:}12$ $= N(13)$	$2_+^{1+8}{:}A_9$ $= C(2)$	$13{:}12$	12	$C_{K_1}(2x)$ order ≥ 9216	3×12	$13,\bullet,3$
${}^2F_4(2)'$	$L_2(25)$	$5^2{:}4A_4$ $= N(\mathrm{Syl}_5)$	$[2^9{\cdot}3]$ $C(2B)$	$5^2{:}12$	12	D_{24}	$4A_4$	$25,2,4$

Table 5

We refer to [Atl] for the existence of subgroups with these properties, and to [GLS, Table 5.3] for tables of normalizers of prime order subgroups of the sporadic groups. The subgroups in Table 5 are described using mostly the notation of [Atl]. However, we write, for example, $N(3)$ or $C(3)$ to denote the normalizer or centralizer in G of a subgroup of order 3 when there is a unique G-conjugacy class of such subgroups; and write $N(3A)$ or $N(3B)$ (or $N(3x)$ when the class is unspecified) only when there is more than one class. Also, Syl_p always denotes a Sylow p-subgroup of G.

In all cases, the results of Section 6 and Appendix F and the minimality assumption on G imply that $K_i \in \mathcal{F}$ for all $i = 1,2,3$. Note in particular the cases $G = \mathrm{HS}$, He, and HN: $K_3 \in \mathcal{F}$ since K_{13} or K_{23} is nonsolvable and in $\mathcal{F}$.

The remarks under the names of the subgroups K_i describe how they are chosen relative to one another. In all cases except $G = M_{22}$, K_1 and K_2 are chosen in one of the following two ways: either

(a) they are the stabilizers of a vertex and an edge (or point pair) of a standard action of G on a graph; or

(b) K_1 is a maximal subgroup of G, and K_2 is the normalizer of some subgroup $X \subseteq K_1$ (as indicated in the table), or a maximal subgroup (not conjugate to K_1) containing $N_G(X)$ and such that $K_{12} = N_{K_1}(X)$.

The subgroup K_3 is then chosen as the normalizer or centralizer of a certain subgroup $Y \subseteq K_{12}$ as indicated. In all cases where K_{12} contains more than one conjugacy class of subgroup of the given order, the choice is either specified under $K = N_{K_{12}}(Y)$, or is clear from the description of K. In many cases, it is unnecessary to identify K_3 more precisely, since the only thing we need know about it is that it must lie in $\mathcal{F}$.

When $G = M_{22}$, $K_3 \cong 2^4{:}A_6$ is the subgroup which leaves invariant some hexad in the Steiner system of order 22, and it has the obvious action on this set of order 6 (cf. [Gr, Theorem 6.8]). Then K_1 is taken to be the stabilizer of some point x in the hexad, and K_2 the stabilizer of some pair of points in the hexad including x.

In all cases, each of the subgroups in the sequence $K \subseteq K_{12} \subseteq K_1 \subseteq G$ is maximal and self-normalizing in the next one. Thus, conditions (b,c,d) in Proposition 5.4 always hold. Condition (e) ((K_1, K_{12}, K) is not G-conjugate to (K_1, K_{13}, K)) is clear except when $G = M_{22}$; in this case K_{12} and K_{13} are distinct parabolic subgroups in $K_1 \cong L_3(4)$ containing the same Borel subgroup K, and hence not conjugate in K_1. Inequality (a1) holds in all cases except when $G = J_1$, as can be checked using the list of indices $[K_{ij}{:}K]$ in the last column (where "$\bullet$" means that the index is >10 and hence large enough not to matter).

We give particular attention to the case $G = J_1$: the first Janko group, and the only sporadic group not handled in [AS]. Fix some $K_1 \cong C_2^3 \rtimes (C_7 \rtimes C_3)$: a maximal subgroup of G by [A2, 16.17] (see also 16.4 and 16.16 in [A2]). Let $K_2 \cong C_7 \rtimes C_6$ be the normalizer of a subgroup of order 7 in K_1, and let $K_3 \cong C_3 \times D_{10}$ be the centralizer in G of a subgroup of order 3 in K_{12}. Then $K_{12} \cong C_7 \rtimes C_3$, $K_{13} \cong C_6 \cong K_{23}$, and $K = K_1 \cap K_2 \cap K_3 \cong C_3$. All of these subgroups are solvable, and hence in $\mathcal{F}$. Also,

$$\sum_{i<j} \frac{1}{[K_{ij}{:}K]} = \frac{1}{7} + \frac{1}{2} + \frac{1}{2} < 1 + \frac{1}{14} + \frac{1}{10} = 1 + \frac{1}{[K_2{:}K]} + \frac{1}{[K_3{:}K]},$$

which proves inequality (a2) in Proposition 5.4. The other conditions in 5.4 are easily checked, and so J_1 has no fixed-point free action on a 2-dimensional acyclic complex.

The remaining twelve sporadic groups can now be handled using Lemma 7.1. In [A2, §28], a p-basis for G is defined to be a regular $(N_G(B), B)$-basis for some $B \subseteq G$ which contains a Sylow p-subgroup T of G, and such that the basis contains representatives for all G-conjugacy classes of maximal subgroups in G which contain T (not only conjugacy class representatives for maximal overgroups of B). Maximal overgroups of the Sylow subgroups of sporadic groups are listed in [A2], and conditions for their forming a p-basis are given in [A2, Theorem 1]. So from [A2, pp. 7-36], we get the following list of sporadic groups G and primes $p = 2$ or 3, where in each case G has a p-basis with at least three elements already known not to have fixed point free actions on 2-dimensional acyclic complexes: J_4 ($p = 2$), McL ($p = 3$), Suz ($p = 3$), Ly ($p = 3, 5$), Co_3 ($p = 2, 5$), Co_2 ($p = 2, 3, 5$), Co_1 ($p = 2, 3$), $\mathrm{Fi}_{22} = M(22)$ ($p = 2, 3$), $\mathrm{Fi}_{23} = M(23)$ ($p = 3$), $\mathrm{Fi}_{24}' = M(24)'$ ($p = 2, 3$), $B = F_2$ ($p = 2, 3, 5$), $M = F_1$ ($p = 2, 3, 5$). Note in particular the case $G = \mathrm{Ly}$ and $p = 3$: the maximal overgroup G_2 ([A2, p.19]) must lie in $\mathcal{F}$ since it surjects onto S_5. This list includes all of the sporadic groups not dealt with in Table 5 or in Proposition E.3, and thus finishes the proof of the proposition. □

8. Proof of Theorem A

We are now ready to prove Theorem A.

Theorem A. *For any finite group G, there is an essential fixed point free 2-dimensional (finite) $\mathbb{Z}$-acyclic G-complex if and only if G is isomorphic to one of the simple groups $L_2(2^k)$ for $k \geq 2$, $L_2(q)$ for $q \equiv \pm 3 \pmod 8$ and $q \geq 5$, or $\mathrm{Sz}(2^k)$ for odd $k \geq 3$. Furthermore, the isotropy subgroups of any such G-complex are all solvable.*

Proof. By Proposition 4.4, if G has an essential action on an acyclic 2-complex X without fixed points, then there is a nonabelian simple normal subgroup $L \lhd G$ whose action also is fixed point free, and such that $C_G(L) = 1$ (i.e., $G \subseteq \mathrm{Aut}(L)$). By the classification theorem, L must be an alternating group, or of Lie type, or the Tits group ${}^2F_4(2)'$, or one of the 26 sporadic simple groups. By [S1, 3.7] (Proposition E.3), L cannot be any of the alternating groups A_n for $n \geq 6$. By [AS, §5] (or Proposition E.4), L cannot be of Lie type and of Lie rank two or more. By Proposition 7.2, L cannot be any of the sporadic simple groups, nor the Tits group (see [AS, §6] for all of these except the first Janko group J_1). Hence L must be of Lie type and of Lie rank one. The groups $U_3(q)$ are eliminated by Propositions 6.5 and 6.6, and the Ree groups ${}^2G_2(q)$ by Proposition 6.7. We are thus reduced to the case where $L \cong L_2(q)$ or $\mathrm{Sz}(q)$; and this was handled in Proposition 6.4. □

Appendix

Throughout the appendix, G will always denote a finite group, though most of the definitions and results stated in Parts A and B apply equally well to actions of an infinite discrete group. A "map" (between spaces or CW complexes) always means a continuous map.

Parts A and B give a brief introduction to (G-) CW complexes and their homology, for readers not already familiar with them. In part C, several results — both elementary and deep — about projective and free $\mathbb{Z}[G]$-modules are given. A survey of of some of the theory of finite simple groups of Lie type is given in part D. Finally, in part E, we sketch some of the results shown in [S1] and [AS] on the nonexistence of fixed point free actions of certain multiply transitive groups, and of certain simple groups of Lie type, on 2-dimensional acyclic complexes.

Appendix A. G-CW complexes

We use [LW] as a general reference for the definition(s) and properties of CW complexes. The following is a combination of [LW, Definitions I.1.1 and II.1.1], but extended to the equivariant case.

Definition A.1. *A G-CW complex is a Hausdorff space X with continuous G-action (i.e., G is represented as a group of homeomorphisms of X), together with a filtration $X^{(0)} \subseteq X^{(1)} \subseteq X^{(2)} \subseteq \cdots$ by closed G-invariant subspaces (the "skeleta" of X), as well as discrete G-sets J_m and G-equivariant "characteristic maps" $\varphi_m \colon J_m \times D^m \to X$ (for all $m \geq 0$), which satisfy the following properties.*

(a) $X = \bigcup_{m=0}^{\infty} X^{(m)}$. *For each* m, φ_m *restricts to a homeomorphism*

$$J_m \times \mathrm{int}(D^m) \xrightarrow{\cong} (X^{(m)} \smallsetminus X^{(m-1)}).$$

(b) For each $m > 0$, $\varphi_m(J_m \times S^{m-1}) \subseteq X^{(m-1)}$. *Moreover, for each* $j \in J_m$, *there are finite subsets* $J'_k \subseteq J_k$ $(0 \le k \le m-1)$ *such that*

$$\varphi_m(j \times S^{m-1}) \subseteq \bigcup_{k=0}^{m-1} \varphi_k(J'_k \times D^k).$$

(c) A subset $U \subseteq X$ *is open if and only if* $\varphi_m^{-1}(U)$ *is open in* $J_m \times D^m$ *for each* $m \ge 0$. *(*X *has the "weak topology" with respect to its cell structure.)*

In the above definition, G is always assumed to act trivially on D^m and S^{m-1}. Usually, a G-CW complex will be called a G-complex for short.

