REDUCED FUSION SYSTEMS OVER 2-GROUPS OF SMALL ORDER

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Abstract. We prove, when S is a 2-group of order at most 2^9, that each reduced fusion system over S is the fusion system of a finite simple group and is tame. It then follows that each saturated fusion system over a 2-group of order at most 2^9 is realizable. What is most interesting about this result is the method of proof: we show that among 2-groups with order in this range, the ones which can be Sylow 2-subgroups of finite simple groups are almost completely determined by criteria based on Bender's classification of groups with strongly 2-embedded subgroups.

A *saturated fusion system* over a finite p-group S is a category whose objects are the subgroups of S, whose morphisms are monomorphisms between subgroups, and which satisfy certain axioms first formulated by Puig [Pg] and motivated in part by conjugacy relations among p-subgroups of a given finite group. A saturated fusion system is *realizable* if it is isomorphic to the fusion system defined by the conjugation relations within a Sylow p-subgroup of some finite group, and is *exotic* otherwise. One of our main goals is to try to understand when and how exotic fusion systems can occur, especially over 2-groups.

A saturated fusion system $\mathcal{F}$ is *reduced* if $O_p(\mathcal{F}) = 1$ and $O^p(\mathcal{F}) = O^{p'}(\mathcal{F}) = \mathcal{F}$ (see Definitions 1.1(c,e) and 1.9(a)). A saturated fusion system $\mathcal{F}$ is *tame* if it is realized by a group G such that the natural homomorphism from $\mathrm{Out}(G)$ to a certain group of outer automorphisms of $\mathcal{F}$ (more precisely, of an associated linking system) is split surjective (Definition 1.10). The main result in our earlier paper [AOV1] says roughly that exotic fusion systems can be detected via tameness of associated reduced fusion systems. More precisely, by [AOV1, Theorems A & B], if the "reduction" of a fusion system $\mathcal{F}$ is tame, then $\mathcal{F}$ is tame and hence realizable, while if a reduced fusion system is not tame, then it is the reduction of an exotic fusion system.

A saturated fusion system is *indecomposable* if it does not split as a product of fusion systems over nontrivial p-groups. We can now state our main result.

Theorem A. *Let $\mathcal{F}$ be a reduced, indecomposable fusion system over a nontrivial 2-group of order at most 2^9. Then $\mathcal{F}$ is the fusion system of a finite simple group, and is tame.*

Proof. This is shown in Theorems 4.1 (for 2-groups of order at most 64), 4.3 (order 2^7), 5.1 (order 2^8), and 6.1 (order 2^9). □

The next theorem follows from Theorem A and the above discussion.

Theorem B. *Each saturated fusion system over a 2-group of order at most 2^9 is realizable.*

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Proof. Let $\mathcal{F}$ be a saturated fusion system over a 2-group S of order at most 2^9. The reduction $\mathfrak{red}(\mathcal{F})$ of $\mathcal{F}$ is defined in [AOV1, § 2] (see Definition 1.9(d) below): it is a reduced fusion system over a subquotient of S. Since $\mathfrak{red}(\mathcal{F})$ is tame by Theorem A, $\mathcal{F}$ is realizable by [AOV1, Theorem A] (Theorem 1.11(a) below). □

Our proofs of these results are based in large part on computer computations. Their starting point is the version of Alperin's fusion theorem (re)stated in Proposition 1.3: each morphism in a saturated fusion system $\mathcal{F}$ is a composite of restrictions of $\mathcal{F}$-automorphisms of S and of certain "$\mathcal{F}$-essential" subgroups. We refer to Definition 1.2(b) for the definition of $\mathcal{F}$-essential.

In [OV], a procedure was developed for determining all reduced fusion systems over a given 2-group, taking as examples two groups of order 2^7 and two of order 2^{10}. The idea was to first determine those subgroups of a given S which could potentially be essential in some fusion system over S (the "critical" subgroups), and then study what their $\mathcal{F}$-automorphism groups could be. In this paper, we first made a computer search (using Magma [Mg] and GAP [Gp]) of all 2-groups of order at most 2^9, to determine which of them have "enough" critical subgroups and satisfy other conditions which are necessary to support a reduced fusion system. These search criteria (listed in Proposition 2.2) are, in fact, satisfied by very few 2-groups. Reduced fusion systems over them are listed individually, using computer computations in some cases and computer-free proofs in others.

The following table shows how close these programs come to restricting attention only to groups which are Sylow 2-subgroups of simple groups.

Group order	2^7	2^8	2^9
Number of groups	2328	56092	$\approx 10^7$
Nr. satisfying conditions in 2.2	9	20	34
Sylows of simple groups	6	6	10
Split as products	2	10	23
Others	1	4	1

More precisely, the number given in the third row of the table is the number of groups of the given order which satisfy the conditions in Proposition 2.2, together with the dihedral and semidihedral groups of that order, and the wreathed groups $C_{2^n} \wr C_2$ if there are any. (These latter were eliminated by condition (a) or (b) in Proposition 2.2, and restored afterwards.) Thus among the groups not eliminated by these formal conditions (based mostly on Bender's theorem [Be, Satz 1]), most are either Sylow 2-subgroups of simple groups, or are products of smaller groups and cannot be Sylow 2-subgroups of simple groups nor of reduced fusion systems. Note that this dichotomy applies only in this range: the group $(D_8 \wr C_2) \times D_8$ of order 2^{10} is a Sylow 2-subgroup of the simple group A_{14} (and its fusion system is reduced and indecomposable).

There are many examples, especially among finite simple groups of Lie type, of different simple groups whose fusion systems (at some given prime p) are reduced and isomorphic. The following theorem gives some examples of this. We do not use this theorem, except to motivate our giving only one example (or one family of examples) of groups which realize any given fusion system. The cases most relevant to this paper are those where $\mathbb{G} = PSL_n$ or PSp_{2n} for some $n \geq 2$.

Theorem 0.1 ([BMO1, Theorem A]). *Fix a prime p, a connected reductive group scheme $\mathbb{G}$ over $\mathbb{Z}$, and a pair of prime powers q and q' both prime to p. Then the following hold, where "$G \sim_p H$" means that the p-fusion systems of G and H are isomorphic.*

(a) *If $\overline{\langle q\rangle} = \overline{\langle q'\rangle}$ as closed subgroups of $\mathbb{Z}_p^\times$, then $\mathbb{G}(q) \sim_p \mathbb{G}(q')$.*

(b) *If $\mathbb{G}$ is of type A_n, D_n, or E_6, τ is a graph automorphism of $\mathbb{G}$, and $\overline{\langle q\rangle} = \overline{\langle q'\rangle} \le \mathbb{Z}_p^\times$, then ${}^\tau\mathbb{G}(q) \sim_p {}^\tau\mathbb{G}(q')$.*

(c) *If the Weyl group of $\mathbb{G}$ contains an element which acts on the maximal torus by inverting all elements, and $\overline{\langle -1, q\rangle} = \overline{\langle -1, q'\rangle} \le \mathbb{Z}_p^\times$, then $\mathbb{G}(q) \sim_p \mathbb{G}(q')$.*

(d) *If $\overline{\langle -q\rangle} = \overline{\langle q'\rangle} \le \mathbb{Z}_p^\times$, then $PSU_n(q) \sim_p PSL_n(q')$ for all $n \ge 2$.*

For example, by (a), if $\mathcal{F}$ is the fusion system of $PSL_3(17)$ (for $p = 2$), then it is also isomorphic to the fusion system of $PSL_3(q)$ for each $q \equiv 17 \pmod{32}$.

Background results on fusion systems are given in Section 1, and the precise criteria which we use in our computer searches are listed in Section 2. In Section 3, we look at the special case of reduced fusion systems over nonabelian 2-groups of the form $S_0 \times A$ with $A \ne 1$ abelian. Afterwards, we handle the individual cases in Theorem A: groups of order at most 2^7 in Section 4, those of order 2^8 in Section 5, and those of order 2^9 in Sections 6–7. At the end, some standard background results about groups and representations are given in an appendix.

When G is a group and $g, h \in G$, we write $[g, h] = ghg^{-1}h^{-1}$ for the commutator. Similarly, if $\alpha \in \mathrm{Aut}(G)$, we set $[\alpha, g] = \alpha(g)g^{-1}$ for $g \in G$. Also, ${}^gx = gxg^{-1}$ and $x^g = g^{-1}xg$ in this situation, and c_x denotes the homomorphism $(g \mapsto {}^xg)$. As usual, when S is a finite p-group, $Z_2(S) \trianglelefteq S$ is defined by $Z_2(S)/Z(S) = Z(S/Z(S))$, its Frattini subgroup is denoted $\Phi(S)$, its rank is denoted $\mathrm{rk}(S)$, and S is said to be *of type G* if it is isomorphic to a Sylow p-subgroup of the finite group G. Also, $\mathrm{Syl}_p(G)$ is the set of Sylow p-subgroups of G. When $n \ge 2$ and q is a prime power, we let $UT_n(q) \le SL_n(q)$ be the subgroup of upper triangular matrices with 1's on the diagonal. Also, we follow the usual notation for extraspecial 2-groups: 2_+^{1+2n} is a central product of n copies of D_8, while 2_-^{1+2n} is a central product of $n-1$ copies of D_8 and one copy of Q_8. We write $G_1 \times_Z G_2$ to denote a central product of groups G_1 and G_2 over $Z \le Z(G_i)$.

In Sections 4, 5, and 6, we frequently refer to the "Magma/GAP numbers" of 2-groups of a given order. These are the numbers given by the "Small Groups library" (see `http://www.icm.tu-bs.de/ag_algebra/software/small/`), and used by both Magma and GAP when referring to 2-groups of order at most 2^9 (as well as groups of other small orders).

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1. Background results about fusion systems

In this section, we collect a few definitions and results needed in the rest of the paper. A *fusion system* over a finite p-group S is a category $\mathcal{F}$ whose objects are the subgroups of S, and whose morphism sets $\mathrm{Hom}_{\mathcal{F}}(P,Q) \subseteq \mathrm{Inj}(P,Q)$ contain all homomorphisms induced by conjugation in S. One also requires that each morphism in $\mathcal{F}$ factors as the composite of an isomorphism (in $\mathcal{F}$) followed by an inclusion.

For such $\mathcal{F}$, a subgroup $P \leq S$ is called *fully normalized* (*fully centralized*) if $|N_S(P)| \geq |N_S(Q)|$ ($|C_S(P)| \geq |C_S(Q)|$) for each Q in the $\mathcal{F}$-isomorphism class of P. The fusion system $\mathcal{F}$ is *saturated* if it satisfies the following two axioms:

(I) (Sylow axiom) If $P \leq S$ is fully normalized, then P is fully centralized and $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$.

(II) (Extension axiom) If $\varphi \in \mathrm{Iso}_{\mathcal{F}}(P,Q)$ where Q is fully centralized, and we set
$$N_\varphi = \bigl\{ g \in N_S(P) \bigm| \varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(Q) \bigr\},$$
then there exists $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ such that $\overline{\varphi}|_P = \varphi$.

We refer to [BLO2, Definition 1.2], [AKO, § I.2], and [Cr, § 4.1] for more details and notation. For example, when G is a finite group with $S \in \mathrm{Syl}_p(G)$, the fusion system of G is the category $\mathcal{F}_S(G)$, where for $P, Q \leq S$, $\mathrm{Hom}_{\mathcal{F}_S(G)}(P,Q) = \mathrm{Hom}_G(P,Q)$: the set of homomorphisms which are induced by conjugation in G.

If $\mathcal{F}$ and $\mathcal{E}$ are saturated fusion systems over S and T, respectively, then we say that $\mathcal{F}$ and $\mathcal{E}$ are *isomorphic* if there is an isomorphism $\alpha\colon S \xrightarrow{\cong} T$ such that for each $P, Q \leq S$, $\mathrm{Hom}_{\mathcal{E}}(\alpha(P), \alpha(Q)) = \alpha \mathrm{Hom}_{\mathcal{F}}(P,Q)\alpha^{-1}$. Thus $\mathcal{F} \cong \mathcal{E}$ as fusion systems if there is an isomorphism of categories induced by an isomorphism between the underlying p-groups. A fusion system $\mathcal{F}$ over S is *realizable* if $\mathcal{F} \cong \mathcal{F}_S(G)$ for some finite group G with $S \in \mathrm{Syl}_p(G)$, and is *exotic* otherwise.

We say that two subgroups $P, Q \leq S$ are *$\mathcal{F}$-conjugate* if they are isomorphic in the category $\mathcal{F}$. Let $P^{\mathcal{F}}$ be the set of all subgroups of S which are $\mathcal{F}$-conjugate to P.

Definition 1.1. *Fix a prime p, a finite p-group S, and a saturated fusion system $\mathcal{F}$ over S. For each subgroup $P \leq S$,*

(a) *P is* $\mathcal{F}$-centric *if $C_S(Q) = Z(Q)$ for each $Q \in P^{\mathcal{F}}$;*

(b) *P is* normal in $\mathcal{F}$ *(denoted $P \trianglelefteq \mathcal{F}$) if $P \trianglelefteq S$, and every morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$ in $\mathcal{F}$ extends to a morphism $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(PQ,PR)$ such that $\overline{\varphi}(P) = P$; and*

(c) *P is* strongly closed *in $\mathcal{F}$ if for each $Q \leq P$ and each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,S)$, $\varphi(Q) \leq P$.*

(d) *The maximal normal p-subgroup of a saturated fusion system $\mathcal{F}$ is denoted $O_p(\mathcal{F})$. This is defined since for $A, B \leq S$ normal in $\mathcal{F}$, AB is also normal in $\mathcal{F}$.*

(e) *For any $\varphi \in \mathrm{Aut}(S)$, ${}^{\varphi}\mathcal{F}$ denotes the fusion system over S defined by*
$$\mathrm{Hom}_{{}^{\varphi}\mathcal{F}}(P,Q) = \varphi \circ \mathrm{Hom}_{\mathcal{F}}(\varphi^{-1}(P), \varphi^{-1}(Q)) \circ \varphi^{-1}$$
for all $P, Q \leq S$.

1.1. Essential and critical subgroups.

We next define the *essential* subgroups in a fusion system $\mathcal{F}$, and describe how their automorphisms and those of S generate $\mathcal{F}$.

Definition 1.2. (a) *If p is a prime and $H < G$ are finite groups, then H is* strongly p-embedded *in G if $p \bigm| |H|$, and $H \cap {}^{x}H$ has order prime to p for each $x \in G \smallsetminus H$.*

(b) *Let S be a finite p-group. A subgroup P of S is* critical *if $P < S$, P is centric in S, and there are subgroups G_0 and G of $\mathrm{Out}(P)$ such that*

$$\mathrm{Out}_S(P) \le G_0 < G \le \mathrm{Out}(P)\,,$$

G_0 is strongly p-embedded in G, and $\mathrm{Out}_S(P) \in \mathrm{Syl}_p(G)$.

(c) *If $\mathcal{F}$ is a saturated fusion system over a finite p-group S, then a subgroup P of S is* $\mathcal{F}$-essential *if $P < S$, P is $\mathcal{F}$-centric and fully normalized in $\mathcal{F}$, and $\mathrm{Out}_{\mathcal{F}}(P)$ contains a strongly p-embedded subgroup. We let $\boldsymbol{E}_{\mathcal{F}}$ denote the set of $\mathcal{F}$-essential subgroups of S.*

Note that by definition, if P is critical in S, then $\alpha(P)$ is critical in S for each $\alpha \in \mathrm{Aut}(S)$.

In the situation of Definition 1.2(b), $O_p(G) = 1$ since G contains a strongly p-embedded subgroup (cf. [AKO, Proposition A.7(c)]), and so $\mathrm{Out}_S(P) \cap O_p(\mathrm{Out}(P)) \le O_p(G) = 1$. Thus our definition of critical subgroup is equivalent to that of [OV, Definition 3.1].

We refer to [Sz2, § 6.4], and also to [AKO, Proposition A.7], for some of the other properties of strongly p-embedded subgroups.

The next proposition, first shown by Puig, is a version of Alperin's fusion theorem for saturated fusion systems. It explains the importance of essential subgroups.

Proposition 1.3 ([O1, Proposition 1.10(a,b)]). *For any saturated fusion system $\mathcal{F}$ over a finite p-group S, each morphism in $\mathcal{F}$ is a composite of restrictions of automorphisms in $\mathrm{Aut}_{\mathcal{F}}(S)$, and of automorphisms in $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))$ for $P \in \boldsymbol{E}_{\mathcal{F}}$.*

The next lemma is an immediate consequence of the definitions: critical subgroups in S are the ones which can be essential in a saturated fusion system over S.

Proposition 1.4 ([OV, Proposition 3.2]). *If $\mathcal{F}$ is a saturated fusion system over a finite p-group S, and $P \in \boldsymbol{E}_{\mathcal{F}}$, then P is a critical subgroup of S.*

The next three propositions give some necessary conditions for a subgroup to be critical, and hence necessary conditions for it to be essential.

Proposition 1.5 ([OV, Lemma 3.4]). *Fix a prime p, a finite p-group S, a subgroup $P \le S$, and a subgroup Θ characteristic in P. Assume there is $g \in N_S(P) \smallsetminus P$ such that*

(a) *$[g, P] \le \Theta{\cdot}\Phi(P)$, and*

(b) *$[g, \Theta] \le \Phi(P)$.*

Then $c_g \in O_p(\mathrm{Aut}(P))$, and hence P is not critical.

The next proposition is a consequence of Bender's classification [Be, Satz 1] of groups with strongly 2-embedded subgroups.

Proposition 1.6 ([OV, Proposition 3.3(a,c,d)]). *Let S be a finite 2-group, and let $P \le S$ be a critical subgroup. Set $S_0 = N_S(P)/P \cong \mathrm{Out}_S(P)$. Then the following hold.*

(a) *Either S_0 is cyclic, or $Z(S_0) = \{g \in S_0 \,|\, g^2 = 1\}$. If $Z(S_0)$ is not cyclic, then $|S_0| = |Z(S_0)|^m$ for $m = 1$, 2, or 3.*

(b) *Set $|S_0| = 2^k$. Then $\mathrm{rk}(P/\Phi(P)) \ge 2k$. If $k \ge 2$, then $\mathrm{rk}([s, P/\Phi(P)]) \ge 2$ for all $1 \ne s \in S_0$.*

(c) *Assume $Z(S_0) \cong (C_2)^n$ with $n \ge 2$, and fix $1 \ne s \in Z(S_0)$. Then $\mathrm{rk}([s, P/\Phi(P)]) \ge n$.*

We also need the following refinement of the last proposition.

Proposition 1.7. *Let S be a finite 2-group, and let $P \leq S$ be a critical subgroup. Let k be such that $2^k = |N_S(P)/P| = |\mathrm{Out}_S(P)|$, and let $\Phi(P) = P_0 < \cdots < P_r = P$ be a sequence of subgroups characteristic in P. Then there is some $1 \leq i \leq r$ such that $\mathrm{rk}(P_i/P_{i-1}) \geq 2k$, and such that if $k \geq 2$, then $\mathrm{rk}([s, P_i/P_{i-1}]) \geq 2$ for each $1 \neq s \in \mathrm{Out}_S(P)$.*

Proof. Let $\varGamma \leq \mathrm{Out}(P)$ be such that $\mathrm{Out}_S(P) \in \mathrm{Syl}_2(\varGamma)$ and $\varGamma$ has a strongly 2-embedded subgroup. Set $K_i = C_\varGamma(P_i/P_{i-1}) \trianglelefteq \varGamma$: the kernel of the $\varGamma$-action on P_i/P_{i-1}. Set $K = \bigcap_{i=1}^r K_i$ (so $K \trianglelefteq \varGamma$). Then $K \leq O_2(\varGamma)$ by Lemma A.1 and $O_2(\varGamma) = 1$ (see [AKO, Proposition A.7(c)]), so $K = 1$.

Fix an involution $t \in \mathrm{Out}_S(P)$. Choose some $g \in \varGamma$ which does not commute with t, and choose i such that $[g, t] \notin K_i$. Then K_i has odd order since all involutions in $\varGamma$ are conjugate to $t \notin K_i$ (see [Sz2, 6.4.4(i)]). Also, $\varGamma/K_i$ has at least two distinct involutions: the images of t and gtg^{-1}. If $\mathrm{Out}_S(P)$ is cyclic or generalized quaternion, it now follows that $\varGamma/K_i$ has a strongly 2-embedded subgroup (the centralizer of one of the involutions). Otherwise, by Bender's theorem (see [Sz2, Theorem 6.4.2]), each strongly 2-embedded subgroup of $\varGamma$ contains $O_{2'}(\varGamma)$ and hence contains K_i, and so $\varGamma/K_i$ still has a strongly 2-embedded subgroup by Lemma A.4(b). So in either case, the proposition follows from [OV, Lemma 1.7], applied with $V = P_i/P_{i-1}$. □

1.2. Reduced fusion systems and tame fusion systems.

Recall that a saturated fusion system is called "exotic" if it is not isomorphic to $\mathcal{F}_S(G)$ for any finite group G with $S \in \mathrm{Syl}_p(G)$. In [AOV1], we described how we could restrict attention to a smaller class of saturated fusion systems which we call *reduced fusion systems*, and still "detect" any exotic fusion systems (reduced or not) which reduce to them. To make this more precise, we list here the main results of [AOV1]: Theorem 1.11 below. We first need some more definitions.

Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. The *focal subgroup* of $\mathcal{F}$ is the subgroup

$$\mathfrak{foc}(\mathcal{F}) \stackrel{\text{def}}{=} \bigl\langle s^{-1}t \bigm| s,t \in S \text{ and } \mathcal{F}\text{-conjugate} \bigr\rangle = \langle [\mathrm{Aut}_\mathcal{F}(P), P] \,|\, P \leq S \rangle,$$

where the last two subgroups are equal by Proposition 1.3 (Alperin's fusion theorem). The *hyperfocal subgroup* of $\mathcal{F}$ is the subgroup

$$\mathfrak{hyp}(\mathcal{F}) = \bigl\langle [O^p(\mathrm{Aut}_\mathcal{F}(P)), P] \bigm| P \leq S \bigr\rangle.$$

Equivalently, in the definition of $\mathfrak{hyp}(\mathcal{F})$, we can restrict to automorphisms of order prime to p. It is not hard to see that the image of the focal subgroup in $S/\mathfrak{hyp}(\mathcal{F})$ is precisely its commutator subgroup $[S,S]\mathfrak{hyp}(\mathcal{F})/\mathfrak{hyp}(\mathcal{F})$ (cf. [AKO, Lemma I.7.2]). Hence $\mathfrak{hyp}(\mathcal{F})$ is a proper subgroup of S if and only if $\mathfrak{foc}(\mathcal{F})$ is a proper subgroup.

As an immediate consequence of Proposition 1.3 and the definition of $\mathfrak{foc}(\mathcal{F})$, we have:

Proposition 1.8. *For any saturated fusion system $\mathcal{F}$ over a finite p-group S,*

$$\mathfrak{foc}(\mathcal{F}) = \bigl\langle [\mathrm{Aut}_\mathcal{F}(P), P] \bigm| P \in \boldsymbol{E}_\mathcal{F} \cup \{S\} \bigr\rangle.$$

By [BCGLO, Theorems 4.3 & 5.4], there is a unique saturated fusion subsystem $O^p(\mathcal{F}) \subseteq \mathcal{F}$ over $\mathfrak{hyp}(\mathcal{F})$ such that $\mathrm{Aut}_{O^p(\mathcal{F})}(P) \geq O^p(\mathrm{Aut}_\mathcal{F}(P))$ for all $P \leq \mathfrak{hyp}(\mathcal{F})$; and a unique saturated fusion subsystem $O^{p'}(\mathcal{F}) \subseteq \mathcal{F}$ over S minimal with respect to the property that $\mathrm{Aut}_{O^{p'}(\mathcal{F})}(P) \geq O^{p'}(\mathrm{Aut}_\mathcal{F}(P))$ for all $P \leq S$. This leads to the following definition.

Definition 1.9. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. Then*

(a) *$\mathcal{F}$ is* reduced *if $O_p(\mathcal{F}) = 1$ and $O^p(\mathcal{F}) = O^{p'}(\mathcal{F}) = \mathcal{F}$;*

(b) *$\mathcal{F}$ is* decomposable *if there are subgroups $1 \neq S_i \trianglelefteq S$ and saturated fusion systems $\mathcal{F}_i$ over S_i ($i = 1, 2$) such that $S = S_1 \times S_2$ and $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$, and $\mathcal{F}$ is* indecomposable *otherwise; and*

(c) *$\mathcal{F}$ is* simple *if there are no proper normal subsystems $\mathcal{E} \trianglelefteq \mathcal{F}$ with $1 \neq \mathcal{E} \subsetneqq \mathcal{F}$ (see* [AKO, Definition I.6.1] *for the definition of a normal fusion subsystem).*

(d) *The* reduction *of $\mathcal{F}$ is the fusion system $\mathfrak{red}(\mathcal{F})$ obtained by first setting $\mathcal{F}_0 = C_{\mathcal{F}}(O_p(\mathcal{F}))/Z(O_p(\mathcal{F}))$, and then letting $\mathfrak{red}(\mathcal{F})$ be the limiting term of the sequence $\mathcal{F}_0 \supseteq \mathcal{F}_1 \supseteq \mathcal{F}_2 \supseteq \dots$, where $\mathcal{F}_{i+1} = O^{p'}(O^p(\mathcal{F}_i))$ for all $i \geq 0$.*

Here, $C_{\mathcal{F}}(Q) \subseteq \mathcal{F}$ denotes the centralizer fusion system of $Q \leq S$: the largest fusion subsystem of $\mathcal{F}$ in which Q is central (cf. [AKO, Definition I.5.3] or [Cr, Definition 4.26(i)]). As the names suggest, the reduction of any saturated fusion system is reduced [AOV1, Proposition 2.2].

Simple fusion systems are always reduced and indecomposable, but the converse need not be true. For example, when $p = 2$, the fusion system of the wreath product $A_6 \wr A_5$ is reduced and indecomposable but not simple. However, a reduced fusion system which has no proper nontrivial strongly closed subgroups is simple (see the proof of Proposition 1.17(d) below).

We next explain how any exotic fusion system $\mathcal{F}$ can be "detected" via its reduction $\mathfrak{red}(\mathcal{F})$. The key to doing this is the idea of a *tame* fusion system. For any finite group G, $BG^\wedge_p$ denotes the p-completion of the classifying space of G, and $\mathrm{Out}(BG^\wedge_p)$ is the group of homotopy classes of homotopy equivalences from the space $BG^\wedge_p$ to itself.

Definition 1.10. *A saturated fusion system $\mathcal{F}$ over a finite p-group is* tamely realized *by a finite group G if*

- *$\mathcal{F}$ is isomorphic to $\mathcal{F}_T(G)$ (where $T \in \mathrm{Syl}_p(G)$); and*
- *the natural homomorphism $\kappa_G \colon \mathrm{Out}(G) \longrightarrow \mathrm{Out}(BG^\wedge_p)$ is split surjective.*

The fusion system $\mathcal{F}$ is tame *if it is tamely realized by some finite group.*

In fact, in the definition in [AOV1], we replace the group $\mathrm{Out}(BG^\wedge_p)$ by one which is defined purely algebraically, as a certain group of outer automorphisms of the centric linking system for G over S. We give the above definition here to avoid a long discussion about linking systems and their automorphisms. The two are equivalent by [BLO1, Theorem B].

The following theorem was shown in [AOV1, Theorems A, B, & C]. It is what motivated Definition 1.10.

Theorem 1.11. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group.*

(a) *If $\mathfrak{red}(\mathcal{F})$ is tame, then $\mathcal{F}$ is also tame, and in particular is realizable as the fusion system of a finite group.*

(b) *If $\mathcal{F}$ is reduced and each indecomposable factor of $\mathcal{F}$ is tame, then $\mathcal{F}$ is tame.*

(c) *If $\mathcal{F}$ is reduced and not tame, then there is an exotic fusion system $\widetilde{\mathcal{F}}$ such that $\mathfrak{red}(\widetilde{\mathcal{F}}) \cong \mathcal{F}$.*

Theorem 1.11 helps to explain why we only look at reduced, indecomposable fusion systems. Points (a) and (b) say that for every exotic fusion system $\mathcal{F}$, the reduction

$\mathfrak{red}(\mathcal{F})$ is not tame, and at least one of its indecomposable factors is also not tame. In other words, each exotic fusion system is detected by some reduced, indecomposable fusion system which is not tame.

Theorem 1.11 shows the importance of determining whether a given reduced fusion system is tame. In general, rather than comparing $\mathrm{Out}(G)$ with $\mathrm{Out}(BG^\wedge_p)$, it is much simpler to compare $\mathrm{Out}(G)$ with a certain group $\mathrm{Out}(\mathcal{F})$ of outer automorphisms of $\mathcal{F}$.

Definition 1.12. *For any saturated fusion system $\mathcal{F}$ over a finite p-group S, let $\mathrm{Aut}(\mathcal{F})$ be the group of those $\alpha \in \mathrm{Aut}(S)$ such that ${}^\alpha\mathcal{F} = \mathcal{F}$ (the "fusion preserving automorphisms"). Set $\mathrm{Out}(\mathcal{F}) = \mathrm{Aut}(\mathcal{F})/\mathrm{Aut}_{\mathcal{F}}(S)$.*

When $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group G with $S \in \mathrm{Syl}_p(G)$, there is a natural homomorphism from $\mathrm{Out}(G)$ to $\mathrm{Out}(\mathcal{F})$ defined by restriction (each class in $\mathrm{Out}(G)$ contains automorphisms of G which normalize S). By [AOV1, §§ 1.3 & 2.2], this map factors as the composite of homomorphisms

$$\mathrm{Out}(G) \xrightarrow{\ \kappa_G\ } \mathrm{Out}(BG^\wedge_p) \xrightarrow{\ \mu_G\ } \mathrm{Out}(\mathcal{F})\,.$$

Since $\mathrm{Out}(\mathcal{F})$ is in general easier to describe than $\mathrm{Out}(BG^\wedge_p)$, the simplest way to prove tameness is usually by showing that $\mu_G \circ \kappa_G$ is split surjective and that μ_G is injective. The following proposition, which is a special case of [AOV1, Proposition 4.2], suffices in all cases considered in this paper for showing that $\mathrm{Ker}(\mu_G) = 1$.

Proposition 1.13. *Fix a finite group G and $S \in \mathrm{Syl}_2(G)$, and set $\mathcal{F} = \mathcal{F}_S(G)$. Assume that at most one subgroup $P \in \boldsymbol{E}_{\mathcal{F}}$ has noncyclic center. Then $\mathrm{Ker}(\mu_G) = 1$.*

Proof. Let $\mathbf{E}^0_{\mathcal{F}}$ be the set of those $P \in \mathbf{E}_{\mathcal{F}}$ such that $C_{Z(P)}(\mathrm{Aut}_{\mathcal{F}}(P)) < C_{Z(P)}(\mathrm{Aut}_S(P))$. If $P \in \mathbf{E}_{\mathcal{F}}$ and $Z(P)$ is cyclic, then each element of odd order in $\mathrm{Aut}_{\mathcal{F}}(P)$ acts trivially on $Z(P)$. Then $C_{Z(P)}(\mathrm{Aut}_S(P)) = C_{Z(P)}(\mathrm{Aut}_{\mathcal{F}}(P))$ (recall P is fully normalized in $\mathcal{F}$), and hence $P \notin \mathbf{E}^0_{\mathcal{F}}$.

Thus $|\mathbf{E}^0_{\mathcal{F}}| \le 1$ under our hypotheses. By [AOV1, Proposition 4.2(d)], $\mathrm{Ker}(\mu_G) = 1$ if $\mathbf{E}^0_{\mathcal{F}} = \varnothing$, so assume $\mathbf{E}^0_{\mathcal{F}} = \{P\}$. Then by the same result, for each $\alpha \in \mathrm{Ker}(\mu_G)$, there is an element $g_P \in C_{Z(P)}(\mathrm{Aut}_S(P)) = Z(S)$ with the property that $\alpha = 1$ if and only if $g_P \in g{\cdot}C_{Z(P)}(\mathrm{Aut}_{\mathcal{F}}(P))$ for some $g \in C_{Z(S)}(\mathrm{Aut}_{\mathcal{F}}(S))$. Note that $P \trianglelefteq S$ and $\mathrm{Aut}_{\mathcal{F}}(S)$ normalizes P (by the uniqueness of P), so by [AOV1, Proposition 4.2(a,c)] (with $P \le S$ in the role of $Q \le P$ in point (c)), $g_P \equiv 1$ modulo $C_{Z(P)}(\mathrm{Aut}_{\mathcal{F}}(S)) = C_{Z(S)}(\mathrm{Aut}_{\mathcal{F}}(S))$. Thus $g_P \in C_{Z(S)}(\mathrm{Aut}_{\mathcal{F}}(S))$, so $\alpha = 1$ by the above remarks, applied with $g = g_P$. □

1.3. **Criteria for detecting reduced fusion systems.**

We now list some conditions on a finite p-group S or on a fusion system $\mathcal{F}$ over S which are necessary for $\mathcal{F}$ to be reduced (or sufficient for $\mathcal{F}$ to not be reduced). We begin with a simple criterion for detecting normal p-subgroups.

Proposition 1.14 ([AKO, Proposition I.4.5])**.** *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and fix $Q \trianglelefteq S$. Assume, for each $P \in \boldsymbol{E}_{\mathcal{F}} \cup \{S\}$, that $Q \le P$ and $\alpha(Q) = Q$ for each $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$. Then $Q \trianglelefteq \mathcal{F}$.*

The next lemma is the starting point for deciding whether or not $O^p(\mathcal{F}) = \mathcal{F}$.

Lemma 1.15 ([AKO, Corollary I.7.5])**.** *For any saturated fusion system $\mathcal{F}$ over a finite p-group S, $\mathcal{F} = O^p(\mathcal{F})$ if and only if $\mathfrak{foc}(\mathcal{F}) = S$. If $\mathrm{Aut}_{\mathcal{F}}(S)$ is a p-group and $O^p(\mathcal{F}) = \mathcal{F}$, then*

$$S = \bigl\langle [\mathrm{Aut}_{\mathcal{F}}(P), P] \bigm| P \in \boldsymbol{E}_{\mathcal{F}} \bigr\rangle.$$

Proof. See, e.g., [AKO, Corollary I.7.5] for a proof of the first statement. The second follows from that, Proposition 1.8, and [G, Theorems 5.1.1(i) & 5.1.3] ($Q[S,S] = S$ implies $Q = S$). □

We next look at cases where there are very few essential subgroups.

