

# THE RATIONAL COHOMOLOGY OF A $p$ -LOCAL COMPACT GROUP

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Let  $p$  be a prime number. In [BLO3], we developed the theory of  $p$ -local compact groups. The theory is modelled on the  $p$ -local homotopy theory of classifying spaces of compact Lie groups and  $p$ -compact groups, and generalises the earlier concept of  $p$ -local finite groups [BLO2]. It provides a coherent context in which classifying spaces of compact Lie groups and  $p$ -compact groups [DW] can be studied, and also gives rise to many exotic examples. In this paper, we study the rational  $p$ -adic cohomology

$$H_{\mathbb{Q}_p}^*(-) \stackrel{\text{def}}{=} H^*(-, \mathbb{Z}_p) \otimes \mathbb{Q}_p$$

of a  $p$ -local compact group. Our main result here is that, as one would expect, the  $p$ -adic rational cohomology of  $p$ -local compact groups behaves the same way as that of a compact Lie group.

**Theorem A.** *Let  $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$  be a  $p$ -local compact group. Let  $S_0 \leq S$  be its maximal torus, and let  $W(\mathcal{G}) \stackrel{\text{def}}{=} \text{Aut}_{\mathcal{F}}(S_0)$  be its Weyl group. Then*

$$H_{\mathbb{Q}_p}^*(B\mathcal{G}) \cong H_{\mathbb{Q}_p}^*(BS_0)^{W(\mathcal{G})}.$$

Of course, the Weyl group of a  $p$ -local compact group need not be a pseudo-reflection group, and hence the rational cohomology of the classifying space is not in general a polynomial algebra.

Like compact Lie groups and  $p$ -compact groups,  $p$ -local compact groups admit unstable Adams operations, which are defined in [JLL], using the internal structure of the  $p$ -local group in question, rather than its rational cohomology. One application of Theorem A is Proposition 3.2, which states that under a mild condition, the obvious cohomological definition of an unstable Adams operation characterises the same family of maps as the one referred to in [JLL] as “*geometric unstable Adams operations*”.

Another easy application of Theorem A is the observation that if  $\mathcal{G}$  is a  $p$ -local compact group with maximal torus  $S_0$ , then the inclusion in  $\mathcal{G}$  of the  $p$ -local subgroup given by the normaliser  $N_{\mathcal{G}}(S_0)$  induces a rational  $p$ -adic cohomology isomorphism.

In Section 1, we recall the basic concepts in the theory of  $p$ -local compact groups which will be needed to prove Theorem A. Section 2 is dedicated to the proof of the theorem. Finally in Section 3 we discuss the applications described above.

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## 1. SOME BASIC CONCEPTS

We recall the definition and some basic properties of  $p$ -local compact groups. The reader is referred to [BLO3] for a comprehensive account of these objects.

We begin by defining discrete  $p$ -toral groups. By  $\mathbb{Z}/p^\infty$  we mean the union of all  $\mathbb{Z}/p^r$  with respect to the natural inclusions.

**Definition 1.1.** *A discrete  $p$ -torus is a group isomorphic to  $(\mathbb{Z}/p^\infty)^r$  for some positive integer  $r$ . A discrete  $p$ -toral group is a group  $S$  which contains a normal discrete  $p$ -torus  $S_0$ , with  $p$ -power index. The normal subgroup  $S_0$  will be referred to as the maximal torus or the identity component of  $S$ , and the quotient group  $\Gamma \cong S/S_0$  will be called the group of components of  $S$ .*

The identity component  $S_0$  of a discrete  $p$ -toral group  $S$  can be characterised as the subset of all infinitely  $p$ -divisible elements in  $S$ , and also as the unique minimal subgroup of finite index in  $S$ . Thus,  $S_0$  is a characteristic subgroup. The rank of  $S$  is the number  $r = \text{rk}(S)$  such that  $S_0 \cong (\mathbb{Z}/p^\infty)^r$ .

Recall that for  $P, Q \leq S$ , the transporter set  $T_S(P, Q)$  is the set of all elements  $g \in S$  such that  $gPg^{-1} \leq Q$ . We denote by  $\text{Hom}_S(P, Q)$  the set of all homomorphisms  $c_g: P \rightarrow Q$ , which are restrictions of an inner automorphism of  $S$ , and by  $\text{Inj}(P, Q)$  denote the set of all the injective homomorphisms  $P \rightarrow Q$ . We are now ready to recall the definition of fusion systems over discrete  $p$ -toral groups.