A CW complex is just a G-CW complex in the case where G is the trivial group. An *open cell* in a (G-)CW complex X is the image $\varphi_m(j \times \mathrm{int}(D^m))$ of one open disk under the characteristic map. Note that if $\sigma = \varphi_m(j \times \mathrm{int}(D^m))$ is any open cell, then $\varphi_m(j \times D^m) = \overline{\sigma}$ (the closure of σ) and $\partial\sigma = \varphi_m(j \times S^{m-1}) = \overline{\sigma} \smallsetminus \sigma$ (the boundary of σ) are determined by σ itself as a subspace of X. By condition (a) in the definition, each point of X lies in exactly one open cell, and the open m-cells of X are the connected components of $X^{(m)} \smallsetminus X^{(m-1)}$.

The following is an alternative way to regard G-complexes, once CW complexes have been defined. Fix a CW complex X with continuous G-action. Call the action *admissible* if it permutes the open cells of X, and sends a cell to itself only via the identity. If X is a G-complex, then by definition the G-action is admissible. Conversely, if the action of G on X is admissible, then the characteristic maps of X can be redefined to yield a G-complex. More precisely, if $\varphi_m \colon J_m \times D^m \to X$ is the given characteristic map for the m-cells of X, then the action of G on the m-cells of X induces an action on J_m. Also, for any orbit Ω of G on J_m and any $j \in \Omega$, one can define φ'_m on $\Omega \times D^m$ by setting $\varphi'_m(gj, x) = g\varphi_m(j, x)$. Upon doing this for all $m \ge 0$ and all orbits of J_m, we get the new characteristic maps which make X into a G-complex.

Note in particular the last part of condition (b). Each cell in a CW complex must be "closure finite": its boundary must be contained in a finite union of closed cells of smaller dimensions. To see the importance of this condition, consider the space $X = D^2$, let J_0 be the circle S^1 with the discrete topology, let J_2 be a set with one element, and set $J_m = \emptyset$ for all $m \ne 0, 2$. Let $\varphi_0 \colon J_0 \times D^0 \to X$ be projection to the first factor (i.e., inclusion of the circle), and let $\varphi_2 \colon J_2 \times D^2 \to X$ be projection to the second factor. These sets and maps satisfy all of the conditions for a CW structure on D^2 except for closure finiteness. But this goes against our intuitive expectations (by analogy with simplicial complexes) that the 0-skeleton of any CW complex should be discrete, and that compact CW complexes should be made up of finitely many cells.

The following lemma describes the principal means of constructing G-complexes (see, e.g., Proposition 0.2).

Lemma A.2. *Let* X *be a* G*-complex, let* J *be any discrete set with* G*-action, and let* $\varphi \colon J \times S^{n-1} \to X^{(n-1)}$ *be any* G*-equivariant map. Then the space*

$$Y = X \cup_\varphi (J \times D^n)$$

is a G-complex.

Proof. Let $\Phi\colon J{\times}D^n \to Y$ be the obvious map; thus $\Phi|_{J\times S^{n-1}} = \varphi$. This, together with the characteristic maps for X, make up the characteristic maps for Y. The other details are the same as in the nonequivariant case; cf. [LW, Proposition II.2.2]. □

If X has been constructed via successively attaching cells, i.e., via successive repetition of the construction of Lemma A.2, starting with a discrete set, then the closure finiteness condition holds automatically. In fact, this is the basis for an alternative definition of a CW complex, described more precisely in [LW, Theorem II.2.4].

A (G-invariant) *subcomplex* of a G-CW complex is a closed (G-invariant) subspace $A \subseteq X$ which is a union of closed cells in X; i.e., a union of images of characteristic maps. A subcomplex is itself a CW complex in an obvious way. Note in particular that if X is a G-complex, then for every $H \subseteq G$, the fixed point set X^H is a subcomplex of X: if $\varphi_m\colon J_m{\times}D^m \to X$ are the characteristic maps for X, then $(\varphi_m)^H\colon (J_m)^H{\times}D^m \to X^H$ are the characteristic maps for X^H.

The following proposition is an immediate consequence of condition (c) in Definition A.1. Roughly, it says that a function defined on a CW complex is continuous if and only if its restriction to each closed cell of the complex is continuous.

Lemma A.3. *Let X be a CW complex, with characteristic maps $\varphi_m\colon J_m{\times}D^m \to X$. Then if Y is any topological space, a function $f\colon X \to Y$ is continuous if and only if $f\circ\varphi_m$ is continuous for each m.* □

Recall (cf. [S1], [AS]) that a simplicial complex X with G action is called *admissible* if the action permutes the simplices linearly, and sends a simplex to itself only via the identity. (If this last condition does not hold, then it does hold for the barycentric subdivision of X.) We claimed in the introduction that Theorem A holds equally well if one replaces "G-complex" by "admissible G-simplicial complex" in the statement. This follows from the following proposition, where simplicial complexes are always assumed to have the metric topology (cf. [LW, Definition IV.4.1]).

Proposition A.4. *Any finite dimensional admissible G-simplicial complex is G-homotopy equivalent to a G-complex of the same dimension. Any countable, finite dimensional G-complex is G-homotopy equivalent to an admissible G-simplicial complex of the same dimension.*

Proof. For any admissible G-simplicial complex X, one can clearly define skeleta and characteristic maps for X which satisfy conditions (a) and (b) in Definition A.1; but for these to also satisfy condition (c) we must replace X with a new space X_{cw} having the same underlying set but a finer topology (more open sets). The identity map $X_{cw} \to X$ is continuous and is a homotopy equivalence by [LW, Proposition IV.4.6] (and the argument in [LW] can easily be fixed to cover the equivariant case).

The second statement is shown, in the nonequivariant case, by Whitehead in [Wh, Theorem 13], and his proof carries over immediately to G-complexes. The idea is the following: once $X^{(m-1)}$ has been replaced by a G-simplicial complex of the same dimension, then approximate the characteristic map $J_m{\times}S^{m-1} \to X^{(m-1)}$ by a simplicial map (possibly after further subdivision of $X^{(m-1)}$), and attach the m-cells after giving them appropriate simplicial structure. □

For any space X, we let ΣX denote its *unreduced suspension*: $\Sigma X \overset{\text{def}}{=} (X\times I)/\sim$, where $(x,0)\sim(x',0)$ and $(x,1)\sim(x',1)$ for all $x,x'\in X$. A G-action on X automatically determines a G-action on ΣX, via the trivial action on the interval I.

Lemma A.5. *The orbit space X/G of a G-complex X inherits a structure of a CW complex, with one n-cell in X/G for each G-orbit of n-cells in X. The unreduced suspension ΣX of any G-complex X is itself a G-complex in a natural way. And if X and Y are any two G-complexes, at least one of which is finite, then their product $X\times Y$ is also a G-complex.*

Proof. If X is a G-complex, with skeleta $X^{(m)}$ and characteristic maps $\varphi_m\colon J_m\times S^m \to X$, then X/G is a CW complex with skeleta $(X/G)^{(m)} = X^{(m)}/G$ and characteristic maps $\varphi_m/G\colon (J_m/G)\times S^m \to X/G$. This follows immediately from Definition A.1. Note in particular that condition (c) holds for X/G by definition of the quotient topology: a subspace is open in X/G if and only if its inverse image is open in X.

The unreduced suspension of a CW complex is again a CW complex by [LW, Corollary II.5.12]. And if X or Y is finite, then $X\times Y$ is a G-complex with the obvious product structure by [LW, Theorem II.5.2]. In each of these last two cases, the arguments in [LW] carry over without change to the equivariant case.

We remark here that if X and Y are arbitrary CW complexes, then there is an obvious way to define skeleta for $X\times Y$: $(X\times Y)^{(m)} = \bigcup_{i+j=m}(X^{(i)}\times Y^{(j)})$. Also, if $\varphi_m\colon J_m\times D^m \to X$ and $\psi_m\colon K_m\times D^m\times Y$ are the characteristic maps for X and Y, then one can define characteristic maps $\omega_m = \coprod_{i+j=m}(\varphi_i\times\psi_j)$ for $X\times Y$. (This requires fixing identifications $D^i\times D^j \cong D^{i+j}$.) Conditions (a) and (b) in Definition A.1 always hold; what can go wrong is condition (c). □

The following lemma is not used in the paper, but does help to motivate the concept of "universal" $(G,\mathcal{F})$-complexes as defined in Section 0.

Proposition A.6. *Fix a family $\mathcal{F}$ of subgroups of G, and let Y be any universal $(G,\mathcal{F})$-complex. Then for any $(G,\mathcal{F})$-complex X, any G-invariant subcomplex $A\subseteq X$, and any equivariant map $f_0\colon A\to Y$, f_0 extends to an equivariant map $f\colon X\to Y$. Furthermore, f is unique up to homotopy, in the sense that if $f'\colon X\to Y$ is any other extension of f_0, then there is an equivariant homotopy $F\colon X\times I\to Y$ such that $F|_{X\times 0}=f$, $F|_{X\times 1}=f'$, and $F|_{A\times I} = f_0\circ\mathrm{proj}_A$.*

Proof. It suffices to prove the existence of $f\colon X\to Y$; the uniqueness then follows by extending the given map on $(X\times\{0,1\})\cup(A\times I)$ to $X\times I$.

We construct $f\colon X\to Y$ one skeleton at a time. The construction of $f_0\colon X^{(0)}\cup A\to Y$ is easy: let $\{x_i\}$ be orbit representatives for the vertices not in A, set $H_i = G_{x_i}$ (the isotropy subgroup), choose any $y_i\in Y^{H_i}$, and define $f_0(gx_i)=gy_i$ for all $g\in G$ and all i (and $f_0|_A = f_A$).

