A KRULL-REMAK-SCHMIDT THEOREM FOR FUSION SYSTEMS

BOB OLIVER

Abstract. We prove that the factorization of a saturated fusion system over a discrete p-toral group as a product of indecomposable subsystems is unique up to normal automorphisms of the fusion system and permutations of the factors. In particular, if the fusion system has trivial center, or if its focal subgroup is the entire Sylow group, then this factorization is unique (up to the ordering of the factors). This result was motivated by questions about automorphism groups of products of fusion systems.

Let $\mathbb{Z}/p^\infty$ denote the union of an ascending sequence $\mathbb{Z}/p \le \mathbb{Z}/p^2 \le \mathbb{Z}/p^3 \le \cdots$ of finite cyclic p-groups. A discrete p-toral group is an extension of a group isomorphic to $(\mathbb{Z}/p^\infty)^r$ (some $r \ge 0$) by a finite p-group. A saturated fusion system $\mathcal{F}$ over a discrete p-toral group S is a category whose objects are the subgroups of S, whose morphisms are injective homomorphisms between the objects, and which satisfies certain axioms first formulated by Puig [Pg] when S is a finite p-group, and by Broto, Levi, and this author [BLO3] in the more general case.

For each compact Lie group G and each prime p, there is a saturated fusion system over a maximal discrete p-toral subgroup $S \le G$ that encodes the G-conjugacy relations between subgroups of S (see [BLO3, § 9]). Likewise, each torsion linear group in characteristic different from p (i.e., each subgroup $G \le GL_n(K)$ such that $n \ge 1$, K is a field with $\mathrm{char}(K) \ne p$, and all elements of G have finite order) has a maximal discrete p-toral subgroup S unique up to conjugacy, and a saturated fusion system over S that encodes G-conjugacy relations among subgroups of S [BLO3, Theorem 8.10].

The Krull-Remak-Schmidt theorem for groups says, in the case of finite groups, that for any two factorizations of G as a product of indecomposable subgroups, there is a normal automorphism of G that sends the one to the other. Here, $\alpha \in \mathrm{Aut}(G)$ is normal if it commutes with all inner automorphisms; equivalently, if α is the identity on $[G, G]$ and induces the identity on $G/Z(G)$. We refer to [Sz1, Theorem 2.4.8] or [Hu, Satz I.12.3] for the complete (much stronger) theorem.

By analogy, if α is an automorphism of a saturated fusion system $\mathcal{F}$ over a discrete p-toral group S (see Definition 2.1), α is normal if $\alpha|_{\mathfrak{foc}(\mathcal{F})} = \mathrm{Id}$ and $[\alpha, S] \le Z(\mathcal{F})$. See Definitions 1.7 and 4.1 and Lemma 4.3(a) for more details (and the more general definition of normal endomorphisms). In these terms, our main theorem is formulated as follows.

Theorem A. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S. Then there exist indecomposable fusion subsystems $\mathcal{E}_1, \ldots, \mathcal{E}_k \le \mathcal{F}$ ($k \ge 1$) such that*

$$\mathcal{F} = \mathcal{E}_1 \times \cdots \times \mathcal{E}_k.$$

If $\mathcal{F} = \mathcal{E}_1^* \times \cdots \times \mathcal{E}_m^*$ is another such factorization, then $k = m$, and there is a normal automorphism $\alpha \in \mathrm{Aut}(\mathcal{F})$ and a permutation $\sigma \in \Sigma_k$ such that $\alpha(\mathcal{E}_i) = \mathcal{E}^*_{\sigma(i)}$ for each i.

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The existence of a factorization into a product of indecomposables is elementary, and is shown in Proposition 2.7. The uniqueness part of Theorem A is a special case of Theorem 5.2 (the case where $\Omega = 1$), and our proof of that theorem is adapted directly from that in [Sz1] of the Krull-Remak-Schmidt theorem for groups. It is mostly a question of finding good definitions and properties of commuting fusion subsystems and normal endomorphisms of fusion systems; once this has been done it is straightforward to translate the proof of Theorem 2.4.8 in [Sz1] into this situation.

As a special case, when $Z(\mathcal{F}) = 1$ or $\mathfrak{foc}(\mathcal{F}) = S$, Theorem A says that the factorization of $\mathcal{F}$ is unique: that $\mathcal{F}$ is the product of all of its indecomposable direct factors. As one consequence of this (Corollary 5.4), if $\mathcal{F} = \mathcal{E}_1 \times \cdots \times \mathcal{E}_k$ where the $\mathcal{E}_i$ are indecomposable (and $Z(\mathcal{F}) = 1$ or $\mathfrak{foc}(\mathcal{F}) = S$), then $\mathrm{Aut}(\mathcal{F})$ is a semidirect product of $\prod_{i=1}^{k} \mathrm{Aut}(\mathcal{E}_i)$ with a certain subgroup of Σ_k.

This work was originally motivated by questions about automorphisms of products of fusion systems that arose during joint work with Carles Broto, Jesper Møller, and Albert Ruiz [BMOR]. It turned out that a special case of Theorem A was sufficient for our purposes in that paper: a case which had been proven earlier in [AOV, Proposition 3.6]. But this led to the question of whether a stronger result of that type might also be true.

One natural question is whether (and how easily) Theorem A, when restricted to fusion systems realized by finite groups, can be proven as a consequence of the Krull-Remak-Schmidt theorem for finite groups. When $p = 2$ and $O^{2'}(\mathcal{F}) = \mathcal{F}$, this can be done using a theorem of Goldschmidt on strongly closed 2-subgroups of a finite group [Gd, Theorem A]. In all other cases, any such argument seems to require the classification of finite simple groups. We refer to the end of Section 5 for a more detailed discussion of this question.

We have tried to write this while keeping in mind those readers who are interested only in the case of fusion systems over finite p-groups. For this reason, as far as possible, the extra complications that arise in the infinite case have been put into Section 1, which can easily be skipped by those interested only in the finite case and familiar with the basic definitions. Morphisms and commuting subsystems of fusion systems are defined and studied in Section 2, sums of endomorphisms in Section 3, and normal endomorphisms in Section 4. The main theorem and two corollaries are proven in Section 5.

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Notation and conventions: Composition of functions and functors is always from right to left. When G is a group and $P, Q \le G$, we let $\mathrm{Hom}_G(P, Q)$ denote the set of (injective) homomorphisms from P to Q induced by conjugation in G, and set $\mathrm{Aut}_G(P) = \mathrm{Hom}_G(P, P) \cap \mathrm{Aut}(P)$.

1. Fusion systems over discrete p-toral groups

In this section, we collect some results that are needed mostly when handling fusion systems over *infinite* discrete p-toral groups. So readers who are already familiar with fusion systems over finite p-groups and only interested in that case can easily skip it.

As defined in the introduction, $\mathbb{Z}/p^\infty$ denotes the union of the chain of cyclic p-groups $\mathbb{Z}/p < \mathbb{Z}/p^2 < \mathbb{Z}/p^3 < \cdots$. It can also be identified with the quotient group $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$, or with the group of complex roots of unity of p-power order.

Definition 1.1. A *discrete p-toral group* is a group S, with normal subgroup $S_0 \trianglelefteq S$, such that S_0 is isomorphic to a finite product of copies of $\mathbb{Z}/p^\infty$ and S/S_0 is a finite p-group. The subgroup S_0 will be called the *identity component* of S, and S will be called *connected* if $S = S_0$. Define $|S| = (\mathrm{rk}(S_0), |S/S_0|)$, where $\mathrm{rk}(S_0) = k$ if $S_0 \cong (\mathbb{Z}/p^\infty)^k$.

The identity component S_0 of a discrete p-toral group S is characterized as the subset of all infinitely p-divisible elements in S, and also as the minimal subgroup of finite index in S. So $|S|$ depends only on S itself as a discrete group. We regard the order of a discrete p-toral group as an element of $\mathbb{N}^2$ with the lexicographical ordering; i.e., $|S| \le |S^*|$ if and only if either $\mathrm{rk}(S) < \mathrm{rk}(S^*)$, or $\mathrm{rk}(S) = \mathrm{rk}(S^*)$ and $|S/S_0| \le |S^*/S^*_0|$. Note that $S^* \le S$ implies $|S^*| \le |S|$, with equality only if $S^* = S$.

Recall that a group is *artinian* if each descending sequence of subgroups of S becomes constant. Discrete p-toral groups can be characterized as follows:

Lemma 1.2 ([BLO3, Proposition 1.2])**.** *A group is discrete p-toral if and only if it is artinian, and every finitely generated subgroup is a finite p-group.*

In particular, each subgroup or quotient group of a discrete p-toral group is again discrete p-toral, and each extension of one discrete p-toral group by another is discrete p-toral [BLO3, Lemma 1.3]. If $Q \le P$ is a pair of discrete p-toral groups, then $\mathrm{Out}_P(Q) \overset{\mathrm{def}}{=} \mathrm{Aut}_P(Q)/\mathrm{Inn}(Q)$ ($\cong N_P(Q)/QC_P(Q)$) is a finite p-group (see [BLO3, Proposition 1.5(c)]).

Before defining fusion systems, we prove two technical results about discrete p-toral groups that are elementary or well known for finite p-groups. The first deals with complications that arise because discrete p-toral groups need not be nilpotent.

Lemma 1.3. *Let S be a discrete p-toral group, and let S_0 be its identity component. Define inductively subgroups $\widetilde{Z}_n(S) \trianglelefteq S$, for $n \ge 0$, by setting $\widetilde{Z}_0(S) = 1$, $\widetilde{Z}_1(S) = \Omega_1(Z(S))$, and $\widetilde{Z}_n(S)/\widetilde{Z}_{n-1}(S) = \Omega_1(Z(S/\widetilde{Z}_{n-1}(S)))$. Set $\widetilde{Z}_\infty(S) = \bigcup_{n=1}^\infty \widetilde{Z}_n(S)$. Then*

(a) *$\widetilde{Z}_\infty(S) \ge S_0$ and $C_S(\widetilde{Z}_\infty(S)) \le \widetilde{Z}_\infty(S)$; and*

(b) *if $\alpha \in \mathrm{Aut}(S)$ has finite order prime to p, and $[\alpha, \widetilde{Z}_n(S)] \le \widetilde{Z}_{n-1}(S)$ for each $n \ge 1$, then $\alpha = \mathrm{Id}_S$.*

Proof. **(a)** For each pair of finite subgroups $P, Q \le S_0$, both normal in S and such that $P \not\le Q$, we have $(PQ/Q) \cap \widetilde{Z}_1(S/Q) \ge \Omega_1(C_{PQ/Q}(S/S_0))$, where $\Omega_1(C_{PQ/Q}(S/S_0)) \ne 1$ since PQ/Q and S/S_0 are both finite p-groups and $PQ/Q \ne 1$. When $Q = \widetilde{Z}_{n-1}(S)$ for $n \ge 1$, this says that $P \cap \widetilde{Z}_n(S) > P \cap \widetilde{Z}_{n-1}(S)$ whenever $\widetilde{Z}_{n-1}(S) \not\ge P$, and hence that $\widetilde{Z}_m(S) \ge P$ for m sufficiently large. In particular, $\widetilde{Z}_\infty(S) \ge \Omega_n(S_0)$ for each n, and so $\widetilde{Z}_\infty(S) \ge S_0$.

Set $T = \widetilde{Z}_\infty(S)$ and $U = C_S(T)$ for short, and assume $U \not\le T$. Then $1 \ne UT/T \trianglelefteq S/T$ where S/T is a finite p-group (recall $T = \widetilde{Z}_\infty(S) \ge S_0$), so $(UT/T) \cap \Omega_1(Z(S/T)) \ne 1$. In other words, there is $x \in S \smallsetminus T$ such that

$$[x, T] = 1, \qquad [x, S] \le T, \qquad \text{and} \qquad x^p \in T.$$

Since $[x, T] = 1$ and S/T is finite (and since $T = \bigcup_{m=1}^\infty \widetilde{Z}_m(S)$), there is $n \ge 1$ such that $[x, S] \le \widetilde{Z}_n(S)$ and $x^p \in \widetilde{Z}_n(S)$. Then $x \in \widetilde{Z}_{n+1}(S) \le T$, contradicting our assumption that $x \notin T$. We conclude that $U \le T$; i.e., that $C_S(\widetilde{Z}_\infty(S)) \le \widetilde{Z}_\infty(S)$.

(b) Assume $\alpha \in \mathrm{Aut}(S)$ has finite order prime to p and induces the identity on each quotient group $\widetilde{Z}_n(S)/\widetilde{Z}_{n-1}(S)$ (all $n \ge 1$). Since each of those quotients is a finite p-group, $\alpha|_{\widetilde{Z}_n(S)} = \mathrm{Id}$ for each n by [G, Theorem 5.3.2], and hence $\alpha|_{\widetilde{Z}_\infty(S)} = \mathrm{Id}$. So by [OV, Lemma 1.2], and since $C_S(\widetilde{Z}_\infty(S)) \le \widetilde{Z}_\infty(S)$ by (a), the class $[\alpha] \in \mathrm{Out}(S)$ is in the image of a certain injective homomorphism $\eta\colon H^1(S/\widetilde{Z}_\infty(S); Z(\widetilde{Z}_\infty(S))) \longrightarrow \mathrm{Out}(S)$. Each element in $H^1(S/\widetilde{Z}_\infty(S); Z(\widetilde{Z}_\infty(S))$ has order dividing $|S/\widetilde{Z}_\infty(S)|$ (see, e.g., Corollary 2 to [Sz1, Theorem 2.7.26]), and hence $[\alpha] = 1$ and $\alpha \in \mathrm{Inn}(S)$. But then $\alpha = 1$, since it has order prime to p while $\mathrm{Inn}(S) \cong S/Z(S)$ is discrete p-toral. □

The following generalization of nilpotent endomorphisms will be needed.

Definition 1.4. Let S be a discrete p-toral group. An endomorphism $f \in \mathrm{End}(S)$ is *locally nilpotent* if for each $x \in S$, there is $n \geq 1$ such that $f^n(x) = 1$.