**Definition 1.2.** *A fusion system  $\mathcal{F}$  over a discrete  $p$ -toral group  $S$  is a category whose objects are the subgroups of  $S$ , and whose morphism sets  $\text{Hom}_{\mathcal{F}}(P, Q)$  satisfy the following conditions:*

- (a)  $\text{Hom}_S(P, Q) \subseteq \text{Hom}_{\mathcal{F}}(P, Q) \subseteq \text{Inj}(P, Q)$  for all  $P, Q \leq S$ .
- (b) *Every morphism in  $\mathcal{F}$  factors as an isomorphism in  $\mathcal{F}$  followed by an inclusion.*

Two subgroups  $P, P' \leq S$  are called  $\mathcal{F}$ -conjugate if  $P$  and  $P'$  are isomorphic as objects in  $\mathcal{F}$ . A subgroup  $P \leq S$  is said to be  $\mathcal{F}$ -centric if for every subgroup  $P' \leq S$  which is  $\mathcal{F}$ -conjugate to  $P$ ,  $C_S(P') = Z(P')$ .

All fusion systems considered in this paper are required to be *saturated* [BLO3, Definition 2.2]. Although the results we present here are based on properties of saturated fusion systems proved in [BLO3], we do not explicitly use the saturation axioms, and thus we will not repeat them here.

Next, we briefly recall what are centric linking systems and  $p$ -local compact groups. The full definition can be found in [BLO3, Definitions 4.1, 4.2]

**Definition 1.3.** *Let  $\mathcal{F}$  be a fusion system over a discrete  $p$ -toral group  $S$ . A centric linking system associated to  $\mathcal{F}$  is a category  $\mathcal{L}$  whose objects are the  $\mathcal{F}$ -centric subgroups of  $S$ , together with a functor*

$$\pi: \mathcal{L} \longrightarrow \mathcal{F}^c,$$

*and “distinguished” monomorphisms  $P \xrightarrow{\delta_P} \text{Aut}_{\mathcal{L}}(P)$  for each  $\mathcal{F}$ -centric subgroup  $P \leq S$ , which satisfy the following conditions.*

- (A)  *$\pi$  is the identity on objects and surjective on morphisms. More precisely, for each pair of objects  $P, Q \in \mathcal{L}$ , the centre  $Z(P)$  acts freely on  $\text{Mor}_{\mathcal{L}}(P, Q)$  by composition (upon identifying  $Z(P)$  with  $\delta_P(Z(P)) \leq \text{Aut}_{\mathcal{L}}(P)$ ), and  $\pi$  induces a bijection*

$$\text{Mor}_{\mathcal{L}}(P, Q)/Z(P) \xrightarrow{\cong} \text{Hom}_{\mathcal{F}}(P, Q).$$

- (B) For each  $\mathcal{F}$ -centric subgroup  $P \leq S$  and each  $g \in P$ ,  $\pi$  sends  $\delta_P(g) \in \text{Aut}_{\mathcal{L}}(P)$  to  $c_g \in \text{Aut}_{\mathcal{F}}(P)$ .
- (C) For each  $f \in \text{Mor}_{\mathcal{L}}(P, Q)$  and each  $g \in P$ , the following square commutes in  $\mathcal{L}$ :

$$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ \delta_P(g) \downarrow & & \downarrow \delta_Q(\pi(f)(g)) \\ P & \xrightarrow{f} & Q. \end{array}$$

A  $p$ -local compact group is a triple  $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ , where  $S$  is a discrete  $p$ -toral group,  $\mathcal{F}$  is a saturated fusion system over  $S$ , and  $\mathcal{L}$  is a centric linking system associated to  $\mathcal{F}$ . The classifying space of  $\mathcal{G}$  is the  $p$ -completed nerve  $|\mathcal{L}|_p^\wedge$ , which we will generally denote by  $B\mathcal{G}$ .

In [BLO3], the authors show that compact Lie groups and  $p$ -compact groups give rise to particular examples of  $p$ -local compact groups. Another large family of examples arises from linear torsion groups. In each case, the respective classifying space coincides up to homotopy (after  $p$ -completion in the case of genuine groups) with the classifying space of the  $p$ -local compact group it gives rise to.

**Definition 1.4.** Let  $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$  be a  $p$ -local compact group. Then the Weyl group  $W(\mathcal{G})$  of  $\mathcal{G}$  is defined to be the automorphism group in  $\mathcal{F}$  of the maximal torus  $S_0 \leq S$ .