Lemma 1.16. *Let S be a nontrivial finite p-group. For any saturated fusion system $\mathcal{F}$ over S, if $|\mathbf{E}_\mathcal{F}| \leq 1$, then $O_p(\mathcal{F}) \neq 1$. If $\mathrm{Out}_\mathcal{F}(S) = 1$ and $\mathbf{E}_\mathcal{F}$ contains exactly one S-conjugacy class, then $O^p(\mathcal{F}) \subsetneqq \mathcal{F}$. In either case, $\mathcal{F}$ is not reduced.*

Proof. If $\mathbf{E}_\mathcal{F} = \varnothing$, then $S \trianglelefteq \mathcal{F}$ by Proposition 1.14, while if $\mathbf{E}_\mathcal{F} = \{P\}$, then $P \trianglelefteq \mathcal{F}$ by the same proposition. In either case, $O_p(\mathcal{F}) \neq 1$, so $\mathcal{F}$ is not reduced. If $\mathrm{Out}_\mathcal{F}(S) = 1$ and $\mathbf{E}_\mathcal{F} = \mathcal{P}$ for some S-conjugacy class $\mathcal{P}$, then $\mathfrak{foc}(\mathcal{F})$ is contained in $\langle \mathcal{P}, [S,S] \rangle < S$, so $O^p(\mathcal{F}) \subsetneqq \mathcal{F}$, and again $\mathcal{F}$ is not reduced. □

In Sections 4–7, our main technique for checking that fusion systems are realizable is to list all reduced fusion systems over a given 2-group S, and then match them with simple groups having Sylow 2-subgroup S. When doing this, it is important to know that the fusion systems of the groups in question are reduced, since this is not the case for all simple groups (e.g., not for A_5). The following proposition, part of which is based on a theorem of Goldschmidt, gives some criteria for showing this.

Proposition 1.17. *Let G be a finite nonabelian simple group. Choose $S \in \mathrm{Syl}_2(G)$, and set $\mathcal{F} = \mathcal{F}_S(G)$.*

(a) *In all cases, $O^2(\mathcal{F}) = \mathcal{F}$ and $\mathcal{F}$ is indecomposable.*

(b) *If S is nonabelian, and if G is not isomorphic to $PSU_3(2^n)$ ($n \geq 2$) nor to $Sz(2^{2n+1})$ ($n \geq 1$), then $O_2(\mathcal{F}) = 1$.*

(c) *If G and S satisfy the hypotheses in (b), and $\mathrm{Aut}(S)$ is a 2-group or $O^{2'}(\mathcal{F}) = \mathcal{F}$, then $\mathcal{F}$ is reduced.*

(d) *If G is a known simple group and $\mathcal{F}$ is reduced, then $\mathcal{F}$ is simple. In particular, this is the case whenever G and S satisfy the hypotheses in (b), and G is an alternating group, a sporadic simple group, a simple group of Lie type in defining characteristic 2, or ${}^2F_4(2)'$.*

Proof. **(a)** By the focal subgroup theorem for groups (cf. [G, Theorem 7.3.4]), $\mathfrak{foc}(\mathcal{F}) = S \cap [G,G]$. Hence $\mathfrak{foc}(\mathcal{F}) = S$ since G is simple, and $O^2(\mathcal{F}) = \mathcal{F}$ by Lemma 1.15.

Assume $\mathcal{F}$ is decomposable: thus $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$, where $\mathcal{F}_i$ is over $S_i \neq 1$ and $S = S_1 \times S_2$. In particular, S_1 and S_2 are both strongly closed in S with respect to G. So by [Gd2, Corollary A1], the normal closures of S_1 and S_2 commute with each other, which is impossible since each has normal closure G. Thus $\mathcal{F}$ is indecomposable.

(b) Set $Q = O_2(\mathcal{F})$ for short. If $Q \neq 1$, then $1 \neq Z(Q) \trianglelefteq \mathcal{F}$ (see [AKO, Proposition I.4.4]), and in particular, $Z(Q)$ is strongly closed with respect to G. But by [Gd1, Theorem A], under the given assumptions and since G is simple, no nontrivial abelian subgroup of S is strongly closed with respect to G, and thus $O_2(\mathcal{F}) = Q = 1$.

(c) If $\mathrm{Aut}(S)$ is a 2-group, then $\mathrm{Aut}_\mathcal{F}(S) = \mathrm{Inn}(S) = \mathrm{Aut}^0_\mathcal{F}(S)$ in the notation of [AKO, Theorem I.7.7], and hence $O^{2'}(\mathcal{F}) = \mathcal{F}$ by that theorem. Together with (a) and (b), this shows that $\mathcal{F}$ is reduced under the above hypotheses.

(d) Assume G is a known simple group such that $\mathcal{F} = \mathcal{F}_S(G)$ is reduced but not simple, and let $1 \neq \mathcal{E} \trianglelefteq \mathcal{F}$ be a proper normal subsystem over $T \trianglelefteq S$. If $T = S$, then

$O^{2'}(\mathcal{F}) \subseteq \mathcal{E} \subsetneqq \mathcal{F}$ (see [AOV1, Lemma 1.26]), which is impossible since $\mathcal{F}$ is reduced. Thus $1 \neq T < S$, where T is strongly closed (this is part of the definition of a normal subsystem [AKO, Definition I.6.1]). By [Ft, Theorem 1], then there is a nontrivial abelian subgroup $Q \leq T$ which is strongly closed in $\mathcal{F}$, and $Q \trianglelefteq \mathcal{F}$ by [AKO, Corollary I.4.7(a)], contradicting the assumption that $\mathcal{F}$ is reduced. Hence $\mathcal{F}$ is simple.

The last statement is shown in [A2, § 16]: in 16.3 (simple groups of Lie type in characteristic 2 and ${}^2F_4(2)'$), 16.5 (A_n), and 16.8 (sporadic simple groups). □

We refer to [FF] and [A2, § 16] for some similar results when p is odd.

2. Computer search criteria

We now list explicitly the criteria which we use to search for 2-groups which could support reduced fusion systems, and to search for critical subgroups of a given 2-group. Throughout this section, when $H \leq G$ are finite groups, we let trf_H^G denote the transfer homomorphism from $G/[G,G]$ to $H/[H,H]$ (see, e.g., [AKO, § I.8]). We will need the following application of these homomorphisms.

Lemma 2.1. *Fix a finite p-group S. Assume there is $g \in S$ which satisfies*

(a) *$g \notin [S,S]$;*

(b) *$[\alpha, g] \in [S,S]$ for each $\alpha \in O^p(\mathrm{Aut}(S))$; and*

(c) *$\mathrm{trf}_P^S([g]) \in P/[P,P]$ is fixed by $O^p(\mathrm{Aut}(P))$ for each critical subgroup $P < S$.*

Alternatively, assume there is $g \in S$ which satisfies

(a′) *$g \notin \Phi(S)$;*

(b′) *$[\alpha, g] \in \Phi(S)$ for each $\alpha \in O^p(\mathrm{Aut}(S))$; and*

(c′) *$\mathrm{trf}_P^S([g]) \in P/\Phi(P)$ is fixed by $O^p(\mathrm{Aut}(P))$ for each critical subgroup $P < S$.*

In either case, every saturated fusion system over S has a normal subsystem of index p, and hence there are no reduced fusion systems over S.

Proof. We prove this for hypotheses (a)–(c). The proof for (a′)–(c′) holds by the same argument, upon replacing $Q/[Q,Q]$ (for $Q \leq S$) by $Q/\Phi(Q)$.

Let $\mathcal{F}$ be a saturated fusion system over S. For each $P, Q \leq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$, we let $\overline{\varphi}$ denote the induced homomorphism from $P/[P,P]$ to $Q/[Q,Q]$. We claim that for each isomorphism $\varphi \in \mathrm{Iso}_{\mathcal{F}}(P,Q)$,

$$\overline{\varphi}(\mathrm{trf}_P^S([g])) = \mathrm{trf}_Q^S([g]) \in Q/[Q,Q]\,. \tag{1}$$

Point (1) holds by (b) (and the naturality properties of the transfer) when $\varphi = \alpha|_P$ for some $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$. If $P, Q \leq R < S$ where $R \in \mathbf{E}_{\mathcal{F}}$ (hence R is critical in S by Proposition 1.4), and $\varphi = \alpha|_P$ for some $\alpha \in O^p(\mathrm{Aut}_{\mathcal{F}}(R))$, then

$$\overline{\varphi}(\mathrm{trf}_P^S([g])) = \overline{\varphi}(\mathrm{trf}_P^R(\mathrm{trf}_R^S([g]))) = \mathrm{trf}_Q^R(\overline{\alpha}(\mathrm{trf}_R^S([g]))) = \mathrm{trf}_Q^R(\mathrm{trf}_R^S([g])) = \mathrm{trf}_Q^S([g]),$$

where the third equality holds by (c). Point (1) now follows from Proposition 1.3: each isomorphism φ is a composite of restrictions of automorphisms of S and of $\mathcal{F}$-essential (hence critical) subgroups.

By [AKO, Proposition I.8.4], there is an injective homomorphism

$$\mathrm{trf}_{\mathcal{F}}\colon S/\mathfrak{foc}(\mathcal{F}) \longrightarrow S/[S,S]\,,$$

together with proper subgroups $P_1, \dots, P_m < S$ and homomorphisms $\varphi_i \in \mathrm{Hom}_{\mathcal{F}}(P_i, S)$ such that for $g \in S$,

$$\mathrm{trf}_{\mathcal{F}}([g]) = \prod_{[\alpha]\in \mathrm{Out}_{\mathcal{F}}(S)} \overline{\alpha}([g])\cdot \prod_{i=1}^{m} \overline{\varphi_i}\big(\mathrm{trf}_{P_i}^{S}([g])\big).$$

Set $Q_i = \varphi_i(P_i)$, and let $\varphi_i' \in \mathrm{Iso}_{\mathcal{F}}(P_i, Q_i)$ be the restriction of φ_i. By (1), if we set $k = |\mathrm{Out}_{\mathcal{F}}(S)|$, then

$$\mathrm{trf}_{\mathcal{F}}([g]) = [g]^k\cdot \prod_{i=1}^{m} \overline{\mathrm{incl}}_{Q_i}^{S}\big(\overline{\varphi_i'}(\mathrm{trf}_{P_i}^{S}([g]))\big) = [g]^k\cdot \prod_{i=1}^{m} \overline{\mathrm{incl}}_{Q_i}^{S}(\mathrm{trf}_{Q_i}^{S}([g])) = \Big[g^k\cdot \prod_{i=1}^{m} g^{[S:Q_i]}\Big] \neq 1$$

since $p\nmid k$ (and $p\big|[S{:}Q_i]$ for each i). Thus $g \notin \mathfrak{foc}(\mathcal{F})$, so $\mathfrak{foc}(\mathcal{F}) < S$, and $\mathcal{F}$ contains a normal subgroup of index p by Lemma 1.15. □

Let S be a finite p-group. A normal subgroup $P \trianglelefteq S$ is called *semicharacteristic* in S if P is normalized by $O^p(\mathrm{Aut}(S))$. If $P_1, P_2 \le S$ are semicharacteristic subgroups, then so is $P_1 \cap P_2$. We can thus define the *semicharacteristic closure* of $Q \le S$ to be the smallest subgroup $P \trianglelefteq S$ containing Q which is semicharacteristic in S.

Proposition 2.2. *Assume $\mathcal{F}$ is a reduced fusion system over a finite nonabelian 2-group S. Assume also that S is not isomorphic to D_{2^n} ($n \ge 3$), SD_{2^n} ($n \ge 4$), or $C_{2^n} \wr C_2$ ($n \ge 2$). Then S satisfies the following conditions.*

(a) *S contains no abelian subgroup of index two.*

(b) *$[S, S]$ is not cyclic.*

(c) *Let A denote the image of $\Omega_1(Z(S))$ in $S/[S, S]$. Then either $A = 1$, or $|A| > 2$ and $\mathrm{Aut}(S)$ is not a 2-group.*

(d) *If $\mathrm{Aut}(S)$ is a 2-group, then*

$$\bigcap_{M<S} \mathrm{Ker}\big[S/[S,S] \xrightarrow{\mathrm{trf}_M^S} M/[M,M]\big] = 1 \quad \textit{and} \quad \bigcap_{M<S} \mathrm{Ker}\big[S/\Phi(S) \xrightarrow{\mathrm{trf}_M^S} M/\Phi(M)\big] = 1,$$

where the intersections are taken over all maximal subgroups $M < S$.

(e) *S has more than one critical subgroup, and S has more than one conjugacy class of critical subgroups if $\mathrm{Aut}(S)$ is a 2-group.*

(f) *For each critical subgroup $P < S$, let Q_P denote the semicharacteristic closure of $[N_S(P), P]$ in P. Then $\big\langle [O^2(\mathrm{Aut}(S)), S], Q_P \big| P \text{ critical}\big\rangle = S$.*

(g) *Let $Q \trianglelefteq S$ be any normal subgroup which is semicharacteristic in S, and which is contained in and semicharacteristic in each critical subgroup $P < S$. Then $Q = 1$.*

(h) *Let $K \le S/[S, S]$ denote the subgroup consisting of those elements $x \in S/[S, S]$ which are fixed by $O^2(\mathrm{Aut}(S))$, and which are such that $\mathrm{trf}_P^S(x) \in P/[P, P]$ is fixed by $O^2(\mathrm{Aut}(P))$ for each critical subgroup $P < S$. Then $K = 1$.*

Similarly, let $K' \le S/\Phi(S)$ denote the subgroup consisting of those elements $x \in S/\Phi(S)$ which are fixed by $O^2(\mathrm{Aut}(S))$, and which are such that $\mathrm{trf}_P^S(x) \in P/\Phi(P)$ is fixed by $O^2(\mathrm{Aut}(P))$ for each critical subgroup $P < S$. Then $K' = 1$.

Proof. By Proposition 1.4, each $\mathcal{F}$-essential subgroup is a critical subgroup of S. This will be used throughout the proof.

Points (a) and (b) follow from [AOV2, Proposition 5.2(a,b)], (c) follows from [AKO, Corollary I.8.5], and (d) and (h) from Lemma 2.1. Since $\mathcal{F}$-essential subgroups are critical, point (e) follows from Lemma 1.16.

(f) Let $P_1, \ldots, P_m$ be conjugacy class representatives for the critical subgroups of S, and let $Q_i = Q_{P_i} \leq P_i$ be the semicharacteristic closure of $[N_S(P_i), P_i]$. For each i, Q_i is $O^2(\mathrm{Aut}_{\mathcal{F}}(P_i))$-invariant and $\mathrm{Aut}_S(P_i)$-invariant, hence $\mathrm{Aut}_{\mathcal{F}}(P_i)$-invariant. So $Q_i \geq [O^{2'}(\mathrm{Aut}_{\mathcal{F}}(P_i)), P_i]$ since $O^{2'}(\mathrm{Aut}_{\mathcal{F}}(P_i))$ is generated by $\mathrm{Aut}_S(P_i) \in \mathrm{Syl}_2(\mathrm{Aut}_{\mathcal{F}}(P_i))$ and its conjugates in $\mathrm{Aut}_{\mathcal{F}}(P_i)$. So if we set

$$H = \bigl\langle [O^2(\mathrm{Aut}_{\mathcal{F}}(S)), S], Q_i \,\bigm|\, 1 \leq i \leq m \bigr\rangle,$$

then

$$\begin{aligned}\mathfrak{foc}(\mathcal{F}) &= \bigl\langle [\mathrm{Aut}_{\mathcal{F}}(S), S], [O^{2'}(\mathrm{Aut}_{\mathcal{F}}(P_i)), P_i] \,\bigm|\, i = 1, \ldots, m \bigr\rangle \\ &\leq [S,S] \cdot \bigl\langle [O^2(\mathrm{Aut}_{\mathcal{F}}(S)), S], Q_i \,\bigm|\, i = 1, \ldots, m \bigr\rangle = [S,S]H\end{aligned}$$

where the first equality follows from Proposition 1.3. Since $\mathcal{F}$ is reduced, $\mathfrak{foc}(\mathcal{F}) = S$ by Lemma 1.15. Thus $[S,S]H = S$, so $H = S$ by [G, Theorems 5.1.1(i) & 5.1.3].

(g) Assume $Q \trianglelefteq S$ is normal and semicharacterisitic in S, and contained in and semicharacteristic in each critical subgroup $P < S$. Then if $P = S$ or $P \in \mathbf{E}_{\mathcal{F}}$, Q is normalized by the action of $O^2(\mathrm{Aut}_{\mathcal{F}}(P)) \cdot \mathrm{Aut}_S(P) = \mathrm{Aut}_{\mathcal{F}}(P)$. Hence $Q \trianglelefteq \mathcal{F}$ by Proposition 1.14, and $Q = 1$ since $\mathcal{F}$ is reduced. $\square$

We now turn to the criteria used in our computer searches to determine (possibly) critical subgroups of a finite 2-group. Note that these are the actual criteria used in our computer program, not necessarily the optimal conditions which must hold when P is critical.

Proposition 2.3. *Let S be a finite 2-group, and let $P \leq S$ be a critical subgroup. Let $S_0 = N_S(P)/P$. Then the following conditions hold.*

(a) *Let $Z' = \bigl\langle x \in Z_2(S) \,\bigm|\, |[x,S]| \leq 2 \bigr\rangle$. Then either $Z' \leq P$, or there exists an involution $h \in S$ such that $P = C_S(h)$, $[N_S(P){:}P] = 2$ and $[h, Z_2(S)] \neq 1$.*

(b) *$P \neq S$ and $C_S(P) \leq P$.*

(c) *Either S_0 is cyclic or $Z(S_0) = \Omega_1(S_0)$.*

(d) *If $Z(S_0)$ is not cyclic, then $|S_0| = |Z(S_0)|^m$ for $m = 1$, 2 or 3.*

(e) *Let k be such that $|S_0| = 2^k$. Then $\mathrm{rk}(P/\Phi(P)) \geq 2k$. If $k \geq 2$, then for all $1 \neq s \in S_0$, $\mathrm{rk}([s, P/\Phi(P)]) \geq 2$.*

(f) *If $Z(S_0) \cong (C_2)^n$ with $n \geq 2$ then $\mathrm{rk}([s, P/\Phi(P)]) \geq n$ for all $s \in Z(S_0)$, $s \neq 1$.*

(g) *Let $\Theta = 1$, $\Theta = Z(P)$ or $\Theta = Z_2(P)$. If $g \in N_S(P)$ satisfies $[g,P] \leq \Theta \cdot \Phi(P)$ and $[g, \Theta] \leq \Phi(P)$ then $g \in P$.*

(h) *$\mathrm{Aut}(P)$ is not a 2-group and $\mathrm{Out}_S(P) \cap O_2(\mathrm{Out}(P)) = 1$.*

(i) *Let k be as in (e). There exists a composition factor M of the $\mathbb{F}_2[\mathrm{Out}(P)]$-module $P/\Phi(P)$ with $\dim(M) \geq 2k$. If $k \geq 2$, then M can be chosen such that $\dim([s,M]) \geq 2$ for all $s \in S_0$, $s \neq 1$.*

(j) *All involutions in S_0 are conjugate in $N_{\mathrm{Out}(P)}(S_0)$.*

Proof. Points (b) and (h) hold by definition of critical subgroups. Point (a) follows from [OV, Lemma 3.6(a)], points (c) and (d) from Proposition 1.6(a), (e) from Proposition

1.6(b), (f) from Proposition 1.6(c), (g) from Proposition 1.5, and (j) from [OV, Proposition 3.3(b)]. Point (i) is shown in Proposition 1.7. □

Definition 2.4. *A subgroup P of a finite 2-group S is* potentially critical *if it satisfies conditions* (a)–(j) *in Proposition 2.3.*

By definition, each automorphism of S sends the set of potentially critical subgroups of S to itself.

3. Fusion systems over 2-groups with abelian direct factor

In this section, we find conditions on a 2-group S_0 which imply that there are no reduced fusion systems over $S = S_0 \times A$ for any abelian 2-group $A \neq 1$. As will be seen, the assumption that a fusion system $\mathcal{F}$ over S has no normal subsystems of 2-power index implies certain extra properties, which allow us to show that A, or some other direct factor in S, is normal in $\mathcal{F}$. Green correspondence plays a key role in the arguments we use, and the following elementary lemma will be useful. We refer to [Bs, §§ 3.10 & 3.12] for more details on vertices and Green correspondence.

Lemma 3.1. *Let G be a finite group and fix $S \in \mathrm{Syl}_p(G)$. Let k be a field of characteristic p, and let V be a finitely generated $k[G]$-module.*

(a) *If V is indecomposable, then for all $H \leq G$, each vertex of each indecomposable direct summand of $V|_H$ is contained in a vertex of V. In particular, if $S \in \mathrm{Syl}_p(G)$ and $V|_S$ has a nonzero direct summand with trivial S-action, then S is a vertex of V.*

(b) *Let $H \leq G$ be a subgroup that contains $N_G(S)$. Let $0 \neq W \leq V|_H$ be a direct summand of V as a $k[H]$-module, and write $W = \bigoplus_{i=1}^n W_i$ where each W_i is indecomposable as a $k[H]$-module. Assume S is a vertex of W_i for each i (in particular, this holds if S acts trivially on W). Let V_i be the $k[G]$-Green correspondent of W_i. Then $V \cong \widetilde{V} \oplus \bigoplus_{i=1}^n V_i$ for some $k[G]$-module $\widetilde{V}$.*

(c) *Let $H < G$ be as in (b), and assume in addition that H is strongly p-embedded in G. Assume also that V is indecomposable as a $k[G]$-module and has vertex S. Let W be the $k[H]$-Green correspondent of V. Then $V|_H \cong W \oplus X$, where X is projective as a $k[H]$-module and hence free as a $k[S]$-module.*

Proof. **(a)** Let P be a vertex of V. Then there is a $k[P]$-module W such that V is a direct summand of $\mathrm{Ind}_P^G(W)$. By the Mackey formula (cf. [Bs, Theorem 3.3.4]), $(\mathrm{Ind}_P^G(W))|_H$ is a direct sum of modules induced up from $H \cap {}^gP$ for elements $g \in G$. Hence each indecomposable direct summand of $V|_H$ has vertex contained in ${}^gP$ for some $g \in G$ (and ${}^gP$ is also a vertex of V).

If $S \in \mathrm{Syl}_p(G)$ and $V|_S$ has a nonzero indecomposable direct summand W with trivial S-action, then S is a vertex of W (see, e.g., [LP, Remark 4.8.11(b)]), and hence also a vertex of V.

(b) Let $V = \bigoplus_{j\in J} \overline{V}_j$ be the decomposition as a sum of indecomposable $k[G]$-modules. Let $J_0 \subseteq J$ be the set of all $j \in J$ such that $\overline{V}_j|_H$ has an indecomposable direct summand $U_j \leq \overline{V}_j$ with vertex S. By Green correspondence and (a), for each $j \in J_0$, U_j is the $k[H]$-Green correspondent of $\overline{V}_j$ as a $k[G]$-module, and is the only indecomposable direct summand of $\overline{V}_j|_H$ with vertex S.

By the Krull-Schmidt theorem applied to $V|_H$ (cf. [Bs, Theorem 1.4.6]), and since $W=\bigoplus_{i=1}^n W_i$ where each W_i is indecomposable and has vertex S, there is an injective map $r\colon \{1,2,\dots,n\} \longrightarrow J_0$ such that $W_i \cong U_{r(i)}$ for each i. The lemma now follows upon setting $V_i=\overline{V}_{r(i)}$ for $1\le i\le n$ and $\widetilde{V}=\bigoplus_{j\in J\smallsetminus \mathrm{Im}(r)}\overline{V}_j$.

(c) By Green correspondence and since V is indecomposable, $V|_H\cong W\oplus X$ where the vertex of each indecomposable direct summand of X is contained in ${}^gS\cap H$ for some $g\in G\smallsetminus H$. Since H is strongly p-embedded, this means that each such direct summand has vertex the trivial group, and hence that X is projective as a $k[H]$-module. In particular, X is free over $k[S]$ (see [CR, Theorem 5.24]). □

We will need the following lemma on Green correspondance.

Lemma 3.2. *Fix a finite group G with a strongly 2-embedded subgroup $H<G$, and choose $S\in \mathrm{Syl}_2(H)\subseteq \mathrm{Syl}_2(G)$. Let V be a finitely generated $\mathbb{F}_2[G]$-module, and assume $0\ne W\le V$ is a direct summand of $V|_H$ upon which S acts trivially. Then there is an $\mathbb{F}_2[G]$-submodule $V^*\le V$ such that $V^*|_H\cong W\oplus X$ as $\mathbb{F}_2[H]$-modules, where X is free as an $\mathbb{F}_2[S]$-module, and where either $X=0$, or $|S|=2$ and $\dim_{\mathbb{F}_2}(X)\ge 4$, or $\dim_{\mathbb{F}_2}(X)\ge 8$.*

Proof. By Lemma 3.1(b), there is an $\mathbb{F}_2[G]$-submodule $V^*\le V$ (in fact, a direct summand) which is isomorphic to the sum of the Green correspondents of the indecomposable direct summands of W. So without loss of generality, we can assume that $V=V^*$ and W are indecomposable as $\mathbb{F}_2[G]$- and $\mathbb{F}_2[H]$-modules, respectively, S is a vertex of both, and V is the Green correspondent of W. By Lemma 3.1(c), $V|_H\cong W\oplus X$ for some X that is free as an $\mathbb{F}_2[S]$-module.

Assume $X\ne 0$. If the action of H on W could be extended to a linear action of G, then that would have vertex S (Lemma 3.1(a)) and hence be the Green correspondent of W, contradicting the uniqueness of Green correspondents. So the action of H on W does not extend to any linear action of G, and in particular, W is not normalized by G.

Set $K=O^{2'}(H)$: the normal closure of S in H. Thus H/K has odd order, and W can be regarded as an $\mathbb{F}_2[H/K]$-module.

Set $N_0=C_G(V)\trianglelefteq G$: the kernel of the G-action on V. Since $X\ne 0$ is free as an $\mathbb{F}_2[S]$-module, $N_0\cap S=1$, and so N_0 has odd order. Also, $HN_0<G$ since the action of HN_0 on V normalizes W while the action of G on V does not. So HN_0/N_0 is strongly 2-embedded in G/N_0 by Lemma A.4(b). Upon replacing G by G/N_0 and H by HN_0/N_0, we can assume that G acts faithfully on V.

Let $k\supseteq \mathbb{F}_2$ be any finite extension, and set $\widehat{V}=k\otimes_{\mathbb{F}_2}V$ and $\widehat{W}=k\otimes_{\mathbb{F}_2}W$. Write $\widehat{W}=\bigoplus_{i=1}^m\widehat{W}_i$, where $\widehat{W}_i$ is irreducible as a $k[H/K]$-module for each i. Let $\widehat{V}_i$ denote the $k[G]$-Green correspondent of $\widehat{W}_i$. By Lemma 3.1(c), $\widehat{V}_i|_H\cong \widehat{W}_i\oplus\widehat{X}_i$ where $\widehat{X}_i$ is free as a $k[S]$-module. Also, $\widehat{V}$ has a direct summand isomorphic to $\bigoplus_{i=1}^m\widehat{V}_i$ by Lemma 3.1(b). By the Krull-Schmidt theorem, and since $\widehat{W}|_{\mathbb{F}_2[H/K]}\cong W^\ell$ as $\mathbb{F}_2[H/K]$-modules where $\ell=\dim_{\mathbb{F}_2}(k)$, $\widehat{W}_i|_{\mathbb{F}_2[H/K]}\cong W^{\ell_i}$ for some $1\le \ell_i\le \ell$. If, for some i, $\widehat{W}_i\cong \widehat{V}_i|_H$, then by Lemma 3.1(b) applied to $\widehat{V}_i|_{\mathbb{F}_2[G]}$, the same must be true for each irreducible direct summand of $\widehat{W}_i|_{\mathbb{F}_2[H/K]}$, which contradicts our assumption that $V>W$. Thus $\dim_k(\widehat{V}_i)-\dim_k(\widehat{W}_i)=\dim_k(\widehat{X}_i)\ge |S|$ for each i, and so $\dim_{\mathbb{F}_2}(X)=\dim_{\mathbb{F}_2}(V/W)\ge \sum_{i=1}^m\dim_k(\widehat{X}_i)\ge m|S|$. This proves the lemma when $m\ge 2$ for some finite extension $k\supseteq\mathbb{F}_2$.

We are left with the case where V is a faithful, indecomposable $\mathbb{F}_2[G]$-module and W is absolutely irreducible as an $\mathbb{F}_2[H/K]$-module (i.e., $k \otimes_{\mathbb{F}_2} W$ is irreducible as a $k[H/K]$-module for each finite extension $k \supseteq \mathbb{F}_2$, see [Is, Theorem 9.2]). There are two cases to consider.

Case 1: $\boldsymbol{O_{2'}(G) \neq 1}$. Since $O_{2'}(G)$ is solvable by the odd order theorem [FT], there is $1 \neq N \trianglelefteq G$ which is an elementary abelian p-group for some odd p. By the Frattini argument, $N_G(SN) = N_G(S){\cdot}N \leq HN$, so $HN/N \geq N_{G/N}(SN/N)$. We claim that $H \cap N \neq 1$ and acts nontrivially on W. Assume otherwise. Regard W as an $\mathbb{F}_2[HN/N]$-module via the isomorphism $HN/N \cong H/(H \cap N)$. Since SN/N acts trivially on W, it is a vertex of W by Lemma 3.1(a). Let V' be its $\mathbb{F}_2[G/N]$-Green correspondent (note that $HN/N \geq N_{G/N}(SN/N)$). Then V' is indecomposable as an $\mathbb{F}_2[G]$-module, and W is a direct summand of $V'|_H$ with vertex S. Hence as an $\mathbb{F}_2[G]$-module, V' has vertex S by Lemma 3.1(a) and is the Green correspondent of W by [Bs, Theorem 3.12.2(i)], so $V' \cong V$ by the uniqueness of Green correspondents. This is impossible, since G acts faithfully on V but not on V', and we conclude that $H \cap N$ acts nontrivially. In particular, $H \cap N \neq 1$.

Let $r > 1$ be the multiplicative order of 2 in $\mathbb{F}_p^\times$. Thus all irreducible $\mathbb{F}_2[C_p]$-modules with nontrivial action have dimension r. Since V is indecomposable, and $V = C_V(N) \oplus [N, V]$ where the summands are $\mathbb{F}_2[G]$-submodules, $V|_N$ is a direct sum of irreducible $\mathbb{F}_2[N]$-modules with nontrivial action, each of which has dimension r. Similarly, since W is irreducible and $H \cap N \trianglelefteq H$ acts nontrivially, each irreducible direct summand of $W|_{H \cap N}$ has dimension r. Thus $r \mid \dim(V)$, $r \mid \dim(W)$, and $r \mid \dim(X)$.

The dimension of each absolutely irreducible $\mathbb{F}_2[H/K]$-module divides $|H/K|$ (cf. [Se, §§ 6.5 & 15.5]). In particular, $\dim(W)$ is odd. Thus $r \mid \dim(W)$ is also odd, so $r \geq 3$, and $r|S| \mid \dim(X)$ in this case.

Case 2: $\boldsymbol{O_{2'}(G) = 1}$. By Bender's theorem [Be, Satz 1 & Lemma 2.6], $O^{2'}(G)$ is isomorphic to $PSL_2(2^n)$, $Sz(2^{2n-1})$, or $PSU_3(2^n)$ $(n \geq 2)$, and hence $Z(S)$ is elementary abelian of rank at least 2. If $\dim(X) < 8$, then $|S| < 8$ since X is free as an $\mathbb{F}_2[S]$-module, and so $S \cong C_2 \times C_2$, and $G \cong A_5$ since $|\mathrm{Out}(A_5)| = 2$ [Sz1, (3.2.17)]. But then $H \cong A_4$, $H/K \cong C_3$, and W has nontrivial H-action since the Green correspondent of the trivial $\mathbb{F}_2[A_4]$-module is trivial. So $\mathbb{F}_4 \otimes_{\mathbb{F}_2} W$ is reducible, which contradicts our assumption that W is absolutely irreducible. □

We now prove four propositions, each showing that under certain (fairly restrictive) hypotheses on a 2-group S_0, there are no reduced fusion systems over $S_0 \times A$ for any abelian 2-group $A \neq 1$. Stronger results of this type will be shown in a later paper.

In the first proposition, we consider certain finite 2-groups of nilpotence class 2. It will be applied, for example, when $S = S_0 \times A$ for S_0 of type $SL_3(2^n)$ $(n \geq 2)$, $2{\cdot}SL_3(4)$, or $Sp_4(2^n)$ $(n \geq 2)$, and $A \neq 1$ is abelian.

Proposition 3.3. *Let S be a finite 2-group containing normal abelian subgroups $P_1, P_2 \trianglelefteq S$ such that*

(i) $S = P_1 P_2$ *and* $P_1 \cap P_2 = Z(S)$*;*

(ii) $[S, S] \cap \Phi(Z(S)) = 1$*;*

(iii) $P_i = Z(S)\Omega_1(P_i)$ *and* $\mathrm{rk}(P_i/Z(S)) \geq 2$ *for* $i = 1, 2$*; and*

(iv) *for each* $i = 1, 2$ *and each* $g \in P_i \smallsetminus Z(S)$, $C_S(g) = P_i$.

Then P_1 and P_2 are the only possible critical subgroups of S, and every elementary abelian subgroup of S is contained in P_1 or P_2. If $\mathcal{F}$ is a saturated fusion system over S such that

$O_2(\mathcal{F})=1$, *then* $\boldsymbol{E}_{\mathcal{F}}=\{P_1,P_2\}$, P_1 *and* P_2 *are elementary abelian,* $\mathrm{rk}(P_1)=\mathrm{rk}(P_2)$, *and* $[S,S]=Z(S)$.

Proof. Set $V_i=\Omega_1(P_i)$ for short ($i=1,2$). Thus $V_i\trianglelefteq S$ and is elementary abelian, and $P_i=V_iZ(S)$ by (iii). Also, $S=V_1V_2Z(S)$, so $[S,S]\le V_1\cap V_2$ is elementary abelian.