Thus $f \in \mathrm{End}(S)$ is locally nilpotent if and only if $S = \bigcup_{n=1}^\infty \mathrm{Ker}(f^n)$.

If S is finite, then clearly all locally nilpotent endomorphisms are nilpotent. As a simple example of an endomorphism that is locally nilpotent but not nilpotent, let S be any discrete p-toral group that is abelian and infinite, and set $f = (x \mapsto x^p) \in \mathrm{End}(S)$.

Lemma 1.5. *Let S be an abelian discrete p-toral group, and assume $f \in \mathrm{End}(S)$ is surjective. Then there are unique subgroups $T, U \leq S$ such that $S = T \times U$, $f|_T \in \mathrm{Aut}(T)$, U is connected, and $f|_U \in \mathrm{End}(U)$ is locally nilpotent.*

Proof. For each $n \geq 1$, set $S_n = \Omega_n(S)$ and $f_n = f|_{S_n}$. Then S_n is a finite abelian p-group, $\{\mathrm{Im}(f_n^i)\}_{i=1}^\infty$ is a decreasing sequence of subgroups of S_n, and $\{\mathrm{Ker}(f_n^i)\}_{i=1}^\infty$ is an increasing sequence. Set $T_n = \bigcap_{i=1}^\infty \mathrm{Im}(f_n^i)$ and $U_n = \bigcup_{i=1}^\infty \mathrm{Ker}(f_n^i)$. Since S_n is finite, there is $k \geq 1$ such that $T_n = \mathrm{Im}(f_n^k)$ and $U_n = \mathrm{Ker}(f_n^k)$, and so $|T_n||U_n| = |S_n|$. Also, $f_n(T_n) = T_n$ (so $f_n \in \mathrm{Aut}(T_n)$), $f_n|_{U_n}$ is a nilpotent endomorphism of U_n, and hence $T_n \cap U_n = 1$ and $S_n = T_n \times U_n$.

From these properties, we see that $T_n \leq T_{n+1}$ and $U_n \leq U_{n+1}$ for all n. Set $U = \bigcup_{i=1}^\infty U_n$ and $T = \bigcup_{n=1}^\infty T_n$. Then $S = T \times U$, $f|_T \in \mathrm{Aut}(T)$, and $f|_U \in \mathrm{End}(U)$ is locally nilpotent. Note that $U = \bigcup_{i=1}^\infty \mathrm{Ker}(f^i)$.

Assume $S = T^* \times U^*$ is a second factorization, where $f|_{T^*} \in \mathrm{Aut}(T^*)$ and $f|_{U^*}$ is a locally nilpotent endomorphism. Then $U^* \leq \bigcup_{i=1}^\infty \mathrm{Ker}(f^i) = U$. For each $n \geq 1$, $f|_{\Omega_n(T^*)} \in \mathrm{Aut}(\Omega_n(T^*))$ since $f|_{T^*}$ is an automorphism, so $\Omega_n(T^*) \leq \bigcap_{i=1}^\infty \mathrm{Im}(f_n^i) = T_n$, and hence $T^* = \bigcup_{i=1}^\infty \Omega_n(T^*) \leq T$. Then $T^* = T$ and $U^* = U$ since $T^* \times U = T \times U$, proving that the decomposition is unique.

It remains to show that U is connected; i.e., that $U \cong (\mathbb{Z}/p^\infty)^r$ for some $r \geq 0$. Let $U_0 \leq U$ be the identity component of U. Set $\psi = f|_U \in \mathrm{End}(U)$ for short. Since ψ is surjective, $U/\mathrm{Ker}(\psi^i) \cong U$ for each $i \geq 1$. Since U is the union of the $\mathrm{Ker}(\psi^i)$ and U_0 has finite index in U, there is $k \geq 1$ such that $U_0\mathrm{Ker}(\psi^k) = U$. So $U/\mathrm{Ker}(\psi^k) \cong U$ is a quotient group of U_0 and hence connected. □

We next consider fusion systems over discrete p-toral groups.

Definition 1.6 ([BLO3, Definitions 2.1–2.2])**.** Fix a discrete p-toral group S.

(a) A *fusion system* $\mathcal{F}$ over S is a category whose objects are the subgroups of S, and whose morphism sets $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ are such that

- $\mathrm{Hom}_S(P, Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P, Q) \subseteq \mathrm{Inj}(P, Q)$ for all $P, Q \leq S$; and
- every morphism in $\mathcal{F}$ factors as an isomorphism in $\mathcal{F}$ followed by an inclusion.

Two subgroups $P, P' \leq S$ are *$\mathcal{F}$-conjugate* if $\mathrm{Iso}_{\mathcal{F}}(P, P') \neq \varnothing$, and two elements $x, y \in S$ are $\mathcal{F}$-conjugate if there is $\varphi \in \mathrm{Hom}_{\mathcal{F}}(\langle x\rangle, \langle y\rangle)$ such that $\varphi(x) = y$. The $\mathcal{F}$-conjugacy classes of $P \leq S$ and $x \in S$ are denoted $P^{\mathcal{F}}$ and $x^{\mathcal{F}}$, respectively.

(b) A subgroup $P \leq S$ is *fully automized* in $\mathcal{F}$ if the index of $\mathrm{Aut}_S(P)$ in $\mathrm{Aut}_{\mathcal{F}}(P)$ is finite and prime to p.

(c) A subgroup $P \leq S$ is *receptive* in $\mathcal{F}$ if the following holds: for each $Q \in P^{\mathcal{F}}$ and each $\varphi \in \mathrm{Iso}_{\mathcal{F}}(Q,P)$, if we set

$$N_\varphi = N_\varphi^{\mathcal{F}} = \{g \in N_S(Q) \,|\, \varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(P)\},$$

then φ extends to a homomorphism $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$.

(d) $\mathcal{F}$ is a *saturated fusion system* if the following two conditions hold:

- For each $P \leq S$, there is $R \in P^{\mathcal{F}}$ such that R is fully automized and receptive in $\mathcal{F}$.
- (Continuity axiom) If $P_1 \leq P_2 \leq P_3 \leq \cdots$ is an increasing sequence of subgroups of S, with $P_\infty = \bigcup_{n=1}^\infty P_n$, and if $\varphi \in \mathrm{Hom}(P_\infty, S)$ is any homomorphism such that $\varphi|_{P_n} \in \mathrm{Hom}_{\mathcal{F}}(P_n, S)$ for all n, then $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P_\infty, S)$.

This definition of saturation is different from that given in [BLO3], but is equivalent to it by [BLO6, Corollary 1.8]. For finite S, it is the definition used in [AKO, § I.2].

Note that $\mathrm{Out}_{\mathcal{F}}(P)$ $(= \mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Inn}(P))$ is finite for each saturated fusion system $\mathcal{F}$ over S and each $P \leq S$. If P is fully automized, then this follows from the definition and since $\mathrm{Out}_S(P)$ is finite ([BLO3, Proposition 1.5(c)]). Otherwise, there is some $R \in P^{\mathcal{F}}$ that is fully automized, and $\mathrm{Out}_{\mathcal{F}}(P) \cong \mathrm{Out}_{\mathcal{F}}(R)$ is finite.

The following additional definitions will be needed.

Definition 1.7. Let $\mathcal{F}$ be a fusion system over a discrete p-toral group S. For $P \leq S$,

- P is *$\mathcal{F}$-centric* if $C_S(Q) \leq Q$ for each $Q \in P^{\mathcal{F}}$;
- P is *$\mathcal{F}$-radical* if $O_p(\mathrm{Out}_{\mathcal{F}}(P)) = 1$;
- P is *central* in $\mathcal{F}$ if each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$, for $Q,R \leq S$, extends to some $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(QP,RP)$ such that $\bar{\varphi}|_P = \mathrm{Id}_P$; and
- P is *strongly closed in $\mathcal{F}$* if for each $x \in P$, $x^{\mathcal{F}} \subseteq P$.

In addition,

- $Z(\mathcal{F}) \trianglelefteq S$ (the *center* of $\mathcal{F}$) is the subgroup generated by all subgroups $Z \leq S$ central in $\mathcal{F}$; and
- $\mathfrak{foc}(\mathcal{F}) = \langle xy^{-1} \,|\, x,y \in S,\ y \in x^{\mathcal{F}} \rangle \trianglelefteq S$ (the *focal subgroup* of $\mathcal{F}$).

It follows immediately from the definitions that $Z(\mathcal{F}) \leq Z(S)$ and is itself central in $\mathcal{F}$, and that $\mathfrak{foc}(\mathcal{F}) \geq [S,S]$.

Lemma 1.8. *Let $\mathcal{F}$ be a fusion system over a discrete p-toral group S. Then*

(a) $Z(\mathcal{F}) \subseteq \{x \in Z(S) \,\big|\, x^{\mathcal{F}} = \{x\}\}$, *with equality if $\mathcal{F}$ is saturated; and*

(b) *if $P \leq S$ is such that $P \leq Z(\mathcal{F})$ or $P \geq \mathfrak{foc}(\mathcal{F})$, then P is strongly closed in $\mathcal{F}$.*

Proof. **(a)** The inclusion is immediate from the definition of a central subgroup. The opposite implication (when $\mathcal{F}$ is saturated) is shown in [AKO, Lemma I.4.2] when S is a finite p-group, and the same argument applies in the discrete p-toral case.

(b) If $P \leq Z(\mathcal{F})$, then by (a), $x^{\mathcal{F}} = \{x\}$ for each $x \in P$, and hence P is strongly closed. If $P \geq \mathfrak{foc}(\mathcal{F})$, then for each $x \in P$ and each $y \in x^{\mathcal{F}}$, $y = x(x^{-1}y) \in P$ since $x^{-1}y \in \mathfrak{foc}(\mathcal{F})$, so P is strongly closed also in this case. □

We next look at fusion subsystems.

Definition-Notation 1.9. Let $\mathcal{F}$ be a fusion system over a discrete p-toral group S.

- A *fusion subsystem* of $\mathcal{F}$ is a subcategory $\mathcal{E}$ of $\mathcal{F}$ whose objects are the subgroups of some $T \le S$, and such that $\mathcal{E}$ is itself a fusion system over T.
- For $T \le S$, $\mathcal{F}|_{\le T}$ denotes the full subcategory of $\mathcal{F}$ whose objects are the subgroups of T, regarded as a fusion subsystem of $\mathcal{F}$ over T.

The fusion subsystem $\mathcal{F}|_{\le T}$ is not, in general, saturated, not even when $\mathcal{F}$ is saturated. But in certain specialized cases this is the case.

Lemma 1.10. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S. Assume $T \le S$ is strongly closed in $\mathcal{F}$, and is such that $S = TC_S(T)$. Then $\mathcal{F}|_{\le T}$ is a saturated fusion subsystem of $\mathcal{F}$.*

Proof. Set $\mathcal{E} = \mathcal{F}|_{\le T}$ for short: by definition, a fusion system over T. Fix $P \le T$, and choose $R \in P^{\mathcal{F}}$ which is fully automized and receptive in $\mathcal{F}$. Then $R \le T$ since T is strongly closed, and $R \in P^{\mathcal{E}}$ since $\mathcal{E}$ is a full subcategory. Also, $\mathrm{Aut}_{\mathcal{E}}(R) = \mathrm{Aut}_{\mathcal{F}}(R)$ (again since $\mathcal{E}$ is a full subcategory), and $\mathrm{Aut}_T(R) = \mathrm{Aut}_S(R)$ since $N_S(R) = N_T(R)C_S(T)$. So R is fully automized in $\mathcal{E}$.

If $\varphi \in \mathrm{Iso}_{\mathcal{E}}(Q,R)$, then since R is receptive in $\mathcal{F}$, φ extends to some $\bar\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi^{\mathcal{F}}, S)$, where $N_\varphi^{\mathcal{F}} \le N_S(Q)$ is as defined in Definition 1.6(c). Then $N_\varphi^{\mathcal{E}} = N_\varphi^{\mathcal{F}} \cap T$, and $\bar\varphi(N_\varphi^{\mathcal{E}}) \le T$ since T is strongly closed. So $\bar\varphi$ restricts to $\widetilde\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi^{\mathcal{E}}, T) = \mathrm{Hom}_{\mathcal{E}}(N_\varphi^{\mathcal{E}}, T)$, and since φ was arbitrary, R is receptive in $\mathcal{E}$.

Thus each subgroup of T is $\mathcal{E}$-conjugate to one that is fully automized and receptive in $\mathcal{E}$. The continuity axiom for $\mathcal{E}$ follows immediately from that for $\mathcal{F}$, and so $\mathcal{E}$ is saturated. $\square$

We end the section with another technical lemma, one that will be needed in Section 4.

Lemma 1.11. *Assume $S = S_1 \times S_2$, where S_1 and S_2 are discrete p-toral groups, and let $\mathcal{F}$ be a saturated fusion system over S. Then for each $P \le S$ that is $\mathcal{F}$-centric and $\mathcal{F}$-radical, there are subgroups $P_i \le S_i$ ($i = 1, 2$) such that $P = P_1 \times P_2$.*

Proof. When S is a finite p-group, this is shown in [AOV, Lemma 3.1]. We adapt that proof to fit this more general situation, while dealing with the extra complications that arise when the groups are infinite.

Let $\mathrm{pr}_i \colon S \longrightarrow S_i$ be the projection ($i = 1, 2$). For each $P \le S$, we write $P_i = \mathrm{pr}_i(P)$, and set $\widehat{P} = P_1 \times P_2 \ge P$. Let $\widetilde{Z}_n(-)$ be as in Lemma 1.3. We first claim that for each $n \ge 1$,

$$\widetilde{Z}_n(P) = P \cap \widetilde{Z}_n(\widehat{P}). \tag{1.1}$$

For $n = 1$, this holds since $Z(P) = P \cap Z(\widehat{P})$. If (1.1) holds for $n \ge 1$, then we can identify $P/\widetilde{Z}_n(P)$ as a subgroup of $\widehat{P}/\widetilde{Z}_n(\widehat{P})$ (which projects surjectively to each factor $P_i/\widetilde{Z}_n(P_i)$), and (1.1) follows for $n+1$ since $\Omega_1(Z(P/\widetilde{Z}_n(P))) = \Omega_1(Z(\widehat{P}/\widetilde{Z}_n(\widehat{P})))$.