Now assume that $n\geq 1$, and that $f_{n-1}\colon X^{(n-1)}\cup A\to Y$ has been constructed. Let $\varphi_n\colon J_n\times D^n\to X$ be the characteristic map for the n-cells of X (where J_n is a discrete set with G-action), and let $J_n'\subseteq J_n$ be the subset of those n-cells not in A. Set

$$u_0 = f_{n-1}\circ\varphi_n|_{J_n'\times S^{n-1}}\colon J_n'\times S^{n-1} \longrightarrow Y.$$

For each $j \in J'_n$, let $G_j = \{g \in G \,|\, gj = j\} \in \mathcal{F}$ be its isotropy subgroup. Then $u_0(j\times S^{n-1}) \subseteq Y^{G_j}$. Also, Y^{G_j} is contractible (since Y is $(G,\mathcal{F})$-universal), the identity map $Y^{G_j} \to Y^{G_j}$ is homotopic to a constant map, and hence any map to Y^{G_j} is homotopic to a constant map. In particular, u_0 can be extended to a (nonequivariant) map $v'_j\colon j\times D^n \to Y^{G_j}$. This can then be extended to a G-map $v_j\colon Gj\times D^n \to Y$ (where Gj is the orbit of j) by setting $v_j(gj,x) = g{\cdot}v'_j(j,x)$. Upon repeating this procedure with one representative from each G-orbit in J'_n, the v_j combine to give a G-map $u\colon J'_n\times D^n \to Y$ whose restriction to $J'_n\times S^{n-1}$ is u_0. If we now set $f_n(x) = f_{n-1}(x)$ for $x \in X^{(n-1)} \cup A$, and $f_n(\varphi_n(j,x)) = u(j,x)$ for $(j,x) \in J'_n\times D^n$, then this is a well defined map of sets from $X^{(n)}$ to Y, which is equivariant by construction, and continuous by Lemma A.3. □

Note that Proposition A.6 implies in particular that any two universal $(G,\mathcal{F})$-complexes are G-homotopy equivalent.

Appendix B. Cellular homology of G-complexes

The *cellular chain complex* $(C_n(X),\partial_n)_{n\geq 0}$ of a CW complex is described in [LW, §V.2]. Formally, this is defined using singular homology (in particular, $C_n(X) = H_n(X^{(n)},X^{(n-1)})$), as in [LW, Definition V.2.1]. By [LW, Proposition V.1.8], $C_n(X)$ is the free abelian group with basis the set of (oriented) n-cells in X; and by [LW, §V.3] each boundary map $\partial_n\colon C_n(X) \to C_{n-1}(X)$ can be described via the matrix whose entries are the degrees of maps between $(n-1)$-spheres induced by the attaching maps for the n-cells. By [LW, Theorem V.2.11], the singular homology $H_*(X)$ is isomorphic to the homology of the complex $(C_n(X),\partial_n)$. Hence, if X is a finite complex, the Euler characteristic $\chi(X)$ is equal to the alternating sum of the numbers of cells in each dimension.

Note that for a map $f\colon X \to Y$ between CW complexes to induce a homomorphism $C_*(X) \to C_*(Y)$, it must be a cellular map, in the sense that $f(X^{(n)}) \subseteq Y^{(n)}$ for all $n \geq 0$. However, since cellular homology $H_*(C_*(X),\partial)$ is isomorphic to singular homology, any continuous map between CW complexes induces a homomorphism between their cellular homology groups.

More generally, if X is any CW complex and $A \subseteq X$ is any subcomplex, then the relative cellular chain complex is defined by setting $C_*(X,A) \overset{\text{def}}{=} C_*(X)/C_*(A)$. Thus, $C_n(X,A)$ is the free abelian group with one generator for each n-cell of X not in A. By [LW, Theorem V.2.11] again, the homology of the complex $(C_*(X,A),\partial)$ is naturally isomorphic to $H_*(X,A)$.

If X is a G-complex and $A \subseteq X$ is a G-invariant subcomplex, then the cellular chain complexes $C_*(X)$ and $C_*(X,A)$, and the homology groups $H_*(X)$ and $H_*(X,A)$, are all $\mathbb{Z}[G]$-modules. In fact, each chain group $C_i(X)$ or $C_i(X,A)$ is a permutation module, in the sense that it has a $\mathbb{Z}$-basis which is permuted by the linear action of G.

Once homology has been defined using the cellular chain complex, then the relative and Mayer-Vietoris exact sequences, and excision, are immediate. (Note, however, that excision in singular homology is needed to establish the basic properties of cellular homology of CW complexes [LW, §V.1–2].) To see this, fix a G-complex X. For any

G-invariant subcomplexes $A_0 \subseteq A \subseteq X$, the short exact sequence of chain complexes

$$0 \longrightarrow C_*(A)/C_*(A_0) \longrightarrow C_*(X)/C_*(A_0) \longrightarrow C_*(X)/C_*(A) \longrightarrow 0$$

induces, via the snake lemma, the relative exact sequence

$$\cdots \longrightarrow H_i(A, A_0) \longrightarrow H_i(X, A_0) \longrightarrow H_i(X, A) \xrightarrow{\partial} H_{i-1}(A, A_0) \longrightarrow \cdots .$$

Similarly, for any pair of G-invariant subcomplexes $A, B \subseteq X$ with $A \cup B = X$, there is a short exact sequence

$$0 \longrightarrow C_*(A \cap B) \longrightarrow C_*(A) \oplus C_*(B) \longrightarrow C_*(A \cup B) \longrightarrow 0$$

which induces the Mayer-Vietoris sequence

$$\cdots \longrightarrow H_i(A \cap B) \longrightarrow H_i(A) \oplus H_i(B) \longrightarrow H_i(X) \xrightarrow{\partial} H_{i-1}(A \cap B) \longrightarrow \cdots .$$

All of these are exact sequences of $\mathbb{Z}[G]$-modules.

Similarly, since $B \smallsetminus (A \cap B)$ contains exactly the same cells as $(A \cup B) \smallsetminus A$, the inclusion map induces an isomorphism

$$H_*(B, A \cap B) \xrightarrow{\cong} H_*(A \cup B, A) \qquad \text{(excision)}$$

since it induces an isomorphism of cellular chain complexes.

The following lemma, used in the proof of Proposition 0.2, is one application of excision and the relative exact sequence. It describes the effect of attaching cells on the homology of the complexes involved.

Lemma B.1. *Let X be a G-complex, let J be a discrete set with G-action, and let $f\colon J \times S^n \to X^{(n)}$ be any G-equivariant map ($n \geq 1$). Set $Y = X \cup_f (J \times D^{n+1})$. Then there is an exact sequence of $\mathbb{Z}[G]$-modules*

$$0 \longrightarrow H_{n+1}(X) \xrightarrow{\text{incl}_*} H_{n+1}(Y) \longrightarrow$$
$$H_n(J \times S^n) \xrightarrow{f_*} H_n(X) \xrightarrow{\text{incl}_*} H_n(Y) \longrightarrow 0;$$

and the inclusion $X \xrightarrow{\text{incl}} Y$ induces isomorphisms $H_i(X) \cong H_i(Y)$ for all $i \neq n, n+1$.

Proof. Let $\alpha\colon J \times D^{n+1} \to Y$ be the characteristic map (so $\alpha|_{J \times S^n} = f$). This induces an isomorphism $C_*(J \times D^{n+1}, J \times S^n) \cong C_*(Y, X)$ of chain complexes, and hence an isomorphism in homology in all degrees. The following square

$$\begin{array}{ccc} H_{n+1}(J \times D^{n+1}, J \times S^n) & \xrightarrow[\cong]{\partial} & H_n(J \times S^n) \\ {\scriptstyle \alpha_*} \big\downarrow {\scriptstyle \cong} & & {\scriptstyle H_n(f)} \big\downarrow \\ H_{n+1}(Y, X) & \xrightarrow{\partial} & H_n(X) \end{array}$$

commutes by the naturality of the relative exact sequences for pairs of CW complexes, and the upper boundary map is an isomorphism since $H_i(J \times D^{n+1}) = 0$ for $i \geq 1$. The lemma now follows from the relative exact sequence for the pair (Y, X), where $H_{n+1}(Y, X)$ is replaced by $H_n(J \times S^n)$ via the above square. □

The following more technical application of excision and the relative exact sequences is needed in the proof of Proposition 1.7.

Lemma B.2. *Fix a CW complex Y and subcomplexes $B, X \subseteq Y$, and set $A = B \cap X$. Assume that the inclusion induces an isomorphism $H_*(A) \to H_*(B)$. Then $H_*(Y, X) \cong H_*(Y, X \cup B)$.*

Proof. It suffices to show that $H_*(X \cup B, X) = 0$; the result then follows from the relative exact sequence for $Y \supseteq X \cup B \supseteq X$. But $H_*(X \cup B, X) \cong H_*(B, A)$ by excision, and this last group vanishes since $H_*(A) \cong H_*(B)$. □

The following result says, roughly, that a union of homology or homotopy equivalences between CW complexes is again a homology or homotopy equivalence.

Proposition B.3. *Let $f: X \to Y$ be a map between CW complexes. Fix subcomplexes $A_1, A_2 \subsetneqq X$ and $B_1, B_2 \subseteq Y$ such that $X = A_1 \cup A_2$ and $Y = B_1 \cup B_2$, and set $A_0 = A_1 \cap A_2$ and $B_0 = B_1 \cap B_2$. Assume that f restricts to homology (homotopy) equivalences $f_i: A_i \to B_i$ for $i = 0, 1, 2$. Then f is itself a homology (homotopy) equivalence.*

Proof. If f_0, f_1, and f_2 are all homology equivalences, then f is a homology equivalence by the Mayer-Vietoris sequences for the two unions (and the 5-lemma).