If $g\in S\smallsetminus(P_1\cup P_2)$, then $g=g_1g_2z$ for some $g_i\in V_i\smallsetminus Z(S)$ and $z\in Z(S)$, so $g^2=[g_1,g_2]z^2\neq 1$ since $[g_1,g_2]\notin\Phi(Z(S))$ by (ii) and (iv), while $z^2\in\Phi(Z(S))$. Thus all elements of order two in S lie in $P_1\cup P_2$.

If $Q\le S$ is elementary abelian, and $Q\nleq Z(S)=P_1\cap P_2$, then choose $g\in Q\smallsetminus Z(S)$. We just showed that $g\in P_i$ for $i=1$ or 2, and so $Q\le C_S(g)=P_i$.

Let $R\le S$ be a critical subgroup. If $R\notin\{P_1,P_2\}$, then $R\nleq P_1$ and $R\nleq P_2$ since R is centric in S. If $x\in\Omega_1(Z(R))$, then $x\in P_i$ for some i as shown above. If $x\notin Z(S)$, then $R\le C_S(x)=P_i$, which is impossible. Thus $\Omega_1(Z(R))\le Z(S)$, so $\Omega_1(Z(S))=\Omega_1(Z(R))$ is characteristic in R. But for $g\in S\smallsetminus R$, $[g,\Omega_1(Z(S))]=1$ and $[g,R]\le[S,S]\le V_1\cap V_2=\Omega_1(Z(S))$, which contradicts Proposition 1.5. We conclude that P_1 and P_2 are the only subgroups of S which could be critical.

Step 1: Let $\mathcal{F}$ be a saturated fusion system over S such that $O_2(\mathcal{F})=1$. Then P_1 and P_2 must both be $\mathcal{F}$-essential by Lemma 1.16. Since $P_i=V_iZ(S)$ where V_i is elementary abelian, $\Phi(P_1)=\Phi(Z(S))=\Phi(P_2)$. So $\Phi(Z(S))\trianglelefteq\mathcal{F}$ by Proposition 1.14, and since $O_2(\mathcal{F})=1$, $\Phi(Z(S))=1$. Thus $P_i=V_i$ is elementary abelian for $i=1,2$.

For $x\in P_1\smallsetminus Z(S)$, we have $\mathrm{rk}(P_1/Z(S))=\mathrm{rk}(S/P_2)\le\mathrm{rk}([x,P_2])=\mathrm{rk}(P_2/Z(S))$, where the inequality holds by Proposition 1.6(c), and the last equality since $C_{P_2}(x)=P_2\cap P_1=Z(S)$ by (iv) and (i). The opposite inequality holds by a similar argument, so $\mathrm{rk}(P_1/Z(S))=\mathrm{rk}(P_2/Z(S))$ and $\mathrm{rk}(P_1)=\mathrm{rk}(P_2)$.

Step 2: It remains to prove that $Z(S)=[S,S]$. By (i), $[S,S]\le P_1\cap P_2=Z(S)$, and it remains to prove the opposite inclusion. Regard $Z(S)$ as an $\mathbb{F}_2[\mathrm{Out}_{\mathcal{F}}(S)]$-module with submodule $[S,S]$. Since $|\mathrm{Out}_{\mathcal{F}}(S)|$ is odd, there is an $\mathbb{F}_2[\mathrm{Out}_{\mathcal{F}}(S)]$-submodule $W\le Z(S)$ which is complementary to $[S,S]$; i.e., $Z(S)=W\times[S,S]$.

For each $i=1,2$, set $G_i=\mathrm{Aut}_{\mathcal{F}}(P_i)$, $T_i=\mathrm{Aut}_S(P_i)\in\mathrm{Syl}_2(G_i)$, and $H_i=N_{G_i}(T_i)$. Then

$$T_i\cong S/P_i\cong P_{3-i}/Z(S)\cong(C_2)^r\quad\text{where}\quad r=\mathrm{rk}(P_1/Z(S))=\mathrm{rk}(P_2/Z(S))\ge 2.\tag{1}$$

Since $\mathrm{Aut}_{\mathcal{F}}(S)$ is generated by $\mathrm{Inn}(S)$ and automorphisms of odd order, each P_i is normalized by $\mathrm{Aut}_{\mathcal{F}}(S)$. By the extension axiom, the homomorphism

$$\psi_i\colon \mathrm{Aut}_{\mathcal{F}}(S)\longrightarrow H_i\,,$$

induced by restriction to P_i, is surjective. In particular, a subgroup of P_i is normalized by H_i if and only if it is normalized by $\mathrm{Aut}_{\mathcal{F}}(S)$.

Consider the $\mathbb{F}_2[\mathrm{Out}_{\mathcal{F}}(S)]$-module $P_i/[S,S]$. Since $|\mathrm{Out}_{\mathcal{F}}(S)|$ is odd, $Z(S)/[S,S]$ has a complement $M/[S,S]$ in $P_i/[S,S]$. Hence $P_i=W\times M$ as $\mathbb{F}_2[\mathrm{Aut}_{\mathcal{F}}(S)]$-modules, and in particular, W is a direct factor of P_i as an $\mathbb{F}_2[H_i]$-module.

Since P_i is maximal among $\mathcal{F}$-essential subgroups, each $\alpha\in\mathrm{Aut}_{\mathcal{F}}(P_i)$ which extends to $\mathrm{Aut}_{\mathcal{F}}(Q)$ for any $Q>P_i$ also extends to $\mathrm{Aut}_{\mathcal{F}}(S)$ (Proposition 1.3), and hence lies in $H_i=N_{G_i}(T_i)$. So by [AKO, Proposition I.3.3(b)] and since $P_i\in\mathbf{E}_{\mathcal{F}}$, H_i is strongly 2-embedded in G_i.

If $W\neq 1$ and U is an indecomposable direct factor of W as an $\mathbb{F}_2[H_i]$-module, then by Lemma 3.1(b,c), the G_i-Green correspondent U^* of U is isomorphic to a direct factor of P_i, and $U^*|_{H_i} \cong U \times X$ for some $\mathbb{F}_2[H_i]$-module X such that $X|_{T_i}$ is free as an $\mathbb{F}_2[T_i]$-module. Since T_i acts trivially on $Z(S)$ and on $P_i/Z(S)$, $P_i|_{T_i}$ contains no nontrivial free $\mathbb{F}_2[T_i]$-submodules (recall that $T_i \cong (C_2)^r$ for $r \geq 2$ by (1)). So $X = 1$, and $U \cong U^*|_{H_i}$.

Thus the Green correspondent of each indecomposable direct factor of W is isomorphic to that direct factor (after restriction to H_i). So by Lemma 3.1(b), there is $W_i \leq P_i$ which is a direct factor of P_i as an $\mathbb{F}_2[G_i]$-module and such that $W_i|_{H_i} \cong W$. Also, $W_i \leq C_{P_i}(T_i) = Z(S)$, $W_i \cap [S,S] = W_i \cap [T_i, P_i] = 1$ since W_i is a direct factor as an $\mathbb{F}_2[G_i]$-module, and so $Z(S) = C_{P_i}(T_i) = [S,S] \times W_i$.

Step 3: Fix $i = 1,2$. By the version of Bender's theorem in [Sz2, Theorem 6.4.2], $O_{2'}(G_i)$ is contained in every strongly 2-embedded subgroup of G_i, and in particular, $O_{2'}(G_i) \leq H_i = N_{G_i}(T_i)$. So $[O_{2'}(G_i), T_i] \leq O_{2'}(G_i) \cap T_i = 1$. Upon setting $\widehat{G}_i = O^{2'}(G_i)$, we get $[\widehat{G}_i, O_{2'}(G_i)] = 1$, and thus $O_{2'}(\widehat{G}_i) \leq Z(\widehat{G}_i)$.

By Bender's theorem [Be, Satz 1], $\widehat{G}_i/O_{2'}(\widehat{G}_i) \cong SL_2(2^r)$ where $r \geq 2$ is as in (1). Then $\widehat{G}_i \cong SL_2(2^r)$, since by [Sch, p. 119, Satz IX], the Schur multiplier of $SL_2(2^r)$ has order 2 (if $r = 2$) or 1 (if $r \geq 3$). Hence there are $T_i^* \in \mathrm{Syl}_2(G_i)$ such that $\langle T_i, T_i^* \rangle = \widehat{G}_i$ and $D_i \stackrel{\text{def}}{=} N_{\widehat{G}_i}(T_i) \cap N_{\widehat{G}_i}(T_i^*) \cong \mathbb{F}_{2^r}^\times$. (Identify $\widehat{G}_i$ with $SL_2(2^r)$ in such a way that T_i is the group of upper triangular matrices with 1's on the diagonal, and let T_i^* be the group of lower triangular matrices with 1's on the diagonal.) Then $C_{P_i}(\widehat{G}_i) = C_{P_i}(T_i) \cap C_{P_i}(T_i^*)$ has index at most 2^{2r} in P_i. Since each faithful $\mathbb{F}_2[\widehat{G}_i]$-module has dimension at least $2r$ (see [OV, Lemma 1.7(a)]), $P_i/C_{P_i}(\widehat{G}_i)$ is $2r$-dimensional and irreducible.

Set $V_i = P_i/C_{P_i}(\widehat{G}_i)$, regarded as an $\mathbb{F}_2[\widehat{G}_i]$-module, and set $K_i = \mathrm{End}_{\mathbb{F}_2[\widehat{G}_i]}(V_i)$. Thus K_i is a finite field extension of $\mathbb{F}_2$, and $\overline{\mathbb{F}}_2 \otimes_{K_i} V_i$ is irreducible as an $\overline{\mathbb{F}}_2[\widehat{G}_i]$-module (see, e.g., [A1, Theorem 26.6.4]). By a theorem of Curtis (see [GLS3, Theorem 2.8.9]), $\dim_{K_i}(C_{V_i}(T_i)) = 1$, so $[K_i{:}\mathbb{F}_2] = \dim_{\mathbb{F}_2}(C_{V_i}(T_i)) \geq r$. Thus $\dim_{K_i}(V_i) = 2r/[K_i{:}\mathbb{F}_2] = 2$ since T_i acts nontrivially on V_i (since its normal closure $\widehat{G}_i$ acts nontrivially), and V_i is the natural $\mathbb{F}_2[SL_2(2^r)]$-module (see [GLS3, Example 2.8.10.b]). In particular, $C_{V_i}(D_i) = 1$, so $C_{P_i}(D_i) = C_{P_i}(\widehat{G}_i)$, and $C_{P_i}(\widehat{H}_i) = C_{P_i}(\widehat{G}_i)$, where $\widehat{H}_i = N_{\widehat{G}_i}(T_i) = T_i D_i$.

Set $H_i^* = \psi_i^{-1}(\widehat{H}_i) \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(S)$ (see Step 2). Set $Q = C_S(H_1^* H_2^*)$, so that

$$Q = C_S(H_1^*) \cap C_S(H_2^*) = C_{P_1}(\widehat{H}_1) \cap C_{P_2}(\widehat{H}_2) = C_{P_1}(\widehat{G}_1) \cap C_{P_2}(\widehat{G}_2)\,.$$

Then Q is normalized by $\mathrm{Aut}_{\mathcal{F}}(S)$ since $H_1^* H_2^* \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(S)$. For $i = 1,2$, Q is normalized by H_i since ψ_i is onto (Step 2), and Q is normalized by $\widehat{G}_i$ since $Q \leq C_{P_i}(\widehat{G}_i)$. Since $G_i = \widehat{G}_i N_{G_i}(T_i) = \widehat{G}_i H_i$ by the Frattini argument, Q is normalized by $G_i = \mathrm{Aut}_{\mathcal{F}}(P_i)$. So $Q \trianglelefteq \mathcal{F}$ by Proposition 1.14 and since $\mathbf{E}_{\mathcal{F}} = \{P_1, P_2\}$, and hence $Q \leq O_2(\mathcal{F}) = 1$.

Set $K = C_{\mathrm{Aut}_{\mathcal{F}}(S)}(Z(S)/[S,S])$. Then $K \geq H_1^* H_2^*$ since, by Step 2, for $i = 1,2$, $\widehat{G}_i$ (hence H_i^*) acts trivially on W_i and $Z(S) = [S,S]W_i$. So $Q[S,S] \geq C_S(K)[S,S] \geq Z(S)$, and thus $[S,S] = Z(S)$ since $Q = 1$. $\square$

Lemma 3.4. *Let S_0 be a finite nonabelian 2-group such that $Z(S_0)$ is cyclic. Let A be a finite abelian 2-group, set $S = S_0 \times A$, and assume $\mathcal{F}$ is a reduced fusion system over S. Set $A^* = [\mathrm{Aut}_{\mathcal{F}}(S), Z(S)]$; thus $\mathrm{Aut}_{\mathcal{F}}(S)$ normalizes A^* and $C_{A^*}(\mathrm{Aut}_{\mathcal{F}}(S)) = 1$. Then $A^* \cong A$, $S = S_0 \times A^*$, and $\Omega_1(A^*) = [\mathrm{Aut}_{\mathcal{F}}(S), \Omega_1(Z(S))]$.*

Proof. Set $Z = Z(S)$, $Z_0 = Z(S_0)$, and $\varGamma = \mathrm{Out}_{\mathcal{F}}(S)$ for short. Thus $\varGamma$ has odd order and acts on Z, so

$$Z_0 \times A = Z = C_Z(\varGamma) \times [\varGamma, Z] = C_Z(\varGamma) \times A^* \tag{2}$$

$$\Omega_1(Z) = C_{\Omega_1(Z)}(\varGamma) \times [\varGamma, \Omega_1(Z)] \tag{3}$$

(see [G, Theorem 5.2.3], and note in particular that $C_{A^*}(\varGamma) = 1$). Since Z_0 is cyclic, $|\Omega_1(Z_0)| = 2$; and since each nontrivial normal subgroup of S_0 intersects nontrivially with Z_0, $\Omega_1(Z_0) \leq [S_0, S_0] = [S, S]$. By [AKO, Corollary I.8.5], and since $\mathfrak{foc}(\mathcal{F}) = S$ by Lemma 1.15 (recall $O^2(\mathcal{F}) = \mathcal{F}$), $\varGamma$ acts with trivial centralizer on

$$\Omega_1(Z)\big/\big(\Omega_1(Z) \cap [S, S]\big) = \Omega_1(Z)/\Omega_1(Z_0)\,.$$

Hence $\Omega_1(Z_0) = C_{\Omega_1(Z_0)}(\varGamma) = C_{\Omega_1(Z)}(\varGamma) = \Omega_1(C_Z(\varGamma))$. Since $[\varGamma, \Omega_1(Z)] \leq \Omega_1(A^*)$, and both are complements to $\Omega_1(Z_0)$ in $\Omega_1(Z)$ by (2) and (3), we have $\Omega_1(A^*) = [\varGamma, \Omega_1(Z)]$.

Since $\Omega_1(C_Z(\varGamma)) = \Omega_1(Z_0)$, and $C_Z(\varGamma) \cap A^* = 1 = Z_0 \cap A$ by (2), we also have $C_Z(\varGamma) \cap A = 1 = Z_0 \cap A^*$ since neither intersection has any elements of order 2. Thus $|A| = |A^*|$, and hence $Z = Z_0 \times A^*$ and $A \cong Z/Z_0 \cong A^*$. Also, $S_0 \cap A^* = 1$, so $S = S_0 \times A^*$. □

Recall that the *rank* of a finite p-group is the largest rank of any of its abelian subgroups.

Lemma 3.5. *Let S_0 be a finite nonabelian 2-group such that $Z(S_0)$ is cyclic. Let $A \neq 1$ be a nontrivial finite abelian 2-group, set $S = S_0 \times A$, and assume $\mathcal{F}$ is a reduced fusion system over S. Then for each $1 \neq \widehat{W} \leq \Omega_1(Z(S))$ which is normalized by $\mathrm{Aut}_{\mathcal{F}}(S)$, there is an $\mathcal{F}$-essential subgroup $P \leq S$ such that*

(a) *$P = C_S(\Omega_1(Z(P)))$ and $\Omega_1(Z(P))$ is fully normalized in $\mathcal{F}$;*

(b) *$\mathrm{Aut}_{\mathcal{F}}(P)(\widehat{W}) \neq \widehat{W}$; and*

(c) *for all $Q \leq S$ such that $|Q| > |P|$, and all $\beta \in \mathrm{Hom}_{\mathcal{F}}(Q, S)$, $\beta(\widehat{W}) = \widehat{W}$.*

Set $W = [\mathrm{Aut}_{\mathcal{F}}(S), \Omega_1(Z(S))] \neq 1$, and let P be a subgroup satisfying (a)–(c) *with $\widehat{W} = W$. Set $V = \Omega_1(Z(P))$.*

(d) *If there is $\varGamma \leq \mathrm{Aut}_{\mathcal{F}}(S)$ such that $C_W(\varGamma) = 1$ and $\varGamma(P) = P$, then either*
   - (i) $\mathrm{rk}(P/A) \geq 6$, *or*
   - (ii) $\mathrm{rk}(C_V(N_S(P))/W) \geq 3$ *and $\varGamma$ acts nontrivially on V/W.*

Proof. **(a,b,c)** Since $\mathcal{F}$ is reduced, $\widehat{W}$ cannot be normal in $\mathcal{F}$. Since $\widehat{W} \leq Z(S)$, it is contained in every $\mathcal{F}$-essential subgroup. Hence by Proposition 1.14 (and since $\mathrm{Aut}_{\mathcal{F}}(S)$ normalizes $\widehat{W}$), there is some $P \in \mathbf{E}_{\mathcal{F}}$ and some $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$ such that $\alpha(\widehat{W}) \neq \widehat{W}$. Choose P to have maximal order among all such $\mathcal{F}$-essential subgroups. Then (c) holds by Proposition 1.3.

Set $V = \Omega_1(Z(P))$. Thus $\widehat{W} \leq \Omega_1(Z(S)) \leq V$. If V is not fully normalized, then there is $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_S(V), S)$ such that $\varphi(V)$ is fully normalized (cf. [AKO, Lemma I.2.6(c)]). Also, $N_S(V) \geq N_S(P) > P$, so $\varphi(\widehat{W}) = \widehat{W}$ by (c), and $\varphi(P)$ is fully normalized (hence $\mathcal{F}$-essential) since P is. Upon replacing P by $\varphi(P)$, we can assume V is fully normalized (and hence fully centralized).

Let $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$ be as above: $\alpha(\widehat{W}) \neq \widehat{W}$. Since V is fully centralized, $\alpha|_V \in \mathrm{Aut}_{\mathcal{F}}(V)$ extends to some $\overline{\alpha} \in \mathrm{Aut}_{\mathcal{F}}(C_S(V))$. If $C_S(V) > P$, then $\overline{\alpha}(\widehat{W}) = \widehat{W}$ by (c), contradicting our assumption on α. Hence $C_S(V) = P$, and in particular, $N_S(V) = N_S(P)$.

(d) By Lemma 3.4, we can assume that $A = [\mathrm{Aut}_{\mathcal{F}}(S), Z(S)]$ and hence is normalized by $\mathrm{Aut}_{\mathcal{F}}(S)$, and that $W = \Omega_1(A) \neq 1$. Fix $V \leq P \in \mathbf{E}_{\mathcal{F}}$ which satisfy conditions (a)–(c) for $\widehat{W} = W$. Let $\varGamma \leq \mathrm{Aut}_{\mathcal{F}}(S)$ be such that $C_W(\varGamma) = 1$ and $\varGamma(P) = P$. Assume $\mathrm{rk}(V/W) \leq 5$ (otherwise (i) holds).

Set $G = \mathrm{Out}_{\mathcal{F}}(P)$ and $T = \mathrm{Out}_S(P) \in \mathrm{Syl}_2(G)$, and let $H \leq G$ be the subgroup generated by the classes of all $\beta \in \mathrm{Aut}_{\mathcal{F}}(P)$ which extend to $\mathcal{F}$-morphisms between subgroups strictly containing P. By [AKO, Proposition I.3.3(b)] and since $P \in \mathbf{E}_{\mathcal{F}}$, $H < G$ and is strongly 2-embedded in G. Since $H \geq T$ by definition, $H \geq N_G(T)$.

Regard $V = \Omega_1(Z(P))$ as an $\mathbb{F}_2[G]$-module. By (c), W is an $\mathbb{F}_2[H]$-submodule of $V|_H$. Set $V_0 = V \cap S_0$. Then $V = V_0 \times W$ since $P = (P \cap S_0) \times A$, and V_0 is acted upon by $T = \mathrm{Out}_{S_0}(P)$. Hence W is a direct factor of $V|_H$ as $\mathbb{F}_2[H]$-modules, since the $\mathbb{F}_2[T]$-linear projection $V \longrightarrow W$ (with kernel V_0) can be made $\mathbb{F}_2[H]$-linear by averaging over the cosets in H/T.

We are thus in the situation of Lemma 3.2. By that lemma, there is an $\mathbb{F}_2[G]$-submodule $V^* \leq V$ such that $V^*|_H \cong W \times X$ for some $\mathbb{F}_2[H]$-module X that is free as an $\mathbb{F}_2[T]$-module, and such that $X = 1$, or $|T| = 2$ and $\mathrm{rk}(X) \geq 4$, or $\mathrm{rk}(X) \geq 8$. Since $\mathrm{rk}(X) \leq \mathrm{rk}(V/W) \leq 5$, the last condition is impossible.

Assume $X = 1$ (i.e., $V^*|_H \cong W$); we show that (ii) holds. Since $\varGamma(P) = P$, we have $[\gamma|_P] \in N_H(T)$ for each $\gamma \in \varGamma$, and hence the action of $\varGamma$ on V factors through a subgroup $\overline{\varGamma} \leq N_H(T)$. Then V^* and W are isomorphic as $\mathbb{F}_2[\overline{\varGamma}]$-modules. If (ii) does not hold, then either $\overline{\varGamma}$ acts trivially on V/W or $\mathrm{rk}(C_V(T)/W) \leq 2$. In the latter case, $\overline{\varGamma}$ acts trivially on $C_V(T)/W$ since it normalizes the subspace $\Omega_1(Z(S))/W \leq C_V(T)/W$ of rank 1, and since the action of $\overline{\varGamma}$ factors through $\varGamma^* = \overline{\varGamma}T/T \leq N_H(T)/T$ of odd order. So in either case, $\varGamma$ acts trivially on $C_V(T)/W$. Also, $C_V(T) = V_1 \times V_2$, where $V_1 = C_{C_V(T)}(\varGamma^*)$ and $V_2 = [\varGamma^*, C_V(T)]$ (cf. [G, Theorem 5.2.3]). Then $W \geq V_2$ since $C_V(T)/W$ has trivial action, and $W = V_2$ since $C_W(\varGamma) = 1$. Also, $V^* \leq C_V(T)$ and $V^* \cap V_1 = C_{V^*}(\overline{\varGamma}) = 1$ since $V^* \cong W$ as $\mathbb{F}_2[\overline{\varGamma}]$-modules, so $V^* = V_2 = W$ since they have the same rank. This is impossible, since $G(W) \neq W$ by (b) while $G(V^*) = V^*$.

Now assume $\mathrm{rk}(X) \geq 4$ and $|T| = |N_S(P)/P| = 2$. Recall that we set $V_0 = V \cap S_0$, so that $V = V_0 \times W$ as $\mathbb{F}_2[T]$-modules. Thus $\mathrm{rk}(V_0) = \mathrm{rk}(V/W) \geq \mathrm{rk}(X)$, and so $4 \leq \mathrm{rk}(V_0) \leq 5$. We have $V^*|_H \cong W \times X$ where $X|_T \cong \mathbb{F}_2[T]^k$ for some k, and $k = 2$ by the assumptions on ranks. Hence $\mathrm{rk}(V^*/C_{V^*}(T)) = 2$, and so $\mathrm{rk}(V_0/C_{V_0}(T)) = \mathrm{rk}(V/C_V(T)) \geq 2$. Since $|T| = 2$,

$$2 \leq \mathrm{rk}(V_0/C_{V_0}(T)) = \mathrm{rk}([T, V_0]) \leq \mathrm{rk}(C_{V_0}(T)).$$

If $P \trianglelefteq S$, i.e., if $|S/P| = 2$, then $\Omega_1(Z(S_0)) = C_{V_0}(T)$ has rank at least 2, which is impossible since $Z(S_0)$ is assumed to be cyclic.

Thus $P \ntrianglelefteq S$. Fix $x \in N_S(P) \smallsetminus P$. Let $\widetilde{P} \neq P$ be another subgroup S-conjugate to P with the same normalizer $P\langle x \rangle$, and set $\widetilde{V} = \Omega_1(Z(\widetilde{P}))$ and $\widetilde{V}_0 = \widetilde{V} \cap S_0$. Then $V \cap \widetilde{V} = \Omega_1(Z(P\langle x \rangle)) = C_V(x)$, so $V_0 \cap \widetilde{V}_0 = C_{V_0}(x) = C_{V_0}(T)$, and we just saw that $\mathrm{rk}(V_0/C_{V_0}(T)) \geq 2$. If $\widetilde{V} \leq P$, then $[V, \widetilde{V}] \leq [V, P] = 1$ and $\mathrm{rk}(V_0\widetilde{V}_0) = \mathrm{rk}(\widetilde{V}_0) + \mathrm{rk}(V_0/C_{V_0}(T)) \geq 4 + 2 = 6$, so (i) holds. If $\widetilde{V} \nleq P$, then we can assume x was chosen so that $x \in \widetilde{V}$. But then $[\widetilde{P}, x] = 1$ and hence $V \cap \widetilde{P} \leq C_V(x)$, which is impossible since $|V/(V \cap \widetilde{P})| \leq |P\langle x \rangle/\widetilde{P}| = 2$ and $|V/C_V(x)| \geq 4$. $\square$

Proposition 3.6. *Let S_0 be a finite 2-group such that $\mathrm{rk}(S_0) \le 5$, $Z(S_0)$ is cyclic, and $\mathrm{Aut}(S_0)$ is a 2-group. Then for any finite abelian 2-group $A \neq 1$, there are no reduced fusion systems over $S_0 \times A$.*

Proof. We can assume S_0 is nonabelian; otherwise the result is clear. Set $S = S_0 \times A$, and assume $\mathcal{F}$ is a reduced fusion system over S. Using Lemma 3.4, we can assume that $A = [\mathrm{Aut}_{\mathcal{F}}(S), Z(S)]$ and hence is normalized by $\mathrm{Aut}_{\mathcal{F}}(S)$. Set $W = [\mathrm{Aut}_{\mathcal{F}}(S), \Omega_1(Z(S))]$; then $W = \Omega_1(A)$ and $C_W(\mathrm{Aut}_{\mathcal{F}}(S)) = 1$ by Lemma 3.4 again.

Fix $\varGamma \le \mathrm{Aut}_{\mathcal{F}}(S)$ of odd order such that $\mathrm{Inn}(S)\varGamma = \mathrm{Aut}_{\mathcal{F}}(S)$ (by the Schur-Zassenhaus theorem, cf. [G, Theorem 6.2.1(i)]). Then $C_W(\varGamma) = C_W(\mathrm{Aut}_{\mathcal{F}}(S)) = 1$ since $[\mathrm{Inn}(S), W] = 1$. For each $\gamma \in \varGamma$, $\gamma(A) = A$, so γ induces an automorphism of $S/A \cong S_0$ which must be the identity since $\mathrm{Aut}(S_0)$ is a 2-group by assumption. Hence $\gamma(P) = P$ for each $P \le S$ which contains A, and in particular, for each $P \in \mathbf{E}_{\mathcal{F}}$. Also, for each such P, $\varGamma$ acts trivially on $Z(P)/A$, and hence on $\Omega_1(Z(P))/W$.

Lemma 3.5(d) now implies that $\mathrm{rk}(S_0) \ge 6$, which contradicts our assumptions. □

We now give two applications of Lemma 3.5 in situations where $\mathrm{Aut}(S_0)$ is not a 2-group. Recall that $UT_n(q) \le SL_n(q)$ denotes the subgroup of upper triangular matrices over $\mathbb{F}_q$ with 1's on the diagonal.

Proposition 3.7. *Let S_0 be a Sylow 2-subgroup of J_2. Then for any finite abelian 2-group $A \neq 1$, there are no reduced fusion systems over $S_0 \times A$.*

Proof. We adopt the notation used in [OV, § 4] and [O2, § 6]. Let $(a \mapsto \bar{a})$ denote the field involution of $\mathbb{F}_4$ (thus $\bar{a} = a^2$). We identify $S_0 = UT_3(4)\langle\theta\rangle$, where $\theta \left(\begin{smallmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{smallmatrix}\right) \theta^{-1} = \left(\begin{smallmatrix} 1 & \bar{c} & \bar{b} \\ 0 & 1 & \bar{a} \\ 0 & 0 & 1 \end{smallmatrix}\right)^{-1}$ for each $a, b, c \in \mathbb{F}_4$, and $\theta^2 = 1$. For $1 \le i < j \le 3$ and $x \in \mathbb{F}_4$, let $e_{ij}^x \in UT_3(4)$ be the elementary matrix with unique nonzero off-diagonal entry x in position (i, j), and set $E_{ij} = \{e_{ij}^x \,|\, x \in \mathbb{F}_4\}$. Set $A_1 = E_{12}E_{13}$ and $A_2 = E_{13}E_{23}$. These are the only subgroups of S_0 isomorphic to $(C_2)^4$, and each involution in $UT_3(4)$ is in $A_1 \cup A_2$ (see [OV, Lemma 4.1(b,c)]).

Set $S = S_0 \times A$, and let $\mathcal{F}$ be a reduced fusion system over S. Using Lemma 3.4, we can assume that $A = [\mathrm{Aut}_{\mathcal{F}}(S), Z(S)]$, and hence is normalized by $\mathrm{Aut}_{\mathcal{F}}(S)$. Set $W = [\mathrm{Aut}_{\mathcal{F}}(S), \Omega_1(Z(S))]$; then $W = \Omega_1(A)$ and $C_W(\mathrm{Aut}_{\mathcal{F}}(S)) = 1$ by Lemma 3.4 again. Let P be as in Lemma 3.5(a–c) for $\widehat{W} = W$, and set $V = \Omega_1(Z(P))$. In particular, $P \in \mathbf{E}_{\mathcal{F}}$ and $P = C_S(V)$. Set $V_0 = V \cap S_0$ and $P_0 = P \cap S_0$.

If $V_0 \not\le UT_3(4)$, then there is $g \in V_0 \smallsetminus UT_3(4)$ of order two, and $C_{UT_3(4)}(g) \cong Q_8$ with center $\langle e_{13}^1 \rangle$ by [OV, Lemma 4.1(a)]. Hence $V_0 \le \Omega_1(C_{S_0}(g)) = \langle g, e_{13}^1 \rangle$, with equality since $V_0 = \Omega_1(Z(P_0)) \ge Z(S_0) = \langle e_{13}^1 \rangle$. So $P = \langle g \rangle \times C_{UT_3(4)}(g) \times A$. Each class in $N_S(P)/P$ is represented by some $x \in UT_3(4)$, $[x, g] \in Z(P) \cap UT_3(4) = \langle e_{13}^1 \rangle$ since $g \in Z(P)$ and $x \in UT_3(4)$, and $[x, C_{UT_3(4)}(g)] \le E_{13} \cap P = \langle e_{13}^1 \rangle$. Hence $[N_S(P), P] = \langle e_{13}^1 \rangle \le \Phi(P)$, so $P \notin \mathbf{E}_{\mathcal{F}}$ by Proposition 1.5 (with $\Theta = 1$), a contradiction.

Thus $V_0 \le UT_3(4)$. If $V_0 \not\le E_{13} = Z(UT_3(4))$, then there is $g \in (V_0 \cap A_i) \smallsetminus E_{13}$ for $i = 1$ or 2, since all elements of order 2 in $UT_3(4)$ lie in $A_1 \cup A_2$. Hence $P \le C_S(g) = A_i \times A$, with equality since P is centric in S. Set $\varGamma = \{\alpha \in \mathrm{Aut}_{\mathcal{F}}(S) \,|\, \alpha(P) = P\}$: a subgroup of index 2 since A_1, A_2 are the only subgroups of S_0 isomorphic to $(C_2)^4$ (and they are S-conjugate). Then $\mathrm{Aut}_{\mathcal{F}}(S) = \mathrm{Inn}(S)\varGamma$, and $C_W(\varGamma) = C_W(\mathrm{Aut}_{\mathcal{F}}(S)) = 1$ since $[\mathrm{Inn}(S), W] = 1$. So the hypothesis in Lemma 3.5(d) holds, which is impossible since $\mathrm{rk}(P/A) = 4$ and $\mathrm{rk}(C_V(N_S(P))/W) = \mathrm{rk}(C_{A_i}(UT_3(4))) = 2$.

The only remaining case is that where $V_0 = E_{13}$ and $P = UT_3(4)\times A$. This is impossible by Lemma 3.5(d) again (applied with $\Gamma = \mathrm{Aut}_{\mathcal{F}}(S)$), and since $\mathrm{rk}(V_0) = 2$ and $UT_3(4) = A_1A_2$ is characteristic in S_0. □

Proposition 3.8. *For any finite abelian 2-group $A \neq 1$, there are no reduced fusion systems over $UT_4(2) \times A$.*

Proof. Set $S_0 = UT_4(2)$ and $S = S_0 \times A$. We need to use the following properties of S_0 (see, e.g., [O2, Lemma C.4(a)]): there is a unique abelian subgroup $B_0 \leq S_0$ of order 16, $B_0 \cong (C_2)^4$, $B_0 \trianglelefteq S_0$, and $S_0/B_0 \cong C_2 \times C_2$ acts on B_0 by permuting a basis freely.

Assume $\mathcal{F}$ is a reduced fusion system over S. Using Lemma 3.4, we can assume that $A = [\mathrm{Aut}_{\mathcal{F}}(S), Z(S)]$ and hence is normalized by $\mathrm{Aut}_{\mathcal{F}}(S)$.