Set $B = \{\alpha \in \mathrm{Aut}_{\mathcal{F}}(P) \,|\, [\alpha, \widetilde{Z}_i(P)] \le \widetilde{Z}_{i-1}(P) \ \forall\, i\}$. Then $\mathrm{Inn}(P) \le B$, and $B \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(P)$ since the $\widetilde{Z}_i(P)$ are all characteristic. By Lemma 1.3, the only element of order prime to p in B is the identity. If $\alpha \in B$ is such that its class $[\alpha] \in B/\mathrm{Inn}(P)$ has order n prime to p, then $\alpha^n \in \mathrm{Inn}(P)$ has p-power order, so α has order np^k for some $k \ge 0$, and there is $\beta \in \langle\alpha\rangle$ of order n such that $[\beta] = [\alpha]$. So the finite group $B/\mathrm{Inn}(P) \trianglelefteq \mathrm{Out}_{\mathcal{F}}(P)$ has p-power order, and hence $B/\mathrm{Inn}(P) \le O_p(\mathrm{Out}_{\mathcal{F}}(P))$. Thus $B = \mathrm{Inn}(P)$, since P is $\mathcal{F}$-radical.

Assume that $P < \widehat{P} = P_1 \times P_2$. Then $P < N_{\widehat{P}}(P)$ (see [BLO3, Lemma 1.8]). Choose $x \in N_{\widehat{P}}(P) \smallsetminus P$, and let $c_x \in \mathrm{Aut}_S(P)$ be conjugation by x. For each $n \geq 1$, c_x induces the identity on $\widetilde{Z}_n(P)/\widetilde{Z}_{n-1}(P)$ by (1.1) and since it induces the identity on $\widetilde{Z}_n(\widehat{P})/\widetilde{Z}_{n-1}(\widehat{P})$. Thus $c_x \in B = \mathrm{Inn}(P)$, and $x \in PC_S(P) = P$ since P is $\mathcal{F}$-centric, contradicting our assumption. We conclude that $P = \widehat{P}$ is a product, as claimed in the lemma. $\square$

2. Morphisms of fusion systems and commuting fusion subsystems

We are now ready to define morphisms of fusion systems.

Definition 2.1. Let $\mathcal{E}$ and $\mathcal{F}$ be fusion systems over discrete p-toral groups T and S, respectively.

(a) A *morphism* from $\mathcal{E}$ to $\mathcal{F}$ is a pair $(f, \widehat{f})$, where $f \in \mathrm{Hom}(T, S)$ and $\widehat{f}\colon \mathcal{E} \longrightarrow \mathcal{F}$ is a functor satisfying

- for each $P \leq T$, $\widehat{f}(P) = f(P) \leq S$; and
- for each $P, Q \leq T$, each $\varphi \in \mathrm{Hom}_{\mathcal{E}}(P, Q)$, and each $x \in P$, we have
$$\widehat{f}(\varphi)(f(x)) = f(\varphi(x)) \in f(Q).$$

We let $\mathrm{Mor}(\mathcal{E}, \mathcal{F}) \subseteq \mathrm{Hom}(T, S)$ denote the set of morphisms from $\mathcal{E}$ to $\mathcal{F}$, set $\mathrm{End}(\mathcal{F}) = \mathrm{Mor}(\mathcal{F}, \mathcal{F})$, and let $\mathrm{Aut}(\mathcal{F}) \leq \mathrm{End}(\mathcal{F})$ be the group of invertible endomorphisms.

(b) For each morphism $(f, \widehat{f}) \in \mathrm{Mor}(\mathcal{E}, \mathcal{F})$, define

- $\mathrm{Ker}(f, \widehat{f}) = \mathrm{Ker}\big(f\colon T \longrightarrow S\big) \trianglelefteq T$ (i.e., the kernel of f as a group homomorphism); and
- $\mathrm{Im}(f, \widehat{f}) = \langle \widehat{f}(\mathcal{E}) \rangle \leq \mathcal{F}$ (the smallest fusion subsystem of $\mathcal{F}$ containing $\widehat{f}(\mathcal{E})$).

By comparison, we write $f(T) \leq S$ to denote the image of f as a group homomorphism. We say that $(f, \widehat{f})$ is *surjective* (or onto) if $\mathrm{Im}(f, \widehat{f}) = \mathcal{F}$; i.e., if each morphism in $\mathcal{F}$ is a composite of morphisms in $\widehat{f}(\mathcal{E})$.

When $(f, \widehat{f}) \in \mathrm{Mor}(\mathcal{E}, \mathcal{F})$, the conditions in Definition 2.1(a) relating f and $\widehat{f}$ make it clear that $\widehat{f}$ is uniquely determined by f. For this reason, when there is no risk of confusion, we usually just write $f \in \mathrm{Mor}(\mathcal{E}, \mathcal{F})$ to represent the pair $(f, \widehat{f})$.

If $\mathcal{E}$ is a saturated fusion system over a *finite* p-group T and $(f, \widehat{f}) \in \mathrm{Mor}(\mathcal{E}, \mathcal{F})$, then by a theorem of Puig, $\mathrm{Im}(f, \widehat{f}) = \widehat{f}(\mathcal{E})$ and is a saturated fusion system. See, e.g., Corollary 5.15 and Proposition 5.11 in [Cr] for details. But we do not know whether this is always the case when T is an infinite discrete p-toral group, nor even whether $\widehat{f}(\mathcal{E})$ is always a subcategory. If one allows $\mathcal{E}$ not to be saturated, then one can easily construct morphisms $(f, \widehat{f}) \in \mathrm{Mor}(\mathcal{E}, \mathcal{F})$ where $\widehat{f}(\mathcal{E})$ is not a subcategory of $\mathcal{F}$.

However, it turns out that none of this is relevant when proving Theorem A, which is why many of the statements in this section and the next involve fusion subsystems that need not be saturated, or morphisms whose domain is not assumed saturated. Later, in Proposition 4.4(d), we will show that in the important case where f is a *normal* endomorphism of a saturated fusion system $\mathcal{F}$, the image $\mathrm{Im}(f)$ is always saturated (and $\mathrm{Im}(f) = \widehat{f}(\mathcal{F})$).

Lemma 2.2. *Let $\mathcal{E}$ and $\mathcal{F}$ be fusion systems over discrete p-toral groups T and S.*

(a) *For each $f \in \mathrm{Mor}(\mathcal{E}, \mathcal{F})$, $\mathrm{Ker}(f)$ is strongly closed in $\mathcal{E}$ (Definition 1.7).*

(b) *If $f \in \mathrm{Mor}(\mathcal{E},\mathcal{F})$ is such that $\mathrm{Ker}(f)=1$ and $\mathrm{Im}(f)=\mathcal{F}$, then f is an isomorphism of fusion systems.*

Proof. **(a)** For each $x \in \mathrm{Ker}(f)$ and $y \in x^{\mathcal{E}}$, there is $\varphi \in \mathrm{Mor}(\mathcal{E})$ such that $y=\varphi(x)$, and hence $f(y)=\widehat{f}(\varphi)(f(x))=1$. So $x^{\mathcal{F}} \subseteq \mathrm{Ker}(f)$, and $\mathrm{Ker}(f)$ is strongly closed in $\mathcal{E}$.

(b) If $(f,\widehat{f}) \in \mathrm{Mor}(\mathcal{E},\mathcal{F})$ and $\mathrm{Ker}(f)=1$, then the functor $\widehat{f}\colon \mathcal{E} \longrightarrow \mathcal{F}$ is injective on objects and on morphisms, and $\widehat{f}(\mathcal{E}) \cong \mathcal{E}$ is a fusion system. If in addition, $\mathrm{Im}(f)=\langle \widehat{f}(\mathcal{E})\rangle = \mathcal{F}$, then $\widehat{f}(\mathcal{E})=\mathcal{F}$, so $\widehat{f}$ is bijective, $\widehat{f}^{-1}$ is also a functor, and hence $(f,\widehat{f})^{-1} \in \mathrm{Mor}(\mathcal{F},\mathcal{E})$. $\square$

We next recall the definition of a direct product of fusion systems.

Definition 2.3. Let $\mathcal{F}_1,\dots,\mathcal{F}_k$ be fusion systems over discrete p-toral groups $S_1,\dots,S_k$. Set $S=S_1\times\cdots\times S_k$, and let $\mathrm{pr}_i\colon S \longrightarrow S_i$ be projection to the i-th factor. The direct product $\mathcal{F}=\mathcal{F}_1\times\cdots\times\mathcal{F}_k$ is the fusion system over S with morphism sets defined by

$$\mathrm{Hom}_{\mathcal{F}}(P,Q)=\bigl\{(\varphi_1,\dots,\varphi_k)|_P \,\bigm|\, \varphi_i \in \mathrm{Hom}_{\mathcal{F}_i}(\mathrm{pr}_i(P),\mathrm{pr}_i(Q)),\ (\varphi_1,\dots,\varphi_k)(P)\le Q\bigr\}.$$

By construction, a product of fusion systems is again a fusion system. In fact, formally, $\mathcal{F}=\mathcal{F}_1\times\cdots\times\mathcal{F}_k$ is a product in the category of fusion systems and morphisms between them. For example, one easily checks that if $\mathcal{F}^*$ is any fusion system over $S=S_1\times\cdots\times S_k$ such that $\mathrm{pr}_i \in \mathrm{Mor}(\mathcal{F}^*,\mathcal{F}_i)$ for all i (in the notation of Definition 2.3), then $\mathcal{F}^*\le\mathcal{F}$.

If S_i is finite and $\mathcal{F}_i$ is saturated for all i, then the product $\mathcal{F}$ is also saturated: see, e.g., [AKO, Theorem I.6.6]. That proof can easily be extended to show that products of saturated fusion systems over discrete p-toral groups are saturated, but since this will not be needed here, we omit it. Note that the converse follows from Lemma 1.10: if a product of fusion systems is saturated, then so is each factor.

The following definition of commuting fusion subsystems is equivalent to that used by Henke [He, Definition 3.1] of "subsystems that centralize each other". The equivalence of the two definitions, at least in the finite case, is essentially the content of [He, Proposition 3.3]. (See also the remarks after Lemma 2.8.)

Definition 2.4. Let $\mathcal{F}$ be a fusion system over a discrete p-toral group S, and let $\mathcal{E}_1,\dots,\mathcal{E}_k\le\mathcal{F}$ be fusion subsystems. We say that $\mathcal{E}_1,\dots,\mathcal{E}_k$ *commute* if there is a morphism of fusion systems $(I,\widehat{I}) \in \mathrm{Mor}(\mathcal{E}_1\times\cdots\times\mathcal{E}_k,\mathcal{F})$ whose restriction to each factor $\mathcal{E}_i$ is the inclusion. In this situation, we set $\mathcal{E}_1\cdots\mathcal{E}_k=\mathrm{Im}(I,\widehat{I})=\widehat{I}(\mathcal{E}_1\times\cdots\times\mathcal{E}_k)$.

The morphism $(I,\widehat{I}) \in \mathrm{Mor}(\mathcal{E}_1\times\cdots\times\mathcal{E}_k,\mathcal{F})$ is uniquely determined whenever it exists. So $\mathcal{E}_1\cdots\mathcal{E}_k$ is well defined, and is the (unique) smallest fusion subsystem of $\mathcal{F}$ in which the $\mathcal{E}_i$ commute. By comparison, $\langle\mathcal{E}_1,\dots,\mathcal{E}_k\rangle$ is defined to be the smallest fusion subsystem of $\mathcal{F}$ containing all of the $\mathcal{E}_i$, and is in general smaller than $\mathcal{E}_1\cdots\mathcal{E}_k$.

This definition of commuting subsystems is, of course, motivated by one characterization of commuting subgroups of a group. But the following examples show that commuting subsystems can behave quite differently from commuting subgroups.

Example 2.5. *Set $p=3$. Fix groups $H_i\cong\Sigma_3$ for $i=1,2,3$, set $T_i=O_3(H_i)\cong C_3$, and choose $b_i\in H_i\smallsetminus T_i$ (so $|b_i|=2$). Set*

$$\widehat{G}=H_1\times H_2\times H_3,\qquad S=T_1\times T_2\times T_3,\qquad G=S\langle b_1b_2,b_1b_3\rangle<\widehat{G},$$

and set $\widehat{\mathcal{F}}=\mathcal{F}_S(\widehat{G})$ and $\mathcal{F}=\mathcal{F}_S(G)$. Also, set $\mathcal{E}_i=\mathcal{F}_{T_i}(H_i)$ (for $i=1,2,3$), so that $\mathcal{E}_i\le\mathcal{F}\le\widehat{\mathcal{F}}$ are all saturated fusion subsystems.

(a) *The subsystems $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_3$ commute pairwise in $\mathcal{F}$, but do not commute as a triple.*

(b) *The saturated fusion subsystems* $\mathcal{E}_1\mathcal{E}_2$ *and* $\mathcal{E}_3$ *commute in* $\widehat{\mathcal{F}}$*, but do not commute in* $\mathcal{F} < \widehat{\mathcal{F}}$.

Of particular interest is the situation where $\mathcal{E}_1$ and $\mathcal{E}_2$ commute in $\mathcal{F}$ and $\mathcal{F} = \mathcal{E}_1\mathcal{E}_2$.

Lemma 2.6. *Let* $\mathcal{F}$ *be a saturated fusion system over a discrete* p*-toral group* S*, and let* $\mathcal{E}_1, \mathcal{E}_2 \leq \mathcal{F}$ *be fusion subsystems over* $T_1, T_2 \leq S$*. Assume that* $\mathcal{E}_1$ *and* $\mathcal{E}_2$ *commute, and that* $\mathcal{E}_1\mathcal{E}_2 = \mathcal{F}$ *(thus* $T_1T_2 = S$*). Then*

(a) $T_1 \cap T_2 \leq Z(\mathcal{F})$*; and*

(b) *if* $T_1 \cap T_2 = 1$*, then* $\mathcal{F} \cong \mathcal{E}_1 \times \mathcal{E}_2$.