Notice that  $H_{\mathbb{Q}_p}^*(X) \stackrel{\text{def}}{=} H^*(X, \mathbb{Z}_p) \otimes \mathbb{Q}$  is not in general isomorphic to  $H^*(X, \mathbb{Q}_p)$ . For instance if  $X = B\mathbb{Z}/p^\infty$ , then  $H_{\mathbb{Q}_p}^*(X)$  is a polynomial ring over the  $p$ -adic rationals on a generator in degree 2, while  $H^*(X, \mathbb{Q}_p)$  is trivial. The use of  $H_{\mathbb{Q}_p}^*$  as the appropriate cohomology theory for our purpose goes back to Dwyer and Wilkerson [DW], in their first paper on  $p$ -compact groups.

## 2. THE RATIONAL COHOMOLOGY

Two preparatory lemmas are needed before we prove our main claim.

**Lemma 2.1.** Let  $P$  be a discrete  $p$ -toral group with maximal torus  $P_0 \leq P$ . Then  $H_{\mathbb{Q}_p}^*(BP) \cong H_{\mathbb{Q}_p}^*(BP_0)^{P/P_0}$ .

*Proof.* This is of course a particular case of a much more general statement. Up to homotopy,  $BP_0$  is a covering space of  $BP$  with group  $P/P_0$ , and so one has the usual transfer map

$$\text{Tr}: H^*(BP_0, \mathbb{Z}_p^\wedge) \longrightarrow H^*(BP, \mathbb{Z}_p^\wedge),$$

where  $\text{Tr} \circ \text{Res}$  is multiplication by  $|P/P_0|$ . Hence after tensoring with  $\mathbb{Q}$  this composite is an isomorphism. On the other hand, the composition the other way  $\text{Res} \circ \text{Tr}$  is norm map for the action of  $P/P_0$  on  $H_{\mathbb{Q}_p}^*(BP_0)$ , and hence the image of restriction is the subgroup of invariants  $H_{\mathbb{Q}_p}^*(BP_0)^{P/P_0}$ .  $\square$

To prove the theorem, we will use the subgroup decomposition for  $p$ -local compact groups [BLO3, Proposition 4.6]. Hence the following lemma is an essential ingredient. In order to state it, we need to recall some notation and terminology.

For a fusion system  $\mathcal{F}$  over a discrete  $p$ -toral group  $S$ , we denote by  $\mathcal{O}(\mathcal{F})$  the orbit category associated to  $\mathcal{F}$ , i.e., the category with the same objects and with

morphisms  $\text{Mor}_{\mathcal{O}(\mathcal{F})}(P, Q) = \text{Rep}_{\mathcal{F}}(P, Q) \stackrel{\text{def}}{=} \text{Hom}_{\mathcal{F}}(P, Q)/\text{Inn}(Q)$ . For  $P, Q \in S$ , let  $N_S(P, Q)$  denote the transporter set consisting of all elements of  $S$  which conjugate  $P$  into  $Q$ . If  $\mathcal{F}'$  is a full subcategory of  $\mathcal{F}$ , we denote by  $\mathcal{O}(\mathcal{F}')$  the full subcategory of  $\mathcal{O}(\mathcal{F})$  whose objects are those of  $\mathcal{F}'$ . If  $\Gamma$  is a finite group, we denote by  $\mathcal{O}_p(\Gamma)$  the category whose objects are the  $p$ -subgroups of  $\Gamma$  and whose morphisms are  $\text{Mor}(P, Q) = C_{\Gamma}(P) \backslash N_{\Gamma}(P, Q)/\text{Inn}(Q)$ .

**Lemma 2.2.** *Let  $\mathcal{F}$  be any saturated fusion system over a discrete  $p$ -toral group  $S$ . Define*

$$F^* : \mathcal{O}(\mathcal{F}^c)^{\text{op}} \longrightarrow \mathbb{Q}\text{-mod}$$

on objects by setting  $F^*(P) = H_{\mathbb{Q}_p}^*(BP)$ . On morphisms,  $F^*$  sends the class of  $\varphi \in \text{Hom}_{\mathcal{F}}(P, P')$  to the homomorphism induced by  $B\varphi$ . Then  $F^*$  is acyclic, namely  $\varprojlim_{\mathcal{O}(\mathcal{F}^c)}^i (F^*) = 0$  for all  $i > 0$ .