Assume now that f_0, f_1, and f_2 are all homotopy equivalences; we must show that f is a homotopy equivalence. By the Van Kampen theorem, f induces an isomorphism of fundamental groups (on each connected component). The map between the universal covers is a homology equivalence, hence a homotopy equivalence; and hence $f: X \to Y$ is itself a homotopy equivalence. For the details of this argument, cf. [Gra, Lemma 16.24 & Theorem 16.22].

Alternatively, and more geometrically, one can show directly that any homotopy inverse $g_0: B_0 \to A_0$ of f_0 can be extended (one cell at a time) to homotopy inverses $g_i: B_i \to A_i$ ($i = 1, 2$), while at the same time extending the homotopies of $g_0 \circ f_0 \simeq \mathrm{Id}_{A_0}$ and $f_0 \circ g_0 \simeq \mathrm{Id}_{B_0}$. The result then follows upon taking $g = g_1 \cup g_2: Y \to X$ (and similarly for the homotopies). The existence of the g_i and the homotopies follows from the proofs of [LW, Theorems IV.3.2–3] (applied to the 2-ads (A_i, A_0) and (B_i, B_0)); although the statements of these theorems are not sufficiently precise to do this. □

Appendix C. Projective $\mathbb{Z}[G]$-modules

Recall that for any G-complex X, $C_*(X)$ and $H_*(X)$ are $\mathbb{Z}[G]$-modules in an obvious way. A finitely generated $\mathbb{Z}[G]$-module M will be called *stably free* if there are *finitely generated* free modules F_0 and F such that $M \oplus F_0 \cong F$. Free $\mathbb{Z}[G]$-modules, and hence (as an intermediate step) stably free $\mathbb{Z}[G]$-modules play a key role when constructing finite G-complexes in Section 1.

Lemma C.1. *If $X \subseteq Y$ are finite G-complexes such that G acts freely on $Y \smallsetminus X$, then $C_*(Y, X)$ is a finite chain complex of free finitely generated $\mathbb{Z}[G]$-modules.*

Proof. By assumption, G permutes freely the cells in Y not in X. Thus, G permutes freely a basis of $C_*(Y, X)$; and this is a finite basis since X and Y have only finitely many cells. □

The following lemma says in particular that if C_* is a finite chain complex of finitely generated free $\mathbb{Z}[G]$-modules all but one of whose homology groups is stably free, then the remaining homology group is also stably free. This does not hold for modules over arbitrary noetherian rings, but uses special properties of group rings.

Proposition C.2. *Let C_* be any finite chain complex of projective $\mathbb{Z}[G]$-modules. Assume, for some k, that $H_i(C_*)$ is projective as a $\mathbb{Z}[G]$-module for all $i \neq k$, and that $H_k(C_*)$ is $\mathbb{Z}$-free. Then $H_k(C_*)$ is also a projective $\mathbb{Z}[G]$-module, and*

$$\bigoplus_{i \text{ even}} H_i(C_*) \oplus \bigoplus_{i \text{ odd}} C_i \cong \bigoplus_{i \text{ odd}} H_i(C_*) \oplus \bigoplus_{i \text{ even}} C_i. \tag{1}$$

Proof. We first claim the following: if $0 \to A \to B \to C \to 0$ is a short exact sequence of finitely generated $\mathbb{Z}$-free $\mathbb{Z}[G]$-modules, and two of the modules A, B, and C are projective (stably free), then so is the third. This is clear if C is projective, since in that case $B \cong A \oplus C$. So assume that A and B are projective (stably free). Since all three groups are $\mathbb{Z}$-free and finitely generated, the dual sequence $0 \to C^* \to B^* \to A^* \to 0$ is also exact. Here, for any $\mathbb{Z}[G]$-module M, $M^* \stackrel{\text{def}}{=} \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ has the obvious structure as a $\mathbb{Z}[G]$-module. Dualization clearly takes finitely generated free $\mathbb{Z}[G]$-modules to free $\mathbb{Z}[G]$-modules, hence the same for projective modules; and so the dualized sequence splits. Thus $B^* \cong A^* \oplus C^*$ as $\mathbb{Z}[G]$-modules; and upon dualizing again we see that $B \cong A \oplus C$. So C is $\mathbb{Z}[G]$-projective (stably free).

Now fix any $m, n \in \mathbb{Z}$ such that $m < k < n$, and $C_i = 0$ for all $i < m$ and all $i > n$. For each i, set $Z_i = \operatorname{Ker}[C_i \xrightarrow{\partial} C_{i-1}]$ and $B_i = \operatorname{Im}[C_{i+1} \xrightarrow{\partial} C_i]$. Consider the short exact sequences

$$0 \longrightarrow Z_i \longrightarrow C_i \longrightarrow B_{i-1} \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow B_i \longrightarrow Z_i \longrightarrow H_i(C_*) \longrightarrow 0.$$

By induction starting at $i = m$, one sees that Z_i is projective for each $i \leq k$, and that B_i is projective for each $i < k$. Similarly, by downward induction starting at $i = n+1$, one sees that B_i is projective for each $i \geq k$, and that Z_i is projective for each $i > k$. In particular, B_k and Z_k are both projective, and so the same holds for $H_k(C_*)$.

In particular, the above short exact sequences split, since all of their terms are projective. Set $C_{\text{ev}} = \bigoplus(C_{2i})$, $C_{\text{od}} = \bigoplus(C_{2i+1})$, $H_{\text{ev}} = \bigoplus(H_{2i}(C_*))$, $H_{\text{od}} = \bigoplus(H_{2i+1}(C_*))$; and similarly for Z_{ev}, Z_{od}, B_{ev}, and B_{od}. Then

$$H_{\text{ev}} \oplus C_{\text{od}} \cong H_{\text{ev}} \oplus B_{\text{ev}} \oplus Z_{\text{od}} \cong Z_{\text{ev}} \oplus Z_{\text{od}},$$

$$H_{\text{od}} \oplus C_{\text{ev}} \cong H_{\text{od}} \oplus B_{\text{od}} \oplus Z_{\text{ev}} \cong Z_{\text{od}} \oplus Z_{\text{ev}},$$

and this proves (1). □

The following property of projective $\mathbb{Z}[G]$-modules is a consequence of a theorem of Swan.

Proposition C.3. *Let P and P' be any two finitely generated projective $\mathbb{Z}[G]$-modules. Then $P \otimes_{\mathbb{Z}} P'$ is stably free as a $\mathbb{Z}[G]$-module.*

Proof. Assume first that P is free. Let $\{a_i\}$ be a $\mathbb{Z}[G]$-basis for P, and let $\{b_j\}$ be a $\mathbb{Z}$-basis for P'. Then $\{a_i \otimes b_j\}$ is a $\mathbb{Z}[G]$-basis for $P \otimes P'$, and this module is free. (Note that we did not need to know that P' is projective, only that it is $\mathbb{Z}$-free.)

Now consider the general case. By [Sw, Theorems 7.1 and 8.1], for any $n > 0$, any finitely generated projective $\mathbb{Z}[G]$-module contains a free submodule of finite index prime to n. In particular, we can choose free submodules $F \subseteq P$ and $F' \subseteq P'$, such that $[P{:}F]$ and $[P'{:}F']$ are finite and relatively prime. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F \otimes F' & \xrightarrow{i_1} & P \otimes F' & \longrightarrow & (P/F) \otimes F' & \longrightarrow & 0 \\ & & {\scriptstyle j_1}\big\downarrow & & {\scriptstyle j_2}\big\downarrow & & {\scriptstyle \alpha}\big\downarrow{\scriptstyle \cong} & & \\ 0 & \longrightarrow & F \otimes P' & \xrightarrow{i_2} & P \otimes P' & \longrightarrow & (P/F) \otimes P' & \longrightarrow & 0, \end{array}$$

where all tensor products are taken over $\mathbb{Z}$. The rows are both exact, and α is an isomorphism since $(P/F) \otimes (P'/F') = 0$. So by an easy diagram chase, the sequence

$$0 \longrightarrow F \otimes F' \xrightarrow{(i_1,j_1)} (P \otimes F') \oplus (F \otimes P') \xrightarrow{j_2 - i_2} P \otimes P' \longrightarrow 0$$

is exact. We have just seen that the first two terms in this sequence are free, and so $P \otimes P'$ is stably free. □

In fact, using stability results of Swan, one can show that the tensor product of any two finitely generated projective $\mathbb{Z}[G]$-modules is free. This is not needed for the constructions in this paper, but the following much deeper stability result *is* used. It is not needed to prove the existence of 2-dimensional acyclic G-complexes, but it is used in Section 3 to show that all of the complexes we construct can be taken to have exactly one free orbit of 2-cells (and no free orbits of cells in other dimensions).

Proposition C.4. *If G is simple, or (more generally) if there is no homomorphism $G \to SU(2)$ $(= SU(2,\mathbb{C}))$ with nonabelian image, then any stably free $\mathbb{Z}[G]$-module is free.*

Proof. By a theorem of Jacobinski [Jac, Theorem 4.1], if $\mathfrak{A}$ is any $\mathbb{Z}$-order in a finite dimensional semisimple $\mathbb{Q}$-algebra A which satisfies the *Eichler condition*, then all finitely generated stably free $\mathfrak{A}$-modules are free. Here, the algebra A satisfies the Eichler condition if it has no simple factor B, with center K, for which every embedding $K \hookrightarrow \mathbb{C}$ has image contained in $\mathbb{R}$ and induces an isomorphism $\mathbb{R} \otimes_K B \cong \mathbb{H}$ (the quaternion algebra over $\mathbb{R}$).

If $\mathbb{Q}[G]$ does not satisfy the Eichler condition — if B is a simple summand of $\mathbb{Q}[G]$ and $\mathbb{R} \otimes_K A \cong \mathbb{H}$ — then the composite

$$\mathbb{Q}[G] \xrightarrow{\text{proj}} B \longrightarrow \mathbb{H}$$

restricts to a multiplicative homomorphism $\alpha\colon G \to S^3 \cong SU(2,\mathbb{C})$. Here, S^3 denotes the group of quaternions of norm 1. And since the image of G in $\mathbb{H}$ generates $\mathbb{H}$ as an $\mathbb{R}$-vector space, $\mathrm{Im}(\alpha)$ must be nonabelian. See also [Re, §38] for more discussion. □

Appendix D. Finite simple groups of Lie type

We give here a very short discussion of groups of Lie type. For more detail, we refer to [St1], [St2], [Ca], or [GLS].