Set $B = B_0 \times A$. Let P_1, P_2, P_3 be the three subgroups of index 2 in S which contain B. Thus $P_i = B\langle x_i\rangle$, and $Z(P_i) = C_B(x_i)$, for some $x_1, x_2, x_3 \in S_0 \smallsetminus B_0$. We first claim that

$$\mathcal{P} \overset{\text{def}}{=} \bigl\{P \in \mathbf{E}_{\mathcal{F}} \bigm| P = C_S(V) \text{ where } V = \Omega_1(Z(P))\bigr\} \subseteq \{B, P_1, P_2, P_3\}\,. \qquad (4)$$

Assume otherwise: then there is $P \in \mathcal{P}$ such that $P \ngeq B$. Set $P_0 = P \cap S_0$. If P_0 is elementary abelian, then $\mathrm{rk}(P_0) \leq 3$ since B_0 is the unique abelian subgroup of order 16 in S_0, so $\mathrm{rk}([x, P]) = \mathrm{rk}([x, P_0]) = 1$ for $x \in N_S(P) \smallsetminus P$ such that $x^2 \in P$. By Proposition 1.6(b), $|N_{S_0}(P_0)/P_0| = |N_S(P)/P| = 2$. Since P is centric in S, $P \nleq B$. Hence either $P_0 \leq \langle g, B_0\rangle$ for some $g \in P_0 \smallsetminus B_0$, in which case $P_0 = \langle g, C_{B_0}(g)\rangle$ and $B \leq N_S(P)$; or there are $g, h \in P_0 \smallsetminus B_0$ with $gh^{-1} \notin B_0$, in which case $P_0 \cap B_0 \leq C_{B_0}(\langle g, h\rangle) = Z(S_0)$ and $Z_2(S_0) \leq N_S(P)$. Thus $|N_{S_0}(P_0)/P_0| \geq 4$ in all cases, which is a contradiction.

Assume P_0 is not elementary abelian, and set $V_0 = \Omega_1(Z(P_0))$. Then $\mathrm{rk}(V_0) \leq 2$ since B_0 is the only abelian subgroup of S_0 of order 16, $V_0 \nleq B_0$ since $P_0 = C_{S_0}(V_0) \ngeq B_0$, so $V_0 = \langle x, Z(S_0)\rangle$ for some $x \in S_0 \smallsetminus B_0$ of order 2. Then $P_0 = C_{S_0}(V_0) = \langle x, y, C_{B_0}(x)\rangle$ for some $y \in C_{S_0}(x) \smallsetminus B_0\langle x\rangle$, $\langle x, C_{B_0}(x)\rangle \cong (C_2)^3$, and hence $P_0 \cong C_2 \times D_8$ and $[N_S(P), P_0] = Z(S_0) = \Phi(P_0)$, which contradicts Proposition 1.5 (with $\Theta = 1$). This proves (4).

Set

$$\begin{aligned} &V = \Omega_1(B) && G = \mathrm{Aut}_{\mathcal{F}}(B) && T = \mathrm{Aut}_S(B) \in \mathrm{Syl}_2(G) \\ &W = \Omega_1(A) && H = N_G(T) && H_0 = C_G(T)\,. \end{aligned}$$

Regard V as an $\mathbb{F}_2[G]$-module. Since each $\beta \in H = N_G(T)$ extends to some $\overline{\beta} \in \mathrm{Aut}_{\mathcal{F}}(S)$ by the extension axiom, W is an $\mathbb{F}_2[H]$-submodule of $V|_H$. For each $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$, $\alpha(B) = B$ and $\alpha|_B \in H$ since B is the unique abelian subgroup of index 4 in S, and hence $W = [\mathrm{Aut}_{\mathcal{F}}(S), \Omega_1(Z(S))] = [H, \Omega_1(Z(S))]$ and $C_W(H) = C_W(\mathrm{Aut}_{\mathcal{F}}(S)) = 1$ by Lemma 3.4. Also, $V = B_0 \times W$ as $\mathbb{F}_2[T]$-modules, the $\mathbb{F}_2[T]$-linear projection $V \longrightarrow W$ can be made $\mathbb{F}_2[H]$-linear by averaging over cosets in H/T, and hence $V = B_0^* \times W$ for some $\mathbb{F}_2[H]$-submodule B_0^*. Since $B_0^*|_T \cong B_0 \cong \mathbb{F}_2[T]$ as $\mathbb{F}_2[T]$-modules, B_0^* is indecomposable as an $\mathbb{F}_2[T]$-module, and hence also indecomposable as an $\mathbb{F}_2[H]$-module.

In particular, if V is decomposable as an $\mathbb{F}_2[G]$-module, then T acts trivially on all but one of its indecomposable direct factors. Let $\widehat{W}$ be an indecomposable direct factor with trivial T-action; then $\widehat{W} \leq C_V(T) \leq \Omega_1(Z(S))$. Also, $\widehat{W}|_H \cong \widehat{W}^*$ as $\mathbb{F}_2[H]$-modules for some $\widehat{W}^* \leq W$, so $C_{\widehat{W}}(H) = 1$, and hence $\widehat{W} = [H, \widehat{W}] \leq [H, \Omega_1(Z(S))] = W$. Moreover, $\widehat{W}$ is normalized by $\mathrm{Aut}_{\mathcal{F}}(P)$ for each $P \in \{B, P_1, P_2, P_3, S\}$ since B is characteristic in each of those subgroups. But this contradicts Lemma 3.5(a,b) and (4).

Thus V is indecomposable as an $\mathbb{F}_2[G]$-module. Since $V|_H = B_0^* \times W$ where T acts trivially on W and $T \in \mathrm{Syl}_2(G)$, Lemma 3.1(b) implies that W is the $\mathbb{F}_2[H]$-Green correspondent of V and is irreducible as an $\mathbb{F}_2[H/T]$-module. Set $\Gamma = \{\alpha \in \mathrm{Aut}_{\mathcal{F}}(S) \,|\, \alpha|_B \in H_0\}$; we already saw that each element of H_0 extends to an element of Γ. If H_0 acts nontrivially on W (equivalently, if Γ acts nontrivially on W), then $C_W(\Gamma) = C_W(H_0) = 1$ ($C_W(H_0)$ is an $\mathbb{F}_2[H]$-submodule of W since $H_0 \trianglelefteq H$), and $\Gamma(P) = P$ for $P \in \{B, P_1, P_2, P_3\}$ by definition of H_0. This contradicts Lemma 3.5(d) and (4).

Thus H_0 acts trivially on W, while H does not since $W = [H, \Omega_1(Z(S))]$. By definition, $H/H_0 \neq 1$ acts as a group of automorphisms of $T \cong C_2 \times C_2$ of odd order, and hence has order 3. So $\mathrm{rk}(W) = 2$ since W is irreducible. Also, $\mathbb{F}_4 \otimes_{\mathbb{F}_2} W$ is decomposable as an $\mathbb{F}_4[H]$-module, and the two direct factors are Galois conjugate, so $\mathbb{F}_4 \otimes_{\mathbb{F}_2} V$ contains the sum of their Green correspondents (Lemma 3.1(b)) which also are Galois conjugate. Since V is indecomposable, $\mathbb{F}_4 \otimes_{\mathbb{F}_2} V$ is the direct sum of at most two indecomposable submodules, and hence is the sum of exactly two indecomposable modules which are Galois conjugate. In particular, $(\mathbb{F}_4 \otimes_{\mathbb{F}_2} V)|_H$ is the direct sum of at least four indecomposable modules. But this is impossible, since $V|_H \cong B_0^* \times W$, and $\mathbb{F}_4 \otimes_{\mathbb{F}_2} B_0^* \cong \mathbb{F}_4[T]$ as $\mathbb{F}_4[T]$-modules and hence is indecomposable. □

4. 2-Groups of order at most 128

Reduced fusion systems over 2-groups of order at most 64 have already been handled in earlier papers. We begin by listing them. Recall that $UT_n(q)$ denotes the group of upper triangular $n \times n$ matrices over $\mathbb{F}_q$ with 1's on the diagonal.

Theorem 4.1. *Let $\mathcal{F}$ be a reduced, indecomposable fusion system over a nontrivial 2-group S of order $2^k \leq 64$. Then one of the following holds:*

(a) *$S \cong D_{2^k}$ for some $3 \leq k \leq 6$, and $\mathcal{F}$ is isomorphic to the fusion system of $PSL_2(q)$ where $q \equiv 2^k \pm 1 \pmod{2^{k+1}}$.*

(b) *$S \cong SD_{2^k}$ for some $4 \leq k \leq 6$, and $\mathcal{F}$ is isomorphic to the fusion system of $PSL_3(q)$ where $q \equiv 2^{k-2} - 1 \pmod{2^{k-1}}$.*

(c) *$S \cong C_4 \wr C_2$, and $\mathcal{F}$ is isomorphic to the fusion system of $SL_3(5)$.*

(d) *$S \cong UT_4(2)$, and $\mathcal{F}$ is isomorphic to the fusion system of $SL_4(2) \cong A_8$ or of $SU_4(2) \cong PSp_4(3)$.*

(e) *$S \cong UT_3(4)$, and $\mathcal{F}$ is isomorphic to the fusion system of $PSL_3(4)$.*

(f) *S is of type M_{12}, and $\mathcal{F}$ is isomorphic to the fusion system of M_{12} or of $G_2(3)$.*

In all cases, $\mathcal{F}$ is tame. Also, the fusion system of each of the groups listed above is simple.

Proof. If S is dihedral, semidihedral, or a wreath product $C_{2^n} \wr C_2$, then by [AOV1, § 4.1] or [AOV2, Proposition 3.1], $\mathcal{F}$ is isomorphic to the fusion system of $PSL_2(q)$ or $PSL_3(q)$ for appropriate odd q, and we are in one of the cases (a), (b), or (c). If not, then $|S| = 64$ by [AOV2, Theorem 5.3], and so S is isomorphic to $UT_4(2)$ or $UT_3(4)$ or is of type M_{12} by [AOV2, Theorem 5.4]. The reduced fusion systems over these three groups are listed in [O2, Propositions 5.1, 6.4, & 4.2], and we are in the situation of (d), (e), or (f).

Tameness for all of these fusion systems is shown in [BMO2, Theorem C], except for that of M_{12}. When $\mathcal{F} = \mathcal{F}_S(G)$ for $G = M_{12}$, then by [O2, Proposition 4.3(a,b)], the composite $\mu_G \circ \kappa_G$ is an isomorphism from $\mathrm{Out}(G)$ to $\mathrm{Out}(\mathcal{F})$. By [AOV2, Proposition

3.2], there are at most two $\mathcal{F}$-essential subgroups, of which only one (R in the notation of [AOV2] and A_+ in [O2, § 4]) has noncyclic center. Hence $\mathrm{Ker}(\mu_G) = 1$ by Proposition 1.13, κ_G is an isomorphism, and $\mathcal{F}$ is tame.

By [O2, Theorem A], the fusion systems of all of the groups listed above are simple. □

As part of the proof of Theorem 4.1, we have also shown:

Lemma 4.2. *If G is isomorphic to the sporadic simple group M_{12}, then (for $p = 2$) κ_G and μ_G are both isomorphisms. In particular, G tamely realizes its fusion system.*

We now turn to groups of order 128, where we begin our use of a computer search to identify 2-groups which potentially could support a reduced fusion system.

Theorem 4.3. *Let $\mathcal{F}$ be a reduced, indecomposable fusion system over a group S of order 128. Then one of the following holds:*

(a) *$S \cong D_{128}$, and $\mathcal{F}$ is isomorphic to the fusion system of $PSL_2(q)$ where $q \equiv \pm 127$ (mod 256).*

(b) *$S \cong SD_{128}$, and $\mathcal{F}$ is isomorphic to the fusion system of $PSL_3(q)$ where $q \equiv 31$ (mod 64).*

(c) *$S \cong C_8 \wr C_2$, and $\mathcal{F}$ is isomorphic to the fusion system of $SL_3(9)$.*

(d) *$S \cong D_8 \wr C_2$, and $\mathcal{F}$ is isomorphic to the fusion system of A_{10} or of $PSL_4(3)$.*

(e) *S is of type M_{22}, and $\mathcal{F}$ is isomorphic to the fusion system of M_{22}, M_{23}, McL, or $PSL_4(5)$.*

(f) *S is of type J_2, and $\mathcal{F}$ is isomorphic to the fusion system of J_2 or J_3.*

In all cases, $\mathcal{F}$ is tame. Also, the fusion system of each of the groups listed above is simple.

Proof. If S is dihedral, semidihedral, or a wreath product $C_8 \wr C_2$, then by [AOV1, § 4.1] or [AOV2, Proposition 3.1], $\mathcal{F}$ is isomorphic to the fusion system of $PSL_2(q)$ or $PSL_3(q)$ for appropriate odd q, and we are in one of the cases (a), (b), or (c). If not, then it satisfies conditions (a)–(h) in Proposition 2.2. By a computer search based on those criteria, this leaves the six possibilities for S listed below, where $\#(-)$ denotes the Magma/GAP identification number.

Tameness for the fusion systems in (a,b,e,f) is shown in [AOV1, Propositions 4.3–5]. Tameness for the fusion system of A_{10} was shown in [AOV1, Proposition 4.8], and for the other fusion systems in (c,d) in [BMO2, Theorem C]. All of these fusion systems are simple by [O2, Theorem A].

#928 : $S \cong D_8 \wr C_2$. By [O2, Theorem A], $\mathcal{F}$ must be as in case (d).

#931 : S is of type M_{22}. Any reduced fusion system over S is as in (e) by [OV, Theorem 5.11]. (If $\mathcal{F}$ is isomorphic to the fusion system of $G \cong P\Sigma L_3(4)$ or $P\Gamma L_3(4)$, then $\mathfrak{foc}(\mathcal{F}) = S \cap [G, G] < S$ by the focal subgroup theorem [G, Theorem 7.3.4].)

#934 : S is of type J_2. By [OV, Theorem 4.8], $\mathcal{F}$ must be isomorphic to the fusion system of J_2 or of J_3, and so we are in the situation of case (f). (If $\mathcal{F}$ is isomorphic to the fusion system of $G \cong PSL_3(4) \rtimes \langle\theta\rangle$ or $PGL_3(4) \rtimes \langle\theta\rangle$, as defined in that theorem, then $\mathfrak{foc}(\mathcal{F}) = S \cap [G, G] < S$ by the focal subgroup theorem [G, Theorem 7.3.4].)

#1411 : This group has two normal elementary abelian subgroups P_1, P_2 of rank 5, where $S = P_1P_2$ and $Z \overset{\text{def}}{=} Z(S) = P_1 \cap P_2 = [S,S]$. Also, for $i = 1,2$ and $x_i \in P_i \smallsetminus Z$, $C_S(x_i) = P_i$, and thus S satisfies the hypotheses of Proposition 3.3 and of Lemma 4.4 below. Let $z_0 \in Z$ be as in Lemma 4.4.

Let $\mathcal{F}$ be a saturated fusion system over S with $O_2(\mathcal{F}) = 1$. By Proposition 3.3, $\mathbf{E}_{\mathcal{F}} = \{P_1, P_2\}$. For $i = 1,2$, by Lemma A.7 and since $\mathrm{Aut}_S(P_i) \cong C_2 \times C_2$, we have $\mathrm{Aut}_{\mathcal{F}}(P_i) = \Delta_i \times H_i$ for some $\Delta_i \cong A_5$ and some H_i of odd order. By Lemma A.8(b), and since $\mathrm{rk}([x,P_i]) = 2$ for $x \in S \smallsetminus P_i$ and $[S,P_i] = C_{P_i}(S)$ has rank 3, $C_{P_i}(\Delta_i) = C_{P_i}(N_{\Delta_i}(\mathrm{Aut}_S(P_i)))$ has rank 1 (and is contained in $Z = Z(S)$). For $\alpha \in N_{\Delta_i}(\mathrm{Aut}_S(P_i)) \cong A_4$ of order 3, α extends to an automorphism of S by the extension axiom, so $C_Z(\alpha) = \langle z_0 \rangle$ by Lemma 4.4 and hence $C_{P_i}(\Delta_i) = \langle z_0 \rangle$. By the same lemma, $z_0 \in C_S(\mathrm{Aut}_{\mathcal{F}}(S))$. So $\langle z_0 \rangle \trianglelefteq \mathcal{F}$ by Proposition 1.14, which contradicts the assumption that $O_2(\mathcal{F}) = 1$. In particular, there are no reduced fusion systems over S.

#2011 : $S \cong D_8 \times D_{16}$. By [O1, Theorem B], $\mathcal{F}$ is decomposable.

#2013 : $S \cong D_8 \times SD_{16}$. By [O1, Theorem B], $\mathcal{F}$ is decomposable. □

The following lemma was needed in the above proof (see group #1411).

Lemma 4.4. *Let S be a group of order 128 generated by two normal elementary abelian subgroups $P_1, P_2 \trianglelefteq S$ of rank 5, where $Z(S) = P_1 \cap P_2 = [S,S]$ has rank 3. Assume also that $[x_1, x_2] \neq 1$ for $x_i \in P_i \smallsetminus Z(S)$ ($i = 1,2$). Then there exists a unique involution $z_0 \in Z(S)$ which is not a commutator. Moreover, for each $\alpha \in \mathrm{Aut}(S)$, $\alpha(z_0) = z_0$, and $C_{Z(S)}(\alpha) = \langle z_0 \rangle$ if for $i = 1$ or $i = 2$, α normalizes P_i and induces an automorphism of order 3 on P_i/Z.*

Proof. Set $Z = Z(S)$. Fix representatives $g_1, g_2, g_3 \in P_1 \smallsetminus Z$ and $h_1, h_2, h_3 \in P_2 \smallsetminus Z$ for the nontrivial cosets in P_1/Z and P_2/Z. For each $i, j \in \{1,2,3\}$, $1 \neq [g_i, h_j] \in Z$, and these nine involutions generate $[S,S] = Z$. Since $|Z \smallsetminus 1| = 7$, there must be two distinct pairs of indices (i,j) and (k,ℓ) such that $[g_i, h_j] = [g_k, h_\ell]$. If $i = k$, then $[g_i, h_jh_\ell] = [g_i, h_j][g_i, h_\ell] = 1$, which is impossible since $h_jh_\ell \notin Z$. So $i \neq k$, $j \neq \ell$ by a similar argument, and without loss of generality, we can assume that $(i,j) = (1,1)$ and $(k,\ell) = (2,2)$. Set $z_1 = [g_1, h_1] = [g_2, h_2]$, and also

$$z_2 = [g_1g_2, h_1] = [g_2, h_1h_2] \quad \text{and} \quad z_3 = [g_1, h_1h_2] = [g_1g_2, h_2].$$

Thus $[g_1, P_2] = \langle z_1, z_3 \rangle$, $[g_2, P_2] = \langle z_1, z_2 \rangle$, $[g_3, P_2] = [g_1g_2, P_2] = \langle z_2, z_3 \rangle$, and $Z = [S,S] = \langle z_1, z_2, z_3 \rangle$. Set $z_0 = z_1z_2z_3$: the unique involution in Z that is not a commutator. Then z_0 is fixed by each $\alpha \in \mathrm{Aut}(S)$. If α acts with order 3 on P_1/Z, then it normalizes P_2 since P_1 and P_2 are the only elementary abelian subgroups of rank 5, so α permutes cyclically the subgroups $[g_i, P_2]$ for $i = 1,2,3$, and hence $C_Z(\alpha) = \langle z_0 \rangle$. A similar argument applies if α acts with order 3 on P_2/Z. □

5. Groups of order 256

The following notation is used in this section to describe certain semidirect products $H \rtimes K$. A superscript λ (for $\lambda \in \mathbb{Z}$) over the "$\rtimes$" means that one generator of K acts on H via $(g \mapsto g^\lambda)$, while a superscript "t" means that a generator acts by exchanging two factors (or central factors) of H. Thus, for example, in (d) below, $(C_8 \times C_8) \overset{-1,t}{\rtimes} (C_2 \times C_2)$

means that one of the factors C_2 acts by inverting $C_8 \times C_8$, while the other acts by exchanging the two C_8's.

Whenever we list potentially critical subgroups of S (Definition 2.4), they were found using computer computations based on the criteria in Proposition 2.3.

Theorem 5.1. *Let $\mathcal{F}$ be a reduced, indecomposable fusion system over a group S of order 256. Then one of the following holds:*

(a) *$S \cong D_{256}$, and $\mathcal{F}$ is isomorphic to the fusion system of $PSL_2(q)$ where $q \equiv \pm 255$ (mod 512).*

(b) *$S \cong SD_{256}$, and $\mathcal{F}$ is isomorphic to the fusion system of $PSL_3(q)$ where $q \equiv 63$ (mod 128).*

(c) *$S \cong (Q_{16} \times_{C_2} Q_{16}) \overset{t}{\rtimes} C_2$, and $\mathcal{F}$ is isomorphic to the fusion system of $PSp_4(7)$.*

(d) *$S \cong (C_8 \times C_8) \overset{-1,t}{\rtimes} (C_2 \times C_2)$, and $\mathcal{F}$ is isomorphic to the fusion system of $G_2(7)$.*

(e) *S is of type Ly, and $\mathcal{F}$ is isomorphic to the fusion system of Lyons' sporadic group.*

(f) *S is of type $Sp_4(4)$, and $\mathcal{F}$ is isomorphic to the fusion system of $Sp_4(4)$.*

In all cases, $\mathcal{F}$ is tame. Also, the fusion system of each of the groups listed above is simple.

Proof. If S is dihedral or semidihedral, then by [AOV1, § 4.1], $\mathcal{F}$ is isomorphic to the fusion system of $PSL_2(q)$ or $PSL_3(q)$ for appropriate odd q, and we are in one of cases (a) or (b). If not, then since a wreath product of the form $C_{2^n} \wr C_2$ cannot have order 2^8, S satisfies conditions (a)–(h) in Proposition 2.2. By a computer search based on those criteria, S must be one of the 18 groups listed below, where $\#(-)$ denotes the Magma/GAP identification number.

All of the fusion systems listed in the theorem are tame by [BMO2, Theorem C], except for the fusion system of Lyons's sporadic group, whose tameness is shown separately (see group #6665 below). If $\mathcal{F}$ is the fusion system of $Sp_4(4)$, then $\mathcal{F}$ is simple by Proposition 1.17(d). All of the other fusion systems listed above are simple by [O2, Theorem A].

#12955, **#12957**, **#12965**, **#15421**, **#26833**, **#26835**, **#55683 :** In the first three cases, $S \cong D_{16} \times D_{16}$, $D_{16} \times SD_{16}$, or $SD_{16} \times SD_{16}$, respectively. In the last four cases, $S \cong D_8 \times S_2$ where $S_2 \cong C_4 \wr C_2$, D_{32}, SD_{32}, or $C_2 \times C_2 \times D_8$, respectively. All of these groups satisfy the hypotheses of [O1, Theorem B], and hence every reduced fusion system over any of them is decomposable.

#5298 : $S \cong (C_8 \times C_8) \overset{-1,t}{\rtimes} (C_2 \times C_2)$. By [O2, Proposition 4.2(c)], every reduced fusion system over S is isomorphic to the fusion system of $G_2(7)$.

#5352 : $S \cong (C_8 \times C_8) \overset{3,t}{\rtimes} (C_2 \times C_2)$. By [O2, Proposition 4.2(a)], there are no reduced fusion systems over S.

#6331 : Here, $S = V_1V_2\langle x\rangle$, where $V_1, V_2 \trianglelefteq S$ are elementary abelian of rank 5, $\widehat{Z} \overset{\text{def}}{=} Z(V_1V_2) = V_1 \cap V_2 = [V_1, V_2]$ has rank 3, $[v_1, v_2] \neq 1$ for $v_i \in V_i \smallsetminus \widehat{Z}$, $x^2 = 1$, and $[x, V_i] \not\leq \widehat{Z}$ for $i = 1, 2$. In particular, V_1V_2 satisfies the hypotheses of Lemma 4.4, and $S/V_i \cong D_8$ for $i = 1, 2$. The group S is, in fact, a Sylow 2-subgroup of $2{\cdot}M_{22}$, although we do not use that here.

For $i=1,2$, choose $u_i\in[x,V_i]\smallsetminus\widehat{Z}$. Thus $[x,u_i]=1$ since $x^2=1$, and $u_{3-i}V_i$ generates the center of $S/V_i\cong D_8$. So the conjugation morphisms c_x and $c_{u_{3-i}}$ commute in $\mathrm{Aut}(V_i)$. If $[x,\widehat{Z}]=1$, then for $v_2\in V_2\smallsetminus\widehat{Z}\langle u_2\rangle$, $[{}^xv_2,u_1]={}^x[v_2,u_1]=[v_2,u_1]$, so $[u_2,u_1]=[[x,v_2],u_1]=1$, which contradicts the above remarks about V_1 and V_2. Thus $[x,\widehat{Z}]\neq1$, and since $\mathrm{rk}(\widehat{Z})=3$ and $x^2=1$, $[x,\widehat{Z}]<C_{\widehat{Z}}(x)$ have rank 1 and 2, respectively. Since each nontrivial coset in S/V_i is represented (up to conjugacy) by an element of V_{3-i}, or by x, or by some g with $g^2\in u_{3-i}V_i$, we have $\mathrm{rk}(C_{V_i}(g))\le3$ for $g\in S\smallsetminus V_i$ $(i=1,2)$. Also, $C_{V_i}(\langle x,u_{3-i}\rangle)=C_{\widehat{Z}}(x)$ has rank 2, and so

$$V\le S \text{ elementary abelian of rank 5} \qquad\Longrightarrow\qquad V=V_1 \text{ or } V_2. \tag{1}$$

For $i=1,2$, define

$$Q_i=V_i\langle x\rangle \qquad\text{and}\qquad R_i=N_S(Q_i)=V_i\langle x,u_{3-i}\rangle\,.$$

By computer computations, there are five S-conjugacy classes of potentially critical subgroups in S: V_1V_2, subgroups S-conjugate to the Q_i, and the R_i. (It is easy to see that these are the only potentially critical subgroups containing V_1 or V_2; the hard part is to show that there are no others.)

Assume there is a reduced fusion system $\mathcal{F}$ over S. Then the following hold.

(i) For $i=1,2$, V_i is normalized by $\mathrm{Aut}_{\mathcal{F}}(V_1V_2)$ and by $\mathrm{Aut}_{\mathcal{F}}(S)=\mathrm{Inn}(S)$:
By (1), $\alpha(V_i)=V_i$ for each $\alpha\in\mathrm{Aut}_{\mathcal{F}}(V_1V_2)$ of odd order. This also holds for $\alpha\in\mathrm{Aut}_S(V_1V_2)$ since $V_i\trianglelefteq S$. The Sylow axiom implies that $\mathrm{Aut}_S(V_1V_2)$ is a Sylow 2-subgroup of $\mathrm{Aut}_{\mathcal{F}}(V_1V_2)$, so $\alpha(V_i)=V_i$ for all $\alpha\in\mathrm{Aut}_{\mathcal{F}}(V_1V_2)$.
Each $\alpha\in\mathrm{Aut}_{\mathcal{F}}(S)$ restricts to an element of $\mathrm{Aut}_{\mathcal{F}}(V_1V_2)$, hence normalizes V_1 and V_2, and normalizes each of the subgroups in the chain
$$\Phi(S)=\widehat{Z}\langle u_1,u_2\rangle<V_1\langle u_2\rangle<V_1V_2<S\,.$$
Since each of these has index 2 in the following, $\mathrm{Aut}_{\mathcal{F}}(S)$ is a 2-group by Lemma A.1, and hence $\mathrm{Aut}_{\mathcal{F}}(S)=\mathrm{Inn}(S)$ by the Sylow axiom.

(ii) For $i=1,2$, if $R_i\in\mathbf{E}_{\mathcal{F}}$, then $Q_i\notin\mathbf{E}_{\mathcal{F}}$, and $Z(R_i)=C_{\widehat{Z}}(x)=Z(S)$ is centralized by $\mathrm{Aut}_{\mathcal{F}}(R_i)$:
To see the last claim, note first that $Z(R_i)=C_{\widehat{Z}}(x)\cong C_2\times C_2$. If $\alpha\in\mathrm{Aut}_{\mathcal{F}}(R_i)$, then $\alpha(Z(R_i))=Z(R_i)$, and $\alpha|_{Z(R_i)}=\alpha|_{Z(S)}$ extends to an automorphism in $\mathrm{Aut}_{\mathcal{F}}(S)$ by the extension axiom. Since $\mathrm{Aut}_{\mathcal{F}}(S)=\mathrm{Inn}(S)$ by (i), $\alpha|_{Z(R_i)}=\mathrm{Id}$.
Consider the following chain of subgroups characteristic in R_i:
$$[R_i,R_i]=[u_{3-i},V_i]\langle u_i\rangle<[R_i,R_i]Z(R_i)<V_i<R_i.$$
Each of the first two inclusions is of index 2, while $R_i/V_i\cong C_2\times C_2$. By Lemma A.1, the kernel of the natural homomorphism from $\mathrm{Aut}(R_i)$ to $\mathrm{Aut}(R_i/V_i)\cong\Sigma_3$ is a 2-group, and since $\mathrm{Out}_{\mathcal{F}}(R_i)$ has a strongly 2-embedded subgroup, it must be isomorphic to Σ_3. Hence $Q_i=V_i\langle x\rangle$ is $\mathcal{F}$-conjugate to $V_i\langle u_{3-i}\rangle\trianglelefteq S$, so Q_i is not fully normalized in $\mathcal{F}$, and hence is not in $\mathbf{E}_{\mathcal{F}}$.

(iii) For $i=1,2$, either $R_i\in\mathbf{E}_{\mathcal{F}}$ or $Q_i\in\mathbf{E}_{\mathcal{F}}$, but not both:
If neither $R_i\notin\mathbf{E}_{\mathcal{F}}$ nor $Q_i\notin\mathbf{E}_{\mathcal{F}}$, then $V_{3-i}\trianglelefteq\mathcal{F}$ by Proposition 1.14, since V_{3-i} is characteristic in Q_{3-i} and R_{3-i} by (1), and is normalized by $\mathrm{Aut}_{\mathcal{F}}(V_1V_2)$ and $\mathrm{Aut}_{\mathcal{F}}(S)$ by (i). By (ii), R_i and Q_i cannot both be in $\mathbf{E}_{\mathcal{F}}$.

(iv) Either $R_1\in\mathbf{E}_{\mathcal{F}}$ or $R_2\in\mathbf{E}_{\mathcal{F}}$:
By (1), for $i=1,2$, V_i is characteristic of index two in Q_i, so $[\mathrm{Aut}_{\mathcal{F}}(Q_i),Q_i]\le V_i$. Recall that $\mathrm{Aut}_{\mathcal{F}}(S)=\mathrm{Inn}(S)$ by (i). Hence if $R_1\notin\mathbf{E}_{\mathcal{F}}$ and $R_2\notin\mathbf{E}_{\mathcal{F}}$, then

$\langle[\mathrm{Aut}_{\mathcal{F}}(P),P]\,\big|\,P\in\mathbf{E}_{\mathcal{F}}\rangle\leq V_1V_2<S$, so $O^2(\mathcal{F})\neq\mathcal{F}$ by Lemma 1.15, contradicting the assumption that $\mathcal{F}$ is reduced.

(v) If $R_1,R_2\in\mathbf{E}_{\mathcal{F}}$, then $O_2(\mathcal{F})\neq 1$:
By (ii), $\mathbf{E}_{\mathcal{F}}\subseteq\{R_1,R_2,V_1V_2\}$, and $\mathrm{Aut}_{\mathcal{F}}(R_i)$ centralizes $Z(S)$ for $i=1,2$. By Lemma 4.4, there is $1\neq z_0\in C_{\widehat{Z}}(\mathrm{Aut}_{\mathcal{F}}(V_1V_2))\leq C_{\widehat{Z}}(\mathrm{Aut}_{\mathcal{F}}(S))$, and in particular, $z_0\in Z(S)$. Hence $1\neq\langle z_0\rangle\trianglelefteq\mathcal{F}$ by Proposition 1.14, and $O_2(\mathcal{F})\neq 1$.

(vi) If $Q_i\in\mathbf{E}_{\mathcal{F}}$ ($i=1$ or 2), then there is $1\neq W_i\leq Z(S)$ that is normalized by $\mathrm{Aut}_{\mathcal{F}}(V_i)$:
By Lemma A.5, and since $2\big||\mathrm{Aut}_{\mathcal{F}}(V_i)|$ and $\mathrm{Aut}_{\mathcal{F}}(V_i)<\mathrm{Aut}(V_i)$ by the Sylow axiom, there is a proper subgroup $1\neq W_i<V_i$ which is normalized by $\mathrm{Aut}_{\mathcal{F}}(V_i)$. As an $\mathbb{F}_2[S/V_i]$-module, V_i/W_i surjects onto C_2 with the trivial action. Hence $W_i\leq[S,V_i]=\widehat{Z}\langle u_i\rangle$ since $\widehat{Z}\langle u_i\rangle$ has index 2 in V_i.
We have seen that $\Phi(Q_i)=[x,V_i]$ and $Z(Q_i)=C_{V_i}(x)$ have rank 2 and 3, respectively, and $[x,V_i]<C_{V_i}(x)$ since c_x has order 2 in $\mathrm{Aut}(V_i)$. So by Lemma A.1 again, applied to the chain $\Phi(Q_i)<Z(Q_i)<V_i<Q_i$, each $\mathrm{Id}\neq\alpha\in\mathrm{Aut}_{\mathcal{F}}(Q_i)$ of odd order acts irreducibly on $V_i/Z(Q_i)\cong C_2\times C_2$. Hence $W_iZ(Q_i)=Z(Q_i)$ or V_i. Since $W_iZ(Q_i)\leq\widehat{Z}\langle u_i\rangle<V_i$, we conclude that $W_i\leq Z(Q_i)=C_{V_i}(x)$.
Since W_i is normalized by $\mathrm{Aut}_S(V_i)\leq\mathrm{Aut}_{\mathcal{F}}(V_i)$, it is normal in S. The cosets xV_i and $u_{3-i}xV_i$ are conjugate in $S/V_i\cong D_8$, and hence $W_i\leq C_{V_i}(\langle x,u_{3-i}\rangle)=C_{\widehat{Z}}(x)=Z(S)$.