*Proof.* Set  $Q = C_S(S_0) \triangleleft S$ , and  $\Gamma = \text{Out}_{\mathcal{F}}(Q)$ . Then  $Q$  is  $\mathcal{F}$ -centric, and is weakly closed in  $\mathcal{F}$  since  $S_0$  is. Let  $\mathcal{F}_{\geq Q}$  denote the full subcategory of  $\mathcal{F}$  whose objects are those  $P \leq S$  which contain  $Q$ , and let

$$\Theta : \mathcal{O}(\mathcal{F}_{\geq Q}) \longrightarrow \mathcal{O}_p(\Gamma)$$

be the functor which sends an object  $P$  to  $\text{Out}_P(Q) \leq \Gamma$ , and a morphism  $\varphi \in \text{Rep}_{\mathcal{F}}(P, P')$  to the class of  $\varphi|_Q \in N_{\Gamma}(\Theta(P), \Theta(P'))$  (see [BLO3, Lemma 5.7]). For each  $p$ -subgroup  $\Pi \leq \Gamma$ , regarded as a group of automorphisms of  $S_0$ , define

$$\Phi^*(\Pi) = H_{\mathbb{Q}_p}^*(BS_0)^{\Pi}.$$

This defines a graded functor  $\Phi^* : \mathcal{O}_p(\Gamma)^{\text{op}} \longrightarrow \mathbb{Q}\text{-mod}$ . Furthermore, for each  $P \leq S$  which contains  $Q$ ,

$$F^*(P) = H_{\mathbb{Q}_p}^*(BQ)^{P/Q} = \Phi^*(\Theta(P)).$$

Thus  $\Phi^* \circ \Theta \cong F^*|_{\mathcal{O}(\mathcal{F}_{\geq Q})}$ .

For each  $P \leq S$ ,  $\text{Out}_Q(P)$  acts trivially on  $F^*(P)$  since  $Q$  centralises  $P_0$ , and  $F^*(P)$  is a subring of  $H_{\mathbb{Q}_p}^*(BP_0)$ . So by [BLO3, Lemma 5.7],

$$\varprojlim_{\mathcal{O}(\mathcal{F}^c)}^* (F^*) \cong \varprojlim_{\mathcal{O}_p(\Gamma)}^* (\Phi^*).$$

The functor  $\Phi^*$  is a Mackey functor on  $\mathcal{O}_p(\Gamma)$ , and hence is acyclic (see [JM, Proposition 5.14] or [JMO, Proposition 5.2]).  $\square$

We are now ready to prove our main theorem.

**Theorem 2.3.** *Let  $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$  be a  $p$ -local compact group. Then*

$$H_{\mathbb{Q}_p}^*(B\mathcal{G}) \cong H_{\mathbb{Q}_p}^*(BS_0)^{W(\mathcal{G})}.$$

*Proof.* Let  $\pi : \mathcal{L} \rightarrow \mathcal{O}(\mathcal{F}^c)$  be the projection, and let  $\tilde{B} : \mathcal{O}(\mathcal{F}^c) \rightarrow \text{Top}$  denote the left homotopy Kan extension of the constant functor on  $\mathcal{L}$  along  $\pi$ . Then there is a homotopy equivalence

$$\text{hocolim}_{\mathcal{O}(\mathcal{F}^c)} \tilde{B} \longrightarrow |\mathcal{L}|,$$

and for each object  $P \in \mathcal{O}(\mathcal{F}^c)$ ,  $\tilde{B}(P) \simeq BP$  [BLO3, Proposition 4.6]. Consider the Bousfield-Kan spectral sequence [BK] for cohomology of the homotopy colimit, with

coefficients in the  $p$ -adic integers  $\mathbb{Z}_p$ . Since  $\mathbb{Q}$  is flat as a  $\mathbb{Z}$ -module, one can tensor the spectral sequence with  $\mathbb{Q}$  to get a spectral sequence for  $p$ -adic rational cohomology

$$E_2^{p,q} = \varprojlim_{\mathcal{O}(\mathcal{F}^c)}^p H_{\mathbb{Q}_p}^q(\tilde{B}(-)) \implies H_{\mathbb{Q}_p}^{p+q}(|\mathcal{L}|).$$

By Lemma 2.2, the higher limits all vanish and we obtain the formula

$$H_{\mathbb{Q}_p}^*(|\mathcal{L}|) \cong \varprojlim_{\mathcal{O}(\mathcal{F}^c)} H_{\mathbb{Q}_p}^*(\tilde{B}(-)). \quad (1)$$

For each  $\mathcal{F}$ -centric subgroup  $P \leq S$ , let  $\iota_P: P \rightarrow S$  denote the inclusion. The inverse limit in (1) consists of all elements  $x \in H_{\mathbb{Q}_p}^*(BS)$  such that  $\varphi^* \circ \iota_Q^*(x) = \iota_P^*(x)$  for all  $\mathcal{F}$ -centric subgroups  $P, Q \leq S$ , and all morphisms  $\varphi \in \text{Hom}_{\mathcal{F}}(P, Q)$ .