The finite simple groups of Lie type consist of the Chevalley groups and their twisted analogs. The finite Chevalley groups are analogs of the (complex or compact) Lie groups, but realized over a finite field. They thus include the four families of classical groups: $A_n(q) \cong L_{n+1}(q) = PSL_{n+1}(q)$, $B_n(q) \cong P\Omega_{2n+1}(q)$ (the commutator subgroup of the projective orthogonal group $PGO_{2n+1}(q)$), $C_n(q) \cong PSp_n(q)$, and $D_n(q) \cong P\Omega^+_{2n}(q)$ (the commutator subgroup of the projective special orthogonal groups with respect to a quadratic form of "plus type"); as well as the exceptional groups $E_6(q)$, $E_7(q)$, $E_8(q)$, $F_4(q)$ and $G_2(q)$. All of these are defined over any finite field; i.e., for any prime power q.

The finite twisted groups of Lie type were first treated systematically by Steinberg in [St1] and [St2], where (very roughly) they are obtained a s fixed points of certain automorphisms of the Chevalley groups — group automorphisms which are associated with automorphisms of the Dynkin diagram. Let $\mathbb{G}$ be one of the symbols A_n, B_n, C_n, etc. Then ${}^m\mathbb{G}(q)$ denotes the fixed subgroup of an automorphism of order m of $\mathbb{G}(q^m)$ (or of $\mathbb{G}(q)$ when $\mathbb{G} = B_2$, G_2, or F_4). The finite twisted groups thus consist of the classical groups ${}^2A_n(q) \cong PSU_{n+1}(q) = U_{n+1}(q)$ and ${}^2D_n(q) \cong \Omega^-_{2n}(q)$ (the commutator subgroup of the projective special orthogonal groups of "minus type"); as well as the Suzuki groups ${}^2B_2(2^{2k+1})$, the Ree groups ${}^2G_2(3^{2k+1})$ and ${}^2F_4(2^{2k+1})$, and the Steinberg groups ${}^2E_6(q)$ and ${}^3D_4(q)$.

To make this more concrete, it is necessary to work with automorphisms of the Chevalley groups over the algebraic closure $\overline{\mathbb{F}}_p$, where p is prime. Let $\mathbb{G}(\overline{\mathbb{F}}_p)$ denote a simple algebraic group of type $\mathbb{G}$ defined over $\overline{\mathbb{F}}_p$. We will always assume that $\mathbb{G}(\overline{\mathbb{F}}_p)$ is of adjoint type (i.e., with trivial center), or equivalently that it is a group of automorphisms of the corresponding Lie algebra. For q a power of p, the finite Chevalley group $\mathbb{G}(q)$ can (roughly) be thought of as the fixed subgroup of the automorphism φ_q of $\mathbb{G}(\overline{\mathbb{F}}_p)$ induced by the field automorphism $(t \mapsto t^q)$. More generally, a *Steinberg endomorphism* of $\overline{G} = \mathbb{G}(\overline{\mathbb{F}}_p)$ is defined to be an algebraic endomorphism of $\overline{G}$ whose fixed subgroup $C_{\overline{G}}(\sigma) = \{x \in \overline{G} \mid \sigma(x) = x\}$ is finite. (In fact, the Steinberg endomorphisms are all automorphisms of $\overline{G}$ as an abstract group, but none of them is invertible as an algebraic endomorphism.) The finite twisted groups of Lie type are (roughly) the fixed subgroups of Steinberg endomorphisms of $\overline{G}$, which are field automorphisms $(t \mapsto t^q)$ "twisted" by graph automorphisms.

More precisely, if σ is a Steinberg endomorphism of $\overline{G} = \mathbb{G}(\overline{\mathbb{F}}_p)$, let $\overline{G}_\sigma$ denote the subgroup of $C_{\overline{G}}(\sigma)$ generated by its Sylow p-subgroups. Equivalently, $\overline{G}_\sigma = \langle C_U(\sigma), C_V(\sigma)\rangle$, where $U, V \subseteq \overline{G}$ are subgroups defined in the next paragraph. If $\widetilde{G}$ is the universal central extension of $\overline{G}$, then $\overline{G}_\sigma \cong C_{\widetilde{G}}(\sigma)/Z$, where Z denotes the center. For example, if q is a power of p, then $SL_n(q) = C_{SL_n(\overline{\mathbb{F}}_p)}(\varphi_q)$, while $PSL_n(q)$ can be a proper subgroup of $C_{PSL_n(\overline{\mathbb{F}}_p)}(\varphi_q)$. For all $\mathbb{G}$ and all $q = p^k$, $\mathbb{G}(q) = \overline{G}_{\varphi_q}$.

To describe the Steinberg endomorphisms, we must first establish notation for certain elements of the Chevalley groups. Fix a prime p, and let $\mathbb{F} \subseteq \overline{\mathbb{F}}_p$ be any subfield. Set $G = \mathbb{G}(\mathbb{F})$, and let Σ be the system of roots of type $\mathbb{G}$. Let $\Sigma_+, \Sigma_- \subseteq \Sigma$ denote the sets of positive and negative roots, respectively. To each $r \in \Sigma$ there corresponds a subgroup (the root subgroup) $X_r = \{x_r(t) \mid t \in \mathbb{F}\} \subseteq G$, isomorphic to the additive group $\mathbb{F}$. Then

$U \stackrel{\text{def}}{=} \langle X_r \,|\, r \in \Sigma_+ \rangle$ and $V \stackrel{\text{def}}{=} \langle X_r \,|\, r \in \Sigma_- \rangle$ are both maximal unipotent subgroups of G; they are closed and connected if $\mathbb{F} = \overline{\mathbb{F}}_p$, and are Sylow p-subgroups of G if $\mathbb{F}$ is finite. Also, $G = \langle U, V \rangle$. The subgroup $H \stackrel{\text{def}}{=} N_G(U) \cap N_G(V)$ is a maximal torus of G if $\mathbb{F} = \overline{\mathbb{F}}_p$, and is called a Cartan subgroup of G when G is finite. This subgroup H is abelian, generated by elements $h_r(t)$ for simple roots r and $t \in \mathbb{F}^*$; and its elements are called "diagonal elements" of G. Also, $N_G(U) = UH$ and $N_G(V) = VH$ (the *Borel subgroups* of G).

For example, when $G = A_n(\mathbb{F}) \cong L_{n+1}(\mathbb{F})$, (of adjoint type), then the roots correspond to the pairs (i,j) for $i \neq j$, and the positive roots correspond to the pairs (i,j) for $i < j$. In this case, $x_{ij}(t) = e_{ij}(t)$, the matrix which has 1's on the diagonal, t in position (i,j), and zeros elsewhere. Thus U and V are the subgroups of (strict) upper and lower triangular matrices, and H is the subgroup of diagonal matrices. Note that when we describe elements and subgroups here in terms of matrices, we mean their images under the surjection of $SL_{n+1}(\mathbb{F})$ onto $L_{n+1}(\mathbb{F}) = PSL_{n+1}(\mathbb{F})$.

Let σ be a Steinberg endomorphism of $\overline{G} = \mathbb{G}(\overline{\mathbb{F}}_p)$ (still assumed of adjoint type). By the Lang-Steinberg theorem [St2, Theorem 10.1], for any $g \in \overline{G}$, there exists $h \in \overline{G}$ such that $g = \sigma(h)h^{-1}$. Hence, all elements in $\mathrm{Inn}(\overline{G}) \circ \sigma$ are conjugate in $\mathrm{Aut}(\overline{G})$. In other words, composing a Steinberg endomorphism σ with an inner automorphism of $\overline{G}$, does not change $\overline{G}_\sigma$ (up to conjugation).

Next, Steinberg showed that for any σ, there is some $g \in \overline{G}$ such that $\mathrm{conj}(g) \circ \sigma$ leaves U and V invariant, and permutes the root subgroups X_r. It thus suffices to consider those σ for which $\sigma(X_r) = X_{\rho(r)}$ for some automorphism ρ of the root system Σ of type $\mathbb{G}$, which preserves the positive roots; i.e., a permutation of Σ which preserves angles between the roots, such that $\rho(\Sigma_+) = \Sigma_+$. Hence ρ permutes the simple roots, and induces a symmetry of the Dynkin diagram of $\mathbb{G}$. By inspection of the Dynkin diagrams, one sees that if $\rho \neq \mathrm{Id}$, then either $\mathbb{G} = A_n$, D_n, or E_6 and ρ is the automorphism of order 2 of the root system; or $\mathbb{G} = D_4$ and ρ is an automorphism of order 3; or $\mathbb{G} = B_2$, F_4, or G_2 and ρ is an automorphism of order 2 which interchanges long and short roots.

If $\sigma(X_r) = X_{\rho(r)}$ for such ρ, then necessarily $\sigma(x_r(t)) = x_{\rho(r)}(\epsilon_r t^{q_r})$ for some $\epsilon_r \in (\overline{\mathbb{F}}_p)^*$ and some q_r powers of p. After composing with conjugation by a diagonal element, we can assume $\epsilon_r = 1$ for all simple roots r (and $\epsilon_r = \pm 1$ for all r). Also, by studying the action of σ on diagonal elements, one can show that the ratio $q_r \cdot \|r\| / \|\rho(r)\|$ is constant, independent of r. In particular, if $\rho = \mathrm{Id}$, then $\sigma = \varphi_q$ ($q = q_r$ for all r) is a field automorphism.