Proof. Let $(I, \widehat{I}) \in \mathrm{Mor}(\mathcal{E}_1 \times \mathcal{E}_2, \mathcal{F})$ be the morphism that extends the inclusions. By assumption, $\mathcal{F} = \mathcal{E}_1\mathcal{E}_2 = \langle \widehat{I}(\mathcal{E}_1 \times \mathcal{E}_2)\rangle$.

(a) Fix $x \in T_1 \cap T_2$; we show that $x^{\mathcal{F}} = \{x\}$. Since $\langle \widehat{I}(\mathcal{E}_1 \times \mathcal{E}_2)\rangle = \mathcal{F}$, it suffices to show that $\widehat{I}(\varphi_1, \varphi_2)(x) = x$ for each $\varphi_i \in \mathrm{Hom}_{\mathcal{E}_i}(P_i, T_i)$ $(i = 1, 2)$ such that $x \in P_1P_2$. Fix such P_i and φ_i, and let $x_i \in P_i \leq T_i$ be such that $x = x_1x_2$. Note that $x_1 = xx_2^{-1} \in T_2$, and similarly $x_2 \in T_1$.

By Definition 2.1(a), $\widehat{I}(\varphi_1, \mathrm{Id}_{T_2}) \in \mathrm{Hom}_{\mathcal{F}}(P_1T_2, S)$ sends $I(x_1, x_1^{-1}) = 1$ to $I(\varphi_1(x_1), x_1^{-1}) = \varphi_1(x_1)x_1^{-1}$. Hence $\varphi_1(x_1) = x_1$. Also, $\varphi_2(x_2) = x_2$ by a similar argument, and so $\widehat{I}(\varphi_1, \varphi_2)(x) = \varphi_1(x_1)\varphi_2(x_2) = x_1x_2 = x$. Hence $x^{\mathcal{F}} = \{x\}$, and $x \in Z(\mathcal{F})$ by Lemma 1.8(a).

(b) If $T_1 \cap T_2 = 1$, then $\mathrm{Ker}(I) = 1$, and I is an isomorphism of fusion systems by Lemma 2.2(b) and since $\mathrm{Im}(I) = \mathcal{E}_1\mathcal{E}_2 = \mathcal{F}$. □

Motivated by Lemma 2.6, we now write $\mathcal{F} = \mathcal{E}_1 \times \mathcal{E}_2$ to mean that $\mathcal{E}_1$ and $\mathcal{E}_2$ are subsystems over T_1 and T_2 that commute in $\mathcal{F}$, such that $T_1 \cap T_2 = 1$ and $\mathcal{E}_1\mathcal{E}_2 = \mathcal{F}$. More generally, if $\mathcal{E}_1, \ldots, \mathcal{E}_k$ is a k-tuple of commuting fusion subsystems of $\mathcal{F}$, then we write $\mathcal{F} = \mathcal{E}_1 \times \cdots \times \mathcal{E}_k$ to mean that the morphism $I \in \mathrm{Mor}(\mathcal{E}_1 \times \cdots \times \mathcal{E}_k, \mathcal{F})$ extending the inclusions is an isomorphism of fusion systems.

A fusion system $\mathcal{F}$ over S is *indecomposable* if there are no fusion subsystems $\mathcal{E}_1, \mathcal{E}_2$ commuting in $\mathcal{F}$, over proper subgroups $T_1, T_2 < S$, such that $\mathcal{F} = \mathcal{E}_1 \times \mathcal{E}_2$. Our goal in the rest of the paper is to prove the essential uniqueness of factorizations of saturated fusion systems as products of indecomposable subsystems. The existence of such a factorization is elementary, based on the fact that discrete p-toral groups are artinian.

Proposition 2.7. *Let* $\mathcal{F}$ *be a fusion system over a discrete* p*-toral group* S*. Then there exist indecomposable fusion subsystems* $\mathcal{E}_1, \ldots, \mathcal{E}_k \leq \mathcal{F}$ *such that* $\mathcal{F} = \mathcal{E}_1 \times \cdots \times \mathcal{E}_k$.

Proof. If there is no such factorization, then there is a descending sequence of fusion subsystems $\mathcal{F} = \mathcal{E}_0 \geq \mathcal{E}_1 \geq \mathcal{E}_2 \geq \cdots$ over subgroups $S = T_0 \geq T_1 \geq T_2 \geq \cdots$, where $\mathcal{E}_i$ is a proper direct factor of $\mathcal{E}_{i-1}$ for each $i \geq 1$ and hence $T_i < T_{i-1}$. But this is impossible, since S is artinian (Lemma 1.2). □

In the next lemma, we give another, equivalent, condition for fusion subsystems to be commuting.

Lemma 2.8. *Let* $\mathcal{F}$ *be a fusion system over a discrete* p*-toral group* S*, and let* $\mathcal{E}_1, \ldots, \mathcal{E}_k \leq \mathcal{F}$ *be fusion subsystems over subgroups* $T_1, \ldots, T_k \leq S$*. Then the following are equivalent.*

(a) *The subsystems* $\mathcal{E}_1, \ldots, \mathcal{E}_k$ *commute.*

(b) *The subgroups T_i commute pairwise, and for each k-tuple of morphisms*

$$\big\{\varphi_i \in \mathrm{Hom}_{\mathcal{E}_i}(P_i,Q_i)\big\}_{i=1}^k \in \mathrm{Mor}(\mathcal{E}_1)\times\cdots\times\mathrm{Mor}(\mathcal{E}_k),$$

there is $\bar\varphi \in \mathrm{Hom}_{\mathcal{F}}(P_1\cdots P_k, Q_1\cdots Q_k)$ such that $\bar\varphi|_{P_i}=\varphi_i$ for each i.

Proof. **(a $\Longrightarrow$ b)** Let $(I,\widehat{I}) \in \mathrm{Mor}(\mathcal{E}_1\times\cdots\times\mathcal{E}_k,\mathcal{F})$ be the morphism that extends the inclusions. Thus $I(x_1,\dots,x_k)=x_1\cdots x_k$ for $x_i\in T_i$, so the subgroups T_i commute pairwise. If $\{\varphi_i \in \mathrm{Hom}_{\mathcal{E}_i}(P_i,Q_i)\}$ is a k-tuple of morphisms and $\bar\varphi=\widehat{I}(\varphi_1,\dots,\varphi_k)$, then for each $(x_1,\dots,x_k)$ with $x_i\in P_i$,

$$\bar\varphi(x_1\cdots x_k)=\widehat{I}(\varphi_1,\dots,\varphi_k)(I(x_1,\dots,x_k))=I(\varphi_1(x_1),\dots,\varphi_k(x_k))=\varphi_1(x_1)\cdots\varphi_k(x_k).$$

So $\bar\varphi\in\mathrm{Hom}_{\mathcal{F}}(P_1\cdots P_k,Q_1\cdots Q_k)$ extends each of the φ_i.

(b $\Longrightarrow$ a) Since the T_i commute pairwise, we can define $I\in\mathrm{Hom}(T_1\times\cdots\times T_k,S)$ by setting $(x_1,\dots,x_k)=x_1\cdots x_k$ for $x_i\in T_i$. Define a functor $\widehat{I}\colon \mathcal{E}_1\times\cdots\times\mathcal{E}_k\longrightarrow\mathcal{F}$ as follows. On objects, we set $\widehat{I}(P)=I(P)$ for each $P\le T_1\times\cdots\times T_k$ as usual. In particular, $\widehat{I}(P_1\times\cdots\times P_k)=P_1\cdots P_k$ for $P_i\le T_i$.

By definition of the product fusion system, for each $\varphi\in\mathrm{Hom}_{\mathcal{E}_1\times\cdots\times\mathcal{E}_k}(P,Q)$, there are morphisms $\varphi_i\in\mathrm{Hom}_{\mathcal{E}_i}(P_i,Q_i)$ (for $1\le i\le k$) such that

$$P\le P_1\times\cdots\times P_k,\qquad Q\le Q_1\times\cdots\times Q_k,\qquad\text{and}\qquad \varphi=(\varphi_1\times\cdots\times\varphi_k)|_P.$$

By assumption (b), there is a morphism $\bar\varphi\in\mathrm{Hom}_{\mathcal{F}}(P_1\cdots P_k,Q_1\cdots Q_k)$ that extends each of the φ_i, this is clearly unique, and we set $\widehat{I}(\varphi_1\times\cdots\times\varphi_k)=\bar\varphi$ and $\widehat{I}(\varphi)=\bar\varphi|_{I(P)}$. By the uniqueness of $\bar\varphi$, these preserve composition, and hence define a functor $\widehat{I}$ from $\mathcal{E}_1\times\cdots\times\mathcal{E}_k$ to $\mathcal{F}$ associated to I. Thus $(I,\widehat{I})$ is a morphism of fusion systems, and the subsystems $\mathcal{E}_i$ commute. □

In [He, Definition 3.1], Henke defined two fusion subsystems $\mathcal{E}_1,\mathcal{E}_2\le\mathcal{F}$ over $T_1,T_2\le S$ to commute in $\mathcal{F}$ if $\mathcal{E}_1\le C_{\mathcal{F}}(T_2)$ and $\mathcal{E}_2\le C_{\mathcal{F}}(T_1)$. (See, e.g., [AKO, Definition I.5.3] for the definition of centralizer fusion systems.) It is not hard to see that this is equivalent to condition (b) in Lemma 2.8.

The next lemma describes some of the elementary relations among commuting subsystems, including a form of associativity.

Lemma 2.9. *Let $\mathcal{F}$ be a fusion system over a discrete p-toral group S, and let $\mathcal{E}_1,\dots,\mathcal{E}_k\le\mathcal{F}$ be fusion subsystems over $T_1,\dots,T_k\le S$.*

(a) *For $2\le\ell<k$, $\mathcal{E}_1,\dots,\mathcal{E}_k$ commute in $\mathcal{F}$ if and only if there is $\mathcal{F}^*\le\mathcal{F}$ such that $\mathcal{E}_1,\dots,\mathcal{E}_\ell$ commute in $\mathcal{F}^*$ and $\mathcal{F}^*,\mathcal{E}_{\ell+1},\dots,\mathcal{E}_k$ commute in $\mathcal{F}$.*

(b) *Assume $f\in\mathrm{Mor}(\mathcal{F},\mathcal{D})$ for some fusion system $\mathcal{D}$. If $\mathcal{E}_1,\dots,\mathcal{E}_k$ commute in $\mathcal{F}$, then the subsystems $\mathrm{Im}(f|_{\mathcal{E}_1}),\dots,\mathrm{Im}(f|_{\mathcal{E}_k})$ commute in $\mathrm{Im}(f)$.*

Proof. **(b)** Let $\{\varphi_i\in\mathrm{Hom}_{\mathcal{E}_i}(P_i,Q_i)\}_{i=1}^k$ be a k-tuple of morphisms. By Lemma 2.8(a⇒b) and since the $\mathcal{E}_i$ commute in $\mathcal{F}$, there is $\bar\varphi\in\mathrm{Hom}_{\mathcal{F}}(P_1\cdots P_k,Q_1\cdots Q_k)$ that extends each of the φ_i. Then $\widehat{f}(\bar\varphi)\in\mathrm{Hom}_{\mathrm{Im}(f)}(f(P_1\cdots P_k),f(Q_1\cdots Q_k))$ extends each of the $\widehat{f}(\varphi_i)$.

Thus each k-tuple in $\prod_{i=1}^k\mathrm{Mor}(\widehat{f}(\mathcal{E}_i))$ extends to a morphism in $\mathrm{Im}(f)$. The same is true for a k-tuple in $\prod_{i=1}^k\mathrm{Mor}(\mathrm{Im}(f|_{\mathcal{E}_i}))$ since each morphism in $\mathrm{Im}(f|_{\mathcal{E}_i})=\langle\widehat{f}(\mathcal{E}_i)\rangle$ is a composite of morphisms in $\widehat{f}(\mathcal{E}_i)$, and so the subsystems $\mathrm{Im}(f|_{\mathcal{E}_1}),\dots,\mathrm{Im}(f|_{\mathcal{E}_k})$ commute in $\mathrm{Im}(f)$ by Lemma 2.8(b⇒a).

(a) Assume first that there is $\mathcal{F}^* \leq \mathcal{F}$ such that $\mathcal{E}_1, \ldots, \mathcal{E}_\ell$ commute in $\mathcal{F}^*$ and also $\mathcal{F}^*, \mathcal{E}_{\ell+1}, \ldots, \mathcal{E}_k$ commute in $\mathcal{F}$. Thus there are morphisms

$$J_1 \in \mathrm{Mor}(\mathcal{F}^* \times \mathcal{E}_{\ell+1} \times \cdots \times \mathcal{E}_k, \mathcal{F}) \qquad \text{and} \qquad J_2 \in \mathrm{Mor}(\mathcal{E}_1 \times \cdots \times \mathcal{E}_\ell, \mathcal{F}^*)$$

each of which extends the inclusions of the different factors into the target. Then $J_1 \circ (J_2 \times \mathrm{Id}_{\mathcal{E}_{\ell+1} \times \cdots \times \mathcal{E}_k})$ is a morphism of fusion systems from $\mathcal{E}_1 \times \cdots \times \mathcal{E}_k$ to $\mathcal{F}$ extending the inclusions of the $\mathcal{E}_i$, so these subsystems commute.