Let  $\varphi: P \rightarrow Q$  be any morphism in  $\mathcal{F}$ , where  $P$  and  $Q$  are  $\mathcal{F}$ -centric. Then by [BLO3, Lemma 2.4] the restriction  $\varphi|_{P_0}$  coincides with the restriction to  $P_0$  of some automorphism  $\sigma \in W(\mathcal{G})$ . Let  $x \in H_{\mathbb{Q}_p}^*(BS_0)^{W(\mathcal{G})} \leq H_{\mathbb{Q}_p}^*(BS)$  be any element. Then  $\iota_P^*(x) \in H_{\mathbb{Q}_p}^*(BP) \leq H_{\mathbb{Q}_p}^*(P_0)$ , and  $\iota_Q^*(x) \in H_{\mathbb{Q}_p}^*(BQ) \leq H_{\mathbb{Q}_p}^*(BQ_0)$ , and

$$\varphi^*(\iota_Q^*(x)) = \sigma^*(\iota_Q^*(x)) = \iota_P^*\sigma^*(x) = \iota_P^*(x).$$

Hence

$$H_{\mathbb{Q}_p}^*(BS_0)^{W(\mathcal{G})} \leq \varprojlim_{\mathcal{O}(\mathcal{F}^c)} H_{\mathbb{Q}_p}^*(\tilde{B}(-)).$$

Conversely, let  $y \in H_{\mathbb{Q}_p}^*(BS) \leq H_{\mathbb{Q}_p}^*(BS_0)$  be an element which is stable under each morphism in  $\mathcal{F}$  between centric subgroups, and let  $\sigma \in W(\mathcal{G})$ . By Alperin's fusion theorem,  $\sigma$  can be decomposed into a sequence  $\sigma = \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_n$ , where each  $\sigma_i \in W(\mathcal{G})$  can be extended to an automorphism of some  $\mathcal{F}$ -centric subgroup  $P_i \leq S$ . But since  $y$  is stable under each of the  $\sigma_i^*$ , it is also stable under  $\sigma^*$ . This shows that

$$\varprojlim_{\mathcal{O}(\mathcal{F}^c)} H_{\mathbb{Q}_p}^*(\tilde{B}(-)) \cong H_{\mathbb{Q}_p}^*(BS_0)^{W(\mathcal{G})}$$

and thus completes the proof of our claim.  $\square$

### 3. APPLICATIONS

For a compact Lie group  $G$ , one defines an unstable Adams operation of degree  $\zeta$  to be a selfmap of the classifying space inducing multiplication by  $\zeta^i$  on rational cohomology in dimension  $2i$ , where  $\zeta$  is an integer. An analogous definition is made for  $p$ -compact groups, except  $\zeta$  is required to be a  $p$ -adic unit, and rational cohomology is replaced by  $p$ -adic rational cohomology. Unstable Adams operations are a very important concept in the homotopy theory of classifying spaces of compact Lie groups and  $p$ -compact groups.

In [JLL], it is shown that  $p$ -local compact groups also admit unstable Adams operations. Let  $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$  be a  $p$ -local compact group and let  $\zeta$  be a  $p$ -adic unit. A *normal Adams automorphism of degree  $\zeta$*  on  $S$  is an automorphism  $\phi \in \text{Aut}(S)$  which restricts to the  $\zeta$ -power map on  $S_0$ , and induces the identity on the group of components  $S/S_0$ . A *geometric unstable Adams operation of degree  $\zeta$  on  $\mathcal{G}$*  is a selfmap  $\Psi$  of  $B\mathcal{G}$ , such that there exist a normal Adams automorphism  $\phi$  of degree  $\zeta$  on  $S$ , with the property that  $\iota \circ B\phi \simeq \Psi \circ \iota$ . Here  $\iota: BS \rightarrow B\mathcal{G}$  is the canonical inclusion. (See [JLL, Definitions 2.3, 3.4]) Theorem A allows us to define geometric unstable Adams

operations of  $p$ -local compact groups, along the lines of the classical cohomological definition.