Assume that $\rho \neq \mathrm{Id}$, and that all roots in Σ have the same length. Then $\sigma = \varphi_q \circ \psi_\rho$, where $q = p^k > 1$ ($q = q_r$ for all r); and where $\psi_\rho(x_r(t)) = x_{\rho(r)}(t)$ for all simple roots r and all $t \in \overline{\mathbb{F}}_p$ (and $\psi_\rho(x_r(t)) = x_{\rho(r)}(\pm t)$ for arbitrary r). The existence of such an automorphism ψ_ρ is shown in [St1, Theorem 29] or [Ca, Proposition 12.2.3]. If m is the order of ρ, then $\sigma^m = \varphi_{q^m}$, so $\overline{G}_{\sigma^m} = \mathbb{G}(q^m)$, and ${}^m\mathbb{G}(q) \stackrel{\text{def}}{=} \overline{G}_\sigma$ can be viewed as the subgroup of $C_{\mathbb{G}(q^m)}(\tau)$ generated by its Sylow-p subgroups, where τ is the restriction of σ to $\mathbb{G}(q^m)$. In other words, we can regard ${}^m\mathbb{G}(q) = \mathbb{G}(q^m)_\tau$, where τ is the field automorphism ($t \mapsto t^q$) "twisted" by the "graph automorphism" of $\mathbb{G}(q^m)$.

As one example, consider the automorphism $\tau(a_{ij}) = (((-1)^{i+j}a_{n+2-j,n+2-i})^q)^{-1}$ of $L_{n+1}(q^2)$. This preserves upper and lower triangular matrices, and sends $x_{ij}(t)$ to $x_{n+2-j,n+2-i}(\pm t^q)$. The signs have been chosen so that $\tau(x_r(t)) = x_{\rho(r)}(t^q)$ when r is a simple root $(i, i+1)$ (but not for all roots). Then ${}^2A_n(q) \overset{\text{def}}{=} (L_{n+1}(q^2))_\tau = PSU_{n+1}(q)$ is the projective special unitary group defined with respect to the hermitian form $(x,y) = u\cdot\sum(-1)^{i+1}x_i(y_{n+2-i})^q$ on $(\mathbb{F}_{q^2})^{n+1}$ (where $u=1$ if n is even and $u^{q-1}=-1$ if n is odd). Note that there can be elements of $PSL_{n+1}(q^2)$ fixed by τ which are not represented by unitary matrices, which is why one must define ${}^2A_n(q) = \langle C_U(\tau), C_V(\tau)\rangle$. If one works in the universal central extension $SL_{n+1}(q^2)$, the subgroup of elements fixed by τ is $SU_{n+1}(q)$.

If Σ has roots of distinct lengths and ρ is nontrivial, then as mentioned above $\mathbb{G} = B_2$, F_4, or G_2 and ρ interchanges long and short roots. Set $p_0 = 2$ if $\mathbb{G} = B_2, F_4$ and $p_0 = 3$ if $\mathbb{G} = G_2$, so that $\frac{\|\rho(r)\|}{\|r\|} = (p_0)^{\pm 1/2}$ for each $r \in \Sigma$. Since $q_r\cdot\frac{\|r\|}{\|\rho(r)\|}$ is independent of r (and the q_r all powers of p), this is possible only if $p = p_0$. Hence, $\sigma = \varphi_q \circ \psi_\rho$ for some $q = p^k \geq 1$, where

$$\psi_\rho(x_r(t)) = \begin{cases} x_{\rho(r)}(t^p) & \text{if } r \text{ is a short root} \\ x_{\rho(r)}(t) & \text{if } r \text{ is a long root.} \end{cases}$$

Then $\sigma^2 = \varphi_{q^2p}$, so $\overline{G}_{\sigma^2} = \mathbb{G}(q^2p) = \mathbb{G}(p^{2k+1})$, and ${}^2\mathbb{G}(p^{2k+1}) \overset{\text{def}}{=} \overline{G}_\sigma$ can be regarded as the fixed subgroup of an involution on $\mathbb{G}(p^{2k+1})$. This group is sometimes denoted ${}^2\mathbb{G}(p^{k+\frac{1}{2}})$.

As an example, Ono [On] carried out this procedure on $Sp_4(2^{2k+1}) = B_2(2^{2k+1})$, regarded as the group of 4×4 matrices which preserve the symplectic form $(x,y) = x_1y_4 + x_2y_3 + x_3y_2 + x_4y_1$. He obtained precisely the matrix presentation of $\mathrm{Sz}(2^{2k+1})$ described in Proposition 3.6, as the fixed points $(Sp_4(2^{2k+1}))_\tau$, where τ is the restriction of the above $\sigma = \varphi_q \circ \psi_\rho$ to $Sp_4(2^{2k+1})$.

The *rank* of a Chevalley group $\mathbb{G}(q)$ is just the rank of $\overline{G} = \mathbb{G}(\overline{\mathbb{F}}_p)$ in the usual sense; i.e., the number of simple roots in its root system, or the number of nodes in its Dynkin diagram. The rank of a twisted group ${}^m\mathbb{G}(q)$ is equal to the number of orbits of roots (or of nodes) under the corresponding automorphism of the root system or the Dynkin diagram of $\mathbb{G}$. There are thus four families of finite simple groups of Lie type and Lie rank 1: the two dimensional projective special linear groups $L_2(q) \cong A_1(q)$, the three dimensional projective special unitary groups $U_3(q) \cong {}^2A_2(q)$, the Suzuki groups $\mathrm{Sz}(q) \cong {}^2B_2(2^{2k+1})$, and the Ree groups $\mathrm{Ree}(3^{2k+1}) \cong {}^2G_2(3^{2k+1})$.

We now return to the internal structure of the groups of Lie type. First let $G = \mathbb{G}(F)$ be a Chevalley group over any field F, and let Σ be a root system of type $\mathbb{G}$. We have already discussed the root subgroups $X_r = \{x_r(t) \,|\, t \in F\}$ for each root $r \in \Sigma$, and the subgroups $U = \langle X_r \,|\, r \in \Sigma_+\rangle$ and $V = \langle X_r \,|\, r \in \Sigma_-\rangle$. For each root r, there is a surjection $\phi_r{:}SL_2(F) \twoheadrightarrow \langle X_r, X_{-r}\rangle$ which sends $\left(\begin{smallmatrix}1 & t\\ 0 & 1\end{smallmatrix}\right)$ to $x_r(t)$ and $\left(\begin{smallmatrix}1 & 0\\ t & 1\end{smallmatrix}\right)$ to $x_{-r}(t)$. This allows the definition of elements $h_r(\lambda) = \phi_r\left(\begin{smallmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{smallmatrix}\right)$ and $n_r = \phi_r\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$. The elements $h_r(\lambda)$, for $r \in \Sigma$ and $\lambda \in F^*$, generate the subgroup H of diagonal elements of $\overline{G}$, and together with the n_r they generate the subgroup $N = \langle H, n_r \,|\, r \in \Sigma\rangle$ of monomial elements. Then $N/H \cong W$, the Weyl group of G (and of its root system), and $B \overset{\text{def}}{=} \langle U, H\rangle = N_G(U)$ is the Borel subgroup of G.

Now set $\overline{G} = \mathbb{G}(\overline{\mathbb{F}}_p)$, and let σ be a Steinberg morphism of $\overline{G}$. Set $U_\sigma = C_U(\sigma)$ and $V_\sigma = C_V(\sigma)$, the subgroups of elements fixed by σ, and let $\widehat{G} = \langle U_\sigma, V_\sigma\rangle$ be the corresponding group of Lie type. Set $\widehat{H} = C_H(\sigma) \cap \widehat{G}$, $\widehat{N} = C_N(\sigma) \cap \widehat{G}$, and $\widehat{B} = C_B(\sigma) \cap \widehat{G}$. Let ρ be the automorphism of the root system Σ associated to σ, as described earlier. In particular ρ permutes the positive roots, and hence the simple roots. By a root (or simple root) of $\widehat{G}$ is meant a ρ-orbit $\hat{r} \subseteq \Sigma$ (or ρ-orbit of simple roots). Note that if $\rho = \mathrm{Id}$, then $\widehat{G}$ is an (untwisted) Chevalley group, and its roots are the roots in the usual sense. We write $\widehat{\Sigma} = \Sigma/\rho$ for the set of roots, $-\hat{r} = \{-r \mid r \in \hat{r}\}$; and $\langle J\rangle$ (when $J \subseteq \widehat{\Sigma}$) for the set of ρ-orbits of roots which are linear combinations of elements $r \in \hat{r} \in J$. The *root subgroup* $X_{\hat{r}}$ corresponding to an orbit $\hat{r}$ is the subgroup $(\prod_{r\in\hat{r}} X_r)_\sigma$ of σ-invariant elements. The Weyl group of $\widehat{G}$ is the group $\widehat{W} = \widehat{N}/\widehat{H}$; or equivalently the subgroup of $W = N/H$ of elements which commute with σ (cf. [Ca, Proposition 13.5.2]) when both are considered as groups of permutations of the roots Σ (or of the real vector space generated by the roots). The Weyl group is generated by elements $w_{\hat{s}}$ of order two, one for each ρ-orbit $\hat{s}$ of simple roots, where the $w_{\hat{s}}$-action on Σ sends s to $-s$ for all $s \in \hat{s}$. The root subgroups of $\widehat{G}$ are discussed in detail in [Ca, Proposition 13.6.3] and [GLS, Table 2.4]; in particular, they need not be abelian. The Weyl groups of the twisted groups are described in [Ca, §13.3]; each is isomorphic to that of some Chevalley group except when $\widehat{G} = {}^2F_4(2^{2k+1})$, in which case $\widehat{W}$ is dihedral of order 16.

For notational convenience we now drop the "hat" from our notation for the finite simple groups of Lie type of the previous paragraph. Thus from now through the end of Appendix D, $G = \widehat{G}$, $U = \widehat{U}$, etc. Also, we'll abuse notation and write $r = \hat{r}$ for a ρ-orbit in Σ.

Tits has axiomatized the properties of the pairs (B, N) in groups of Lie type. These permit, for example, uniform proofs of the simplicity of these groups in all cases where they are simple. See, e.g., [Ca, §8.2] or [GLS, §1.11] for more detail about such BN-pairs.

By definition, any group of Lie type is generated by its root subgroups (for a given choice of root system). In fact, it suffices to take the simple roots.