(vii) If $Q_i,R_{3-i}\in\mathbf{E}_{\mathcal{F}}$ for $i=1$ or $i=2$, then $O_2(\mathcal{F})\neq 1$:
By (vi), there is $1\neq W_i\leq Z(S)$ that is normalized by $\mathrm{Aut}_{\mathcal{F}}(V_i)$, and hence by $\mathrm{Aut}_{\mathcal{F}}(Q_i)$. Also, W_i is normalized by $\mathrm{Aut}_{\mathcal{F}}(V_1V_2)$ by (i), and is centralized by $\mathrm{Aut}_{\mathcal{F}}(R_{3-i})$ by (ii). Since each $\mathcal{F}$-essential subgroup is S-conjugate to V_1V_2, Q_i, or R_{3-i} by (iii) (and since $\mathrm{Aut}_{\mathcal{F}}(S)=\mathrm{Inn}(S)$), $W_i\trianglelefteq\mathcal{F}$ by Proposition 1.14, and hence $O_2(\mathcal{F})\neq 1$.

By (iii) and (iv), either R_1 and R_2 are both essential, or R_i and Q_{3-i} are essential for $i=1$ or 2. Hence $O_2(\mathcal{F})\neq 1$ by (v) or (vii), and $\mathcal{F}$ is not reduced.

#6661 : $S\cong(Q_{16}\times_{C_2}Q_{16})\overset{t}{\rtimes}C_2$. By [O2, Proposition 5.6], every reduced fusion system over S is isomorphic to the fusion system of $PSp_4(7)$.

#6662 : $S\cong(SD_{16}\times_{C_2}SD_{16})\overset{t}{\rtimes}C_2$. By [O2, Proposition 5.6], there are no reduced fusion systems over S.

#6665 : S is of type Ly. Let G be Lyons's group, assume $S\in\mathrm{Syl}_2(G)$, and set $\mathcal{F}=\mathcal{F}_S(G)$. By [O2, Proposition 6.6], each reduced fusion system over S is isomorphic to $\mathcal{F}$. By the same proposition, there is exactly one $\mathcal{F}$-essential subgroup with noncyclic center, and $\mathrm{Out}(\mathcal{F})=1$. Hence $\mu_G\colon\mathrm{Out}(BG^\wedge_2)\longrightarrow\mathrm{Out}(\mathcal{F})$ is injective by Proposition 1.13, so $\mathrm{Out}(BG^\wedge_2)=1$, and $\mathcal{F}$ is tame. In fact, since $\mathrm{Out}(G)=1$ by [Ly1, Proposition 5.8], κ_G and μ_G are both isomorphisms.

#6666 : S is the nonsplit extension of $UT_3(4)$ by the group $\langle\phi,\tau\rangle$ of field and graph automorphisms (denoted $S^*_{\phi,\tau}$ in [O2, p. 71]). By [O2, Proposition 6.6], there are no reduced fusion systems over S.

#8935 : In this case, $S=P_1P_2$, where P_1 and P_2 are elementary abelian subgroups of rank 6 such that $Z(S)=P_1\cap P_2=[S,S]$ has rank 4. By Proposition 5.3 below, every reduced fusion system over S is isomorphic to the fusion system of $Sp_4(4)$.

#53366, **#53380 :** $S \cong C_2 \times C_2 \times S_0$, where S_0 has type M_{12} or $S_0 \cong UT_4(2)$. By Proposition 3.6 or 3.8, respectively, there are no reduced fusion systems over S. (When S_0 has type M_{12}, $\mathrm{Aut}(S_0)$ is a 2-group by [AOV2, Proposition 3.2].)

#55676 : $S \cong C_2 \times C_2 \times UT_3(4)$. By Proposition 3.3, applied with $P_i = C_2 \times C_2 \times A_i \cong (C_2)^6$ where $A_1, A_2 < UT_3(4)$ are the two (normal) elementary abelian subgroups of rank 4, there are no reduced fusion systems over S.

This finishes the proof of the theorem. □

Within the proof of Theorem 5.1, we have also shown:

Lemma 5.2. *If G is isomorphic to the sporadic simple group of Lyons, then (for $p = 2$) κ_G and μ_G are both isomorphisms. In particular, G tamely realizes its fusion system.*

It remains to describe fusion systems over the group #8935, which is a Sylow 2-subgroup of $Sp_4(4)$.

Proposition 5.3. *Let S be a group of order 256 with subgroups $P_1, P_2 \trianglelefteq S$ such that*

$$P_1 \cong P_2 \cong (C_2)^6, \;\; P_1 \cap P_2 = Z(S) = [S,S] \cong (C_2)^4 . \tag{2}$$

Then every saturated fusion system $\mathcal{F}$ over S such that $O_2(\mathcal{F}) = 1$ is isomorphic to the fusion system of $Sp_4(4)$.

Proof. By comparing orders, we see that $S = P_1P_2$. Set $Z = Z(S)$ and $\overline{P}_i = P_i/Z \cong C_2 \times C_2$. Thus $S/Z = \overline{P}_1 \times \overline{P}_2$. Choose subgroups $P_i^0 < P_i$ ($i = 1,2$) such that $P_i = Z \times P_i^0$, and set $\widehat{S} = P_1^0 * P_2^0$ (the free product). The commutator subgroup of $\widehat{S}/[\widehat{S},[\widehat{S},\widehat{S}]]$ is central in $\widehat{S}/[\widehat{S},[\widehat{S},\widehat{S}]]$ of order 16 (isomorphic via the commutator pairing to the tensor product $P_1^0 \otimes P_2^0$ when we regard P_1^0 and P_2^0 as $\mathbb{F}_2$-vector spaces), and hence the natural homomorphism from $\widehat{S}$ to S is surjective with kernel $[\widehat{S},[\widehat{S},\widehat{S}]]$.

Assume S^* is another group of order 2^8, with subgroups $P_1^*, P_2^* \le S^*$ which also satisfy (2). Choose subgroups $P_i^{*0} < P_i^*$ complementary to $Z(S^*)$, and set $\widehat{S}^* = P_1^{*0} * P_2^{*0}$. Then $S^* \cong \widehat{S}^*/[\widehat{S}^*,[\widehat{S}^*,\widehat{S}^*]]$ by the above argument. Any pair of isomorphisms $\psi_i \in \mathrm{Iso}(P_i^0, P_i^{*0})$ ($i = 1,2$) extends to an isomorphism $\widehat{\psi} \in \mathrm{Iso}(\widehat{S},\widehat{S}^*)$ and hence to an isomorphism $\psi \in \mathrm{Iso}(S,S^*)$. The conditions (2) thus determine S uniquely up to isomorphism.

Let $\mathrm{Aut}^0(S/Z) < \mathrm{Aut}(S/Z)$ be the subgroup of those automorphisms which normalize the set $\{\overline{P}_1, \overline{P}_2\}$. The above argument, when applied with $S^* = S$ and $\{P_1^{*0}, P_2^{*0}\} = \{P_1^0, P_2^0\}$, shows that the projection $S \to S/Z$ induces a (split) surjection

$$\Psi\colon \mathrm{Aut}(S) \xrightarrow{\quad\quad} \mathrm{Aut}^0(S/Z) \cong \big(\mathrm{Aut}(\overline{P}_1) \times \mathrm{Aut}(\overline{P}_2)\big) \rtimes C_2 \cong \Sigma_3 \wr C_2 .$$

(Note that $\Psi(\mathrm{Aut}(S)) \le \mathrm{Aut}^0(S/Z)$ since by Proposition 3.3, P_1 and P_2 are the unique maximal elementary abelian subgroups of S.) It follows that

$$\text{for each } \varGamma \in \mathrm{Syl}_3(\mathrm{Aut}(S)),\ \Psi(N_{\mathrm{Aut}(S)}(\varGamma)) = \mathrm{Aut}^0(S/Z) \tag{3}$$

by the Frattini argument applied to the normal subgroup $\mathrm{Ker}(\Psi)\varGamma \trianglelefteq \mathrm{Aut}(S)$.

Let $\mathcal{F}$ be a saturated fusion system over S such that $O_2(\mathcal{F}) = 1$. By Proposition 3.3, $\mathbf{E}_{\mathcal{F}} = \{P_1, P_2\}$. Since $|\mathrm{Out}_{\mathcal{F}}(S)|$ is odd and $\mathrm{Ker}(\Psi)$ is a 2-group by Lemma A.1,

$$\mathrm{Out}_{\mathcal{F}}(S) \cong (C_3)^r \quad \text{for some } 0 \le r \le 2. \tag{4}$$

Fix $i = 1,2$. By Lemma A.7, and since $\mathrm{Aut}_S(P_i) \cong S/P_i \cong C_2 \times C_2$, $\mathrm{Aut}_{\mathcal{F}}(P_i) = \Delta_i \times H_i$ for some $\Delta_i \cong A_5$ and some H_i of odd order. Also, $\mathrm{rk}([s,P_i]) = 2$ for $s \in S \smallsetminus P_i$, and

$[S,P_i]=[S,S]=Z=C_{P_i}(S)$ has rank 4. So by Lemma A.8(b), P_i is indecomposable as an $\mathbb{F}_2[\Delta_i]$-module, $Z_i \overset{\text{def}}{=} C_{P_i}(\Delta_i)$ has rank 2, and the action of $\Delta_i\cong A_5$ on $P_i/Z_i\cong (C_2)^4$ is isomorphic to the canonical action of $SL_2(4)$ on $\mathbb{F}_4^2$. Thus for $g\in\Delta_i$ of order three, $C_{P_i/Z_i}(g)=1$, and hence $C_{P_i}(g)=Z_i$. In summary, and since $\mathrm{Aut}_S(P_i)\le\Delta_i$,

$$C_{P_i}(g)=Z_i=C_{P_i}(\Delta_i)\le C_{P_i}(\mathrm{Aut}_S(P_i))=Z\,.$$

For $i=1,2$, the homomorphism

$$\mathrm{Aut}_{\mathcal{F}}(S) \xrightarrow{\qquad\quad} N_{\mathrm{Aut}_{\mathcal{F}}(P_i)}(\mathrm{Aut}_S(P_i))\cong A_4\times H_i$$

induced by restriction is surjective by the extension axiom, and its kernel is a 2-group by Lemma A.1 (or Lemma A.2). Hence $r\ge 1$ and $H_i\cong (C_3)^{r-1}$ (with r as in (4)).

Fix some $\Gamma\in\mathrm{Syl}_3(\mathrm{Aut}_{\mathcal{F}}(S))$. Choose $\gamma_1,\gamma_2\in\Gamma$ of order 3 such that $\gamma_i|_{P_i}\in\Delta_i$. Then $Z_i=C_{P_i}(\Delta_i)=C_{P_i}(\gamma_i)=C_Z(\gamma_i)$ has rank 2 and is normalized by Γ. If $Z_1=Z_2$, then $Z_1\trianglelefteq\mathcal{F}$ by Proposition 1.14, which contradicts our assumption that $O_2(\mathcal{F})=1$. Thus $Z_1\ne Z_2$ and $\langle\gamma_1\rangle\ne\langle\gamma_2\rangle$, so $\Gamma=\langle\gamma_1,\gamma_2\rangle\cong(C_3)^2$ by (4), and $r=2$. In particular, $\mathrm{Aut}_{\mathcal{F}}(P_i)\cong A_5\times C_3$ for $i=1,2$. Also, upon regarding Z as an $\mathbb{F}_2[\Gamma]$-module with $Z_i=C_Z(\gamma_i)$, we see that $Z=Z_1\times Z_2$ where the Z_i are $\mathbb{F}_2[\Gamma]$-submodules. We will show that the choice of the pair $\bigl(\langle\gamma_1\rangle,\langle\gamma_2\rangle\bigr)$ completely determines $\mathcal{F}$.

For $i=1,2$, choose any $\eta_i\in\Gamma$ of order 3 such that $\eta_i|_{P_i}\in H_i$. Since $\eta_i|_{P_i}$ commutes with $\mathrm{Aut}_S(P_i)$, η_i acts trivially on S/P_i, and hence trivially on $\overline{P}_{3-i}$. So η_i must act nontrivially on $\overline{P}_i$ since $\mathrm{Ker}(\Psi)$ is a 2-group. Also, since $\gamma_i|_{P_i}$ does not commute with $\mathrm{Aut}_S(P_i)$, γ_i acts nontrivially on $S/P_i\cong\overline{P}_{3-i}$, and it acts nontrivially on $\overline{P}_i=P_i/Z$ since $C_{P_i}(\gamma_i)=Z_i<Z$. Thus $\langle\eta_1\rangle$, $\langle\eta_2\rangle$, $\langle\gamma_1\rangle$, and $\langle\gamma_2\rangle$ are the four distinct subgroups of order 3 in Γ, and $\langle\eta_1\rangle$ and $\langle\eta_2\rangle$ are the unique subgroups of Γ of order 3 which induce the identity on $\overline{P}_2$ and $\overline{P}_1$, respectively.

Fix $\omega\in\mathbb{F}_4\smallsetminus\mathbb{F}_2$. For $i=1,2$, $C_{P_i}(\eta_i)=1$, since $C_{\overline{P}_i}(\eta_i)=1$ and $C_Z(\eta_i)=C_Z(\gamma_1^{\varepsilon_{i1}}\gamma_2^{\varepsilon_{i2}})=1$. Hence we can make P_i into an $\mathbb{F}_4$-vector space by defining multiplication by ω to be η_i. Thus $C_{\mathrm{Aut}(P_i)}(\eta_i)=\mathrm{Aut}_{\mathbb{F}_4}(P_i)\cong GL_3(4)$. Set

$$G_i=\bigl\{\alpha\in\mathrm{Aut}_{\mathbb{F}_4}(P_i)\,\big|\,\alpha(Z_i)=Z_i\bigr\}\qquad\text{and}\qquad Q_i=O_2(G_i)\,.$$

Then Q_i is the group of all $\alpha\in\mathrm{Aut}_{\mathbb{F}_4}(P_i)$ which induce the identity on Z_i and on P_i/Z_i, $Q_i\cong\mathbb{F}_4^2\cong(C_2)^4$, and $G_i/Q_i\cong GL_2(4)\times C_3\cong A_5\times(C_3)^2$. Set $K_i=\mathrm{Aut}_S(P_i)\langle\gamma_i|_{P_i}\rangle\cong A_4$. Thus $K_i<\Delta_i<G_i$ for $i=1,2$.

Let $\Delta_i^*<G_i$ be *any* subgroup such that $\Delta_i^*\cong A_5$ and $K_i<\Delta_i^*$. Then $Q_i\cap\Delta_i^*=1$, and $\Delta_i^*Q_i=\Delta_iQ_i=O^3(G_i)$ since $G_i/Q_i\cong A_5\times(C_3)^2$. So by Proposition A.3, applied with $G=G_i$, $Q=Q_i$, $H=\Delta_i$, and $H_0=K_i$, we have $\Delta_i^*={}^{\psi}(\Delta_i)$ for some $\psi\in C_{Q_i}(K_i)$. Since $C_{Q_i}(K_i)\le C_{Q_i}(\gamma_i)=1$ (recall $C_{P_i}(\gamma_i)=Z_i$), it follows that $\Delta_i^*=\Delta_i$.

Thus $\mathrm{Aut}_{\mathcal{F}}(P_1)$ and $\mathrm{Aut}_{\mathcal{F}}(P_2)$ are determined by the choice of $\Gamma=\langle\gamma_1\rangle\times\langle\gamma_2\rangle$. So by Proposition 1.3, $\mathcal{F}$ is determined by this choice. We just saw that $\langle\gamma_1\rangle$ and $\langle\gamma_2\rangle$ are the two subgroups of order 3 in Γ which act nontrivially on both $\overline{P}_1$ and $\overline{P}_2$. If $\mathcal{F}'$ is another saturated fusion system over S with $O_2(\mathcal{F}')=1$, and $\mathcal{F}'$ is determined by $\Gamma'=\langle\gamma_1'\rangle\times\langle\gamma_2'\rangle$, then there is $\varphi\in\mathrm{Aut}(S)$ such that ${}^{\varphi}\Gamma'=\Gamma$. Either ${}^{\varphi}\langle\gamma_i'\rangle=\langle\gamma_i\rangle$ for $i=1,2$, or ${}^{\varphi}\langle\gamma_i'\rangle=\langle\gamma_{3-i}\rangle$. By (3), there is $\psi\in N_{\mathrm{Aut}(S)}(\Gamma)$ which exchanges $\langle\gamma_1\rangle$ and $\langle\gamma_2\rangle$, and so either ${}^{\varphi}\mathcal{F}'=\mathcal{F}$ or ${}^{\psi\varphi}\mathcal{F}'=\mathcal{F}$.

Fix $S^*\in\mathrm{Syl}_2(Sp_4(4))$, and set $\mathcal{F}^*=\mathcal{F}_{S^*}(Sp_4(4))$. By the Chevalley commutator formula (see [Ca, Theorem 5.2.2]), the two unipotent radical subgroups $P_1^*,P_2^*<S^*$, are both isomorphic to $\mathbb{F}_4^3\cong(C_2)^6$, and are such that $P_1^*\cap P_2^*=[S^*,S^*]=Z(S^*)\cong(C_2)^4$.

Hence S^* satisfies (2), so $S^* \cong S$. Also, $O_2(\mathcal{F}^*) = 1$ by Proposition 1.17(b), so $\mathcal{F}^* \cong \mathcal{F}$ since there is up to isomorphism at most one such saturated fusion system over S. $\square$

6. Groups of order 512

Throughout this section again, whenever we list potentially critical subgroups of S, they were found using computer computations based on the criteria in Proposition 2.3.

Theorem 6.1. *Let $\mathcal{F}$ be a reduced, indecomposable fusion system over a 2-group S of order 512. Then one of the following holds:*

(a) *$S \cong D_{512}$, and $\mathcal{F}$ is isomorphic to the fusion system of $PSL_2(q)$ where $q \equiv \pm 511$ (mod 1024).*

(b) *$S \cong SD_{512}$, and $\mathcal{F}$ is isomorphic to the fusion system of $PSL_3(q)$ where $q \equiv 127$ (mod 256).*

(c) *$S \cong C_{16} \wr C_2$, and $\mathcal{F}$ is isomorphic to the fusion system of $SL_3(17)$.*

(d) *$S \cong D_{16} \wr C_2$, and $\mathcal{F}$ is isomorphic to the fusion system of $PSL_4(7)$.*

(e) *$S \cong SD_{16} \wr C_2$, and $\mathcal{F}$ is isomorphic to the fusion system of $SL_5(3)$.*

(f) *$S \cong UT_3(8)$, and $\mathcal{F}$ is isomorphic to the fusion system of $SL_3(8)$.*

(g) *S is of type A_{12}, and $\mathcal{F}$ is isomorphic to the fusion system of A_{12}, or of $Sp_6(2)$, or of $\Omega_7(3)$.*

(h) *$S \cong (Q_8 \wr C_2) \times_{C_2} Q_8$, and $\mathcal{F}$ is isomorphic to the fusion system of $PSp_6(3)$.*

(i) *S is of type* HS, *and $\mathcal{F}$ is isomorphic to the fusion system of the Higman-Sims sporadic group.*

(j) *S is of type* O'N, *and $\mathcal{F}$ is isomorphic to the fusion system of O'Nan's sporadic group.*

In all cases, $\mathcal{F}$ is tame. Also, the fusion system of each of the groups listed above is simple.

Proof. If S is dihedral, semidihedral, or a wreath product $C_{16} \wr C_2$, then by [AOV1, § 4.1] or [AOV2, Proposition 3.1], $\mathcal{F}$ is isomorphic to the fusion system of $PSL_2(q)$ or $PSL_3(q)$ for appropriate odd q, and we are in one of the cases (a), (b), or (c). If not, then it satisfies conditions (a)–(h) in Proposition 2.2. By a computer search based on those criteria, we are left with the 31 possibilities for S listed below, where #(−) denotes the Magma/GAP identification number.

The tameness of the fusion systems of the sporadic groups of O'Nan and Higman-Sims (#58362 and #60329 below) is shown in Proposition 6.3. All of the other fusion systems listed in the theorem are tame by [BMO2, Theorem C], or (for the fusion system of A_{12}) by [AOV1, Proposition 4.8].

By [O2, Theorem A], the fusion systems listed above in cases (a–e) are all simple. By Proposition 1.17(a,b,d), if $\mathcal{F}$ is any of the fusion systems in (f–j), then $O^2(\mathcal{F}) = \mathcal{F}$, $O_2(\mathcal{F}) = 1$, $\mathcal{F}$ is indecomposable, and $\mathcal{F}$ is simple except possibly when $\mathcal{F}$ is the fusion system of $\Omega_7(3)$ or $PSp_6(3)$. In case (g), $\mathrm{Aut}(S)$ is a 2-group (see the proof of Proposition 7.4), so those fusion systems are simple by Proposition 1.17(c,d). By Proposition 6.4, when S is of type $PSp_6(3)$, there is a unique saturated fusion system $\mathcal{F}$ over S such that

$O_2(\mathcal{F}) = 1$, and since $O_2(O^{2'}(\mathcal{F})) = 1$ [AOV1, Lemma 1.20(e) & Proposition 1.25(b)], $\mathcal{F} = O^{2'}(\mathcal{F})$ must be simple by Proposition 1.17(c,d).

#128270, **#128271**, **#399715**, **#399717**, **#399770**, **#399771** : In these six cases, $S = S_1 \times S_2$ where $S_1 \cong D_{16}$ or SD_{16}, and $S_2 \cong D_{32}$, SD_{32}, or $C_4 \wr C_2$. By [O1, Theorem B], each reduced fusion system over any of these groups is decomposable.

#420360, **#420362**, **#6480905**, **#7998954**, **#6480855** : In these cases, $S \cong D_8 \times S_0$, where $S_0 \cong D_{64}$, SD_{64}, $UT_4(2)$, $UT_3(4)$, or is of type M_{12}. By [O1, Theorem B], each reduced fusion system over S is decomposable.

#10483221, **#10483222**, **#10493114** : In these cases, $S \cong D_8 \times S_0$ where $S_0 \cong C_2 \times C_2 \times D_{16}$, $C_2 \times C_2 \times SD_{16}$, or $(C_2)^3 \times D_8$, respectively. By [O1, Theorem B], there are no reduced fusion systems over S.

#7606661 : $S \cong D_8 \times D_8 \times D_8$, and each reduced fusion system over S is decomposable by [O1, Theorem C].

#7530050, **#7530054**, **#7530055**, **#10482003** : In these cases, $S \cong C_2 \times C_2 \times S_0$ where $S_0 \cong D_8 \wr C_2$ or has type M_{22} or M_{12}:2; or $S \cong (C_2)^3 \times S_0$ where S_0 has type M_{12}. In all cases, $|Z(S_0)| = 2$ and $\mathrm{rk}(S_0) \leq 4$. Also, $\mathrm{Aut}(S_0)$ is a 2-group by [O2, Corollary A.10(c)], [OV, Lemma 5.5], the proof of [O2, Proposition 4.3(c)], or [AOV2, Proposition 3.2], respectively. So by Proposition 3.6, there are no reduced fusion systems over S.

#7530088, **#10482065** : $S \cong C_2 \times C_2 \times S_0$ where S_0 has type J_2, or $S \cong (C_2)^3 \times UT_4(2)$. By Proposition 3.7 or 3.8, respectively, there are no reduced fusion systems over S.

#10493307 : Here, $S \cong (C_2)^3 \times UT_3(4)$. By Proposition 3.3, applied with $P_i = (C_2)^3 \times A_i \cong (C_2)^7$ where $A_1, A_2 < UT_3(4)$ are the two (normal) elementary abelian subgroups of rank 4, there are no reduced fusion systems over S.

#58362 : S is of type O'N. By Proposition 6.3 below, every reduced fusion system over S is isomorphic to the fusion system of O'Nan's group.

#60329 : S is of type HS. By Proposition 6.3 below, every reduced fusion system over S is isomorphic to the fusion system of the Higman-Sims group.

#60809 : $S \cong D_{16} \wr C_2$. By [O2, Proposition 5.5(a)], every reduced fusion system over S is isomorphic to $\mathcal{F}_S(PSL_4(7))$.

#60833 : $S \cong SD_{16} \wr C_2$. By [O2, Proposition 5.5(b)], every reduced fusion system over S is isomorphic to $\mathcal{F}_S(SL_5(3))$.

#406983 : S is of type A_{12}. By Proposition 7.4 below, each reduced fusion system over S is isomorphic to the fusion system of one of the groups A_{12}, $Sp_6(2)$, or $\Omega_7(3)$.

#6407070 : By computer computations, S has two potentially critical subgroups P_1 and P_2, both normal, where $P_1 \cong 2^{1+6}_+$, $S/P_1 \cong C_2 \times C_2$, and $|P_2| = 2^8$. (In terms of the Magma/GAP generators, $P_1 = \langle s_1, s_4, s_5, s_6, s_7, s_8, s_9 \rangle$ and $P_2 = \langle s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9 \rangle$.) Set $V = P_1/Z(P_1)$, regarded as a 6-dimensional $\mathbb{F}_2[S/P_1]$-module. Then $[S/P_1, V] = C_V(S/P_1)$ has rank 3, $[x, V]$ has rank 2 for each $1 \neq x \in S/P_1$, and $\bigcap_{1 \neq x \in S/P_1}[x, V] \neq 1$.

Let $\mathcal{F}$ be a reduced fusion system over S. If $P_1 \in \mathbf{E}_{\mathcal{F}}$, then by Lemma A.7, there is $\varGamma \leq \mathrm{Out}_{\mathcal{F}}(P_1)$ such that $\varGamma \cong A_5$ and $\mathrm{Out}_S(P_1) \in \mathrm{Syl}_2(\varGamma)$. This in turn induces an action of $\varGamma$ on V that contains the action of S/P_1 on V as a Sylow 2-subgroup. But this is impossible by Lemma A.8(b) and the above remarks, and hence $P_1 \notin \mathbf{E}_{\mathcal{F}}$.

Thus $\mathbf{E}_{\mathcal{F}} \subseteq \{P_2\}$, contradicting Lemma 1.16.

#7530110 : S is of type $PSp_6(3)$. By Proposition 6.4 below, every reduced fusion system over S is isomorphic to the fusion system of $PSp_6(3)$.

#7540630 : $S \cong C_2 \times C_2 \times S_0$, where S_0 is the group of order 128 with Magma/GAP number 1411 described in the proof of Theorem 4.3. In particular, there are two subgroups $P_1, P_2 < S$ isomorphic to $(C_2)^7$, and they satisfy the hypotheses of Proposition 3.3. Since $[S,S] < Z(S)$, there are no reduced fusion systems over S by that proposition.

#10481201 : $S \cong UT_3(8)$. By Proposition 6.2 below, every reduced fusion system over S is isomorphic to $\mathcal{F}_S(SL_3(8))$. □

It remains to handle some of the individual cases.

Proposition 6.2. *Fix $k \geq 1$. A saturated fusion system $\mathcal{F}$ over $UT_3(2^k)$ is reduced if and only if it is isomorphic to the fusion system of $PSL_3(2^k)$.*

Proof. Since this holds for $k=1$ by [AOV1, Proposition 4.3] ($UT_3(2) \cong D_8$), we assume $k \geq 2$ from now on. Set $q = 2^k$ and $S = UT_3(q)$ for short. For each $1 \leq i < j \leq 3$, let $E_{ij} \leq S$ be the subgroup of upper triangular matrices with 1's on the diagonal and nonzero off-diagonal entry only in position (i,j). Thus $E_{13} = Z(S) = [S,S]$. Set $A_1 = E_{12}E_{13}$, and $A_2 = E_{13}E_{23}$. Then $A_1 \cong A_2 \cong (\mathbb{F}_q)^2 \cong (C_2)^{2k}$, and we regard these groups as $\mathbb{F}_q$-vector spaces.

Fix a reduced fusion system $\mathcal{F}$ over S. By Proposition 3.3, $\mathbf{E}_{\mathcal{F}} = \{A_1, A_2\}$. For each $i = 1,2$, set $\varGamma_i = \mathrm{Aut}_{\mathcal{F}}(A_i)$.

We first examine $\varGamma_1 = \mathrm{Aut}_{\mathcal{F}}(A_1)$. Fix a strongly 2-embedded subgroup $H < \varGamma_1$ which contains $\mathrm{Aut}_S(A_1)$. Consider the following sets of subgroups of A_1:

$$\mathscr{V}_0 = \bigl\{C_{A_1}(T) \bigm| T \in \mathrm{Syl}_2(H)\bigr\}, \qquad \mathscr{V}_1 = \bigl\{C_{A_1}(T) \bigm| T \in \mathrm{Syl}_2(\varGamma_1) \smallsetminus \mathrm{Syl}_2(H)\bigr\},$$
$$\mathscr{V} = \mathscr{V}_0 \cup \mathscr{V}_1 = \bigl\{C_{A_1}(T) \bigm| T \in \mathrm{Syl}_2(\varGamma_1)\bigr\}.$$

In particular, $E_{13} = C_{A_1}(\mathrm{Aut}_S(A_1)) \in \mathscr{V}_0$. Since $\varGamma_1$ permutes the elements of $\mathscr{V}$ transitively (since $\varphi(C_{A_1}(T)) = C_{A_1}({}^{\varphi}T)$ for $\varphi \in \varGamma_1$ and $T \in \mathrm{Syl}_2(\varGamma_1)$), $\mathrm{rk}(V) = \mathrm{rk}(E_{13}) = k$ for each $V \in \mathscr{V}$.

For each $V_0 \in \mathscr{V}_0$ and each $V_1 \in \mathscr{V}_1$, $V_0 = C_{A_1}(T_0)$ and $V_1 = C_{A_1}(T_1)$ for some $T_0 \in \mathrm{Syl}_2(H)$ and $T_1 \in \mathrm{Syl}_2(\varGamma_1) \smallsetminus \mathrm{Syl}_2(H)$, and $\langle T_0, T_1\rangle$ contains the strongly 2-embedded subgroup $\langle T_0, T_1\rangle \cap H$. In particular, $O_2(\langle T_0, T_1\rangle) = 1$, so $\langle T_0, T_1\rangle$ acts faithfully on $A_1/C_{A_1}(\langle T_0, T_1\rangle)$ (Lemma A.1). By [OV, Lemma 1.7(a)], $\mathrm{rk}\bigl(A_1/C_{A_1}(\langle T_0, T_1\rangle)\bigr) \geq 2k$, so $V_0 \cap V_1 = C_{A_1}(\langle T_0, T_1\rangle) = 1$, and $A_1 = V_0 \times V_1$.

In particular, $\mathscr{V}_0 \cap \mathscr{V}_1 = \varnothing$. Also, when $V_0 = E_{13}$, this shows that each $V \in \mathscr{V}_1$ contains a representative for each coset of E_{13} in $A_1 \smallsetminus E_{13}$. Since $\mathrm{Aut}_S(A_1)$ acts transitively on each such coset, we conclude that $\bigcup_{V \in \mathscr{V}_1} V \supseteq A_1 \smallsetminus E_{13}$. Hence $V_0 \cap (A_1 \smallsetminus E_{13}) = \varnothing$ for each $V_0 \in \mathscr{V}_0$, so $\mathscr{V}_0 = \{E_{13}\}$. Since $\varGamma_1$ acts transitively on $\mathscr{V}$ and $E_{13} \cap V = 1$ for each $V \in \mathscr{V} \smallsetminus \{E_{13}\}$, we see that $V \cap W = 1$ for each pair V, W of distinct elements in $\mathscr{V}$. Thus $A_1 \smallsetminus 1$ is the disjoint union of the sets $V \smallsetminus 1$ for all $V \in \mathscr{V}$, so $|\mathscr{V}| = q+1$, and $\mathrm{Aut}_S(A_1)$ normalizes the set $\mathscr{V}_1 = \mathscr{V} \smallsetminus \{E_{13}\}$.

Fix some $V \in \mathscr{V} \smallsetminus \{E_{13}\}$. Since $A_1 = E_{13} \times V$, there is $\varphi_1 \in \mathrm{Aut}(A_1)$ which induces the identity on E_{13} and on A_1/E_{13}, and such that $\varphi_1(V) = E_{12}$. Set $\mathscr{V}^* = \varphi_1(\mathscr{V})$. Thus $E_{13} = \varphi_1(E_{13})$ and $E_{12} = \varphi_1(V)$ are both in $\mathscr{V}^*$. Also, $[\varphi_1, \mathrm{Aut}_S(A_1)] = 1$ since the group of automorphisms of A_1 which induce the identity on E_{13} and on A_1/E_{13} is abelian, so $\mathrm{Aut}_S(A_1)$ also normalizes the set $\mathscr{V}^* \smallsetminus \{E_{13}\}$. Since $|\mathscr{V}^* \smallsetminus \{E_{13}\}| = q = |\mathrm{Aut}_S(A_1)|$ and E_{12} is normalized only by the identity in $\mathrm{Aut}_S(A_1)$, $\mathrm{Aut}_S(A_1)$ acts transitively on this set. Hence $\mathscr{V}^* = \varphi_1(\mathscr{V})$ is precisely the set of all 1-dimensional $\mathbb{F}_q$-linear subspaces of A_1.

Let $\Theta_i^{SL} < \Theta_i^{GL} < \Theta_i^{\Gamma L} < \mathrm{Aut}(A_i)$ be the subgroups $SL_2(q)$, $GL_2(q)$, and $\Gamma L_2(q)$, respectively, defined with respect to the canonical $\mathbb{F}_q$-vector space structure on A_i. By the fundamental theorem of affine geometry [Bg, Théorème 2.6.3], $N_{\mathrm{Aut}(A_1)}(\varphi_1(\mathscr{V})) = \Theta_1^{\Gamma L}$, and hence ${}^{\varphi_1}(\Gamma_1) \le \Theta_1^{\Gamma L}$. All Sylow 2-subgroups of ${}^{\varphi_1}(\Gamma_1)$ are $\Theta_1^{\Gamma L}$-conjugate to $\mathrm{Aut}_S(A_1)$ and hence contained in Θ_1^{SL}. So $O^{2'}({}^{\varphi_1}(\Gamma_1)) = \Theta_1^{SL}$ since $SL_2(q)$ is generated by any two of its Sylow 2-subgroups.