The following lemma is an analogue of a theorem of Notbohm [N, Proposition 4.1].

**Lemma 3.1.** *Let  $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$  be a  $p$ -local compact group, and let  $T$  be a discrete  $p$ -torus. Then there is an isomorphism*

$$\mathrm{Hom}(T, S_0)/W(\mathcal{G}) \xrightarrow{\cong} [BT, B\mathcal{G}],$$

where  $W(\mathcal{G})$  acts by left translation. Also, two maps  $f, h: BT \rightarrow B\mathcal{G}$  are homotopic if and only if they induce the same homomorphism on  $H_{\mathbb{Q}_p}^*(-)$ .

*Proof.* By [BLO3, Theorem 6.3 (a)] there is an isomorphism of sets

$$\mathrm{Rep}(T, \mathcal{L}) \xrightarrow{\cong} [BT, B\mathcal{G}],$$

where  $\mathrm{Rep}(T, \mathcal{L}) \stackrel{\mathrm{def}}{=} \mathrm{Hom}(T, S)/\sim$ , with  $\alpha \sim \beta$  if and only if there is some  $\varphi \in \mathrm{Hom}_{\mathcal{F}}(\alpha(T), \beta(T))$  such that  $\varphi \circ \alpha = \beta$ . Since  $T$  is a discrete  $p$ -torus, the image of every homomorphism from it to  $S$  is contained in  $S_0$ , and by [BLO3, Lemma 2.4 (b)], every homomorphism in  $\mathcal{F}$  between subgroups of  $S_0$  is the restriction of some element in  $W(\mathcal{G})$ . Thus

$$\mathrm{Rep}(T, \mathcal{L}) \cong \mathrm{Hom}(T, S_0)/W(\mathcal{G}),$$

as claimed.

It remains to prove the last statement. Two maps  $f, h: BT \rightarrow B\mathcal{G}$  that are homotopic clearly induce the same map on cohomology. Conversely, assume that  $f, h: BT \rightarrow B\mathcal{G}$  are two maps such that  $f^* = g^*: H_{\mathbb{Q}_p}^*(B\mathcal{G}) \rightarrow H_{\mathbb{Q}_p}^*(BT)$ . Let  $\alpha, \beta: T \rightarrow S_0$  be homomorphisms such that  $f = \iota \circ B\alpha$  and  $g = \iota \circ B\beta$ , where  $\iota: BS_0 \rightarrow B\mathcal{G}$  is the inclusion of the maximal torus.

We will show that there is  $w \in W(\mathcal{G})$  such that  $w \circ \alpha = \beta$ , following the argument used by Adams and Mahmud to prove [AM, Theorem 1.7]: an argument based on the uniqueness of factorisation in the polynomial ring  $H_{\mathbb{Q}_p}^*(BS_0)$ . For simplicity, write  $V = H_{\mathbb{Q}_p}^2(BS_0)$  and  $V' = H_{\mathbb{Q}_p}^2(BT)$ . For each  $w \in W(\mathcal{G})$ , define

$$V(w) = \{x \in V \mid B\beta^*(x) = B(w \circ \alpha)^*(x)\} = \mathrm{Ker}((B\beta^* - B(w \circ \alpha)^*)|_V).$$

For each  $x \in V$ , set

$$\hat{x} = \prod_{w \in W(\mathcal{G})} Bw^*(1+x) \in S(V) \cong H_{\mathbb{Q}_p}^*(BS_0)$$

where  $S(V)$  denotes the symmetric algebra on the  $\mathbb{Q}_p$ -vector space  $V$ . Since  $\hat{x}$  is  $W(\mathcal{G})$ -invariant, Theorem 2.3 implies that  $\hat{x} \in \mathrm{Im}(\iota^*)$ , and hence that  $B\alpha^*(\hat{x}) = B\beta^*(\hat{x})$ . In other words,

$$\prod_{w \in W(\mathcal{G})} (1 + B\alpha^*Bw^*x) = \prod_{w \in W(\mathcal{G})} (1 + B\beta^*Bw^*x) \in S(V').$$

Since  $S(V')$  is a unique factorization domain, there is  $w \in W(\mathcal{G})$  such that  $(1 + B\beta^*x) = \lambda(1 + B\alpha^*Bw^*x)$ , for some  $\lambda \in \mathbb{Q}_p^\times$ . Then  $\lambda = 1$  and hence  $B\beta^*x = B\alpha^*Bw^*x$ . In particular,  $x \in V(w)$ .