Lemma D.1. *Let G be a finite simple group of Lie type, with root system Σ. Then G is generated by the root subgroups X_s and X_{-s} for simple roots $s \in \Sigma_+$.*

Proof. See [Ca, Proposition 13.6.5]. Very briefly, when G is a Chevalley group, this holds since conjugation by elements of N (or of $W = N/H$) permutes the root subgroups in the same way as the Weyl group permutes the roots, and each root is in the W-orbit of a simple root. Since N/H is generated by the elements $n_s \in \langle X_s, X_{-s}\rangle$ for simple roots s, this shows that $\langle X_s, X_{-s} \mid s \text{ simple}\rangle$ contains all of the X_r for $r \in \Sigma$, and hence is all of G. The same argument works for the twisted groups. □

We now turn attention to parabolic subgroups: proper subgroups of G which contain a Borel subgroup. For convenience, set $B' = VH$ (and $B = UH$ as usual). Let Σ be the root system corresponding to G. For each proper subset J of simple roots of G, let $\langle J\rangle \subseteq \Sigma$ be as defined above, and set

$$P_J = \langle B, n_s \mid s \in J\rangle = \langle B, X_r \mid r \in \langle J\rangle\rangle \quad \text{and} \quad P'_J = \langle B', n_s \mid s \in J\rangle = \langle B', X_r \mid r \in \langle J\rangle\rangle.$$

By [Ca, Theorem 8.3.2], these are precisely the overgroups of B in G (i.e., the parabolic subgroups containing B).

Lemma D.2. *Let G be a finite simple group of Lie type. Let Σ be the root system associated with G, and let Σ_+ and Σ_- be the sets of positive and negative roots. Fix a set J of simple roots which does not contain all of them, and let L_J be the subgroup generated by the diagonal subgroup H together with the root subgroups X_r for all $r \in \langle J\rangle$. Let U_J and V_J be the subgroups generated by all X_r for roots $r \in \Sigma_+$ or $r \in \Sigma_-$, respectively, which are not in $\langle J\rangle$. Then $U_J \lhd P_J = U_J L_J$ and $V_J \lhd P'_J = V_J L_J$, U_J and V_J are nilpotent, and $\langle U_J, V_J\rangle = G$.*

Proof. When G is a Chevalley group, the nilpotency of $U \supseteq U_J$ and $V \supseteq V_J$ follows from [Ca, Theorem 5.3.3], and L_J normalizes U_J and V_J by [Ca, Theorem 8.5.2]. Both of these are consequences of Chevalley's commutator formula, which says that for any pair of roots $r, s \in \Sigma$, $[X_r, X_s]$ is generated by the subgroups X_t for all roots $t = ir + js$ where $i, j > 0$. The twisted group case follows immediately by restriction. And $P_J = U_J L_J$ and $P'_J = V_J L_J$ since U and V are generated by their root subgroups: by definition when G is a Chevalley group, and by [Ca, Proposition 13.6.1] when G is a twisted group.

This also shows that $\langle L_J, U_J, V_J\rangle = \langle U, V\rangle = G$. Thus $\langle U_J, V_J\rangle \lhd G$, since L_J normalizes U_J and V_J; and so $G = \langle U_J, V_J\rangle$ since G is simple. □

The decomposition $P_J = U_J L_J$ of Lemma D.2 is called the Levi decomposition of P_J, and L_J is called the Levi subgroup.

We now return to looking at group actions on 2-dimensional acyclic complexes.

Lemma D.3. *Let G be a finite simple group of Lie type, and let $P \subsetneqq G$ be one of the parabolic subgroups P_J or P'_J of Lemma D.2. Then for any action of G on an acyclic 2-complex X, $X^P \neq \emptyset$.*

Proof. We can assume $X^G = \emptyset$. By Lemma D.2, there are subgroups $U_J \lhd P_J$, $V_J \lhd P'_J$, and $L_J = P_J \cap P'_J$, such that U_J and V_J are nilpotent, $P_J = U_J L_J$, $P'_J = V_J L_J$, and $\langle U_J, V_J\rangle = G$. In particular, X^{U_J} and X^{V_J} are acyclic, disjoint, and L_J-invariant. Then $X^{L_J} \neq \emptyset$ by Corollary 4.2, applied to the action of L_J on X with invariant subspaces $A = X^{U_J}$ and $B = X^{V_J}$; and so X^{P_J} and $X^{P'_J}$ are nonempty by Lemma 4.3(a).

To see this more directly, let Y be the complex obtained by collapsing X^{U_J} and X^{V_J} to separate points. Then Y is still acyclic, L_J acts on Y, and Y^{L_J} contains at least the two collapse points. Thus, Y^{L_J} is acyclic by Theorem 4.1, is in particular connected, and hence X^{L_J} must intersect with both subcomplexes X^{U_J} and X^{V_J}. It follows that $X^{P_J} = X^{L_J} \cap X^{U_J} \neq \emptyset$ and $X^{P'_J} = X^{L_J} \cap X^{V_J} \neq \emptyset$. □

Appendix E. The four-subgroup criterion

In [S1] and [AS], very strong restrictions were placed on the finite simple groups which could possibly have actions on 2-dimensional acyclic complexes without fixed points. The main tool for doing this was a "four subgroup criterion", which for the sake of completeness we present here as Proposition E.1. To illustrate its use, we then describe how it was applied to certain multiply transitive groups, and to simple groups of Lie type and Lie rank at least two — those cases of the proof of Theorem A which were not dealt with in Sections 6 and 7.

Proposition E.1 [S1, Theorem 3.2]**.** *Fix a finite group G and a 2-dimensional acyclic G-complex X. Let $H_1, H_2, H_3, H_4 \subseteq G$ be subgroups such that $X^{\langle H_i,H_j,H_k\rangle} \neq \emptyset$ for any i, j, k. Then $X^{\langle H_1,H_2,H_3,H_4\rangle} \neq \emptyset$.*

Proof. Assume otherwise: that $X^{\langle H_1,H_2,H_3,H_4\rangle} = \emptyset$. Set $\mathcal{H} = \{H_1, H_2, H_3, H_4\}$. By Theorem 4.1, $X^{\mathcal{H}}$ is the union of the acyclic subcomplexes X^{H_i}, which have the property that any two or three of them have acyclic intersection, but the four have empty intersection. This implies that $H_2(X^{\mathcal{H}}) \cong H_2(S^2) \cong \mathbb{Z}$ (see Lemma 0.1, applied using the poset S of nonempty proper subsets of $\{1, 2, 3, 4\}$). But this is impossible, since $X^{\mathcal{H}}$ must be homologically 1-dimensional by Lemma 1.6. □

The simplest application of Proposition E.1 is to multiply transitive groups.

Corollary E.2. *Assume that G acts 4-transitively on a set S with point stabilizer $H \subseteq G$. Let X be a 2-dimensional acyclic G-complex such that $X^H \neq \emptyset$. Then $X^G \neq \emptyset$.*

Proof. If $|S| = 4$, then this follows from Theorem B. So assume $|S| \geq 5$, and fix four elements $s_1, s_2, s_3, s_4 \in S$. For each $i = 1, 2, 3, 4$, let $H_i \subseteq G$ be the subgroup of elements which fix s_j for all $j \neq i$. For each $\{i, j, k, r\} = \{1, 2, 3, 4\}$, $\langle H_i, H_j, H_k\rangle$ is the point stabilizer of s_r, and hence fixes a point in X by assumption. So $X^G \neq \emptyset$ by Proposition E.1. □

This is now applied to the alternating groups, as well as most of the Mathieu groups.

Proposition E.3 [S1, 3.6]**.** *If $G \cong A_n$ for $n \geq 6$, or if G is one of the Mathieu groups M_{11}, or M_{12}, then every G-action on an acyclic 2-complex has fixed points. The same holds for M_{23} and M_{24} if it holds for M_{22}.*

Proof. Let X be a 2-dimensional acyclic G-complex. If $G = A_n$ for $n \geq 6$, then by Corollary E.2, $X^G \neq \emptyset$ if $X^{A_{n-1}} \neq \emptyset$. By Proposition 6.4 above, $A_6 \cong L_2(9)$ must have nonempty fixed point set, and the result now follows by induction on n.

Each of the simple Mathieu groups M_n for $n = 11, 12, 23, 24$ acts 4-transitively on a set with point stabilizer M_{n-1} (cf. [A3, 18.9–10 & 19.4], [Gr, 5.33 & 6.18], [Mat], or [Wt]). So by Corollary E.2, the proposition holds for M_n if it holds for M_{n-1}. Since M_{10} contains a subgroup A_6 of index 2, this proves the proposition when $n = 11$ or 12; and it will follow for the other simple Mathieu groups once it has been shown for M_{22}. □

Proposition E.1 can also be applied to simple groups of Lie type of Lie rank at least two. In this case, the subgroups in question come from the root system of the group. Note that the following proof applies only to groups of Lie type which are themselves simple. The Tits group ${}^2F_4(2)'$, which has index two in ${}^2F_4(2)$, is dealt with here in Proposition 7.2, as well as in [AS, 5.2].