Conversely, assume $\mathcal{E}_1, \ldots, \mathcal{E}_k$ commute in $\mathcal{F}$, let $I \in \mathrm{Mor}(\mathcal{E}_1 \times \cdots \times \mathcal{E}_k, \mathcal{F})$ extend the inclusions, and set $\mathcal{F}^* = \mathcal{E}_1 \cdots \mathcal{E}_\ell$. The subsystems $(\mathcal{E}_1 \times \cdots \times \mathcal{E}_\ell), \mathcal{E}_{\ell+1}, \ldots, \mathcal{E}_k$ commute in $\mathcal{E}_1 \times \cdots \times \mathcal{E}_k$. So by (b), applied with I in the role of f, $(\mathcal{E}_1 \cdots \mathcal{E}_\ell), \mathcal{E}_{\ell+1}, \ldots, \mathcal{E}_k$ commute in $\mathrm{Im}(I)$ and hence in $\mathcal{F}$. □

The next lemma gives conditions for a saturated fusion system to factorize as a product when the underlying discrete p-toral group factorizes.

Lemma 2.10. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S. Assume $T, U \leq S$ are such that $S = T \times U$, and set $\mathcal{E} = \mathcal{F}|_{\leq T}$ and $\mathcal{D} = \mathcal{F}|_{\leq U}$. Assume also that T and U are strongly closed in $\mathcal{F}$, and that $\mathcal{E}$ and $\mathcal{D}$ commute in $\mathcal{F}$. Then $\mathcal{F} = \mathcal{E} \times \mathcal{D}$.*

Proof. Let $I \in \mathrm{Mor}(\mathcal{E} \times \mathcal{D}, \mathcal{F})$ be the morphism that extends the inclusions. Then $\mathrm{Ker}(I) = T \cap U = 1$, so by Lemma 2.2, $\mathrm{Im}(I) \leq \mathcal{F}$ is a fusion subsystem over $S = TU$. We will show that $\mathrm{Im}(I) = \mathcal{F}$. Once this has been shown, then $\mathcal{F} = \mathcal{E} \times \mathcal{D}$ by Lemma 2.6(b).

Assume $P \leq S$ is $\mathcal{F}$-centric and $\mathcal{F}$-radical (Definition 1.7). By Lemma 1.11, we have $P = P_1 \times P_2$, where $P_1 = P \cap T$ and $P_2 = P \cap U$. For each $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$, $\alpha|_{P_i} \in \mathrm{Aut}_{\mathcal{F}}(P_i)$ for $i = 1, 2$ since T and U are strongly closed, and hence $\alpha|_{P_1} \in \mathrm{Aut}_{\mathcal{E}}(P_1)$ and $\alpha|_{P_2} \in \mathrm{Aut}_{\mathcal{D}}(P_2)$ since $\mathcal{E}$ and $\mathcal{D}$ are full subcategories. So $\alpha = \widehat{I}(\alpha|_{P_1}, \alpha|_{P_2}) \in \mathrm{Aut}_{\mathrm{Im}(I)}(P)$.

By the version of Alperin's fusion theorem shown in [BLO3, Theorem 3.6], each morphism in $\mathcal{F}$ is a (finite) composite of restrictions of $\mathcal{F}$-automorphisms of $\mathcal{F}$-centric $\mathcal{F}$-radical subgroups of S. So every morphism in $\mathcal{F}$ is in $\mathrm{Im}(I)$, and $\mathcal{F} = \mathrm{Im}(I) = \mathcal{E} \times \mathcal{D}$. □

3. Sums and summability of endomorphisms

We next define sums of morphisms between a pair of fusion systems.

Definition 3.1. Let $\mathcal{E}$ and $\mathcal{F}$ be fusion systems over discrete p-toral groups T and S.

(a) A k-tuple of morphisms $f_1, \ldots, f_k \in \mathrm{Mor}(\mathcal{E}, \mathcal{F})$ (for $k \geq 2$) is *summable* if the fusion subsystems $\mathrm{Im}(f_1), \ldots, \mathrm{Im}(f_k)$ commute. When this is the case, $f_1 + \cdots + f_k \in \mathrm{Hom}(T, S)$ is the morphism

$$(f_1 + \cdots + f_k)(x) = f_1(x) \cdots f_k(x) \in S \qquad\qquad (\text{all } x \in T).$$

(b) Let $\mathbf{0} = \mathbf{0}_{\mathcal{E},\mathcal{F}} \in \mathrm{Mor}(\mathcal{E}, \mathcal{F})$ be the neutral element for sums of morphisms: the homomorphism sending T to $1 \in S$. Write $\mathbf{0}_{\mathcal{F}} = \mathbf{0}_{\mathcal{F},\mathcal{F}} \in \mathrm{End}(\mathcal{F})$ for short.

We first check that a sum of summable morphisms from $\mathcal{E}$ to $\mathcal{F}$ is, in fact, a morphism from $\mathcal{E}$ to $\mathcal{F}$.

Lemma 3.2. *Let $\mathcal{E}$ and $\mathcal{F}$ be fusion systems over discrete p-toral groups T and S, and let $f_1, \ldots, f_k \in \mathrm{Mor}(\mathcal{E}, \mathcal{F})$ be a summable k-tuple of morphisms. Then $f_1 + \cdots + f_k \in \mathrm{Mor}(\mathcal{E}, \mathcal{F})$, and $\mathrm{Im}(f_1 + \cdots + f_k) \subseteq \mathrm{Im}(f_1) \cdots \mathrm{Im}(f_k)$.*

Proof. Set $f = f_1 + \cdots + f_k$ for short. As a homomorphism of groups, f is the composite

$$T \xrightarrow{(f_1,\ldots,f_k)} f_1(T) \times \cdots \times f_k(T) \xrightarrow{\ I\ } S,$$

where $I(x_1,\ldots,x_k) = x_1 \cdots x_k$ for $x_i \in f_i(T) \le S$. By assumption, there are functors $\widehat{f_i}$ associated to f_i and $\widehat{I}$ associated to I, and we let $\widehat{f}^*$ denote the composite functor

$$\widehat{f}^*\colon \mathcal{E} \xrightarrow{(\widehat{f}_1,\ldots,\widehat{f}_k)} \mathrm{Im}(f_1) \times \cdots \times \mathrm{Im}(f_k) \xrightarrow{\ \widehat{I}\ } \mathcal{F}.$$

Then $\widehat{f}^*(P) = f_1(P) \cdots f_k(P)$ for $P \le T$, and the following diagram of groups commutes for each $\varphi \in \mathrm{Hom}_{\mathcal{E}}(P,Q)$:

$$\begin{tikzcd}
P \arrow[r, "f|_P"] \arrow[d, "\varphi"] & f_1(P)\cdots f_k(P) \arrow[d, "\widehat{f}^*(\varphi)"] \\
Q \arrow[r, "f|_Q"] & f_1(Q)\cdots f_k(Q).
\end{tikzcd}$$

Thus $\widehat{f}^*(\varphi)(f(P)) \le f(Q)$. So if we define $\widehat{f}\colon \mathcal{E} \longrightarrow \mathcal{F}$ to be the functor that sends P to $f(P)$ and sends $\varphi \in \mathrm{Hom}_{\mathcal{E}}(P,Q)$ to the restriction $\widehat{f}^*(\varphi)|_{f(P)}$, then this is a functor associated to f. So $f \in \mathrm{Mor}(\mathcal{E},\mathcal{F})$.

In particular, every morphism in $\mathrm{Im}(f)$ is the restriction of a morphism in $\mathrm{Im}(I)$, and hence is also in $\mathrm{Im}(I)$. Since $\mathrm{Im}(f_1)\cdots\mathrm{Im}(f_k) = \mathrm{Im}(I)$ by definition, this proves that $\mathrm{Im}(f) \le \mathrm{Im}(f_1)\cdots\mathrm{Im}(f_k)$. □

We next check that a composite of sums of morphisms is a sum of composites in the way one expects. Since this is clear on the level of group homomorphisms, the main problem is to check summability.

Lemma 3.3. *Let $\mathcal{D}$, $\mathcal{E}$, and $\mathcal{F}$ be fusion systems over discrete p-toral groups, and assume that $f_1,\ldots,f_n \in \mathrm{Mor}(\mathcal{E},\mathcal{F})$ and $g_1,\ldots,g_m \in \mathrm{Mor}(\mathcal{D},\mathcal{E})$ are summable m- and n-tuples of morphisms (some $m,n \ge 1$). Then $\{f_ig_j \,|\, 1 \le i \le n,\ 1 \le j \le m\}$ is summable, and*

$$(f_1 + \cdots + f_n) \circ (g_1 + \cdots + g_m) = \sum_{i=1}^{n}\sum_{j=1}^{m} f_ig_j. \tag{3.1}$$

Proof. By assumption, the fusion subsystems $\mathrm{Im}(g_1),\ldots,\mathrm{Im}(g_m)$ commute in $\mathcal{E}$. So for each $1 \le i \le n$, the subsystems $\mathrm{Im}(f_ig_1),\ldots,\mathrm{Im}(f_ig_m)$ commute in $\mathrm{Im}(f_i)$ by Lemma 2.9(b). Since the $\mathrm{Im}(f_i)$ commute in $\mathcal{F}$, the subsystems $\mathrm{Im}(f_ig_j)$ for all $1 \le i \le n$ and $1 \le j \le m$ commute in $\mathcal{F}$ by repeated applications of Lemma 2.9(a).

Thus the f_ig_j are summable. Equation (3.1) holds since for each $x \in U$,

$$\begin{aligned}
(f_1 + \cdots + f_n)(g_1 + \ldots g_m)(x) &= f_1(g_1(x)\cdots g_m(x))\cdots f_n(g_1(x)\cdots g_m(x)) \\
&= \prod_{i=1}^{n}\prod_{j=1}^{m} f_ig_j(x) = \Bigl(\sum_{i=1}^{n}\sum_{j=1}^{m} f_ig_j\Bigr)(x).
\end{aligned}$$ □

4. Normal endomorphisms

An endomorphism of a group G is defined to be normal if it commutes with all inner automorphisms of G, and it is not at all obvious how to translate this directly to an analogous definition for fusion systems. We refer to Remark 4.5 below for more discussion about the difficulties with such a definition.

Instead, we use a different property of normal endomorphisms of groups. It is an easy exercise to show that $f \in \mathrm{End}(G)$ is normal if and only if there is $\chi \in \mathrm{End}(G)$ such that $[\mathrm{Im}(f), \mathrm{Im}(\chi)] = 1$ and $f + \chi = \mathrm{Id}_G$, and this criterion is easily adapted to endomorphisms of fusion systems.

Definition 4.1. Let $\mathcal{F}$ be a fusion system over a discrete p-toral group S. An endomorphism $f \in \mathrm{End}(\mathcal{F})$ is *normal* if there is $\chi \in \mathrm{End}(\mathcal{F})$ such that f and χ are summable and $f + \chi = \mathrm{Id}_{\mathcal{F}}$. Let $\mathrm{End}^N(\mathcal{F}) \geq \mathrm{Aut}^N(\mathcal{F})$ denote the sets of normal endomorphisms and automorphisms of $\mathcal{F}$.

As one example, assume G is a finite group with $S \in \mathrm{Syl}_p(G)$, and let $f \in \mathrm{End}(G)$ be an endomorphism such that $f(S) \leq S$. If f is normal, then there is $\chi \in \mathrm{End}(G)$ such that $f + \chi = \mathrm{Id}_G$, and the corresponding relation holds for $f|_S$ and $\chi|_S$ as endomorphisms of the fusion system $\mathcal{F}_S(G)$. So $f|_S$ is a normal endomorphism of $\mathcal{F}_S(G)$ if f is a normal endomorphism of G. However, the converse is not true, as will be shown in Remark 4.5.

We first check some of the most basic properties of normal endomorphisms.

Lemma 4.2. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S.*

(a) *If $f, f' \in \mathrm{End}^N(\mathcal{F})$, then $f \circ f'$ is normal.*

(b) *If $f, f' \in \mathrm{End}^N(\mathcal{F})$ and $f \circ f' = \mathbf{0}_{\mathcal{F}}$, then f and f' are summable and $f + f'$ is normal.*

(c) *If $\mathcal{F} = \mathcal{E}_1 \times \mathcal{E}_2$, and $f \in \mathrm{End}(\mathcal{F})$ is the identity on $\mathcal{E}_1$ and trivial on $\mathcal{E}_2$, then f is normal.*

Proof. **(a,b)** Fix a pair f, f' of normal endomorphisms of $\mathcal{F}$, and let $\chi, \chi' \in \mathrm{End}(\mathcal{F})$ be such that f and χ are summable, f' and χ' are summable, and $f + \chi = \mathrm{Id}_{\mathcal{F}} = f' + \chi'$. By Lemma 3.3, $\mathrm{Id}_{\mathcal{F}} = (f + \chi) \circ (f' + \chi') = ff' + (f\chi' + \chi f' + \chi\chi')$, and so ff' is normal.

If $ff' = \mathbf{0}_{\mathcal{F}}$, then by Lemma 3.3,

$$\mathrm{Id}_{\mathcal{F}} = (f + \chi) \circ (f' + \chi') = f(f' + \chi') + (f + \chi)f' + \chi\chi' = f + f' + \chi\chi'.$$

Hence f and f' are summable, and $f + f'$ is normal.

(c) If $\mathcal{F} = \mathcal{E}_1 \times \mathcal{E}_2$, and $f_1, f_2 \in \mathrm{End}(\mathcal{F})$ are the projections with images $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively, then $f_1 + f_2 = \mathrm{Id}_{\mathcal{F}}$. So f_1 and f_2 are normal. □

We do not know whether or not the sum of each summable pair of normal endomorphisms is normal. (This is clearly true for normal endomorphisms of a group.) One can at least weaken the extra hypothesis in Lemma 4.2(b): it suffices to assume that $ff'(S) \leq Z(\mathcal{F})$.

By point (a) above, $\mathrm{End}^N(\mathcal{F})$ is a monoid for each fusion system $\mathcal{F}$. In the next lemma, we show that $\mathrm{Aut}^N(\mathcal{F})$ is always a group.