This proves that  $V = \bigcup_{w \in W(\mathcal{G})} V(w)$ . Since  $\mathbb{Q}_p$  is infinite,  $V$  finite dimensional, and  $W(\mathcal{G})$  finite, there is  $w \in W(\mathcal{G})$  such that  $V = V(w)$  (cf. [AM, Lemma 3.1]). Hence  $B\beta^* = B(w \circ \alpha)^*$ . Since  $\mathrm{Hom}(T, S_0)$  injects into  $\mathrm{Hom}(H_{\mathbb{Q}_p}^*(BS_0), H_{\mathbb{Q}_p}^*(T))$ , it now follows that  $w \circ \alpha = \beta \in \mathrm{Hom}(T, S_0)$ , and hence that  $f \simeq g$  as maps  $BT \rightarrow B\mathcal{G}$ .  $\square$

**Proposition 3.2.** *Let  $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$  be a  $p$ -local compact group, and let  $\zeta$  be a  $p$ -adic unit. Then any geometric unstable Adams operation  $\Psi$  of degree  $\zeta$  induces multiplication by  $\zeta^i$  on  $H_{\mathbb{Q}_p}^{2i}(B\mathcal{G})$ . If  $S_0$  is self centralising in  $S$ , then any self equivalence  $\Psi$  of  $B\mathcal{G}$  which induces multiplication by  $\zeta^i$  on  $H_{\mathbb{Q}_p}^{2i}(B\mathcal{G})$  for each  $i$  is a geometric unstable Adams operation on  $\mathcal{F}$ .*

*Proof.* Let  $\iota: BS \rightarrow B\mathcal{G}$  be the canonical inclusion (induced by the distinguished monomorphism  $\delta_S: S \rightarrow \text{Aut}_{\mathcal{L}}(S)$ ), and set  $\iota_0 = \iota|_{S_0}$ . If  $\psi$  is a geometric unstable Adams operation on  $\mathcal{G}$  of degree  $\zeta$ , then by definition, there exists a normal Adams automorphism  $\phi$  of  $S$  such that  $\Psi \circ \iota \simeq \iota \circ B\phi$ , and hence  $\Psi \circ \iota_0 \simeq \iota_0 \circ B(\phi|_{S_0})$ . For each  $i \geq 0$ ,  $\phi|_{S_0}$  induces multiplication by  $\zeta^i$  on  $H_{\mathbb{Q}_p}^{2i}(BS_0)$ , and hence  $\Psi$  does the same on  $H_{\mathbb{Q}_p}^{2i}(B\mathcal{G})$ .

Assume now that  $S_0$  is self centralising in  $S$ . Let  $\Psi: B\mathcal{G} \rightarrow B\mathcal{G}$  be a self equivalence such that  $\Psi^*$  is multiplication by  $\zeta^i$  on  $H_{\mathbb{Q}_p}^{2i}(B\mathcal{G})$ . By [BLO3, Theorem 6.3(a)] and Lemma 3.1, the natural maps

$$\text{End}(S)/\text{Aut}_{\mathcal{F}}(S) \xrightarrow{\cong} [BS, B\mathcal{G}] \quad \text{and} \quad \text{End}(S_0)/W(\mathcal{G}) \xrightarrow{\cong} [BS_0, B\mathcal{G}] \quad (2)$$

are bijections. Hence there is  $\varphi \in \text{End}(S)$  such that  $\iota \circ B\varphi \simeq \Psi \circ \iota$ , and  $\varphi \in \text{Aut}(S)$  since  $\Psi$  is a homotopy equivalence. Let  $\varphi_0 \in \text{Aut}(S_0)$  be the restriction of  $\varphi$  to  $S_0$ , let  $\widehat{\zeta}$  denote the  $\zeta$ -power map on  $S_0$ , and set  $\rho = \widehat{\zeta} \circ \varphi_0^{-1} \in \text{Aut}(S_0)$ . Then

$$B\varphi_0^* \circ \iota_0^* = B\widehat{\zeta}^* \circ \iota_0^*: H_{\mathbb{Q}_p}^*(B\mathcal{G}) \longrightarrow H_{\mathbb{Q}_p}^*(BS_0),$$

and by Lemma 3.1, there is  $w \in W(\mathcal{G})$  such that  $w \circ \varphi_0 = \widehat{\zeta} \in \text{Aut}(S_0)$ .