Proposition E.4 [AS, §5]**.** *If G is a simple group of Lie type and Lie rank at least 2, then every G-action on an acyclic 2-complex has fixed points.*

Proof. We use the notation of Lemma D.2. Fix a root system $\Sigma = \Sigma_+ \amalg \Sigma_-$ for G, and let $J_1 \amalg J_2$ be a decomposition of the set of simple roots as a disjoint union of nonempty subsets. For each $i = 1, 2$, set

$$H_i^+ = \langle H, X_s \,|\, s \in J_i \rangle \qquad \text{and} \qquad H_i^- = \langle H, X_{-s} \,|\, s \in J_i \rangle.$$

The subgroup generated by any three of the $H_i^{\pm}$ is contained in one of the parabolic subgroups P_{J_i} or P'_{J_i} (in the notation of Lemma D.2), and hence has nonempty fixed point set in X by Lemma D.3. But $\langle H_1^{\pm}, H_2^{\pm} \rangle = G$ by Lemma D.1, since it contains all subgroups X_s and X_{-s} for simple roots s, and hence $X^G \neq \emptyset$ by Proposition E.1. $\square$

List of notation:

<u>Groups:</u>

C_m: a cyclic group of order m

D_{2m}: a dihedral group of order $2m$

A_n: the alternating group on n letters

Σ_n: the symmetric group on n letters

$PGL_n(q) = GL_n(q)/(\text{center})$: the projective general linear group over $\mathbb{F}_q$

$L_n(q) = PSL_n(q)$: the projective special linear group over $\mathbb{F}_q$

$PGU_n(q)$: the projective general unitary group over $\mathbb{F}_{q^2}$

$U_n(q) = PSU_n(q)$: the projective special unitary group over $\mathbb{F}_{q^2}$

<u>Topological spaces:</u>

$I = [0, 1]$: the unit interval

$D^n = \{x \in \mathbb{R}^n \,\big|\, \|x\| \leq 1\}$: the unit ball in $\mathbb{R}^n$

$S^n = \{x \in \mathbb{R}^{n+1} \,\big|\, \|x\| = 1\}$: the unit sphere in $\mathbb{R}^{n+1}$

$X \cong Y$: X and Y are homeomorphic

$X \simeq Y$: X and Y are homotopy equivalent

$X \simeq *$: X is contractible

$H_*(X) \overset{\text{def}}{=} H_*(X; \mathbb{Z})$

<u>Acyclic</u> means $\mathbb{Z}$-acyclic: X is acyclic iff $H_*(X; \mathbb{Z}) \cong H_*(\text{pt}, \mathbb{Z})$

<u>Families and sets of subgroups of G:</u>

$\mathcal{S}(G)$: the family of all subgroups of G

(H): the conjugacy class of $H \subseteq G$

$\mathcal{F} \subseteq \mathcal{S}(G)$ is a <u>family</u> $\iff$ $H \in \mathcal{F}$ implies $(H) \subseteq \mathcal{F}$

$\mathcal{SLV}(G)$: the family of solvable subgroups of G

$\mathcal{MAX}(G)$: the maximal separating family of subgroups of G

$(G, \mathcal{F}) \in \mathcal{U}_2 \iff \exists$ a 2-dimensional $\mathbb{Z}$-acyclic $(G, \mathcal{F})$-complex

<u>For any families $\mathcal{F}$, $\mathcal{F}'$ of subgroups of G:</u>

$\mathcal{F}_{\max}$: the set of maximal subgroups of $\mathcal{F}$

$\mathcal{F}_{\geq H} = \{K \in \mathcal{F} \mid K \supseteq H\} \qquad \forall H \subseteq G$

$\mathcal{F}_{>H} = \{K \in \mathcal{F} \mid K \supsetneqq H\} \qquad \forall H \subseteq G$

$\mathcal{F}_{>H}^{<M} = \{K \in \mathcal{F} \mid H \subsetneqq K \subsetneqq M\} \qquad \forall H \subsetneqq M \subseteq G$

$\mathcal{F}_{\geq \mathcal{H}} = \{K \in \mathcal{F} \mid K \supseteq H, \text{ some } H \in \mathcal{H}\} \qquad \forall \mathcal{H} \subseteq \mathcal{S}(G)$

$\mathcal{F}_{[n]} = \big\{H \in \mathcal{F} \bigm| n \big| |H|\big\} \qquad \forall n > 1$

$\mathcal{F} \wedge \mathcal{F}' = \{H \cap H' \mid H \in \mathcal{F},\ H' \in \mathcal{F}'\}$

$H \in \mathcal{F}$ is <u>critical</u> in $\mathcal{F} \iff \mathcal{N}(\mathcal{F}_{>H}) \not\simeq *$

$\mathcal{F}_c = \{H \in \mathcal{F} \mid H \text{ critical in } \mathcal{F}\}$

<u>If X is a G-complex:</u>

$G_x = \{g \in G \mid gx = x\} \qquad \forall x \in X$

X is a $(G, \mathcal{F})$-complex $\iff G_x \in \mathcal{F}\ \forall x \in X$

$X^H = \{x \in X \mid hx = x\, \forall h \in H\}$: the fixed point set

$X^{>H} = \{x \in X \mid G_x \supsetneqq H\} = \bigcup_{K \supsetneqq H} X^K$

$X^{\mathcal{H}} = \bigcup_{H \in \mathcal{H}} X^H \qquad \forall \mathcal{H} \subseteq \mathcal{S}(G)$

$X^{[n]} = \bigcup_{n \big| |H|} X^H = \{x \in X \bigm| |G_x \in n\mathbb{Z}\} \qquad \forall n > 1$

$X^{(H)} = \bigcup_{g \in G} X^{gHg^{-1}}$

$X_s = \bigcup_{1 \neq H \subseteq G} X^H = \{x \in X \mid G_x \neq 1\}$: the "singular set" of X

References

[Art] E. Artin, Geometric algebra, Interscience (1957)

[A1] M. Aschbacher, Finite group theory, Cambridge Univ. Press (1986)

[A2] M. Aschbacher, Overgroups of Sylow subgroups in sporadic groups, Memoirs Amer. Math. Soc. <u>343</u> (1986)

[A3] M. Aschbacher, Sporadic groups, Cambridge Univ. Press (1994)

[AS] M. Aschbacher & Y. Segev, A fixed point theorem for groups acting on finite 2-dimensional acyclic simplicial complexes, Proc. London Math. Soc. <u>67</u> (1993), 329–354

[Bl] D. Bloom, The subgroups of $PSL(3, q)$ for odd q, Trans. Amer. Math. Soc. <u>127</u> (1967), 150–178

[Atl] J. Conway, R. Curtis, S. Norton, R. Parker, & R. Wilson, Atlas of finite groups, Oxford Univ. Press (1985)

[Br] G. Bredon, Introduction to compact transformation groups, Academic Press (1972)

[Ca] R. Carter, Simple groups of Lie type, Wiley (1972)

[FR] E. Floyd & R. Richardson, An action of a finite group on an n-cell without stationary points, Bull. A.M.S. <u>65</u> (1959), 73–76

[Go] D. Gorenstein, Finite groups, Harper & Row (1968)

[GLS] D. Gorenstein, R. Lyons, & R. Solomon, The classification of the finite simple groups, nr. 3, Amer. Math. Soc. (1998)

[Gra] B. Gray, Homotopy theory, Academic Press (1975)

[Gr] R. Griess, Twelve sporadic groups, Springer-Verlag (1998)

[Ha] R. Hartley, Determination of the ternary collineation groups whose coefficients lie in the $GF(2^n)$, Annals of Math. <u>27</u> (1925), 140–158

[Hu] S.-T. Hu, Homotopy theory, Academic Press (1959)

[H1] B. Huppert, Endliche Gruppen I, Springer-Verlag (1967)

[HB2] B. Huppert & N. Blackburn, Finite groups II, Springer-Verlag (1982)

[HB3] B. Huppert & N. Blackburn, Finite groups III, Springer-Verlag (1982)

[Jac] H. Jacobinski, Genera and decompositions of lattices over orders, Acta math. 121 (1968), 1–29

[Jan] Z. Janko, A characterization of the smallest group of Ree associated with the simple Lie algebra of type (G_2), J. Algebra 4 (1966), 293–299

[KS] R. Kirby & M. Scharlemann, Eight faces of the Poincaré homology 3-sphere, Geometric topology (Proc. Georgia topology conference 1977), Academic Press (1979), 113–146

[Kl1] P. Kleidman, Ph.D. thesis, Imperial College (1987)

[Kl2] P. Kleidman, The maximal subgroups of the Chevalley grous $G_2(q)$ with q odd, the Ree groups ${}^2G_2(q)$ and their automorphism groups, J. Algebra 117 (1988), 30–71

[LW] A. Lundell & S. Weingram, The topology of CW complexes, Van Nostrand (1969)

[Mat] E. Mathieu, Sur la fonction cinq fois transitive de 24 quantités, J. Math. Pures Appl. 18 (1873), 25–46

[O1] R. Oliver, Fixed point sets of group actions on finite acyclic complexes, Comment. Math. Helv. 50 (1975), 155–177

[O2] R. Oliver, Smooth compact group actions on disks, Math. Z. 149 (1976), 79–96

[O3] R. Oliver, A proof of the Conner conjecture, Annals of Math. 103 (1976), 637–644

[On] T. Ono, An identification of Suzuki groups with groups of generalized Lie type, Annals of Math. 75 (1962), 251–259; Corrigendum, Annals of Math. 77 (1963), 413

[Q1] D. Quillen, Higher algebraic K-theory I, Lecture note in mathematics 341 (1973), 85–147

[Q2] D. Quillen, Homotopy properties of posets of nontrivial p-subgroups of a group, Adv. in Math. 28 (1978), 101–108

[Re] I. Reiner, Maximal orders, Academic Press (1975)

[Rim] D. S. Rim, Modules over finite groups, Annals of Math. 69 (1959), 700–712

[S1] Y. Segev, Group actions on finite acyclic simplicial complexes, Israel J. Math. 82 (1993), 381–393

[S2] Y. Segev, Some remarks on finite 1-acyclic and collapsible complexes, Jour. combinatorial theory 65 (1994), 137–150

[Se] J.-P. Serre, Trees, Springer-Verlag (1980)

[St1] R. Steinberg, Lectures on Chevalley groups, Mimeographed notes, Yale Univ. (1968)

[St2] R. Steinberg, Endomorphisms of linear algebraic groups, Memoirs Amer. Math. Soc. 80 (1968)

[Sz1] M. Suzuki, On a class of doubly transitive groups, Annals of Math. 75 (1962), 105–145

[Sz2] M. Suzuki, Group theory, Springer-Verlag (1982)

[Sw] R. Swan, Induced representations and projective modules, Annals of math. 71 (1960), 552–578

[Wh] J. H. C. Whitehead, Combinatorial homotopy I, Bull. Amer. Math. Soc. 55 (1949), 213–245

[Wt] E. Witt, Die 5-fach transitiven Gruppen von Mathieu, Abh. Math. Sem. Hansischen Univ. 12 (1938), 256–264

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