Lemma 4.3. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and assume $f \in \mathrm{End}(\mathcal{F})$ is surjective. Then*

(a) *f is normal if and only if $[f, S] \leq Z(\mathcal{F})$ and $f|_{\mathfrak{foc}(\mathcal{F})} = \mathrm{Id}$; and*

(b) *if f is normal and invertible, then f^{-1} is also normal.*

Proof. If f is normal, then there is $\chi \in \mathrm{End}(\mathcal{F})$ such that f and χ are summable and $f + \chi = \mathrm{Id}_{\mathcal{F}}$, and in particular, $\mathrm{Im}(f) = \mathcal{F}$ commutes with $\mathrm{Im}(\chi)$. So $[f, S] = \chi(S) = f(S) \cap \chi(S) \leq Z(\mathcal{F})$ by Lemma 2.6(a). Hence $x \in S$ and $y \in x^{\mathcal{F}}$ imply that $\chi(y) = \chi(x)$, so $\mathfrak{foc}(\mathcal{F}) \leq \mathrm{Ker}(\chi)$, and $f|_{\mathfrak{foc}(\mathcal{F})} = \mathrm{Id}$.

Conversely, if $[f,S] \le Z(\mathcal{F})$, then we can define χ by setting $\chi(x) = f(x)^{-1}x \in Z(\mathcal{F})$ for $x \in S$. Then $\chi \in \mathrm{End}(S)$, $\chi(S) \le Z(\mathcal{F})$, and $f+\chi = \mathrm{Id}_S$ as endomorphisms of S. If in addition, $f|_{\mathfrak{foc}(\mathcal{F})} = \mathrm{Id}$, then $\mathrm{Ker}(\chi) \ge \mathfrak{foc}(\mathcal{F})$, so χ is constant on $\mathcal{F}$-conjugacy classes. Hence there is a functor $\widehat{\chi}\colon \mathcal{F} \longrightarrow \mathcal{F}$ associated to χ, defined on objects by setting $\widehat{\chi}(P) = \chi(P) \le Z(\mathcal{F})$ for each $P \le S$, and on morphisms by setting $\widehat{\chi}(\varphi) = \mathrm{Id}_{\chi(P)}$ for each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$. So $\chi \in \mathrm{End}(\mathcal{F})$, and f is normal.

(b) If $f \in \mathrm{End}(\mathcal{F})$ is normal and invertible, then $[f^{-1},S] = [f,S] \le Z(\mathcal{F})$, and $f^{-1}|_{\mathfrak{foc}(\mathcal{F})} = \mathrm{Id}$ since $f|_{\mathfrak{foc}(\mathcal{F})} = \mathrm{Id}$. So f^{-1} is normal by (a). □

By comparison, an automorphism of a group G is normal if and only if it induces the identity on $[G,G]$ and on $G/Z(G)$ (see [Sz1, 6.18.ii]). This gives another way to see that when G is finite and $S \in \mathrm{Syl}_p(G)$, each normal automorphism of G that sends S to itself induces a normal automorphism of the fusion system $\mathcal{F}_S(G)$.

We can now show that the image of a normal endomorphism is always saturated and is a full subcategory.

Proposition 4.4. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, let $f \in \mathrm{End}^N(\mathcal{F})$ be a normal endomorphism, and set $T = f(S)$. Let $\chi \in \mathrm{End}^N(\mathcal{F})$ be such that $f+\chi = \mathrm{Id}_{\mathcal{F}}$, and set $U = \chi(S)$. Then*

(a) *$f\chi = \chi f$ and $f(U) = \chi(T) = T \cap U \le Z(\mathcal{F})$;*

(b) *T is strongly closed in $\mathcal{F}$ (see Definition 1.7);*

(c) *f commutes in $\mathrm{End}(\mathcal{F})$ with all elements of $\mathrm{Aut}_{\mathcal{F}}(S)$; and*

(d) *$\mathrm{Im}(f) = \widehat{f}(\mathcal{F}) = \mathcal{F}|_{\le T}$ and is a saturated fusion subsystem of $\mathcal{F}$.*

Proof. Since f and χ are summable, we have $[T,U]=1$. Since $x = f(x)\chi(x)$ for each $x \in S$, we have $TU = S$. As usual, $\widehat{f}, \widehat{\chi}\colon \mathcal{F} \longrightarrow \mathcal{F}$ denote the functors associated to f and χ.

(a) For each $x \in S$,
$$f(f(x))\chi(f(x)) = f(x) = f(f(x)\chi(x)) = f(f(x))f(\chi(x)).$$
So $\chi f(x) = f\chi(x)$ for each $x \in S$, and $\chi f = f\chi \in \mathrm{End}(S)$.

In particular, $f(U) = f\chi(S) = \chi f(S) = \chi(T) \le T \cap U$. For each $x \in T \cap U$, $x = f(x)\chi(x) \in f(U)\chi(T) = f(U)$, and so $f(U) = \chi(T) = T \cap U$.

Since f and χ are summable and $f+\chi = \mathrm{Id}_{\mathcal{F}}$, the subsystems $\mathrm{Im}(f)$ and $\mathrm{Im}(\chi)$ commute, and $\mathrm{Im}(f)\mathrm{Im}(\chi) = \mathcal{F}$ by Lemma 3.2. Hence $T \cap U \le Z(\mathcal{F})$ by Lemma 2.6(a).

(b) Assume $t \in T$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(\langle t\rangle, S)$. Then $\chi(t) \in \chi(T) = T \cap U$. Also, $\varphi(t) = f(\varphi(t))\chi(\varphi(t))$, where $f(\varphi(t)) \in f(S) = T$, and where $\chi(\varphi(t)) = \widehat{\chi}(\varphi)(\chi(t)) = \chi(t) \in T \cap U$ by (a) and since $\chi(t) \in T \cap U \le Z(\mathcal{F})$. Thus $\varphi(t) \in T$, so $t^{\mathcal{F}} \subseteq T$, and T is strongly closed in $\mathcal{F}$.

(c) Fix $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$; we must show that $f\alpha = \alpha f$. For each $x \in S$,
$$f(\alpha(x))\chi(\alpha(x)) = \alpha(x) = \alpha(f(x)\chi(x)) = \alpha f(x)\alpha\chi(x),$$
so $f\alpha(x) = \alpha f(x)$ if and only if $\chi\alpha(x) = \alpha\chi(x)$.

For each $t \in T$, $\chi(t) \in \chi(T) \le Z(\mathcal{F})$ by (a), so
$$\chi(\alpha(t)) = \widehat{\chi}(\alpha)(\chi(t)) = \chi(t) = \alpha(\chi(t)).$$
Thus $\chi\alpha|_T = \alpha\chi|_T$, and by the above remarks, this implies $f\alpha|_T = \alpha f|_T$. The restrictions to U commute by a similar argument. Since $S = TU$, we have $f\alpha = \alpha f$ and $\chi\alpha = \alpha\chi$.

(d) Fix $P, Q \leq T$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$. We must show that $\varphi \in \mathrm{Hom}_{\widehat{f}(\mathcal{F})}(P, Q)$.

If $\varphi(P)$ is receptive in $\mathcal{F}$, then since $U \leq C_S(P) \leq N_\varphi^{\mathcal{F}}$, φ extends to some $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(PU, QU)$, where $\bar{\varphi}(U) = U$ since U is strongly closed by (b). If $\varphi(P)$ is not receptive, let $R \in P^{\mathcal{F}}$ be such that R is receptive; then $R \leq T$ since T is strongly closed, and there are isomorphisms $\bar{\varphi}_1 \in \mathrm{Hom}_{\mathcal{F}}(PU, RU)$ and $\bar{\varphi}_2 \in \mathrm{Hom}_{\mathcal{F}}(\varphi(P)U, RU)$ such that $\bar{\varphi}_i(U) = U$, $\bar{\varphi}_1(P) = R$, $\bar{\varphi}_2(\varphi(P)) = R$, and $\bar{\varphi}_2^{-1}\bar{\varphi}_1 \in \mathrm{Hom}_{\mathcal{F}}(PU, \varphi(P)U)$ extends φ. So in all cases, we get $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(PU, QU)$ extending φ.

Set $\psi = \widehat{f}(\bar{\varphi}) \in \mathrm{Hom}_{\widehat{f}(\mathcal{F})}(f(PU), f(QU))$. We claim that φ is a restriction of ψ (in particular, that $P \leq f(PU)$). For each $x \in P$, $x = f(x)\chi(x)$ where $f(x) \in f(P)$ and $\chi(x) \in \chi(T) = f(U) \leq Z(\mathcal{F})$, so $x \in f(PU)$ and

$$\begin{aligned} \psi(x) &= \psi(f(x)\chi(x)) = \widehat{f}(\bar{\varphi})(f(x)) \cdot \psi(\chi(x)) \\ &= f(\bar{\varphi}(x)) \cdot \chi(x) = f(\varphi(x)) \cdot \widehat{\chi}(\varphi)(\chi(x)) && (\chi(x) \in Z(\mathcal{F})) \\ &= f(\varphi(x))\chi(\varphi(x)) = \varphi(x). \end{aligned}$$

So $\psi|_P = \varphi$, and hence $\varphi \in \mathrm{Hom}_{\widehat{f}(\mathcal{F})}(P, Q)$.

Thus $\mathrm{Im}(f) = \widehat{f}(\mathcal{F}) = \mathcal{F}|_{\leq T}$: the full subcategory of $\mathcal{F}$ with objects the subgroups of T. Since $S = TU$, $[T, U] = 1$, and T is strongly closed in $\mathcal{F}$, Lemma 1.10 now implies that $\mathrm{Im}(f)$ is a saturated fusion subsystem of $\mathcal{F}$. □

Remark 4.5. *Recall that an endomorphism of a group is defined to be normal if it commutes with all inner automorphisms. When $\mathcal{F}$ is a saturated fusion system over S, the closest we can come to inner automorphisms are the elements of $\mathrm{Aut}_{\mathcal{F}}(S)$, and Proposition 4.4(c) says that each normal endomorphism of $\mathcal{F}$ commutes with all of these. However, the converse is not true. If, for example, S and $\mathrm{Aut}_{\mathcal{F}}(S)$ are both abelian, then each $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ lies in $\mathrm{Aut}(\mathcal{F})$ and commutes with $\mathrm{Aut}_{\mathcal{F}}(S)$. But α need not be the identity on $[S, \mathrm{Aut}_{\mathcal{F}}(S)] \leq \mathfrak{foc}(\mathcal{F})$, and hence by Lemma 4.3(a) need not be normal.*

Recall (Definition 1.4) that when $\mathcal{F}$ is a fusion system over S, an endomorphism $f \in \mathrm{End}(\mathcal{F}) \leq \mathrm{End}(S)$ is locally nilpotent if $S = \bigcup_{i=1}^\infty \mathrm{Ker}(f^i)$. If S is finite, then clearly all locally nilpotent endomorphisms are nilpotent. The next lemma, which is modeled on (2.4.9) in [Sz1], implies among other things that this is also true whenever $Z(\mathcal{F})$ is finite.

Proposition 4.6. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and let $f \in \mathrm{End}^N(\mathcal{F})$ be a normal endomorphism. Then there is a unique pair of saturated fusion subsystems $\mathcal{E}, \mathcal{D} \leq \mathcal{F}$, over $T, U \leq S$, such that*

- *$\mathcal{F} = \mathcal{E} \times \mathcal{D}$;*
- *$f|_T \in \mathrm{Aut}^N(\mathcal{E})$; and*
- *$f|_U \in \mathrm{End}^N(\mathcal{D})$ is locally nilpotent, and is nilpotent if $Z(\mathcal{F})$ is finite.*

If f is surjective, then U is connected and central in $\mathcal{F}$, and hence $U = 1$ if $|Z(\mathcal{F})| < \infty$.

Proof. Since S is artinian (Lemma 1.2) and $\{f^i(S)\}_{i=1}^\infty$ is a descending sequence of subgroups of S, there is $n \geq 1$ such that $f^i(S) = f^n(S)$ for all $i \geq n$. Consider the following subgroups of S and fusion subsystems of $\mathcal{F}$:

$$T_+ = \bigcap_{i=1}^\infty f^i(S) = f^n(S), \qquad \mathcal{E}_+ = \mathcal{F}|_{\leq T_+}, \qquad U = \bigcup_{i=1}^\infty \mathrm{Ker}(f^i), \qquad \mathcal{D} = \mathcal{F}|_{\leq U}.$$

We first check that

$$S = T_+\mathrm{Ker}(f^n) = T_+U; \quad T_+, U \text{ strongly closed in } \mathcal{F}; \quad \mathcal{E}_+, \mathcal{D} \text{ commute in } \mathcal{F}. \tag{4.1}$$

For each $x \in S$, since $f^{2n}(S) = f^n(S)$, there is $y \in T_+$ such that $f^n(y) = f^n(x)$ and $x = y(y^{-1}x) \in T_+\mathrm{Ker}(f^n)$. Thus $S = T_+\mathrm{Ker}(f^n) = T_+U$. Also, T_+ and U are strongly closed in $\mathcal{F}$ by Proposition 4.4(b) and Lemma 2.2(a), respectively.

Let $\psi_n \in \mathrm{End}(\mathcal{F})$ be such that $f^n + \psi_n = \mathrm{Id}_{\mathcal{F}}$. Then $\mathrm{Im}(f^n)$ and $\mathrm{Im}(\psi_n)$ commute in $\mathcal{F}$, where $\mathrm{Im}(f^n) = \mathcal{E}_+$ and $\mathrm{Im}(\psi_n) = \mathcal{F}|_{\le\psi_n(S)}$ by Proposition 4.4(d). Also, $U \le \psi_n(S)$, since for $u \in U$, $f^{(m+1)n}(u) = 1$ for some $m \ge 1$, and hence

$$u = (uf^n(u)^{-1})(f^n(u)f^{2n}(u)^{-1})\cdots f^{mn}(u) = \psi_n(uf^n(u)\cdots f^{mn}(u)) \in \psi_n(S).$$

Thus $\mathcal{D} \le \mathcal{F}|_{\psi_n(S)} = \mathrm{Im}(\psi_n)$ commutes with $\mathcal{E}_+$, finishing the proof of (4.1).