By a similar argument, there is $\varphi_2 \in \mathrm{Aut}(A_2)$ which induces the identity on E_{13} and on A_2/E_{13}, and such that $\Theta_2^{SL} \le {}^{\varphi_2}(\Gamma_2) \le \Theta_2^{\Gamma L}$ and $O^{2'}({}^{\varphi_2}(\Gamma_2)) = \Theta_2^{SL}$. Let $\varphi \in \mathrm{Aut}(S)$ be the unique automorphism such that $\varphi|_{A_i} = \varphi_i$ for $i = 1, 2$. (Note that φ has the form $\varphi(g) = g\chi(g)$ for some $\chi \in \mathrm{Hom}(S, Z(S))$.) Upon replacing $\mathcal{F}$ by ${}^{\varphi}\mathcal{F}$, we can assume that $\Theta_i^{SL} \le \mathrm{Aut}_{\mathcal{F}}(A_i) \le \Theta_i^{\Gamma L}$ and $O^{2'}(\mathrm{Aut}_{\mathcal{F}}(A_i)) = \Theta_i^{SL}$ for each $i = 1, 2$.

Set $G = PSL_3(q)$ for short, and identify $S = UT_3(q)$ with its image in G. Fix a generator $\lambda \in \mathbb{F}_q^\times$. Let $\beta_1, \beta_2 \in \mathrm{Aut}_G(S)$ be conjugation by the classes of the diagonal matrices $\mathrm{diag}(\lambda, \lambda^{-1}, 1)$ and $\mathrm{diag}(1, \lambda, \lambda^{-1})$ respectively. Then $\mathrm{Aut}_G(S) = \mathrm{Inn}(S)\langle\beta_1, \beta_2\rangle$ and for $a, b, c \in \mathbb{F}_q$,

$$\beta_1\left(\left(\begin{smallmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} 1 & \lambda^2 a & \lambda b \\ 0 & 1 & \lambda^{-1}c \\ 0 & 0 & 1 \end{smallmatrix}\right) \qquad \text{and} \qquad \beta_2\left(\left(\begin{smallmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} 1 & \lambda^{-1} a & \lambda b \\ 0 & 1 & \lambda^{2}c \\ 0 & 0 & 1 \end{smallmatrix}\right).$$

Thus $\beta_1|_{A_2} \in \Theta_2^{SL} \le \mathrm{Aut}_{\mathcal{F}}(A_2)$. By the extension axiom, there is $\beta_1' \in \mathrm{Aut}_{\mathcal{F}}(S)$ such that $\beta_1'|_{A_2} = \beta_1|_{A_2}$. Then $\beta_1'|_{A_1} \in \mathrm{Aut}_{\mathcal{F}}(A_1)$, so $\beta_1'|_{A_1} \in \Theta_1^{\Gamma L}$ by our assumptions, and $\beta_1|_{A_1} \in \Theta_1^{\Gamma L}$ by construction. Set $\tau = \beta_1^{-1}\beta_1'$; then τ induces the identity on A_2 and hence on S/A_2 (since $C_S(A_2) = A_2$) and $\tau|_{A_1} \in \Theta_1^{\Gamma L}$. It follows that $\tau|_{A_1} \in \mathrm{Aut}_{A_2}(A_1)$, and hence that $\tau \in \mathrm{Inn}(S)$. So $\beta_1 \in \mathrm{Aut}_{\mathcal{F}}(S)$.

By a similar argument, $\beta_2 \in \mathrm{Aut}_{\mathcal{F}}(S)$. Hence $\mathrm{Aut}_G(S) \le \mathrm{Aut}_{\mathcal{F}}(S)$. Also, $\mathrm{Aut}_G(A_i) = \Theta_i^{SL}\langle\beta_i|_{A_i}\rangle$ for $i = 1, 2$ (the subgroup of index $(3, q-1)$ in Θ_i^{GL}), so $O^{2'}(\mathrm{Aut}_{\mathcal{F}}(A_i)) = \Theta_i^{SL} \le \mathrm{Aut}_G(A_i) \le \mathrm{Aut}_{\mathcal{F}}(A_i)$. Since $\mathbf{E}_{\mathcal{F}} = \{A_1, A_2\}$, we have $\mathcal{F}_S(G) \subseteq \mathcal{F}$ by Proposition 1.3 (Alperin's fusion theorem). In addition, the A_i are minimal $\mathcal{F}$-centric subgroups, so $\mathrm{Aut}_G(P) \ge O^{2'}(\mathrm{Aut}_{\mathcal{F}}(P))$ for each $P \in \mathcal{F}^c$. (If $P \in \mathcal{F}^c \smallsetminus \{A_1, A_2\}$, then $O^{2'}(\mathrm{Aut}_{\mathcal{F}}(P)) = \mathrm{Aut}_S(P)$ by Proposition 1.3 and since there are no larger essential subgroups.) So by [AKO, Lemma I.7.6(a)], $\mathcal{F}$ contains $\mathcal{F}_S(G)$ as a subsystem of odd index in the sense of [AKO, Definition I.7.3]. Thus $\mathcal{F} = \mathcal{F}_S(G)$ since $O^{2'}(\mathcal{F}) = \mathcal{F}$ ($\mathcal{F}$ is reduced).

Conversely, $\mathcal{F}_S(G)$ is simple by Proposition 1.17(d). □

The 2-groups of type O'N and HS will be handled together.

Proposition 6.3. (a) *If S is a group of order 512 of type* O'N*, then every reduced fusion system over S is isomorphic to the fusion system of O'Nan's group, and is tame.*

(b) *If S is a group of order 512 of type* HS*, then every reduced fusion system over S is isomorphic to the fusion system of the Higman-Sims group, and is tame.*

If G is either of the groups O'N *or* HS*, then κ_G and μ_G are both isomorphisms, and hence G tamely realizes its fusion system.*

Proof. We let $S^{\mathrm{O'N}}$ and S^{HS} be 2-groups of type O'N or HS, respectively, and set $S = S^{\mathrm{O'N}}$ or S^{HS} when we do not need to distinguish them. By [O'N] or [Alp], these groups have a presentation with generators v_1, v_2, v_3, s, t, where

$$A = \langle v_1, v_2, v_3\rangle \cong (C_4)^3,$$

and with additional relations

$$\begin{array}{llll} {v_1}^t = v_3^{-1} & {v_2}^t = v_2^{-1} & {v_3}^t = v_1^{-1} & t^2 = 1 \\ {v_1}^s = v_2 & {v_2}^s = v_3 & {v_3}^s = v_1 v_2^{-1} v_3 & s^t = s^{-1} \end{array}$$

(in both cases), and

$$s^4 = \begin{cases} v_1 v_3 & \text{if } S = S^{\text{O'N}} \\ 1 & \text{if } S = S^{\text{HS}}. \end{cases}$$

Thus S is an extension of A by D_8. Also, A is the unique subgroup of S isomorphic to $(C_4)^3$, and hence is characteristic in all subgroups of S which contain it.

By computer computations, there are three conjugacy classes of potentially critical subgroups in each case, with representatives

$$\begin{aligned} P_1 &= A\langle s^2, t\rangle = C_S(v_1 v_2^2 v_3^{-1}) \\ P_2^{\text{O'N}} &= \langle s^2 v_1, t, v_1 v_3, v_1^2, v_2^2\rangle \cong C_4 \times_{C_2} 2_+^{1+4} \\ P_2^{\text{HS}} &= \langle s v_1 v_2, t, v_1 v_3, v_1^2, v_2^2\rangle \\ P_3 &= A\langle st, s^2\rangle = C_S(v_1^2 v_2^2)\,. \end{aligned}$$

Note that $P_1, P_3 \trianglelefteq S$ (they have index two), while $N_S(P_2) = P_1$ when $S = S^{\text{O'N}}$, and $N_S(P_2) = P_2\langle s\rangle$ when $S = S^{\text{HS}}$. Since P_1 and P_3 are the only potentially critical subgroups of index 2 in S (and are not isomorphic to each other), they are both characteristic in S.

Set $w = v_1 v_3$ and $z = w^2 = v_1^2 v_3^2 \in Z(S) \cong C_2$. By direct computation, $\text{Out}(S) \cong (C_2)^4$, with explicit generators $[\chi_i]$ $(i = 1, 2, 3, 4)$, where the $\chi_i \in \text{Aut}(S)$ act as follows.

χ	$\chi(v_1)$	$\chi(v_2)$	$\chi(v_3)$	$\chi^{\text{O'N}}(s)$	$\chi^{\text{O'N}}(t)$	$\chi^{\text{HS}}(s)$	$\chi^{\text{HS}}(t)$
χ_1	$v_1 z$	$v_2 z$	$v_3 z$	s	t	s	t
χ_2	v_1	v_2	v_3	$v_1^{-1} v_2^2 v_3 s$	t	$w^{-1}s$	zt
χ_3	v_1	v_2	v_3	$v_1^{-1} v_2^{-1} s$	t	ws	$w^{-1}t$
χ_4	v_1^{-1}	v_2^{-1}	v_3^{-1}	$v_3 s$	$v_2 t$	zs	t

Note that $\chi_i(P_j) = P_j$ for all i, j, except for the case $\chi_4(P_2^{\text{O'N}}) = {}^s(P_2^{\text{O'N}}) \neq P_2^{\text{O'N}}$. Also, $\text{Out}(P_2^{\text{O'N}}) \cong C_2 \times \Sigma_6$, $\text{Out}(P_2^{\text{HS}}) \cong C_2 \times C_2 \times \Sigma_4$, and $\text{Out}(P_1)$ and $\text{Out}(P_3)$ are described by the following table (where the second factor in $\text{Out}(P_3)$ has trivial center):

P	$H^1(P/A; A)$	$N_{\text{Aut}(A)}(\text{Aut}_P(A))/\text{Aut}_P(A)$	$\text{Out}(P)$
P_1	$\mathbb{Z}/4$	$C_2 \times \Sigma_4$	$D_8 \times \Sigma_4$
P_3	$(\mathbb{Z}/2)^3$	$C_2 \times \Sigma_4$	$C_2 \times C_2 \times ((C_2)^4 \rtimes \Sigma_3)$

Fix a reduced fusion system $\mathcal{F}$ over S. Then

- $\text{Out}_{\mathcal{F}}(S) = 1$ since $\text{Out}(S)$ is a 2-group;
- $P_2 \in \mathbf{E}_{\mathcal{F}}$ by Proposition 1.14, since A is characteristic in P_1 and P_3 and $A \ntrianglelefteq \mathcal{F}$; and
- $P_3 \in \mathbf{E}_{\mathcal{F}}$ since $Z(S)$ is characteristic in P_1 and in P_2 ($Z(S) \leq Z(P_2) \leq Z(P_1) = \langle v_1 v_2^2 v_3^{-1}\rangle \cong C_4$) and $Z(S) \ntrianglelefteq \mathcal{F}$.

If $\boldsymbol{S = S^{\text{O'N}}}$, then since $P_2 \in \mathbf{E}_{\mathcal{F}}$ and $\text{Out}_S(P_2) \cong C_2 \times C_2$, we have $\text{Out}_{\mathcal{F}}(P_2) \cong A_5 \times H$ for some H of odd order by Lemma A.7. In particular, there is $\xi \in N_{\text{Aut}_{\mathcal{F}}(P_2)}(\text{Aut}_S(P_2))$

of order three. By the extension axiom, ξ extends to an element $\widehat{\xi} \in \mathrm{Aut}_{\mathcal{F}}(P_1)$ (recall $P_1 = N_S(P_2)$). Thus $\mathrm{Aut}_{\mathcal{F}}(P_1)$ is not a 2-group, and so $P_1 \in \mathbf{E}_{\mathcal{F}}$ by Proposition 1.3.

If $S = S^{\mathbf{HS}}$, set $T = N_S(P_2) \trianglelefteq S$. Then $(P_3 \cap T)/[P_3, P_3] = \Omega_1(P_3/[P_3, P_3])$, so $P_3 \cap T$ is characteristic of index two in P_3. Thus $[\mathrm{Aut}(P_3), P_3] \le T$ and $[\mathrm{Aut}(P_2), P_2] \le T$, so $P_1 \in \mathbf{E}_{\mathcal{F}}$ by Lemma 1.15.

In both cases, set $A_0 = \Omega_1(A) = \langle v_1^2, v_2^2, v_3^2 \rangle$. Since $C_S(A_0) = A$,

$$\mathrm{Aut}_S(A_0) \cong S/A \cong D_8 \qquad \text{and} \qquad \mathrm{Aut}_{P_i}(A_0) \cong P_i/A \cong C_2 \times C_2 \ \text{ for } i = 1, 3\,.$$

So for $i = 1, 3$, by Lemma A.2 (applied with $G = P_i$ and $H = A$), restriction induces a homomorphism from $\mathrm{Out}_{\mathcal{F}}(P_i)$ into the group $N_{\mathrm{Aut}(A_0)}(\mathrm{Aut}_{P_i}(A_0))/\mathrm{Aut}_{P_i}(A_0) \cong \Sigma_3$ with kernel a 2-group. This restriction homomorphism is an isomorphism since $P_i \in \mathbf{E}_{\mathcal{F}}$, so $\mathrm{Aut}_{\mathcal{F}}(A_0)$ contains the normalizers in $\mathrm{Aut}(A_0)$ of $\mathrm{Aut}_{P_1}(A_0)$ and $\mathrm{Aut}_{P_3}(A_0)$; i.e., the two maximal parabolic subgroups in $\mathrm{Aut}(A_0)$. Thus $\mathrm{Aut}_{\mathcal{F}}(A_0) = \mathrm{Aut}(A_0) \cong GL_3(2)$.

Now, $O_2(\mathrm{Aut}(A)) = \{\alpha \in \mathrm{Aut}(A) \,|\, \alpha|_{A_0} = \mathrm{Id}\} \cong (C_2)^9$, so $\mathrm{Aut}(A)/O_2(\mathrm{Aut}(A)) \cong \mathrm{Aut}(A_0)$. Let $\mathcal{G}$ be the set of all subgroups $\Gamma < \mathrm{Aut}(A)$ such that $\Gamma \cong GL_3(2)$ and $\mathrm{Aut}_S(A) < \Gamma$. Since $\mathrm{Aut}_S(A) \cap O_2(\mathrm{Aut}(A)) = 1$, $\mathrm{Aut}_{\mathcal{F}}(A) \cap O_2(\mathrm{Aut}(A)) = 1$ by the Sylow axiom, and hence $\mathrm{Aut}_{\mathcal{F}}(A) \in \mathcal{G}$ by the extension axiom. By Proposition A.3, applied with $G = \mathrm{Aut}(A)$, $Q = O_2(G)$, $H = \mathrm{Aut}_{\mathcal{F}}(A)$, and $H_0 = \mathrm{Aut}_S(A)$, the group $C_{O_2(\mathrm{Aut}(A))}(\mathrm{Aut}_S(A)) = \langle \chi_1|_A, \chi_4|_A \rangle \cong C_2 \times C_2$ acts transitively on $\mathcal{G}$ via conjugation. For $\Gamma \in \mathcal{G}$ and $\chi \in O_2(\mathrm{Aut}(A))$, ${}^{\chi}\Gamma = \Gamma$ if and only if $[\chi, \Gamma] \le \Gamma \cap O_2(\mathrm{Aut}(A)) = 1$, in which case $[\chi, \mathrm{Aut}(A)] = [\chi, O_2(\mathrm{Aut}(A))] = 1$ since $O_2(\mathrm{Aut}(A))$ is abelian. Thus $\chi_4|_A$ normalizes all elements in $\mathcal{G}$, while $\chi_1|_A \notin Z(\mathrm{Aut}(A))$ normalizes none of them. Hence $|\mathcal{G}| = 2$, and its elements are exchanged by $\chi_1|_A$.

Now fix some $\Gamma \in \mathcal{G}$. We can assume, after replacing $\mathcal{F}$ by ${}^{\chi_1}\mathcal{F}$ if necessary, that $\mathrm{Aut}_{\mathcal{F}}(A) = \Gamma$. We claim that there is a unique possibility for $\mathrm{Aut}_{\mathcal{F}}(P_1)$ whose elements restrict to elements of Γ, and exactly two such possibilities for $\mathrm{Aut}_{\mathcal{F}}(P_3)$. To see this, note that for $P = P_1$ or P_3, the image of $\mathrm{Out}_{\mathcal{F}}(P)$ in $N_{\mathrm{Aut}(A)}(\mathrm{Aut}_P(A))/\mathrm{Aut}_P(A)$ is precisely $N_\Gamma(\mathrm{Aut}_P(A))/\mathrm{Aut}_P(A) \cong \Sigma_3$ by the extension axiom. When $P = P_1$, $O_3\big(N_\Gamma(\mathrm{Aut}_P(A))/\mathrm{Aut}_P(A)\big)$ lifts to a cyclic subgroup of order 12 in $\mathrm{Out}(P_1)$, and hence there is only one choice for the subgroup of order 3 in $\mathrm{Out}_{\mathcal{F}}(P_1)$. When $P = P_3$, it lifts to a subgroup isomorphic to $C_2 \times A_4$, so there are four subgroups of order 3, of which just two are normalized by $\mathrm{Out}_S(P_3)$. By direct computations, these two possibilities for $\mathrm{Out}_{\mathcal{F}}(P_3)$ are exchanged by χ_2, and fixed by χ_3 and χ_4.

If $S = S^{\mathbf{O'N}}$, then $\mathrm{Out}(P_2) \cong C_2 \times \Sigma_6$. We already saw that $O^{2'}(\mathrm{Out}_{\mathcal{F}}(P_2)) \cong A_5$, and hence $\mathrm{Out}_{\mathcal{F}}(P_2) \cong A_5$. Also, $\mathrm{Out}_{\mathcal{F}}(P_2) \le O^2(\mathrm{Out}(P_2)) \cong A_6$, A_6 contains 12 subgroups isomorphic to A_5 (six which act fixing a point and six which act transitively), and they are all conjugate in $\mathrm{Aut}(A_6)$ (see [Sz1, (3.2.19)]). Since each of those has five Sylow 2-subgroups all lying in the same A_6-conjugacy class, and A_6 has 30 subgroups isomorphic to $C_2 \times C_2$ (15 in each of two classes), we see by counting that each $C_2 \times C_2 \le A_6$ is contained in exactly two subgroups isomorphic to A_5. Thus there are two possibilities for $\mathrm{Out}_{\mathcal{F}}(P_2)$, and they are exchanged by χ_3 and fixed by $c_s \circ \chi_4$. (Recall that χ_4 does not normalize P_2.) We now conclude that there is (up to isomorphism) at most one reduced fusion system $\mathcal{F}$ over $S^{\mathrm{O'N}}$, and that $\mathrm{Out}(\mathcal{F}) = \langle [\chi_4] \rangle \cong C_2$. In particular, $\mathcal{F}$ is isomorphic to the fusion system of O'Nan's simple group, which is reduced by Proposition 1.17(d).

If $S = S^{\mathbf{HS}}$, then $\mathrm{Out}(P_2) \cong C_2 \times C_2 \times \Sigma_4$ contains exactly four subgroups of order three of which two are normalized by $\mathrm{Out}_S(P_2)$. Those two are exchanged by χ_3 and fixed by χ_4. Thus there is, up to isomorphism, at most one reduced fusion system $\mathcal{F}$ over

S^{HS}, and $\mathrm{Out}(\mathcal{F}) = \langle[\chi_4]\rangle \cong C_2$. In particular, $\mathcal{F}$ isomorphic to the fusion system of the Higman-Sims simple group, which is reduced by Proposition 1.17(d).

In both cases, since P_3 is the only $\mathcal{F}$-essential subgroup with noncyclic center, $\mathrm{Ker}(\mu_G) = 1$ for $G = \mathrm{O'N}$ or HS by Proposition 1.13. When $G = \mathrm{O'N}$, then by [O'N, Lemma 11.2], $|\mathrm{Out}(G)| \leq 2$. Also, G contains two conjugacy classes of subgroups isomorphic to $L_3(7)^*$ by [O'N, Lemma 10.6(iii)], where $L_3(7)^*$ denotes the extension of $PSL_3(7)$ by its graph automorphism (cf. [O'N, p. 471]). If $\mu_G \circ \kappa_G$ is not injective, then there is $\alpha \in \mathrm{Aut}(G) \smallsetminus \mathrm{Inn}(G)$ such that $\alpha|_S = \mathrm{Id}$, α exchanges the two G-conjugacy classes of subgroups isomorphic to $L_3(7)^*$ by [O'N, Lemma 11.1], and hence exchanges the two G-conjugacy classes of cyclic subgroups of order 16 ([O'N, Lemma 10.13]). (By [O'N, Lemma 4.3(vi)], G contains four classes of elements of order 16. Each element of order 16 in S has the form $(sa)^{\pm 1}$ for $a \in A$, $|\mathrm{Aut}_S(\langle sa\rangle)| = 4$ for each a, and thus there are only two classes of cyclic subgroups of order 16.) So $N_{\mathrm{Aut}(G)}(A) = N_{\mathrm{Inn}(G)}(A)\langle\alpha\rangle$ satisfies the hypotheses of [O'N, Lemma 11.3] (where V in [O'N] corresponds to A here), which is impossible by point (ii) in that lemma. Thus $\mu_G \circ \kappa_G$ is injective. Since $|\mathrm{Out}(G)| = 2$ (cf. [JW]), μ_G and κ_G are isomorphisms, and $\mathcal{F} = \mathcal{F}_S(G)$ is tame.

Now assume that $G = \mathrm{HS}$. Then $|\mathrm{Out}(G)| \geq 2$ by the construction of G in [HS], with equality by, e.g., [Ly2, pp. 10–11]. By [Fr, Table IIIb], there is no element $\alpha \in \mathrm{Aut}(G) \smallsetminus \mathrm{Inn}(G)$ such that $C_G(\alpha)$ contains a Sylow 2-subgroup of G. Thus $\mu_G \circ \kappa_G$ is injective, and hence (since μ_G is injective and $|\mathrm{Out}(G)| = |\mathrm{Out}(\mathcal{F})|$) μ_G and κ_G are both isomorphisms. So $\mathcal{F} = \mathcal{F}_S(G)$ is tame. $\square$

It remains only to consider 2-groups of type $PSp_6(3)$.

Proposition 6.4. *Let S be a group of order 512 of type $PSp_6(3)$, and thus isomorphic to $(Q_8 \wr C_2) \times_{C_2} Q_8$. Then every saturated fusion system over S with $O_2(\mathcal{F}) = 1$ is isomorphic to the fusion system of $PSp_6(3)$.*

Proof. Let $\mathbf{Q}_1, \mathbf{Q}_2 \leq S$ be the the two quaternion factors in $Q_8 \wr C_2$, and fix $t \in S$ such that $t^2 = 1$ and ${}^t\mathbf{Q}_1 = \mathbf{Q}_2$. Let $\mathbf{Q}_3 \leq S$ be the other factor, and set $R_0 = \mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3 \cong (Q_8)^3/C_2$. For $i = 1, 2, 3$, let $z_i \in Z(\mathbf{Q}_i)$ be the generator. Thus for distinct $i, j \in \{1, 2, 3\}$, $[\mathbf{Q}_i, \mathbf{Q}_j] = 1$ and $\mathbf{Q}_i \cap \mathbf{Q}_j = 1$. Also, $z_1z_2z_3 = 1$, $Z(R_0) = \langle z_1, z_2, z_3\rangle \cong C_2 \times C_2$, and $Z(S) = \langle z_1z_2\rangle = \langle z_3\rangle$.

Choose $\gamma \in \mathrm{Aut}(R_0)$ of order 3 that permutes the $\mathbf{Q}_i$ cyclically, and such that $\langle\gamma, c_t\rangle \cong \Sigma_3$. For $i = 1, 2, 3$, choose $\eta_i \in \mathrm{Aut}(R_0)$ such that $\eta_i|_{\mathbf{Q}_i}$ is an automorphism of order 3 and $\eta_i|_{\mathbf{Q}_j} = \mathrm{Id}$ for $i \neq j$. Assume also that we have done this in such a way that conjugation by γ and c_t in $\mathrm{Aut}(R_0)$ permutes the set $\{\eta_1, \eta_2, \eta_3\}$. In particular, $\langle\eta_1, \eta_2, \eta_3, \gamma, c_t\rangle \cong C_3 \wr \Sigma_3$. Also, since $\eta_1\eta_2$ and η_3 both commute with c_t, there are automorphisms $\eta_{12}^S, \eta_3^S \in \mathrm{Aut}(S)$ such that

$$\eta_{12}^S|_{R_0} = \eta_1\eta_2, \quad \eta_3^S|_{R_0} = \eta_3, \quad \text{and} \quad \eta_{12}^S(t) = t = \eta_3^S(t).$$

The groups $\mathrm{Out}(R_0)$ and $\mathrm{Out}(S)$ were described in [OV, Lemma 7.4(b,c,e)] (where they are denoted R_0 and R_2, respectively). Each automorphism of R_0 permutes the three subgroups $\mathbf{Q}_iZ(R_0)$. Let $\mathrm{Aut}^0(R_0) \trianglelefteq \mathrm{Aut}(R_0)$ be the subgroup of those elements which are the identity on $Z(R_0)$; equivalently, which normalize each $\mathbf{Q}_iZ(R_0)$. Set $\mathrm{Out}^0(R_0) = \mathrm{Aut}^0(R_0)/\mathrm{Inn}(R_0)$. Then

$$\mathrm{Out}^0(R_0) \cong \Sigma_4 \times \Sigma_4 \times \Sigma_4 \qquad \text{and} \qquad \mathrm{Out}(R_0) \cong \Sigma_4 \wr \Sigma_3. \tag{1}$$

The homomorphism

$$\mathrm{Out}(S) \xrightarrow{\ \cong\ } N_{\mathrm{Out}(R_0)}(\mathrm{Out}_S(R_0))/\mathrm{Out}_S(R_0) \cong \Sigma_4 \times \Sigma_4$$

induced by restriction is an isomorphism (by the proof of [OV, Lemma 7.4(e)]). In particular,

$$\langle \eta_1, \eta_2, \eta_3, \gamma\rangle \in \mathrm{Syl}_3(\mathrm{Aut}(R_0)) \quad \text{and} \quad \langle \eta_{12}^S, \eta_3^S\rangle \in \mathrm{Syl}_3(\mathrm{Aut}(S)). \tag{2}$$

We also need to work with the extraspecial subgroups

$$\mathbf{B}_0 = \langle z_1, z_2, t\rangle \times_{C_2} \mathbf{Q}_{12} \cong 2_-^{1+4} \qquad \text{and} \qquad \mathbf{B} = \mathbf{B}_0 \times_{C_2} \mathbf{Q}_3 \cong 2_+^{1+6}\,,$$

where $\mathbf{Q}_{12} = C_{\mathbf{Q}_1\mathbf{Q}_2}(t)$ is the "diagonal" subgroup in $\mathbf{Q}_1\mathbf{Q}_2$. Consider the following subgroups and inclusions in $\mathrm{Out}(\mathbf{B})$:

$$\underbrace{\mathrm{Out}_S(\mathbf{B})\langle[\eta_{12}^S|_{\mathbf{B}}]\rangle}_{\cong A_4} \times \underbrace{\langle[\eta_3^S|_{\mathbf{B}}]\rangle}_{\cong A_3} \le \underbrace{\mathrm{Out}(\mathbf{B}_0)}_{\cong \Sigma_5} \times \underbrace{\mathrm{Out}(\mathbf{Q}_3)}_{\cong \Sigma_3} \le \underbrace{\mathrm{Out}(\mathbf{B})}_{\cong \Sigma_8}\,. \tag{3}$$

Here, $\mathrm{Out}(\mathbf{B}_0) \cong SO_4^-(2) \cong \Sigma_5$ by [A1, Exercises 8.5(3) & 7.7(5)], and $\mathrm{Out}(\mathbf{B}) \cong SO_6^+(2) \cong \Sigma_8$ by [A1, Exercises 8.5(3) & 7.7(7)]. We are regarding $\mathrm{Out}(\mathbf{B_0})$ and $\mathrm{Out}(\mathbf{Q}_3)$ as subgroups of $\mathrm{Out}(\mathbf{B})$: those classes of automorphisms that are the identity on the other factor.

Fix a saturated fusion system $\mathcal{F}$ over S such that $O_2(\mathcal{F}) = 1$. By computer computations, R_0 and $\mathbf{B}$ are the only potentially critical subgroups of S, so $\mathbf{E}_{\mathcal{F}} = \{R_0, \mathbf{B}\}$ by Lemma 1.16.

Set $M = \langle \eta_{12}^S, \eta_3^S\rangle \in \mathrm{Syl}_3(\mathrm{Aut}(S))$. Choose $\varphi \in \mathrm{Aut}(S)$ such that ${}^{\varphi}\mathrm{Aut}_{\mathcal{F}}(S) \cap M \in \mathrm{Syl}_3({}^{\varphi}\mathrm{Aut}_{\mathcal{F}}(S))$. Upon replacing $\mathcal{F}$ by ${}^{\varphi}\mathcal{F}$, we can assume that $\mathrm{Aut}_{\mathcal{F}}(S)\cap M \in \mathrm{Syl}_3(\mathrm{Aut}_{\mathcal{F}}(S))$. Equivalently, since $\mathrm{Out}_{\mathcal{F}}(S)$ has odd order, it must be a 3-group, and hence $\mathrm{Aut}_{\mathcal{F}}(S) \le \mathrm{Inn}(S){\cdot}M$.

Set $\overline{\mathbf{B}} = \mathbf{B}/\langle z_3\rangle$, and more generally $\overline{P} = P/\langle z_3\rangle$ when $z_3 \in P \le \mathbf{B}$. By Lemma A.7, $\mathrm{Out}_{\mathcal{F}}(\mathbf{B}) = \Delta \times X$, where $\Delta \cong A_5$ and X has odd order. By Lemma A.8(a), and since $[\mathrm{Out}_S(\mathbf{B}), \overline{\mathbf{B}}] \nleq C_{\overline{\mathbf{B}}}(\mathrm{Out}_S(\mathbf{B}))$ (and $\mathrm{rk}([x, \overline{\mathbf{B}}]) = 2$ for $x \in S \smallsetminus \mathbf{B}$), we have $\overline{\mathbf{B}} = \overline{B}_1 \times \overline{B}_2$, where $\overline{B}_1 = [\Delta, \overline{\mathbf{B}}]$ is 4-dimensional and irreducible as an $\mathbb{F}_2[\Delta]$-module and $\overline{B}_2 = C_{\overline{\mathbf{B}}}(\Delta)$ is 2-dimensional. For $i = 1, 2$, let $B_i \le \mathbf{B}$ be such that $z_3 \in B_i$ and $B_i/\langle z_3\rangle = \overline{B}_i$. Thus $\overline{B}_2 \le C_{\overline{\mathbf{B}}}(\mathrm{Out}_S(\mathbf{B})) = \overline{\mathbf{Q}_3\langle z_1\rangle}$, so $B_2 \le \mathbf{Q}_3\langle z_1\rangle \cong Q_8 \times C_2$, and $z_1 \in [\mathrm{Aut}_S(\mathbf{B}), \mathbf{B}] \le B_1$. Hence $B_2 \cong Q_8$. Also, $B_1 = C_{\mathbf{B}}(B_2)$ since $\overline{B}_1 \cap \overline{C_{\mathbf{B}}(B_2)}$ is a nontrivial $\mathbb{F}_2[\Delta]$-submodule of $\overline{B}_1$. Hence $B_1 \cong 2_-^{1+4}$, so $\mathrm{Out}(B_1) \cong \Sigma_5$, and Δ has index two in $C_{\mathrm{Out}(\mathbf{B})}(\overline{B}_2) \cong \mathrm{Out}(B_1)$. In particular, $[\eta_{12}^S|_{\mathbf{B}}] \in \Delta \le \mathrm{Out}_{\mathcal{F}}(\mathbf{B})$ and hence $\eta_{12}^S|_{\mathbf{B}} \in \mathrm{Aut}_{\mathcal{F}}(\mathbf{B})$.

By the extension axiom, $\eta_{12}^S|_{\mathbf{B}} \in \mathrm{Aut}_{\mathcal{F}}(\mathbf{B})$ extends to an element $\eta \in \mathrm{Aut}_{\mathcal{F}}(S)$. Then $\eta|_{\mathbf{Q}_3} = \mathrm{Id}$, and since $\mathrm{Aut}_{\mathcal{F}}(S) \le \mathrm{Inn}(S){\cdot}M$, we have $\eta \in \mathrm{Inn}(S){\cdot}\eta_{12}^S$. Thus $\eta_{12}^S \in \mathrm{Aut}_{\mathcal{F}}(S)$, and $\mathrm{Out}_{\mathcal{F}}(S) = \langle[\eta_{12}^S]\rangle$ or $\langle[\eta_{12}^S], [\eta_3^S]\rangle$.

Now, $\langle z_3\rangle \ntrianglelefteq \mathcal{F}$ since $O_2(\mathcal{F}) = 1$. Since $\langle z_3\rangle = Z(S) = Z(\mathbf{B}) \cong C_2$, there is by Proposition 1.14 an element of $\mathrm{Aut}_{\mathcal{F}}(R_0)$ which does not fix z_3. Also, $Z(R_0) = \langle z_1, z_2\rangle \cong C_2 \times C_2$, where $z_3 = z_1z_2$, and $c_t \in \mathrm{Aut}_S(R_0)$ exchanges z_1 and z_2. So $\mathrm{Aut}_{\mathcal{F}}(Z(R_0)) \cong \Sigma_3$. By the extension axiom, each element of $\mathrm{Aut}_{\mathcal{F}}(Z(R_0))$ is the restriction of an element of $\mathrm{Aut}_{\mathcal{F}}(R_0)$. Hence there is $\gamma' \in \mathrm{Aut}_{\mathcal{F}}(R_0)$ of odd order such that $\gamma'|_{Z(R_0)} = \gamma|_{Z(R_0)}$. In particular, $\gamma' \notin \mathrm{Aut}^0(R_0)$.