Fix a morphism  $\widehat{\iota} \in \text{Mor}_{\mathcal{L}}(S_0, S)$  such that  $\pi(\widehat{\iota})$  is the inclusion, and regard this as the inclusion of  $S_0$  in  $S$  in the category  $\mathcal{L}$ . By [BLO3, Lemma 4.3(a)], for each  $g \in S$ , there is a unique restriction  $\delta_{S_0}(g) \in \text{Aut}_{\mathcal{L}}(S_0)$  of  $\delta_S(g) \in \text{Aut}_{\mathcal{L}}(S)$ ; i.e., a unique morphism such that  $\widehat{\iota} \circ \delta_{S_0}(g) = \delta_S(g) \circ \widehat{\iota}$ . Identify  $S$  and  $S_0$  with their images in  $\text{Aut}_{\mathcal{L}}(S_0)$ . Let  $\alpha \in \text{Aut}_{\mathcal{L}}(S_0)$  be a lift of  $w$ , i.e.,  $\pi(\alpha) = w$ . By Axiom (C), for each  $t \in S_0$ ,  $\alpha \circ \delta_{S_0}(t) = \delta_{S_0}(w(t)) \circ \alpha$ . Hence,  $c_\alpha|_{S_0} = w$ , and so  $\chi \stackrel{\text{def}}{=} c_\alpha|_S \circ \varphi: S \rightarrow \text{Aut}_{\mathcal{L}}(S_0)$  restricts to  $w \circ \varphi_0 = \widehat{\zeta}$  on  $S_0$ .

Now, for each  $g \in S$ ,  $\chi \circ c_g = c_{\chi(g)} \circ \chi$  as automorphisms of  $S_0$ , and since  $\chi|_{S_0} = \widehat{\zeta}$  is central in  $\text{Aut}(S_0)$ ,  $c_g|_{S_0} = c_{\chi(g)}|_{S_0}$ . Since  $S_0$  is self centralising in  $S$ , it follows that for each  $g \in S$ ,  $g \equiv \chi(g) \pmod{S_0}$ . In particular,  $\chi(S) = S$ , and  $\chi$  induces the identity on  $S/S_0$ . Thus  $\chi$  is a normal Adams automorphism of  $S$  of degree  $\zeta$ . Also,

$$\iota \circ B\chi \simeq \iota \circ B(c_\alpha|_S) \circ B\varphi \simeq \iota \circ B\varphi \simeq \Psi \circ \iota,$$

and thus  $\Psi$  is a geometric unstable Adams operation on  $\mathcal{G}$  as claimed.  $\square$

If  $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$  is a  $p$ -local compact group, and  $P \leq S$  is a subgroup satisfying a certain mild condition (fully normalised), then one can define the *normaliser fusion system*,  $\mathcal{N}_{\mathcal{F}}(P)$ , which is shown in [BLO6, Theorem 2.3] to be a saturated fusion system. The normaliser linking system  $N_{\mathcal{L}}(P)$  can be defined in exactly the same way as in [BLO2, Definition 6.1], and the proof of [BLO2, Lemma 6.2] applies verbatim to show that  $N_{\mathcal{L}}(P)$  is a centric linking system associated to  $\mathcal{N}_{\mathcal{F}}(P)$ . Thus in this case  $N_{\mathcal{G}}(P) = (N_S(P), \mathcal{N}_{\mathcal{F}}(P), N_{\mathcal{L}}(P))$  is a  $p$ -local compact subgroup of  $\mathcal{G}$ .

In particular, the maximal torus  $S_0$ , is fully normalised, since it is unique in its  $\mathcal{F}$ -conjugacy class, and we may consider the inclusion

$$\mathcal{N}_{\mathcal{G}}(S_0) \longrightarrow \mathcal{G}. \quad (3)$$

Then,  $S_0$  is the maximal torus in both  $\mathcal{G}$  and  $\mathcal{N}_{\mathcal{G}}(S_0)$ , and from the definition of morphisms in the normaliser fusion system [BLO6, Definition 2.1],

$$W(\mathcal{G}) = \text{Aut}_{\mathcal{F}}(S_0) = \text{Aut}_{\mathcal{N}_{\mathcal{F}}(S_0)}(S_0) = W(N_{\mathcal{G}}(S_0)).$$

Thus one obtains as an immediate corollary of Theorem A, that the inclusion (3) induces an isomorphism in  $p$ -adic rational cohomology. This is analogous to the corresponding statements for compact Lie groups and  $p$ -compact groups.

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