If S is finite, then set $T = T_+ = f^n(S)$ and $\mathcal{E} = \mathcal{E}_+$. Since $|S| = |f^i(S)| \cdot |\mathrm{Ker}(f^i)|$ for all i, we have $U = \mathrm{Ker}(f^n) = \mathrm{Ker}(f^i)$ for all $i \ge n$, and $|S| = |T| \cdot |U|$. By (4.1), $S = TU$, $[T,U] = 1$, T and U are strongly closed, and $\mathcal{E}$ and $\mathcal{D}$ commute, so $S = T \times U$, and $\mathcal{F} = \mathcal{E} \times \mathcal{D}$ by Lemma 2.10. By construction, $f|_T \in \mathrm{Aut}(\mathcal{E})$ and $f|_U$ is nilpotent. This proves the existence statement in the finite case, and uniqueness will be shown below.

In the general case, set $Z = T_+ \cap U$. Since $f(T_+) = T_+$ and $f(U) \le U$, we have $f(Z) \le Z$. Also, $Z \le f^n(S) \cap \psi_n(S) \le Z(\mathcal{F})$ by Proposition 4.4(a). For $z \in Z$, $z = f(t)$ for some $t \in T_+$, $f^i(z) = f^{i+1}(t) = 1$ for some i, and hence $t \in T_+ \cap U = Z$. Thus $f|_Z \in \mathrm{End}(Z)$ is surjective and locally nilpotent. So Z is connected by Lemma 1.5. If f is surjective, then $T_+ = S$, and so $U = Z$ is connected. If $|Z(\mathcal{F})| < \infty$, then $Z = 1$.

Case 1: If $Z = 1$ (in particular, if $|Z(\mathcal{F})| < \infty$), then set $T = T_+$ and $\mathcal{E} = \mathcal{E}_+$. By (4.1) together with Lemma 2.10, we have $S = T \times U$, $U = \mathrm{Ker}(f^n)$, and $\mathcal{F} = \mathcal{E} \times \mathcal{D}$. Thus $f|_U$ is nilpotent. Also, $\mathrm{Ker}(f|_{T_+}) = 1$, so $f|_{T_+} \in \mathrm{Aut}(\mathcal{E}_+)$ by Lemma 2.2(b).

Case 2: If $Z \ne 1$, set $f_0 = f|_{T_+} \in \mathrm{End}(\mathcal{E}_+)$. Then f_0 is surjective by definition of T_+ and $\mathcal{E}_+$. Since there is $\chi \in \mathrm{End}(\mathcal{F})$ such that $f + \chi = \mathrm{Id}_{\mathcal{F}}$, we have $f_0 + \chi|_{T_+} = \mathrm{Id}_{\mathcal{E}_+}$ where $\chi(T_+) \le T_+$, and so $f_0 \in \mathrm{End}^N(\mathcal{E}_+)$.

By Lemma 4.3(a) and since f_0 is normal and surjective, $\mathrm{Ker}(f_0^i) \le [f_0^i, T_+] \le Z(\mathcal{E}_+)$ for each i and $f|_{\mathfrak{foc}(\mathcal{E}_+)} = \mathrm{Id}$. Hence $Z \le Z(\mathcal{E}_+)$ and $Z \cap \mathfrak{foc}(\mathcal{E}_+) = 1$.

Set $\bar{S} = T_+/\mathfrak{foc}(\mathcal{E}_+)$, and let $\bar{f} \in \mathrm{End}(\bar{S})$ be the endomorphism induced by f_0. Then $\bar{S}$ is abelian and $\bar{f}$ is surjective, so by Lemma 1.5, there is a unique factorization $\bar{S} = \bar{T} \times \bar{Z}$ such that $\bar{f}|_{\bar{T}} \in \mathrm{Aut}(\bar{T})$ and $\bar{f}|_{\bar{Z}} \in \mathrm{End}(\bar{Z})$ is locally nilpotent.

Let $T \le T_+$ be such that $T \ge \mathfrak{foc}(\mathcal{E}_+)$ and $T/\mathfrak{foc}(\mathcal{E}_+) = \bar{T}$. Then $f|_T \in \mathrm{Aut}(T)$ since $f_0 = f|_{T_+}$ induces the identity on $\mathfrak{foc}(\mathcal{E}_+)$ and an automorphism of the quotient. Hence $T \cap \mathrm{Ker}(f^i) = 1$ for all i, so $T \cap U = 1$. For each $x \in T_+$, since $\bar{S} = \bar{T}\bar{Z}$, there are $x_1, x_2 \in T_+$ such that $x = x_1x_2$, $x_1 \in T$, and $f_0^i(x_2) = y \in \mathfrak{foc}(\mathcal{E}_+)$ for some $i \ge 1$. Since $f|_{\mathfrak{foc}(\mathcal{E}_+)} = \mathrm{Id}$, we have $y^{-1}x_2 \in \mathrm{Ker}(f_0^i) \le Z$, and so $x = (x_1y)(y^{-1}x_2) \in TZ$. Thus $T_+ = TZ$, so $S = T_+U = TZU = TU$ since $Z \le U$, and $S = T \times U$ since $[T,U] \le [T_+,U] = 1$.

Set $\mathcal{E} = \mathcal{F}|_{\le T}$. By Lemma 1.8(b), T is strongly closed in $\mathcal{E}_+$ since it contains $\mathfrak{foc}(\mathcal{E}_+)$. Since $\mathcal{E}_+ = \mathcal{F}|_{\le T_+}$ where T_+ is strongly closed in $\mathcal{F}$, this implies that T is strongly closed in $\mathcal{F}$. We already saw that U is strongly closed in $\mathcal{F}$, and $\mathcal{E}$ commutes with $\mathcal{D}$ in $\mathcal{F}$ since $\mathcal{E}_+$ commutes with $\mathcal{D}$. So $\mathcal{F} = \mathcal{E} \times \mathcal{D}$ by Lemma 2.10. Also, $\mathrm{Ker}(f|_T) = 1$, so $f|_T \in \mathrm{Aut}(\mathcal{E})$ by Lemma 2.2(b), finishing the proof of existence of such a factorization.

Uniqueness: Assume $\mathcal{F} = \mathcal{E}^* \times \mathcal{D}^*$ over $S = T^* \times U^*$ is a second factorization, where $f|_{T^*} \in \mathrm{Aut}^N(\mathcal{E}^*)$, and $f|_{U^*} \in \mathrm{End}^N(\mathcal{D}^*)$ and is locally nilpotent. Then $U^* = \bigcup_{i=1}^{\infty} \mathrm{Ker}(f^i) = U$, and $T^* \le \bigcap_{i=1}^{\infty} f^i(S) = T_+$. Also, $\mathfrak{foc}(\mathcal{E}_+) \le T^*$ since $f|_{\mathfrak{foc}(\mathcal{E}_+)} = \mathrm{Id}$. So $T_+/\mathfrak{foc}(\mathcal{E}_+) = (T^*/\mathfrak{foc}(\mathcal{E}_+)) \times V$ where V is the image of $(U^* \cap T_+)$ in the quotient, f induces an isomorphism on the first factor, and induces a locally nilpotent endomorphism on the second factor. By the uniqueness statement in Lemma 1.5, applied again with $T_+/\mathfrak{foc}(\mathcal{E}_+)$ in the role of S, we have $T^*/\mathfrak{foc}(\mathcal{E}_+) = \bar{T} = T/\mathfrak{foc}(\mathcal{E}_+)$ and hence $T^* = T$. Also, $\mathcal{E}^* = \mathcal{F}|_{\le T^*} = \mathcal{E}$ and $\mathcal{D}^* = \mathcal{F}|_{\le U^*} = \mathcal{D}$, and so the factorization is unique. □

5. A Krull-Remak-Schmidt theorem for fusion systems

By analogy with the Krull-Remak-Schmidt theorem for groups, we look at a slightly more general version of Theorem A where we assume all direct factors are invariant under a given group Ω of automorphisms. If one ignores the next paragraph, and takes $\Omega = 1$ in Lemma 5.1 and Theorem 5.2, then one gets Theorem A as stated in the introduction.

Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and let $\Omega \le \mathrm{Aut}(\mathcal{F})$ be a group of automorphisms. We say that

- a fusion subsystem $\mathcal{E} \le \mathcal{F}$ is Ω-*invariant* if $\widehat{\omega}(\mathcal{E}) = \mathcal{E}$ for each $\omega \in \Omega$;
- an Ω-invariant subsystem $\mathcal{E} \le \mathcal{F}$ is Ω-*indecomposable* if there are no proper Ω-invariant subsystems $\mathcal{E}_1, \mathcal{E}_2 < \mathcal{E}$ such that $\mathcal{E} = \mathcal{E}_1 \times \mathcal{E}_2$; and
- an endomorphism of $\mathcal{F}$ is Ω-*normal* if it is normal and commutes in $\mathrm{End}(\mathcal{F})$ with all $\omega \in \Omega$.

The sets of all Ω-normal endomorphisms and Ω-normal automorphisms of $\mathcal{F}$ form a monoid and a group, respectively, which we denote

$$\mathrm{End}^{\Omega}(\mathcal{F}) = C_{\mathrm{End}^N(\mathcal{F})}(\Omega) \qquad \text{and} \qquad \mathrm{Aut}^{\Omega}(\mathcal{F}) = C_{\mathrm{Aut}^N(\mathcal{F})}(\Omega).$$

Here, Ω acts on $\mathrm{End}^N(\mathcal{F})$ by conjugation. Note that sums of Ω-normal endomorphisms are Ω-normal when the the hypotheses of Lemma 4.2(b) hold.

Lemma 5.1. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S. Fix a subgroup $\Omega \le \mathrm{Aut}(\mathcal{F})$, and assume that $\mathcal{F}$ is Ω-indecomposable. Then*

(a) *each Ω-normal endomorphism $f \in \mathrm{End}^{\Omega}(\mathcal{F})$ is either nilpotent or an isomorphism, or possibly (if $Z(\mathcal{F})$ is infinite) locally nilpotent; and*

(b) *if $f_1, \dots, f_k \in \mathrm{End}^{\Omega}(\mathcal{F})$ are summable, and $f_1 + \cdots + f_k \in \mathrm{Aut}(\mathcal{F})$, then f_i is an automorphism for some $i = 1, \dots, k$.*

Proof. **(a)** By Proposition 4.6, there is a unique factorization $\mathcal{F} = \mathcal{E} \times \mathcal{D}$ over $S = T \times U$ such that $f|_T \in \mathrm{Aut}(\mathcal{E})$ and $f|_U \in \mathrm{End}(\mathcal{D})$ is locally nilpotent. For each $\omega \in \Omega$, since f commutes with ω, we get another factorization $\mathcal{F} = \widehat{\omega}(\mathcal{E}) \times \widehat{\omega}(\mathcal{D})$ with the same properties. Hence $\widehat{\omega}(\mathcal{E}) = \mathcal{E}$ and $\widehat{\omega}(\mathcal{D}) = \mathcal{D}$ by the uniqueness of factorization, so $\mathcal{E}$ and $\mathcal{D}$ are Ω-invariant. Since $\mathcal{F}$ is Ω-indecomposable, $\mathcal{F} = \mathcal{E}$ or $\mathcal{F} = \mathcal{D}$, and so $f \in \mathrm{Aut}(\mathcal{F})$ or f is locally nilpotent. By Proposition 4.6 again, if f is locally nilpotent and $|Z(\mathcal{F})| < \infty$, then f is nilpotent.

(b) By induction, it suffices to prove this when $k = 2$. Assume otherwise: let $f_1, f_2 \in \mathrm{End}^{\Omega}(\mathcal{F})$ be summable, and such that $f_1 + f_2$ is an automorphism but neither f_1 nor f_2 is one. We claim that this is impossible. Set $\alpha = f_1 + f_2 \in \mathrm{Aut}(\mathcal{F})$: then $\alpha\omega = \omega\alpha$ for all $\omega \in \Omega$ (but we do not assume α is normal). Set $f_1' = f_1\alpha^{-1}$ and $f_2' = f_2\alpha^{-1}$. By Lemma 3.3, f_1' and f_2' are summable and $f_1' + f_2' = \mathrm{Id}_{\mathcal{F}}$. So f_1' and f_2' are normal, and $f_1', f_2' \in \mathrm{End}^{\Omega}(\mathcal{F})$ since the f_i and α all commute with Ω. Upon replacing f_i by f_i', we can now assume that $f_1 + f_2 = \mathrm{Id}_{\mathcal{F}}$. Since neither f_1 nor f_2 is an isomorphism, they are both locally nilpotent by (a).

Thus for $1 \neq x \in S$, there are $n, m \geq 1$ such that $f_1^n(x) = 1 = f_2^m(x)$. Also, $f_1f_2 = f_2f_1$ by Proposition 4.4(a), so $(f_1 + f_2)^{n+m}(x) = 1$, which is impossible when $f_1 + f_2 = \mathrm{Id}_{\mathcal{F}}$. $\square$

We are now ready to prove Theorem A, in the following, stronger version. As noted above, Theorem A (the uniqueness part) is the special case of Theorem 5.2 where $\Omega = 1$.

Theorem 5.2. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and fix $\Omega \leq \mathrm{Aut}(\mathcal{F})$. Let $\mathcal{E}_1, \dots, \mathcal{E}_k$ and $\mathcal{E}_1^*, \dots, \mathcal{E}_m^*$ be Ω-indecomposable Ω-invariant fusion subsystems of $\mathcal{F}$ such that*

$$\mathcal{F} = \mathcal{E}_1 \times \cdots \times \mathcal{E}_k = \mathcal{E}_1^* \times \cdots \times \mathcal{E}_m^*.$$

Then $k = m$, and there are $\alpha \in \mathrm{Aut}^{\Omega}(\mathcal{F})$ and $\sigma \in \Sigma_k$ such that $\alpha(\mathcal{E}_i) = \mathcal{E}_{\sigma(i)}^*$ for each i.

Proof. Set $\mathcal{E}^* = \mathcal{E}_1^*$ and $\mathcal{D} = \mathcal{E}_2^* \times \cdots \times \mathcal{E}_m^*$; thus $\mathcal{F} = \mathcal{E}^* \times \mathcal{D}$. Let $T_1, \dots, T_k, T^*, U \leq S$ be such that $\mathcal{E}_i$, $\mathcal{E}^*$, and $\mathcal{D}$ are fusion subsystems over T_i, T^*, and U, respectively.