Let $H_0 < H_i < H \trianglelefteq \mathrm{Aut}(R_0)$ ($i = 1, 2, 3$) be the subgroups

$$H_0 = O_2(\mathrm{Aut}(R_0))\,, \qquad H_i = H_0\langle \eta_i\rangle\,, \qquad H = H_1H_2H_3 \le \mathrm{Aut}^0(R_0).$$

By (1), $H/\mathrm{Inn}(R_0) \cong (A_4)^3$, and

$$H/H_0 = O_3(\mathrm{Aut}(R_0)/H_0) = (H_1/H_0) \times (H_2/H_0) \times (H_3/H_0) \cong (C_3)^3.$$

Consider the homomorphism

$$\Psi\colon \mathrm{Aut}(R_0) \xrightarrow{\qquad} \mathrm{Aut}(H/H_0) \cong GL_3(3)$$

induced by conjugation. Then $\mathrm{Ker}(\Psi) = H$, and $\mathrm{Im}(\Psi) \cong C_2 \wr \Sigma_3$ acts on the set $\{\eta_1^{\pm1}H_0, \eta_2^{\pm1}H_0, \eta_3^{\pm1}H_0\}$ as the group of signed permutations. Also, conjugation by c_t exchanges η_1 and η_2 and sends η_3 to itself. Since $\Psi(\gamma') \neq 1$ has odd order and $\langle [c_t] \rangle \in \mathrm{Syl}_2(\mathrm{Out}_{\mathcal{F}}(R_0))$ by the Sylow axiom, $\Psi(\mathrm{Aut}_{\mathcal{F}}(R_0)) = \langle \Psi(c_t), \Psi(\gamma') \rangle \cong \Sigma_3$, and $\Psi(\gamma')$ permutes cyclically the three cosets $\eta_1 H_0$, $\eta_2 H_0$, and $\eta_3^{\varepsilon} H_0$ for some $\varepsilon = \pm1$.

Choose $\beta \in \mathrm{Aut}(S)$ such that

$$\beta|_{\mathbf{Q}_1\mathbf{Q}_2} = \mathrm{Id}\,, \qquad \beta(t) = t\,, \qquad \beta(\mathbf{Q}_3) = \mathbf{Q}_3\,,$$

and such that β induces an automorphism of order 2 on $\mathbf{Q}_3/\langle z_3 \rangle$. Then conjugation by $[\beta|_{R_0}]$ in $\mathrm{Out}(R_0)$ fixes $[\eta_1]$ and $[\eta_2]$ and inverts $[\eta_3]$. So upon replacing $\mathcal{F}$ by ${}^{\beta}\mathcal{F}$ and γ' by $\beta\gamma'\beta^{-1}$ if necessary, we can assume that $\varepsilon = 1$ (i.e., $\Psi(\gamma')$ permutes cyclically the three cosets $\eta_i H_0 \in H/H_0$). Since $\beta \in N_{\mathrm{Aut}(S)}(\mathrm{Inn}(S)M)$, we still have $\mathrm{Out}_{\mathcal{F}}(S) = \langle [\eta_{12}^S] \rangle$ or $\langle [\eta_{12}^S], [\eta_3^S] \rangle$. In particular, $\eta_1\eta_2 = \eta_{12}^S|_{R_0} \in \mathrm{Aut}_{\mathcal{F}}(R_0)$.

Set $\mathrm{Aut}_{\mathcal{F}}^*(R_0) = \mathrm{Aut}_{\mathcal{F}}(R_0) \cap H$ for short. Since $1 \neq \eta_1\eta_2 H_0 \in \mathrm{Aut}_{\mathcal{F}}^*(R_0)H_0/H_0$, and each nontrivial subgroup of H/H_0 normalized by $\Psi(\gamma')$ contains the coset $\eta_1\eta_2\eta_3 H_0$ (since $C_{H/H_0}(\Psi(\gamma')) = \langle \eta_1\eta_2\eta_3 H_0 \rangle$), there is $\eta_3' \in \mathrm{Aut}_{\mathcal{F}}^*(R_0) \cap \eta_3 H_0$. Upon replacing η_3' by some appropriate power of η_3', if necessary, we can assume $|\eta_3'| = 3$. Since γ' normalizes $\mathrm{Aut}_{\mathcal{F}}^*(R_0)$, there are elements $\eta_i' \in \mathrm{Aut}_{\mathcal{F}}^*(R_0) \cap \eta_i H_0$ of order 3 for $i = 1, 2$. Thus $\mathrm{Aut}_{\mathcal{F}}(R_0) \geq \mathrm{Inn}(R_0)\langle \eta_1', \eta_2', \eta_3', \gamma', c_t \rangle$, with equality because $H_0 \cap \mathrm{Aut}_{\mathcal{F}}(R_0) = \mathrm{Inn}(R_0)$.

Since c_t normalizes $\mathrm{Aut}_{\mathcal{F}}^*(R_0) = \mathrm{Inn}(R_0)\langle \eta_1', \eta_2', \eta_3' \rangle$, conjugation by $[c_t]$ exchanges the classes $[\eta_1']$ and $[\eta_2']$ in $\mathrm{Out}_{\mathcal{F}}(R_0)$ and centralizes $[\eta_3']$. In particular, $\eta_1'\eta_2'$ and η_3' both normalize $\mathrm{Aut}_S(R_0)$, and hence by the extension axiom, both extend to elements of $\mathrm{Aut}_{\mathcal{F}}(S)$. Thus $\mathrm{Out}_{\mathcal{F}}(S) = \langle [\eta_{12}^S], [\eta_3^S] \rangle$, and so $\eta_1\eta_2, \eta_3 \in \mathrm{Aut}_{\mathcal{F}}(R_0)$ by restriction. Since $H_0 \cap \mathrm{Aut}_{\mathcal{F}}(R_0) = \mathrm{Inn}(R_0)$, it follows that $[\eta_1'\eta_2'] = [\eta_1\eta_2]$ and $[\eta_3'] = [\eta_3]$ in $\mathrm{Out}^0(R_0) \cong (\Sigma_4)^3$.

Recall that the $[\eta_i]$ and $[\eta_i']$ $(i = 1, 2, 3)$ all lie in $H/\mathrm{Inn}(R_0) \cong (A_4)^3$. For each i, there are four subgroups of order 3 in $H_i/\mathrm{Inn}(R_0) \cong A_4 \times (C_2)^4$, and they generate a subgroup $H_i^* \cong A_4$. Thus $[\eta_i], [\eta_i'] \in H_i^*$ for $i = 1, 2$. Also, $H_1^* \cap H_2^* = 1$ and $[\eta_1\eta_2] = [\eta_1'\eta_2']$, and we conclude that $[\eta_i] = [\eta_i'] \in \mathrm{Out}(R_0)$ for each $i = 1, 2, 3$. In particular, $\eta_1, \eta_2, \eta_3 \in \mathrm{Aut}_{\mathcal{F}}(R_0)$.

Set

$$K = \langle [\eta_1], [\eta_2], [\eta_3] \rangle = \mathrm{Aut}_{\mathcal{F}}^*(R_0)/\mathrm{Inn}(R_0) \leq \mathrm{Out}_{\mathcal{F}}(R_0).$$

Both $[\gamma]$ and $[\gamma']$ permute the elements $[\eta_i]$ cyclically under conjugation, and $\gamma|_{Z(R_0)} = \gamma'|_{Z(R_0)}$ by assumption. So $[\gamma^{-1}\gamma'] \in C_{\mathrm{Out}(R_0)}(K)$, where $C_{\mathrm{Out}(R_0)}(K) = K$ by (1) and since a 3-cycle is self-centralizing in Σ_4. Hence $\gamma \in \mathrm{Aut}_{\mathcal{F}}(R_0)$, and so

$$\mathrm{Aut}_{\mathcal{F}}(R_0) = \mathrm{Inn}(R_0)\langle \eta_1, \eta_2, \eta_3, \gamma, c_t \rangle, \quad \text{and} \quad \mathrm{Aut}_{\mathcal{F}}(S) = \mathrm{Inn}(S)\langle \eta_{12}^S, \eta_3^S \rangle.$$

Also, by (3) and since $[\eta_3^S|_{\mathbf{B}}] \in \mathrm{Out}(\mathbf{B})$ corresponds to a 3-cycle in Σ_8, $\mathrm{Out}_{\mathcal{F}}(\mathbf{B}) \cong A_5 \times C_3$ is the unique subgroup of this isomorphism type in $\mathrm{Out}(\mathbf{B}) \cong \Sigma_8$ which contains $\mathrm{Out}_S(\mathbf{B})$, $[\eta_{12}^S|_{\mathbf{B}}]$, and $[\eta_3^S|_{\mathbf{B}}]$.

We have now shown that each saturated fusion system $\mathcal{F}^*$ over S with $O_2(\mathcal{F}^*) = 1$ is isomorphic to $\mathcal{F}$. In particular, if $\mathcal{F}^*$ is the fusion system of $PSp_6(3)$, then $O_2(\mathcal{F}^*) = 1$ by Proposition 1.17(b), and so $\mathcal{F} \cong \mathcal{F}^*$. □

7. Fusion systems over a 2-group of type A_{12}

Throughout this section, we fix the following notation for certain elements of A_{12}:

$$a_1 = (1\,2)(3\,4) \qquad a_2 = (5\,6)(7\,8) \qquad a_3 = (9\,10)(11\,12)$$
$$b_1 = (1\,3)(2\,4) \qquad b_2 = (5\,7)(6\,8) \qquad b_3 = (9\,11)(10\,12)$$
$$\mu_{12} = (1\,2)(5\,6) \qquad \mu_{23} = (5\,6)(9\,10) \qquad \tau = (1\,5)(2\,6)(3\,7)(4\,8)$$

and set $S = \big\langle a_1, a_2, a_3, b_1, b_2, b_3, \mu_{12}, \mu_{23}, \tau \big\rangle \in \mathrm{Syl}_2(A_{12})$. We need to consider the following subgroups of S:

$$A = \langle a_1, a_2, a_3, b_1, b_2, b_3 \rangle \cong (C_2)^6 \qquad Q = \langle a_1, a_2, b_1b_2, \mu_{12}, \tau \rangle \cong 2^{1+4}_+$$
$$N_1 = A\langle \mu_{12}, \tau \rangle \cong ((C_2)^4 \rtimes (C_2 \times C_2)) \times C_2 \times C_2 \qquad H_1 = A\langle \tau \rangle$$
$$N_2 = A\langle \mu_{12}, \mu_{23} \rangle \qquad H_2 = A\langle \mu_{23} \rangle$$
$$N_3 = Q\langle a_3, b_3, \mu_{23} \rangle \qquad N_{13} = N_1 \cap N_3 = Q \times \langle a_3, b_3 \rangle.$$

Note that A, Q, N_1, N_2, and N_3 are all normal in S.

Lemma 7.1. *Assume the above notation. Then A is the unique elementary abelian subgroup of rank 6 in S, and hence is characteristic in all subgroups which contain it. The subgroups N_1, N_2, N_3, and N_{13} are all characteristic in S, and N_{13} is characteristic in N_1 and in N_3.*

Proof. Assume A is not unique: let $P < S$ be such that $P \neq A$ and $P \cong (C_2)^6$. Then $P \cap A \leq C_A(PA/A)$, and hence $\mathrm{rk}(C_A(PA/A)) \geq 6 - \mathrm{rk}(PA/A)$. It is straightforward to check that $S/A \cong D_8$, that $\mathrm{rk}(C_A(g)) = 4$ for $g \in S/A$ of order 2, and that $\mathrm{rk}(C_A(V)) = 3$ if $V < S/A$ and $V \cong C_2 \times C_2$. So there is no such subgroup P, and A is the unique elementary abelian subgroup of rank 6. In particular, A is characteristic in every subgroup of S that contains it.

If $N < S$ is such that $N \geq A$ and $N/A \cong C_2 \times C_2$, then $N = N_1$ or N_2. Since $N_1 \not\cong N_2$ ($[N_2, N_2] = Z(N_2)$ while $[N_1, N_1] \neq Z(N_1)$), both subgroups are characteristic in S.

Set $Z = \langle a_1a_2 \rangle = [N_1, N_1] \cap Z(N_1)$: a subgroup characteristic in N_1 and in S. Then $N_{13}/Z \cong (C_2)^6$, $S/N_{13} \cong C_2 \times C_2$, $\mathrm{rk}(C_{N_{13}/Z}(g)) \leq 4$ for each $g \in S \smallsetminus N_{13}$, and $\mathrm{rk}(C_{N_{13}/Z}(S)) = 2$. So N_{13}/Z is the unique elementary abelian subgroup of S/Z of rank 6, and N_{13} is characteristic in N_1 and in S. Also, the three subgroups of S containing N_{13} with index 2 (N_1, N_3, and $N_{13}\langle b_1\mu_{23} \rangle$) are pairwise nonisomorphic — they have commutator subgroups $\langle a_1, a_2, b_1b_2 \rangle$, $\langle a_1, a_2, a_3, \mu_{12} \rangle$, and $\langle a_1, a_2, a_3, b_1b_2\mu_{12} \rangle$, respectively — and hence all three are characteristic in S.

Since $N_{13}/Z(N_3) \cong (C_2)^5$ is the unique abelian subgroup of index two in $N_3/Z(N_3)$, N_{13} is characteristic in N_3. □

Lemma 7.2. *Assume the above notation.*

(a) *There is an automorphism $\nu_3 \in \mathrm{Aut}(N_3)$ of order 3 which takes values as follows:*

g	a_1	a_2	a_3	b_3	b_1b_2	μ_{12}	μ_{23}	τ
$\nu_3(g)$	$a_2\mu_{12}$	$a_1\mu_{12}$	a_3	b_3	$a_1a_2b_1b_2\tau$	a_2	μ_{23}	b_1b_2

The action of ν_3 on

$$Q = \langle a_1b_1b_2\mu_{12}, a_1\tau \rangle \times_{\langle a_1a_2 \rangle} \langle a_1b_1b_2\tau, b_1b_2\mu_{12} \rangle \cong Q_8 \times_{C_2} Q_8$$

has order three on each factor. Also, ν_3 commutes in $\mathrm{Aut}(Q)$ with $c_{\mu_{23}}$, where $c_{\mu_{23}}$ exchanges the two quaternion factors.

(b) $\mathrm{Aut}(N_3)/O_2(\mathrm{Aut}(N_3)) \cong \Sigma_3$, *generated by the classes of* ν_3 *and* c_{b_1}.

(c) *If* $\alpha \in \mathrm{Aut}(N_3)$ *has order* 3, *then* $\alpha|_{Z(N_{13})} = \mathrm{Id}$, *and* α *acts on* $N_{13}/Z(N_{13}) \cong Q/Z(Q)$ *with* $C_{N_{13}/Z(N_{13})}(\alpha) = 1$. *If in addition,* $\alpha(Q) = Q$, *then* $\langle {}^{\varphi}\alpha\rangle = \langle \nu_3\rangle$ *for some* $\varphi \in \mathrm{Aut}_Q(N_3)$.

(d) *Let* $\Delta \leq \mathrm{Aut}(N_3)$ *be such that* $\Delta > \mathrm{Aut}_S(N_3)$ *and* $\Delta/\mathrm{Inn}(N_3) \cong \Sigma_3$. *Then there is* $\varphi \in \mathrm{Aut}(S)$ *such that* ${}^{\varphi}\Delta = \langle \mathrm{Aut}_S(N_3), \nu_3\rangle$.

Proof. **(a)** The first two statements are easily checked. Also, $\big[\nu_3|_Q, c_{\mu_{23}}\big] = \mathrm{Id}$ in $\mathrm{Aut}(Q)$ since ν_3 is a homomorphism and $\nu_3(\mu_{23}) = \mu_{23}$.

(b) Consider the chain of subgroups $P_0 < P_1 < P_2 < N_3$, where

$$P_0 = \Phi(N_3) = \langle a_1, a_2, a_3, \mu_{12}\rangle, \quad P_1 = P_0 Z(N_{13}) = P_0\langle b_3\rangle, \quad P_2 = N_{13} = P_1\langle b_1b_2, \tau\rangle.$$

Since N_{13} is characteristic in N_3 by Lemma 7.1, each of the P_i is characteristic in N_3. So by Lemma A.1, the kernel of the induced homomorphism $\mathrm{Aut}(N_3) \longrightarrow \mathrm{Aut}(P_2/P_1) \cong \Sigma_3$ is contained in $O_2(\mathrm{Aut}(N_3))$. Also, the images of ν_3 and c_{b_1} in $\mathrm{Aut}(P_2/P_1)$ generate this group, so $\mathrm{Aut}(N_3)/O_2(\mathrm{Aut}(N_3)) \cong \Sigma_3$.

(c) Assume $\alpha \in \mathrm{Aut}(N_3)$ has order 3. Since α normalizes the chain

$$\underset{=\langle a_1a_2\rangle}{[N_{13}, N_{13}]} < \underset{=\langle a_1a_2, a_3\rangle}{Z(N_3)} < \underset{=\langle a_1a_2, a_3, b_3\rangle}{Z(N_{13})}$$

of characteristic subgroups, $\alpha|_{Z(N_{13})} = \mathrm{Id}$ by Lemma A.1.

Consider the subgroups $P_1 = Z(N_{13})\langle a_2, \mu_{12}\rangle$ and $N_{13} = P_1\langle b_1b_2, \tau\rangle$. Since

$$[\mu_{23}, b_1b_2] = a_2 \quad \text{and} \quad [\mu_{23}, \tau] = \mu_{12}\,,$$

$[\mu_{23}, -]$ sends N_{13}/P_1 isomorphically to $P_1/Z(N_{13})$, and this isomorphism commutes with α since $\alpha(\mu_{23}) \in \mu_{23}N_{13}$. So α induces an automorphism of order 3 on $P_1/Z(N_{13})$ since it induces an automorphism of order 3 on N_{13}/P_1 by the proof of (b). Thus α acts on $N_{13}/Z(N_{13}) \cong Q/Z(Q)$ with $C_{N_{13}/Z(N_{13})}(\alpha) = 1$.

Now assume in addition that $\alpha(Q) = Q$. Then $C_Q(\alpha) = Z(Q)$, so α and ν_3 both act nontrivially on each of the quaternion factors of Q. They also commute with $c_{\mu_{23}}|_Q$ modulo $\mathrm{Aut}_{N_{13}}(Q) = \mathrm{Inn}(Q)$, where $c_{\mu_{23}}$ exchanges the two quaternion factors of Q. Thus $\alpha|_Q$ is congruent to $\nu_3|_Q$ or $\nu_3^{-1}|_Q$ modulo $\mathrm{Inn}(Q)$. Hence $\langle \alpha|_Q\rangle$ and $\langle \nu_3|_Q\rangle$ are Sylow 3-subgroups of $\mathrm{Inn}(Q)\langle \alpha|_Q\rangle$, so upon replacing α by α^{-1} if necessary, we can assume that ${}^{\varphi}\alpha|_Q = \nu_3|_Q$ for some $\varphi \in \mathrm{Aut}_Q(N_3)$. We already showed that $\alpha|_{Z(N_{13})} = \mathrm{Id}$, and so ${}^{\varphi}\alpha|_{N_{13}} = \nu_3|_{N_{13}}$.

Thus ${}^{\varphi}\alpha = \beta \circ \nu_3$ for some $\beta \in \mathrm{Aut}(N_3)$ such that $\beta|_{N_{13}} = \mathrm{Id}$. Then $\beta(\mu_{23}) = g\mu_{23}$ for some $g \in Z(N_{13})$, and in particular, β has order at most 2 and commutes with ν_3. Thus $\beta = \mathrm{Id}$ since $\beta \circ \nu_3$ has order 3, and hence ${}^{\varphi}\alpha = \nu_3$.

(d) Set $T_3 = \langle a_3, b_3\rangle \trianglelefteq S$; thus $N_{13} = Q \times T_3$. For each $\chi \in \mathrm{Hom}(Q, T_3)$, set $Q_\chi = \{g\chi(g) \,|\, g \in Q\}$. Each subgroup of $N_{13} = Q \times T_3$ isomorphic to Q is sent isomorphically to Q by projection to the first factor, and hence is equal to Q_χ for some unique χ.

Let $\mathcal{Q} \supseteq \mathcal{Q}_0$ be the sets of subgroups of N_{13} isomorphic to Q which are normal in N_3 or in S, respectively. For example, $Q \in \mathcal{Q}_0$. Then

$$\mathcal{Q} = \{Q_\chi \,|\, \chi \in \mathrm{Hom}_{\mathrm{Aut}_{N_3}(N_{13})}(Q, T_3)\} \qquad \text{and} \qquad \mathcal{Q}_0 = \{Q_\chi \,|\, \chi \in \mathrm{Hom}_{\mathrm{Aut}_S(N_{13})}(Q, T_3)\},$$

where $\mathrm{Hom}_X(Q, T_3)$ denotes the set of group homomorphisms from Q to T_3 which commute with the action of X.

If $\chi \in \mathrm{Hom}(Q, T_3)$ is such that $\nu_3(Q_\chi) = Q_\chi$, then $\chi(\nu_3(g)) = \chi(g)$ for each $g \in Q$ (recall $\nu_3|_{T_3} = \mathrm{Id}$), and $\chi = 1$ since $Q = [\nu_3, Q]$. Thus Q is the only member of $\mathcal{Q}$ normalized by ν_3.

Fix $\Delta \le \mathrm{Aut}(N_3)$ such that $\Delta > \mathrm{Aut}_S(N_3)$ and $\Delta/\mathrm{Inn}(N_3) \cong \Sigma_3$. Choose $\alpha \in \Delta$ of order three. Since $\mathrm{Aut}(N_3)/O_2(\mathrm{Aut}(N_3)) \cong \Sigma_3$ by (b), there is $\psi \in O_2(\mathrm{Aut}(N_3))$ such that $\langle {}^\psi\alpha \rangle = \langle \nu_3 \rangle$. Let χ be such that $Q_\chi = \psi^{-1}(Q) \in \mathcal{Q}$. Since Q is the only member of $\mathcal{Q}$ normalized by ν_3, Q_χ is the only member normalized by α. Also, c_{b_1} normalizes $\mathrm{Inn}(N_3)\langle \alpha \rangle$, so ${}^{b_1}Q_\chi = Q_\chi$ by the uniqueness of Q_χ, and hence $Q_\chi \trianglelefteq S$ and $Q_\chi \in \mathcal{Q}_0$. In particular, χ commutes with the action of $\mathrm{Aut}_S(N_{13})$.

We claim that there is $\varphi \in \mathrm{Aut}(S)$ such that $\varphi|_A = \mathrm{Id}$, $\varphi(\mu_{23}) = \mu_{23}$, $\varphi(\mu_{12}) = \mu_{12}\chi(\mu_{12})$, and $\varphi(\tau) = \tau\chi(\tau)$. Since χ commutes with the action of $c_{b_1} \in \mathrm{Aut}_S(N_{13})$ and $[b_1, T_3] = 1$, we have $Q \cap A = [b_1, Q] \le \mathrm{Ker}(\chi)$. Thus $\varphi(g) = g\chi(g)$ for all $g \in Q$. Since $T_3 \le Z(N_1)$ (where $N_1 = AQ$), $\varphi|_{N_1}$ is a well defined automorphism. To see that φ is well defined, it remains to check that $\chi({}^{\mu_{23}}g) = {}^{\mu_{23}}\chi(g)$ for all $g \in Q$, and this holds since χ commutes with the action of $\mathrm{Aut}_S(N_{13})$.

By construction, $\varphi(Q_\chi) = Q$. So upon replacing Δ by ${}^\varphi\Delta$ and α by ${}^\varphi\alpha$, we can assume that $\alpha(Q) = Q$. By (c), $\langle {}^\eta\alpha \rangle = \langle \nu_3 \rangle$ for some $\eta \in \mathrm{Aut}_Q(N_3)$. Thus $\alpha \equiv \nu_3^{\pm 1}$ (mod $\mathrm{Inn}(N_3)$), so $\Delta = \langle \mathrm{Aut}_S(N_3), \alpha \rangle = \langle \mathrm{Aut}_S(N_3), \nu_3 \rangle$. □

Lemma 7.3. *Assume the above notation. Set $A_0 = Z(N_2) = \langle a_1, a_2, a_3 \rangle$, and let*

$$R_0 \colon \mathrm{Aut}(N_2) \longrightarrow \mathrm{Aut}(A_0)$$

be the homomorphism induced by restriction.

(a) *There is an automorphism $\nu_2 \in \mathrm{Aut}(N_2)$ of order 3 which takes values as follows:*

g	a_1	b_1	a_2	b_2	a_3	b_3	μ_{12}	μ_{23}
$\nu_2(g)$	a_2	b_2	a_3	b_3	a_1	b_1	μ_{23}	$\mu_{12}\mu_{23}$

(b) *For each $\beta \in \mathrm{Ker}(R_0)$, β induces the identity on N_2/A_0. Furthermore, $\mathrm{Ker}(R_0) = O_2(\mathrm{Aut}(N_2))$ is an elementary abelian 2-group, and $\mathrm{Im}(R_0) = \langle \nu_2|_{A_0}, c_\tau|_{A_0} \rangle \cong \Sigma_3$.*

(c) *Let $\Delta \le \mathrm{Aut}(N_2)$ be such that $\Delta > \mathrm{Aut}_S(N_2)$ and $\Delta/\mathrm{Inn}(N_2) \cong \Sigma_3$. Then there is $\xi \in \mathrm{Aut}(S)$ such that ${}^\xi\Delta = \langle \mathrm{Aut}_S(N_2), \nu_2 \rangle$ and $\xi(\tau) = \tau$.*

Proof. Throughout the proof, we set $K = \mathrm{Ker}(R_0)$.

(a) This is easily checked.

(b) Fix $\beta \in K$. Then $\beta(A) = A$ since A is characteristic in N_2 by Lemma 7.1. The commutators

$$[\mu_{12}, A] = \langle a_1, a_2 \rangle, \qquad [\mu_{23}, A] = \langle a_2, a_3 \rangle, \qquad \text{and} \qquad [\mu_{12}\mu_{23}, A] = \langle a_1, a_3 \rangle$$

are all distinct, so for $x, y \in N_2$ such that $[x, A] = [y, A]$, we have $x \equiv y$ (mod A). The equalities $[\beta(x), A] = \beta([x, A]) = [x, A]$ for $x \in N_2$ now show that β induces the identity on N_2/A. Thus β normalizes $C_A(\mu_{12}) = A_0\langle b_3 \rangle$, $C_A(\mu_{23}) = A_0\langle b_1 \rangle$, and $C_A(\mu_{12}\mu_{23}) = A_0\langle b_2 \rangle$, and hence induces the identity on A/A_0.

Let $h_{12}, h_{23} \in A$ be such that $\beta(\mu_{ij}) = \mu_{ij}h_{ij}$. Then $h_{12} \in C_A(\mu_{12}) = A_0\langle b_3 \rangle$ since $(\mu_{12}h_{12})^2 = 1$, and $h_{23} \in A_0\langle b_1 \rangle$ and $h_{12}h_{23} \in A_0\langle b_2 \rangle$ by similar arguments. Hence $h_{12}, h_{23} \in A_0$, and so β induces the identity on N_2/A_0. Also, $\beta \in O_2(\mathrm{Aut}(N_2))$ by Lemma A.1.

Thus $K \le O_2(\mathrm{Aut}(N_2))$. Each $\alpha \in \mathrm{Aut}(N_2)$ permutes the subgroups $[x, A]$ for $x \in \{\mu_{12}, \mu_{23}, \mu_{12}\mu_{23}\}$; and thus permutes their pairwise intersections $\langle a_i\rangle$ for $i = 1, 2, 3$. So $\mathrm{Im}(R_0) \cong \Sigma_3$, generated by $\nu_2|_{A_0}$ and $c_\tau|_{A_0}$, and $K = O_2(\mathrm{Aut}(N_2))$. Also, K is elementary abelian, since each $\beta \in K$ has the form $\beta(g) = g\chi(g)$ for some $\chi \in \mathrm{Hom}(N_2, A_0) \cong (C_2)^{15}$ (and the resulting bijection $K \cong \mathrm{Hom}(N_2, A_0)$ is an isomorphism).

(c) Let $\Delta \le \mathrm{Aut}(N_2)$ be a subgroup such that $\Delta > \mathrm{Aut}_S(N_2)$ and $\Delta/\mathrm{Inn}(N_2) \cong \Sigma_3$. Thus $\Delta \cap K = \mathrm{Inn}(N_2)$. By Proposition A.3, applied with $\mathrm{Out}(N_2)$, $K/\mathrm{Inn}(N_2)$, $\langle \mathrm{Out}_S(N_2), [\nu_2]\rangle$, and $\mathrm{Out}_S(N_2)$ in the roles of G, Q, H, and H_0, there is $\xi_0 \in K$ such that $[\xi_0, c_\tau] \in \mathrm{Inn}(N_2)$ and ${}^{\xi_0}\Delta = \langle \nu_2, \mathrm{Aut}_S(N_2)\rangle$. Since $K = C_K(\nu_2) \times [\nu_2, K]$ (see [G, Theorem 5.2.3]) and both factors are normalized by c_τ, we can choose $\xi_0 \in [\nu_2, K]$.

Now, $[\nu_2, \mathrm{Inn}(N_2)]$ has an $\mathbb{F}_2$-basis $\{c_{b_1b_3}, c_{b_2b_3}, c_{\mu_{12}\mu_{23}}, c_{\mu_{23}}\}$ permuted freely by c_τ. Hence each element of $[\nu_2, K]/[\nu_2, \mathrm{Inn}(N_2)]$ which is centralized by c_τ lifts to an element of $[\nu_2, K]$ which commutes with c_τ in $\mathrm{Aut}(N_2)$. In particular, there is $\xi_1 \equiv \xi_0 \pmod{[\nu_2, \mathrm{Inn}(N_2)]}$ such that $\xi_1 \in C_K(c_\tau)$. Since ξ_1 commutes with c_τ in $\mathrm{Aut}(N_2)$, it extends to an element $\xi \in \mathrm{Aut}(S)$ such that $\xi(\tau) = \tau$, and we still have ${}^{\xi}\Delta = \langle \nu_2, \mathrm{Aut}_S(N_2)\rangle$. □

The following proposition is essentially a special case of the main theorem in Ron Solomon's paper [So1], where he lists the finite simple groups (more generally, the "fusion-simple" groups) with Sylow 2-subgroups isomorphic to S. (See also [So2, Theorem 1.1].) But our method of proof, based on analysis of the possible essential subgroups, is somewhat different.

Proposition 7.4. *Let S be a 2-group of type A_{12}. Then every reduced fusion system over S is isomorphic to the fusion system of one of the groups A_{12}, $Sp_6(2)$, or $\Omega_7(3)$. The subgroups N_1, N_2, and N_3 are essential in all three fusion systems, H_1 is essential in the fusion system of A_{12}, and these are the only essential subgroups up to S-conjugacy.*

Proof. By a computer search, the only potentially critical subgroups of S (up to conjugacy) are N_1, N_2, N_3, H_1, and H_2. Of these, the N_i are characteristic of index two in S (Lemma 7.1), while $N_S(H_i) = N_i$ for $i = 1, 2$. Also, $\mathrm{Aut}(S)$ is a 2-group by Lemma A.1, applied to the sequence $\Phi(S) < N_1 \cap N_2 \cap N_3 < N_1 \cap N_2 < N_1 < S$ of characteristic subgroups of S.

Assume $\mathcal{F}$ is a reduced fusion system over S. Since $\mathrm{Out}(S)$ is a 2-group, $\mathrm{Out}_{\mathcal{F}}(S) = 1$. By Lemma 1.15,

$$S = \big\langle [\mathrm{Aut}_{\mathcal{F}}(P), P] \,\big|\, P \in \mathbf{E}_{\mathcal{F}} \big\rangle. \tag{1}$$

Step 1: $\mathbf{E}_{\mathcal{F}} \supsetneqq \{N_2, N_3\}$. By Lemma 7.1, A is a characteristic subgroup of each subgroup of S that contains A. Since N_3 is the only potentially critical subgroup that does not contain A, it must be essential, since otherwise $A \trianglelefteq \mathcal{F}$ by Proposition 1.14.

Since N_{13} is characteristic of index 2 in N_3 by Lemma 7.1, $[\mathrm{Aut}_{\mathcal{F}}(N_3), N_3] \le N_{13} \le N_1$. Since A is characteristic of index two in H_2 by Lemma 7.1, $[\mathrm{Aut}_{\mathcal{F}}(H_2), H_2] \le A \le N_1$. So $N_2 \in \mathbf{E}_{\mathcal{F}}$, since otherwise $[\mathrm{Aut}_{\mathcal{F}}(P), P] \le N_1$ for all $P \in \mathbf{E}_{\mathcal{F}}$, contradicting (1).

Since $\mathrm{Aut}(N_3)/O_2(\mathrm{Aut}(N_3)) \cong \Sigma_3$ by Lemma 7.2(b), $\mathrm{Out}_{\mathcal{F}}(N_3) \cong \Sigma_3$. By Lemma 7.2(d), there is $\varphi \in \mathrm{Aut}(S)$ such that ${}^{\varphi}\mathrm{Aut}_{\mathcal{F}}(N_3) = \langle \mathrm{Aut}_S(N_3), \nu_3\rangle$. So upon replacing $\mathcal{F}$ by ${}^{\varphi}\mathcal{F}$, we can assume that $\mathrm{Aut}_{\mathcal{F}}(N_3) = \langle \mathrm{Aut}_S(N_3), \nu_3\rangle$.

Since $\mathrm{Aut}(N_2)/O_2(\mathrm{Aut}(N_2)) \cong \Sigma_3$ by Lemma 7.3(b), $\mathrm{Out}_{\mathcal{F}}(N_2) \cong \Sigma_3$. By Lemma 7.3(c), there is $\xi \in \mathrm{Aut}(S)$ such that $\xi(\tau) = \tau$ and ${}^{\xi}\mathrm{Aut}_{\mathcal{F}}(N_2) = \langle \nu_2, \mathrm{Aut}_S(N_2)\rangle$. Upon replacing $\mathcal{F}$ by ${}^{\xi}\mathcal{F}$, we can assume that $\mathrm{Aut}_{\mathcal{F}}(N_2) = \langle \nu_2, \mathrm{Aut}_S(N_2)\rangle$. Also, ξ normalizes

$Q = \langle \tau, [\tau, S] \rangle$, so $({}^{\xi}\nu_3)(Q) = Q$, and ${}^{\xi}\nu_3 \in \mathrm{Inn}(N_3)\langle \nu_3 \rangle$ by Lemma 7.2(c). So we still have $\mathrm{Aut}_{\mathcal{F}}(N_3) = \langle \mathrm{Aut}_S(N_3), \nu_3 \rangle$.

By a direct computation, $[\nu_3, N_3] = Q$, and hence

$$[\mathrm{Aut}_{\mathcal{F}}(N_2), N_2][\mathrm{Aut}_{\mathcal{F}}(N_3), N_3] = \langle a_1, a_2, a_3, b_1b_2, b_2b_3, \mu_{12}, \mu_{23}, \tau \rangle < S\,. \qquad (2)$$

So by (1), $\mathbf{E}_{\mathcal{F}} \supsetneqq \{N_2, N_3\}$.