Let $f_1, \dots, f_k, g, g' \in \mathrm{End}^{\Omega}(\mathcal{F})$ be the projections to $\mathcal{E}_1, \dots, \mathcal{E}_k, \mathcal{E}^*, \mathcal{D}$, respectively. Thus $f_1 + \cdots + f_k = \mathrm{Id}_{\mathcal{F}} = g + g'$. Also,

$$g|_{T^*} = g \circ (f_1 + \cdots + f_k)|_{T^*} = gf_1|_{T^*} + \cdots + gf_k|_{T^*}$$

by Lemma 3.3, and we regard this as a sum of endomorphisms of $\mathcal{E}^*$. Since $g|_{T^*}$ is an automorphism, at least one of the summands is an automorphism by Lemma 5.1(b) (applied with $\{\omega|_{T^*} \,|\, \omega \in \Omega\} \leq \mathrm{Aut}(\mathcal{E}^*)$ in the place of Ω).

Let j be such that $gf_j|_{T^*} \in \mathrm{Aut}(\mathcal{E}^*)$. Thus $f_j|_{T^*}$ is an injective morphism from $\mathcal{E}^*$ to $\mathcal{E}_j$, and g restricts to a surjection from $\langle \widehat{f_j}(\mathcal{E}^*) \rangle$ onto $\mathcal{E}^*$.

Consider $f_jg|_{T_j} \in \mathrm{End}(\mathcal{E}_j)$. For each $n \geq 1$, $(f_jg)^n|_{T_j} = (f_j|_{T^*})((gf_j)^{n-1}|_{T^*})(g|_{T_j})$, and since $gf_j|_{T^*} \in \mathrm{Aut}(\mathcal{E}^*)$ and $f_j|_{T^*}$ is injective, we see that $\mathrm{Ker}((f_jg)^n|_{T_j}) = \mathrm{Ker}(g|_{T_j})$. Since $g|_{T_j}$ is nontrivial, this shows that $f_jg|_{T_j}$ is not locally nilpotent, and hence is an automorphism of $\mathcal{E}_j$ by Lemma 5.1(a). Then $g|_{T_j} \in \mathrm{Iso}(\mathcal{E}_j, \mathcal{E}^*)$ and $f_j|_{T^*} \in \mathrm{Iso}(\mathcal{E}^*, \mathcal{E}_j)$.

Now, f_jg and g' are both Ω-normal endomorphisms of $\mathcal{F}$, and $(f_jg)g' = \mathbf{0}_{\mathcal{F}}$ since $gg' = \mathbf{0}_{\mathcal{F}}$. So by Lemma 4.2(b), f_jg and g' are summable and $f_jg + g'$ is Ω-normal. Set $h_1 = f_jg + g' \in \mathrm{End}^{\Omega}(\mathcal{F})$. Then $h_1|_{T^*} = f_jg|_{T^*} = f_j|_{T^*}$ sends $\mathcal{E}^*$ isomorphically to $\mathcal{E}_j$, while $h_1|_U = \mathrm{Id}_{\mathcal{D}}$.

Assume $x \in \mathrm{Ker}(h_1) \leq S$. Then $1 = gh_1(x) = (gf_jg + gg')(x) = gf_j(g(x))$, and $g(x) = 1$ since $g(S) = T^*$ and $gf_j|_{T^*}$ is an automorphism. Hence $1 = h_1(x) = (f_jg + g')(x) = g'(x)$, so $x = (g + g')(x) = g(x)g'(x) = 1$. This proves that $\mathrm{Ker}(h_1) = 1$, and hence (since $|h_1(S)| = |S|$) that $h_1(S) = S$. Then $\mathrm{Im}(h_1) = \mathcal{F}|_{\leq S} = \mathcal{F}$ by Proposition 4.4(d), so $h_1 \in \mathrm{Aut}^{\Omega}(\mathcal{F})$ by Lemma 2.2(b).

We have now constructed $h_1 \in \mathrm{Aut}^{\Omega}(\mathcal{F})$ that sends $\mathcal{E}^* = \mathcal{E}_1^*$ isomorphically to $\mathcal{E}_j$ (where j is as above), and is the identity on $\mathcal{E}_i^*$ for $2 \leq i \leq m$. In particular, $\mathcal{F} = \mathcal{E}_j \times \mathcal{E}_2^* \times \cdots \times \mathcal{E}_m^*$. Upon repeating this construction, but with $\mathcal{E}^* = \mathcal{E}_2^*$ and $\mathcal{D} = \mathcal{E}_j \times \mathcal{E}_3^* \times \cdots \times \mathcal{E}_m^*$, we obtain $h_2 \in \mathrm{Aut}^{\Omega}(\mathcal{F})$ that sends $\mathcal{E}_2^*$ isomorphically to $\mathcal{E}_{j_2}$ for some $j_2 \in \{1, \dots, k\}$, and $\mathcal{D}$ to itself via the identity. Also, $j_2 \neq j$ since h_2 is injective ($\mathcal{E}_{j_2} = h_2(\mathcal{E}_2^*) \neq h_2(\mathcal{E}_j) = \mathcal{E}_j$).

Upon continuing this process, we obtain Ω-normal automorphisms $h_1, \dots, h_m$ of $\mathcal{F}$ such that for each $i = 1, \dots, m$, $h = h_m \circ \cdots \circ h_1$ sends $\mathcal{E}_i^*$ isomorphically to $\mathcal{E}_{j_i}$ for some $j_i \in \{1, \dots, k\}$. The j_i are distinct since h is injective, and $\{j_1, \dots, j_m\} = \{1, \dots, k\}$ since h is an isomorphism. So $m = k$, and $h \in \mathrm{Aut}^{\Omega}(\mathcal{F})$ sends each $\mathcal{E}_i^*$ to some $\mathcal{E}_j$. $\square$

The first corollary is a special case of Theorem 5.2.

Corollary 5.3. *If $\mathcal{F}$ is a saturated fusion system over a discrete p-toral group S such that either $Z(\mathcal{F}) = 1$ or $\mathfrak{foc}(\mathcal{F}) = S$, then $\mathcal{F}$ factors as a product of indecomposable fusion* *subsystems in a unique way. Thus $\mathcal{F}$ is the direct product of all of its indecomposable direct factors.*

Proof. If $Z(\mathcal{F})=1$ or $\mathfrak{foc}(\mathcal{F})=S$, then $\mathrm{Aut}^N(\mathcal{F})=\{\mathrm{Id}_{\mathcal{F}}\}$ by Lemma 4.3. So the result follows immediately from Theorem 5.2 (applied with $\Omega=1$). □

Our original interest in this problem arose from trying to describe automorphism groups of product fusion systems in terms of those of their indecomposable factors. Our last corollary is a very simple application of this type. (Compare it with Propositions 3.4 and 3.8 in [BMOR].)

Corollary 5.4. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S such that either $Z(\mathcal{F})=1$ or $\mathfrak{foc}(\mathcal{F})=S$. Assume $\mathcal{F}=\mathcal{E}_1\times\cdots\times\mathcal{E}_k$ where each $\mathcal{E}_i$ is an indecomposable fusion subsystem over $T_i\trianglelefteq S$, and set*

$$\mathrm{Aut}^0(\mathcal{F})=\bigl\{\alpha\in\mathrm{Aut}(\mathcal{F})\bigm|\alpha(T_i)=T_i \text{ for each } 1\le i\le k\bigr\}.$$

Let $\Gamma\le\Sigma_k$ be the subgroup of all permutations σ such that $\mathcal{E}_{\sigma(i)}\cong\mathcal{E}_i$ for each i. Then $\mathrm{Aut}^0(\mathcal{F})\cong\prod_{i=1}^k\mathrm{Aut}(\mathcal{E}_i)$ and is normal in $\mathrm{Aut}(\mathcal{F})$, and there is a subgroup $K\le\mathrm{Aut}(\mathcal{F})$ such that $K\cong\Gamma\cong\mathrm{Aut}(\mathcal{F})/\mathrm{Aut}^0(\mathcal{F})$ and $\mathrm{Aut}(\mathcal{F})=\mathrm{Aut}^0(\mathcal{F})\rtimes K$.

Proof. The isomorphism $\mathrm{Aut}^0(\mathcal{F})\cong\prod_{i=1}^k\mathrm{Aut}(\mathcal{E}_i)$ is clear. By Corollary 5.3, for each $\alpha\in\mathrm{Aut}(\mathcal{F})$, there is $\sigma\in\Sigma_k$ such that $\alpha(T_i)=T_{\sigma(i)}$ for each i, and $\sigma\in\Gamma$ since $\mathcal{E}_{\sigma(i)}=\widehat{\alpha}(\mathcal{E}_i)\cong\mathcal{E}_i$ for each i. This defines a homomorphism $\rho\colon\mathrm{Aut}(\mathcal{F})\longrightarrow\Gamma$ with kernel $\mathrm{Aut}^0(\mathcal{F})$.

To see that $\mathrm{Aut}(\mathcal{F})$ is a semidirect product and ρ is surjective, choose isomorphisms $\beta_{i,j}\in\mathrm{Mor}(\mathcal{E}_i,\mathcal{E}_j)$ for each $1\le i<j\le k$ such that $\mathcal{E}_i\cong\mathcal{E}_j$, in such a way that $\beta_{h,j}=\beta_{i,j}\circ\beta_{h,i}$ whenever $h<i<j$ are such that $\mathcal{E}_h\cong\mathcal{E}_i\cong\mathcal{E}_j$. Set $\beta_{i,i}=\mathrm{Id}_{\mathcal{E}_i}$ for all i, and let $\beta_{j,i}=\beta_{i,j}^{-1}$ whenever $\beta_{i,j}$ is defined. Set

$$K=\bigl\{\alpha\in\mathrm{Aut}(\mathcal{F})\bigm|\forall\, i=1,\dots,k,\ \alpha|_{\mathcal{E}_i}=\beta_{i,j}\text{ where } j=\rho(\alpha)(i)\bigr\}.$$

Then $\rho|_K$ is an isomorphism from K to Γ, and hence $\mathrm{Aut}(\mathcal{F})=\mathrm{Aut}^0(\mathcal{F})\rtimes K$. □

We now return to the question raised in the introduction: how easily can Theorem 5.2, after restriction to fusion systems realized by finite groups, be derived from the Krull-Remak-Schmidt theorem for finite groups? In most cases (at least), this seems to require tools much more sophisticated than those used here to prove Theorem 5.2.

The simplest case seems to be that when $p=2$ and $O^{2'}(\mathcal{F})=\mathcal{F}$. Here, one can apply the following lemma, which is based on a theorem of Goldschmidt.

Lemma 5.5. *Let $\mathcal{F}$ be a realizable fusion system over a finite 2-group S such that $O^{p'}(\mathcal{F})=\mathcal{F}$. Then there is a finite group G such that $O_{2'}(G)=1$, $O^{2'}(G)=G$, $S\in\mathrm{Syl}_2(G)$ and $\mathcal{F}=\mathcal{F}_S(G)$. For each such G, if $\mathcal{F}=\mathcal{E}_1\times\cdots\times\mathcal{E}_k$ is a factorization of $\mathcal{F}$ over $S=T_1\times\cdots\times T_k$, then there are subgroups $H_1,\dots,H_k\le G$ such that $T_i\in\mathrm{Syl}_2(H_i)$ and $\mathcal{E}_i=\mathcal{F}_{T_i}(H_i)$ for each i, and such that $G=H_1\times\cdots\times H_k$.*

Proof. If Γ is an arbitrary finite group that realizes $\mathcal{F}$, then $\mathcal{F}$ is also realized by $\Gamma/O_{2'}(\Gamma)$, and by $G\overset{\text{def}}{=}O^{2'}(\Gamma/O_{2'}(\Gamma))$ since $O^{2'}(\mathcal{F})=\mathcal{F}$.

For each $1\le i\le k$, let H_i be the normal closure of T_i in G. By [Gd, Theorem A] and since $O_{2'}(G)=1$, the H_i commute pairwise in G. Hence there is a homomorphism I from $H_1\times\cdots\times H_k$ to G, its kernel has odd order since $S=T_1\times\cdots\times T_k$, and I is injective since $O_{2'}(H_i)=1$ for each i (since $O_{2'}(G)=1$).

By construction, $\mathrm{Im}(I)$ is the normal closure of S in G, hence equal to G since $O^{2'}(G) = G$. Thus $G = H_1 \times \cdots \times H_k$. Also, $T_i \le H_i$ for each i by construction, and $T_i \in \mathrm{Syl}_2(H_i)$ since $S \in \mathrm{Syl}_2(G)$. Also, $\mathcal{F}_{T_i}(H_i) \le \mathcal{E}_i$ for each i, with equality since $\mathcal{F} = \prod_{i=1}^k \mathcal{E}_i$ and $\mathcal{F} = \mathcal{F}_S(G) = \prod_{i=1}^k \mathcal{F}_{T_i}(H_i)$. □

Thus under these assumptions ($p = 2$ and $O^{2'}(\mathcal{F}) = \mathcal{F}$), two distinct direct factor decompositions of $\mathcal{F}$ give rise to two distinct factorizations of G, and the factors of G are indecomposable if those of $\mathcal{F}$ are. Since the two factorizations of G are linked by a normal automorphism, so are those of $\mathcal{F}$.

This argument can be extended to one that applies to arbitrary realizable fusion systems over finite 2-groups, but as far as we can see, only with the help of additional tools such as the notion of tameness of a fusion system and [BMOR, Theorem C] (all realizable fusion systems are tame). When p is odd, a result analogous to Lemma 5.5 can be shown using Theorem 1.2 in [FF]. The proofs of [BMOR, Theorem C] and [FF, Theorem 1.2] both depend on the classification of finite simple groups.

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Université Sorbonne Paris Nord, LAGA, UMR 7539 du CNRS, 99, Av. J.-B. Clément, 93430 Villetaneuse, France.

Email address: `bobol@math.univ-paris13.fr`