Step 2: $H_2 \notin \mathbf{E}_{\mathcal{F}}$, and at least one of N_1, H_1 lies in $\mathbf{E}_{\mathcal{F}}$. Since $\nu_2 \in \mathrm{Aut}_{\mathcal{F}}(N_2)$, H_2 is $\mathcal{F}$-conjugate to $\nu_2^{-1}(H_2) = A\langle \mu_{12} \rangle$. Since $A\langle \mu_{12} \rangle$ is normal in S, H_2 is not fully normalized in $\mathcal{F}$, and hence is not $\mathcal{F}$-essential. Thus at least one of the subgroups N_1 or H_1 is $\mathcal{F}$-essential.

Step 3: $\mathrm{Aut}_{\mathcal{F}}(N_1)$ normalizes Q. Set $T_3 = \langle a_3, b_3 \rangle$, so that $N_{13} = N_1 \cap N_3 = Q \times T_3$. Recall that N_{13} is characteristic in N_1 by Lemma 7.1.

By [O1, Proposition 3.2(b,c)], $\mathrm{Out}(N_{13})/O_2(\mathrm{Out}(N_{13})) \cong \mathrm{Out}(Q) \times \mathrm{Aut}(T_3) \cong (\Sigma_3 \wr C_2) \times \Sigma_3$. The automorphisms c_{b_1} and $c_{\mu_{23}}$ both act nontrivially on $Q/Z(Q)$, and only the second acts nontrivially on T_3. Hence $\mathrm{Out}_S(N_{13}) = \langle [c_{b_1}], [c_{\mu_{23}}] \rangle$ embeds into the quotient group $\mathrm{Out}(N_{13})/O_2(\mathrm{Out}(N_{13}))$. Since $N_{13} \trianglelefteq S$, $\mathrm{Out}_S(N_{13}) \in \mathrm{Syl}_2(\mathrm{Out}_{\mathcal{F}}(N_{13}))$, and so $\mathrm{Out}_{\mathcal{F}}(N_{13}) \cap O_2(\mathrm{Out}(N_{13})) = 1$. Hence $\mathrm{Out}_{\mathcal{F}}(N_{13})$ also embeds into $(\Sigma_3 \wr C_2) \times \Sigma_3$.

Thus any two elements of odd order in $\mathrm{Out}_{\mathcal{F}}(N_{13})$ commute. So if $\alpha \in \mathrm{Aut}_{\mathcal{F}}(N_1)$ has odd order, then $\big[\alpha|_{N_{13}}, \nu_3|_{N_{13}}\big] \in \mathrm{Inn}(N_{13})$. Since $[N_{13}, N_{13}] \le Q$, this shows that α normalizes $Q = [\nu_3, N_{13}]$. So $O^2(\mathrm{Aut}_{\mathcal{F}}(N_1))$ normalizes Q, $\mathrm{Aut}_S(N_1) \in \mathrm{Syl}_2(\mathrm{Aut}_{\mathcal{F}}(N_1))$ normalizes Q since $Q \trianglelefteq S$, and hence $\mathrm{Aut}_{\mathcal{F}}(N_1)$ normalizes Q.

We claim that in fact,

$$\mathrm{Aut}_{\mathcal{F}}(N_1) = \big\{ \alpha \in \mathrm{Aut}(N_1) \,\big|\, \alpha|_A \in \mathrm{Aut}_{\mathcal{F}}(A),\ \alpha(Q) = Q \big\}. \qquad (3)$$

Since A is characteristic in N_1 by Lemma 7.1, $\mathrm{Aut}_{\mathcal{F}}(N_1)$ normalizes A as well as Q, and hence $\mathrm{Aut}_{\mathcal{F}}(N_1)$ is contained in the right hand side of (3).

If $\alpha \in \mathrm{Aut}(N_1)$ is such that $\alpha|_A \in \mathrm{Aut}_{\mathcal{F}}(A)$ and $\alpha(Q) = Q$, then $\alpha|_A$ normalizes $\mathrm{Aut}_{N_1}(A)$. So by the extension axiom (and since $C_S(A) = A \le N_1$), there is $\beta \in \mathrm{Aut}_{\mathcal{F}}(N_1)$ such that $\beta|_A = \alpha|_A$. Set $P = \langle a_1, b_1, a_2, b_2, \tau, \mu_{12} \rangle = Q\langle b_1 \rangle$. Then $\alpha^{-1}\beta$ induces the identity on A, hence on $P \cap A$, and it normalizes P since it normalizes Q. So by Lemma A.2, and since $P \cap A$ is centric in P and $\mathrm{Aut}_P(P \cap A) = \langle c_\tau, c_{\mu_{12}} \rangle$ permutes freely the basis $\{b_1, a_1b_1, b_2, a_2b_2\}$ of $P \cap A$, we have $(\alpha^{-1}\beta)|_P \in \mathrm{Inn}(P)$. Let $x \in P$ be such that $(\alpha^{-1}\beta)|_P = c_x|_P$; then $x \in C_P(P \cap A) = P \cap A$ since $(\alpha^{-1}\beta)|_A = \mathrm{Id}$. Hence $(\alpha^{-1}\beta)|_A = \mathrm{Id} = c_x|_A$, and since $N_1 = AP$, we have $\alpha^{-1}\beta = c_x \in \mathrm{Inn}(N_1)$. Thus $\alpha \in \mathrm{Aut}_{\mathcal{F}}(N_1)$, and this finishes the proof of (3).

Step 4: The group $\mathrm{Aut}_{\mathcal{F}}(A)$. Set

$$\mathit{\Gamma} = \mathrm{Aut}_{\mathcal{F}}(A) \qquad \text{and} \qquad \mathit{\Gamma}_0 = \langle \mathrm{Aut}_S(A), \nu_2|_A \rangle \cong \Sigma_4\,. \qquad (4)$$

Thus $O_2(\mathit{\Gamma}_0) = \langle c_{\mu_{12}}, c_{\mu_{23}} \rangle$, and $\mathrm{Aut}_S(A) \cong D_8$ is a Sylow 2-subgroup of $\mathit{\Gamma}$. We will show that

$$\mathit{\Gamma} \cong A_7, \quad GL_3(2), \quad \text{or} \quad (C_3)^3 \rtimes \Sigma_4\,. \qquad (5)$$

By the Gorenstein-Walter theorem on groups with dihedral Sylow 2-subgroups [GW, Theorem 1], $\mathit{\Gamma}/O_{2'}(\mathit{\Gamma})$ is isomorphic to a subgroup of $P\mathit{\Gamma}L_2(q)$ which contains $PSL_2(q)$ (q odd), to A_7, or to D_8. In the first case, $|PSL_2(q)|$ divides $|\mathit{\Gamma}|$ and $|\mathit{\Gamma}|$ divides $|GL_6(2)|$, and since $16 \nmid |PSL_2(q)|$, we have $q \le 9$. The third case is impossible, since $\mathit{\Gamma}_0/O_{2'}(\mathit{\Gamma}_0) \cong \Sigma_4$ is not a subquotient of D_8. Hence $\mathit{\Gamma}/O_{2'}(\mathit{\Gamma})$ is isomorphic to one of the groups $PGL_2(3) \cong \Sigma_4$, $PGL_2(5) \cong \Sigma_5$, $PSL_2(7) \cong GL_3(2)$, $PSL_2(9) \cong A_6$, or A_7.

Assume $O_{2'}(\mathit{\Gamma}) \neq 1$. By Lemma A.6, for some odd prime p, there is a normal elementary abelian p-subgroup $1 \neq K \trianglelefteq \mathit{\Gamma}$, and $N_{\mathrm{Aut}(A)}(K) \geq \mathit{\Gamma} \geq \mathit{\Gamma}_0 \cong \Sigma_4$. Also, $O_2(\mathit{\Gamma}_0)$ cannot centralize K, since with respect to the $\mathbb{F}_2$-basis $\{a_1, a_2, a_3, b_1, b_2, b_3\}$ of A,

$$C_{\mathrm{Aut}(A)}\bigl(\langle c_{\mu_{12}}, c_{\mu_{23}}\rangle\bigr) = \bigl\{\bigl(\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\bigr) \bigm| X \in M_3(\mathbb{F}_2)\bigr\} \cong (C_2)^9. \tag{6}$$

Thus $O^2(N_{\mathrm{Aut}(A)}(K)) \geq O^2(\mathit{\Gamma}_0) \geq O_2(\mathit{\Gamma}_0)$ contains involutions which do not centralize K, and from the list of normalizers in Lemma A.6, this is possible only if $K \cong (C_3)^3$ and $\mathit{\Gamma} \leq N_{\mathrm{Aut}(A)}(K) \cong \Sigma_3 \wr \Sigma_3$. Since $[\mathit{\Gamma}{:}\mathit{\Gamma}_0]$ is odd, $\mathit{\Gamma} = K\mathit{\Gamma}_0 \cong (C_3)^3 \rtimes \Sigma_4$.

Now assume $O_{2'}(\mathit{\Gamma}) = 1$. We already showed that $\mathit{\Gamma} \cong \Sigma_4$, Σ_5, A_6, A_7, or $GL_3(2)$, and it remains to eliminate the first three possibilities. Since A is characteristic in all subgroups of S which contain it (Lemma 7.1), $C_P(\mathrm{Aut}_{\mathcal{F}}(P)) \geq C_A(\mathit{\Gamma})$ for all $P \leq S$ containing A. Since N_3 is the only $\mathcal{F}$-essential subgroup that does not contain A, and $C_{N_3}(\mathrm{Aut}_{\mathcal{F}}(N_3)) \geq \langle a_1a_2a_3\rangle = C_A(\mathit{\Gamma}_0) \geq C_A(\mathit{\Gamma})$ by a direct check, Proposition 1.14 implies that $C_A(\mathit{\Gamma}) \trianglelefteq \mathcal{F}$, and hence that

$$C_A(\mathit{\Gamma}) \leq O_2(\mathcal{F}) = 1\,. \tag{7}$$

In particular, $\mathit{\Gamma} > \mathit{\Gamma}_0$, and $\mathit{\Gamma} \not\cong \Sigma_4$.

We claim that

$$\mathit{\Gamma}_0 < \mathit{\Gamma}_1 \leq \mathit{\Gamma}, \quad \mathit{\Gamma}_1 \cong \Sigma_5 \text{ or } A_6 \quad \Longrightarrow \quad C_A(\mathit{\Gamma}_1) = \langle a_1a_2a_3\rangle\,. \tag{8}$$

Fix $\mathit{\Gamma}_1$ as on the left side of (8). Assume there is $H < \mathit{\Gamma}_1$ such that $H \cong A_5$, $\mathit{\Gamma}_1 = \langle \mathit{\Gamma}_0, H\rangle$, and $\mathit{\Gamma}_0 \cap H \cong A_4$. Under this assumption, since $C_A(\langle \mu_{12}, \mu_{23}\rangle) = \langle a_1, a_2, a_3\rangle = [\langle \mu_{12}, \mu_{23}\rangle, A]$, and since $\mathrm{rk}([x, A]) = \mathrm{rk}([\mu_{12}, A]) = 2$ for each involution $x \in H$, Lemma A.8(b) implies that $C_A(H) = C_A(H \cap \mathit{\Gamma}_0) = C_A(\mathit{\Gamma}_0) = \langle a_1a_2a_3\rangle$, and hence that (8) holds.

We now check that such an H exists under the hypotheses of (8). If $\mathit{\Gamma}_1 \cong \Sigma_5$, then $H = O^2(\mathit{\Gamma}_1)$ satisfies the above conditions. If $\mathit{\Gamma}_1 \cong A_6$, then there is at least one pair of subgroups $\mathit{\Gamma}_0^*, H^* < \mathit{\Gamma}$ such that $\mathit{\Gamma}_0^* \cong \Sigma_4$, $H^* \cong A_5$, $\mathit{\Gamma}_1 = \langle \mathit{\Gamma}_0^*, H^*\rangle$, and $\mathit{\Gamma}_0^* \cap H^* \cong A_4$; and we can take $\mathit{\Gamma}_0^* = \mathit{\Gamma}_0$ since the subgroups of A_6 isomorphic to Σ_4 are permuted transitively by $\mathrm{Aut}(A_6)$. (This last claim holds by [Sz1, (3.2.20)], and since the two A_6-conjugacy classes of subgroups isomorphic to Σ_4 contain elements of order 3 in different classes.) This finishes the proof of (8). By (7) and (8), $\mathit{\Gamma} \not\cong \Sigma_5$ and $\mathit{\Gamma} \not\cong A_6$, so (5) holds.

Step 5: Three distinct fusion systems over S. Set $\widehat{a} = a_1a_2a_3$ for short. Consider the automorphisms $\sigma_1, \sigma_2 \in \mathrm{Aut}(S)$, defined by setting (for $i = 1, 2, 3$):

$$\begin{array}{lllll} \sigma_1(a_i) = a_i & \sigma_1(b_i) = a_ib_i & \sigma_1(\mu_{12}) = \mu_{12} & \sigma_1(\mu_{23}) = \mu_{23} & \sigma_1(\tau) = \tau \\ \sigma_2(a_i) = a_i & \sigma_2(b_i) = \widehat{a}b_i & \sigma_2(\mu_{12}) = \mu_{12} & \sigma_2(\mu_{23}) = \mu_{23} & \sigma_2(\tau) = \tau. \end{array}$$

For each $j = 1, 2$, $\sigma_j|_{N_2}$ commutes with ν_2, and $\sigma_j|_{N_3}$ commutes with ν_3 modulo $\mathrm{Inn}(N_3)$. Commutativity with ν_2 is easily checked. Commutativity with ν_3 modulo $\mathrm{Inn}(N_3)$ follows from Lemma 7.2(c), applied with ${}^{\sigma_j}\nu_3$ in the role of α. Hence we can replace $\mathcal{F}$ by ${}^{\sigma_1}\mathcal{F}$ or ${}^{\sigma_2}\mathcal{F}$ without changing $\mathrm{Aut}_{\mathcal{F}}(N_2)$ or $\mathrm{Aut}_{\mathcal{F}}(N_3)$ (or $\mathit{\Gamma}_0$).

Consider the $\mathbb{F}_2$-basis $\mathcal{B} = \{b_1, a_1b_1, b_2, a_2b_2, b_3, a_3b_3\}$ for A, and set $\widehat{\mathcal{B}} = \mathcal{B} \cup \{\widehat{a}\}$. Each element of $\mathit{\Gamma}_0 < \mathrm{Aut}(A)$ permutes the elements of $\mathcal{B}$ and fixes $\widehat{a}$.

We claim that

$$H_1 \text{ fully normalized in } \mathcal{F} \quad \Longrightarrow \quad \mathrm{Out}_{\mathcal{F}}(H_1) \cong C_{\mathit{\Gamma}}(c_\tau)/\langle c_\tau\rangle. \tag{9}$$

By Lemma A.2, restriction induces a homomorphism $R\colon \mathrm{Out}_{\mathcal{F}}(H_1) \longrightarrow C_{\mathit{\Gamma}}(c_\tau)/\langle c_\tau\rangle$, and $\mathrm{Ker}(R)$ is a 2-group. Also, R is surjective by the extension axiom. If H_1 is fully normalized, then $\mathrm{Out}_S(H_1) = \bigl\langle [c_{\mu_{12}}]\bigr\rangle \in \mathrm{Syl}_2(\mathrm{Out}_{\mathcal{F}}(H_1))$, and so R is injective.

Assume $\varGamma \cong A_7$. Then $\varGamma$ contains two conjugacy classes of subgroups isomorphic to $C_2 \times C_2$, represented by the two subgroups of this type in any Sylow 2-subgroup. Of these, each subgroup in one of the classes contains a subgroup of order three in its centralizer, while those in the other class have normalizer isomorphic to Σ_4. By (6), $O_2(\varGamma_0) = \langle c_{\mu_{12}}, c_{\mu_{23}} \rangle$ must be of the latter type, so $\varGamma_0 = N_\varGamma(\langle c_{\mu_{12}}, c_{\mu_{23}} \rangle)$, and there is a unique subgroup $\varGamma_1 < \varGamma$ such that $\varGamma_0 < \varGamma_1 \cong A_6$. By (8), $\widehat{a} \in C_A(\varGamma_1)$, and hence lies in a $\varGamma$-orbit X of length $|\varGamma/\varGamma_1| = 7$. By the above remarks about subgroups of A_7, $\varGamma_0$ acts on X with two orbits: of lengths 1 and 6. By a direct check, the action of $\varGamma_0$ on A has exactly two orbits of length 6: the orbits $\mathcal{B}$ of b_1 and $\sigma_2(\mathcal{B})$ of $\widehat{a}b_1$. So after replacing $\mathcal{F}$ by ${}^{\sigma_2}\mathcal{F}$ if necessary, we can assume that $X = \widehat{\mathcal{B}}$, and hence that $\varGamma$ is the group of even permutations of $\widehat{\mathcal{B}}$. Since $\varGamma$ determines $\mathrm{Aut}_\mathcal{F}(N_1)$ by (3), and since H_1 is not fully normalized (hence not $\mathcal{F}$-essential) by (9) ($C_\varGamma(c_\tau)/\langle c_\tau \rangle \cong C_2 \times \Sigma_3$), $\varGamma$ determines $\mathcal{F}$. More precisely, there is at most one reduced fusion system $\mathcal{F}$ over S for which $\varGamma = \mathrm{Aut}_\mathcal{F}(A)$, $\mathrm{Aut}_\mathcal{F}(N_2)$, and $\mathrm{Aut}_\mathcal{F}(N_3)$ are as just described; and any other reduced fusion system over S with $\mathrm{Aut}_\mathcal{F}(A) \cong A_7$ is isomorphic to it. Also, by Step 2, N_1 is $\mathcal{F}$-essential in this case since H_1 is not.

Assume $\varGamma \cong GL_3(2)$. Since $\widehat{a} = a_1a_2a_3$ is fixed by $\varGamma_0$ but not by $\varGamma$, it must lie in an orbit X of length $7 = |\varGamma/\varGamma_0|$. We claim that any transitive action of $GL_3(2)$ on a set X of 7 elements (with isotropy subgroups isomorphic to Σ_4) is the group of all automorphisms of $(X \cup \{0\}, +)$ under some group structure such that $(X \cup \{0\}, +) \cong \mathbb{F}_2^3$. To see this, note that the isotropy subgroup must be the normalizer of some $P \cong C_2 \times C_2$ (since it is maximal), and hence the stabilizer subgroup of a 1- or 2-dimensional subspace in $\mathbb{F}_2^3$ (since P is conjugate to one of two subgroups in any given Sylow 2-subgroup). Hence the elements of X are in natural correspondence with the elements of $\mathbb{F}_2^3 \smallsetminus \{0\}$ or $(\mathbb{F}_2^3)^* \smallsetminus \{0\}$, where $(\mathbb{F}_2^3)^*$ denotes the dual vector space.

Since the six elements of $X \smallsetminus \{\widehat{a}\}$ are permuted transitively by $\varGamma_0$, we can assume (after replacing $\mathcal{F}$ by ${}^{\sigma_2}\mathcal{F}$ if necessary) that they form the basis $\mathcal{B}$, and hence that $\varGamma$ permutes the set $\widehat{\mathcal{B}}$. Thus $\varGamma$ is the group of all automorphisms of $(\widehat{\mathcal{B}} \cup \{0\}, +)$ for some group structure such that $(\widehat{\mathcal{B}} \cup \{0\}, +) \cong (\mathbb{F}_2)^3$. Since the fixed set of an automorphism is a subgroup, we have $\widehat{a} + b_i = a_ib_i$ for each $i = 1, 2, 3$ (corresponding to automorphisms in $\langle c_{\mu_{12}}|_A, c_{\mu_{23}}|_A \rangle$). Hence there are exactly two such structures on which $\varGamma_0$ acts via automorphisms: one in which $b_1 + b_2 = b_3$, and the other in which $b_1 + b_2 = a_3b_3$. So we can assume, after replacing $\mathcal{F}$ by ${}^{\sigma_1}\mathcal{F}$ if necessary, that $\varGamma$ is the group of automorphisms of $(\widehat{\mathcal{B}} \cup \{0\}, +)$ where $b_1 + b_2 = b_3$. Again in this case, $\varGamma$ determines $\mathrm{Aut}_\mathcal{F}(N_1)$ by (3), H_1 is not fully normalized (hence not $\mathcal{F}$-essential) by (9) ($C_\varGamma(c_\tau)/\langle c_\tau \rangle \cong C_2 \times C_2$), and thus $\varGamma$ determines $\mathcal{F}$. Also, by Step 2, N_1 is $\mathcal{F}$-essential since H_1 is not.

If $\varGamma \cong (C_3)^3 \rtimes \Sigma_4$, then by Lemma A.6, applied with $K \cong (C_3)^3$, there is a decomposition $A = A_1 \times A_2 \times A_3$ such that $\mathrm{rk}(A_i) = 2$ for each i and each element of $\varGamma$ permutes the subgroups A_1, A_2, A_3. The subgroup $O_2(\varGamma_0) = \langle c_{\mu_{12}}, c_{\mu_{23}} \rangle < \mathrm{Aut}_S(A)$ normalizes each A_i and acts on it nontrivially, so there is a unique basis $\mathcal{B}_i$ of A_i which is permuted by $O_2(\varGamma_0)$. Then $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$ is a basis of A which is permuted transitively by $\varGamma_0$. After replacing $\mathcal{F}$ by ${}^{\sigma_2}\mathcal{F}$ if necessary, we can assume that $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 = \mathcal{B}$, and hence (possibly after permuting the indices) that $\mathcal{B}_i = \{b_i, a_ib_i\}$ for each i. Thus $\varGamma = \langle \mathrm{Aut}_S(A), \nu_2|_A, \alpha_1, \alpha_2, \alpha_3 \rangle$, where the α_i are defined by setting

$$\alpha_i(a_i) = b_i, \quad \alpha_i(b_i) = a_ib_i, \qquad \alpha_i(a_j) = a_j, \quad \alpha_i(b_j) = b_j \qquad \text{(for } j \neq i.\text{)}$$

By (3), $\mathrm{Aut}_\mathcal{F}(N_1)$ is uniquely determined by this choice of $\varGamma = \mathrm{Aut}_\mathcal{F}(A)$.

Since $C_{O_3(\Gamma)}(c_\tau) = \langle \alpha_1\alpha_2, \alpha_3\rangle$ is not isomorphic to $C_{O_3(\Gamma)}(c_{\mu_{12}}) = \langle \alpha_3\rangle$, c_τ and $c_{\mu_{12}}$ are not conjugate in $\Gamma = \mathrm{Aut}_{\mathcal{F}}(A)$, and hence $H_1 = A\langle\tau\rangle$ is not $\mathcal{F}$-conjugate to $A\langle\mu_{12}\rangle$. So H_1 is fully normalized in $\mathcal{F}$. By (9), $\mathrm{Out}_{\mathcal{F}}(H_1) \cong C_\Gamma(c_\tau)/\langle c_\tau\rangle \cong C_3 \times \Sigma_3$, where the quotient is generated by the classes of α_3, $\alpha_{12} \overset{\text{def}}{=} \alpha_1\alpha_2$, and $c_{\mu_{12}}$. Thus $H_1 \in \mathbf{E}_{\mathcal{F}}$ and

$$\mathrm{Aut}_{\mathcal{F}}(H_1) = \langle \mathrm{Aut}_S(H_1), \nu_1|_{H_1}, \eta\rangle$$

for some $\nu_1 \in \mathrm{Aut}_{\mathcal{F}}(N_1)$ and $\eta \in \mathrm{Aut}_{\mathcal{F}}(H_1)$ of order three such that $\nu_1|_A = \alpha_3$ and $\eta|_A = \alpha_{12}$. (Note that for $P = N_1$ or H_1, $\mathrm{Ker}\big[\mathrm{Aut}_{\mathcal{F}}(P) \longrightarrow \mathrm{Aut}_{\mathcal{F}}(A)\big]$ is a 2-group, by Lemma A.2 and since A is centric in P.) Also, $[\nu_1, c_{\mu_{23}}] \neq 1$ in $\mathrm{Out}_{\mathcal{F}}(N_1)$ (since they do not commute after restriction to $Z(N_1) = \langle a_1a_2, a_3, b_3\rangle$), so $[c_{\mu_{23}}] \notin Z(\mathrm{Out}_{\mathcal{F}}(N_1))$, and $N_1 \in \mathbf{E}_{\mathcal{F}}$.

By Step 3, $\nu_1(Q) = Q$, so ν_1 induces the identity on $A\cap Q$ and (since $C_Q(A\cap Q) = A\cap Q$) on $Q/(A\cap Q) \cong C_2 \times C_2$. Hence $\nu_1|_Q = \mathrm{Id}_Q$ by Lemma A.1 and since $|\nu_1| = 3$. Also, $\eta(\tau) = \tau g$ for some $g \in A$, $g{\cdot}\alpha_{12}(g){\cdot}\alpha_{12}^2(g) = 1$ since $|\eta| = 3$, $g \in C_A(\tau)$ since $|\eta(\tau)| = 2$, and thus $g \in \langle a_1a_2, b_1b_2\rangle$. So after replacing η by $c_{a_1^ib_1^j} \circ \eta$ for appropriate i, j, we can assume that $\eta(\tau) = \tau$. Thus $\mathrm{Aut}_{\mathcal{F}}(H_1)$, and hence $\mathcal{F}$, is uniquely determined in this case.

Step 6: Identifying the fusion system $\boldsymbol{\mathcal{F}}$. We have now shown that $\mathcal{F}$ is isomorphic to one of three fusion systems, $\mathcal{F}_1$, $\mathcal{F}_2$, or $\mathcal{F}_3$, where $\mathrm{Aut}_{\mathcal{F}_1}(A) \cong A_7$, $\mathrm{Aut}_{\mathcal{F}_2}(A) \cong GL_3(2)$, and $\mathrm{Aut}_{\mathcal{F}_3}(A) \cong (C_3)^3 \rtimes \Sigma_4$.

When $G = \Omega_7(3)$, the group of all elements of G which act up to sign on an orthonormal basis of $\mathbb{F}_3^7$ is isomorphic to $(C_2)^6 \rtimes A_7$ and has odd index in G. This follows from the formula for $|G|$ (cf. [Ta, p. 166]). Since S splits over A and the two A_7-actions are isomorphic (see Step 5), S is isomorphic to any $S_1 \in \mathrm{Syl}_2(G)$. Since $\mathrm{Out}(S_1)$ is a 2-group, $\mathcal{F}_{S_1}(G)$ is reduced by Proposition 1.17(c). Hence $\mathrm{Aut}_G(A) \cong A_7$ (it can't be any bigger by Step 4), and so $\mathcal{F}_{S_1}(G) \cong \mathcal{F}_1$ in this case.

The group $G = Sp_6(2)$ has a maximal parabolic subgroup $H \cong M_3^s(2) \rtimes GL_3(2)$ (the stabilizer of a maximal isotropic subspace), where $M_3^s(2) \cong (C_2)^6$ is the group of symmetric 3×3 matrices over $\mathbb{F}_2$, and $A \in GL_3(2)$ acts on it via $X \mapsto AXA^t$. By the order formulas for symplectic groups [Ta, p. 70], H has odd index in G. Also, the $GL_3(2)$-orbit of $\left(\begin{smallmatrix} 1&0&0\\0&0&0\\0&0&0 \end{smallmatrix}\right)$ has order 7 and stabilizer subgroup isomorphic to Σ_4, and the other six elements form a basis for $M_3^s(2)$. So this is the same action as that described in Step 5, $S_2 \cong S$ for $S_2 \in \mathrm{Syl}_2(G)$, and $\mathcal{F}_{S_2}(G) \cong \mathcal{F}_2$ by Step 5 (and since $\mathcal{F}_{S_2}(G)$ is reduced by Proposition 1.17(c)).

Finally, when $G = A_{12}$ and $A < S$ are as defined here, then $S \in \mathrm{Syl}_2(G)$ by definition, $\mathcal{F}_S(G)$ is reduced by Proposition 1.17(c), $\mathrm{Aut}_{\Sigma_{12}}(A) \cong \Sigma_3 \wr \Sigma_3$ and hence $\mathrm{Aut}_G(A) \cong (C_3)^3 \rtimes \Sigma_4$, and so $\mathcal{F}_S(G) \cong \mathcal{F}_3$.

This finishes the proof that every reduced fusion system over S is isomorphic to the 2-fusion system of $\Omega_7(3)$, $Sp_6(2)$, or A_{12}. □

Appendix A. Background results on groups and representations

We collect here some background results on groups and their representations which are needed elsewhere in the paper. The first ones involve automorphisms and extensions of p-groups.

Lemma A.1. *Fix a prime p, a finite p-group P, a subgroup $P_0 \le \Phi(P)$, and a sequence of subgroups*
$$P_0 < P_1 < \cdots < P_k = P$$
all normal in P. Set
$$\mathcal{A} = \bigl\{\alpha \in \mathrm{Aut}(P) \bigm| [\alpha, P_i] \le P_{i-1},\ \text{all } i = 1, \ldots, k\bigr\} \le \mathrm{Aut}(P):$$
the group of automorphisms which leave each P_i invariant, and which induce the identity on each quotient group P_i/P_{i-1}. Then $\mathcal{A}$ is a p-group. If the P_i are all characteristic in P, then $\mathcal{A} \trianglelefteq \mathrm{Aut}(P)$, and hence $\mathcal{A} \le O_p(\mathrm{Aut}(P))$.

Proof. If $\alpha \in \mathcal{A}$ has order prime to p, then α induces the identity on P/P_0 and hence on $P/\Phi(P)$ by [G, Theorem 5.3.2], and so $\alpha = \mathrm{Id}$ by [G, Theorem 5.1.4]. Thus $\mathcal{A}$ is a p-group. If the P_i are all characteristic, then $\mathcal{A}$ is the kernel of a homomorphism from $\mathrm{Aut}(P)$ to $\prod_{i=1}^{k} \mathrm{Aut}(P_i/P_{i-1})$, and hence is normal in $\mathrm{Aut}(P)$. □

When $H \trianglelefteq G$, we let $\mathrm{Aut}(G, H)$ be the group of automorphisms of G which normalize H, and set $\mathrm{Out}(G, H) = \mathrm{Aut}(G, H)/\mathrm{Inn}(G)$.

Lemma A.2. *Fix a group G and a normal subgroup $H \trianglelefteq G$ such that $C_G(H) \le H$ (i.e., H is centric in G). Then there is an exact sequence*

$$1 \longrightarrow H^1(G/H; Z(H)) \xrightarrow{\ \eta\ } \mathrm{Out}(G, H) \xrightarrow{\ R\ } \\ N_{\mathrm{Out}(H)}(\mathrm{Out}_G(H))/\mathrm{Out}_G(H) \xrightarrow{\ \chi\ } H^2(G/H; Z(H)), \quad (1)$$

where R is induced by restriction, and where all maps except (possibly) χ are homomorphisms. If H is abelian and the extension of H by G/H is split, then R is onto. If $Z(H)$ has exponent p for some prime p, and has a basis which is permuted freely by G/H under conjugation, then R is an isomorphism.

Proof. See [OV, Lemma 1.2 & Corollary 1.3]. □

Proposition A.3 ([OV, Proposition 1.8])**.** *Fix a prime p, a finite group G, and a normal abelian p-subgroup $Q \trianglelefteq G$. Let $H \le G$ be such that $Q \cap H = 1$, and let $H_0 \le H$ be of index prime to p. Consider the set*
$$\mathcal{H} = \bigl\{H' \le G \bigm| H' \cap Q = 1,\ QH' = QH,\ H_0 \le H'\bigr\}.$$
Then for each $H' \in \mathcal{H}$, there is $g \in C_Q(H_0)$ such that $H' = gHg^{-1}$.

We next note the following elementary properties of strongly p-embedded subgroups (Definition 1.2(a)).

Lemma A.4. *Let H be a strongly p-embedded subgroup of a finite group G.*

(a) *For each subgroup $H^* < G$ such that $H^* \ge H$, H^* is also strongly p-embedded in G.*

(b) *For each normal subgroup $K \trianglelefteq G$ of order prime to p such that $HK < G$, HK/K is strongly p-embedded in G/K.*

Proof. Point (a) follows from the definition as an easy exercise, and also follows immediately from the equivalence (when $k = 1$) of points (1) and (3) in [A1, 46.4]. In the situation of (b), HK is strongly p-embedded in G by (a), and hence HK/K is strongly p-embedded in G/K by definition. □

The next lemma involves subgroups of $GL_5(2)$.

Lemma A.5. *If G acts linearly, faithfully, and irreducibly on $\mathbb{F}_2^5$, then either G has odd order, or $G \cong GL_5(2)$.*

Proof. See [Wg, Theorem 1.1]. Wagner's theorem deals more generally with irreducible subgroups of $PSL_5(2^a)$ for arbitrary $a \geq 1$, but only cases (ii) and (v) in the theorem apply when $a = 1$. □

The next three lemmas involve representations over $\mathbb{F}_2$.

Lemma A.6. *Assume $G \leq GL_6(2)$ is such that $O_{2'}(G) \neq 1$. Then for some odd prime p, G contains a normal elementary abelian p-subgroup $1 \neq K \trianglelefteq G$ characteristic in $O_{2'}(G)$, and hence G is contained up to conjugation in one of the following normalizers:*

$K \cong$	$N_{GL_6(2)}(K) \cong$
C_3	$\Sigma_3 \times GL_4(2)$, $(GL_2(4) \rtimes C_2) \times \Sigma_3$, or $GL_3(4) \rtimes C_2$
$(C_3)^2$	$(\Sigma_3 \wr C_2) \times \Sigma_3$, $(GL_2(4) \rtimes C_2) \times \Sigma_3$, or $(C_3)^3 \rtimes (C_2 \times \Sigma_3)$
$(C_3)^3$	$\Sigma_3 \wr \Sigma_3$
C_5	$(C_{15} \rtimes C_4) \times \Sigma_3$
C_7	$(C_7 \rtimes C_3) \times GL_3(2)$, $(C_7)^2 \rtimes C_6$, or $GL_2(8) \rtimes C_3$
$(C_7)^2$	$(C_7 \rtimes C_3) \wr C_2$
C_{31}	$C_{31} \rtimes C_5$

When $K \cong (C_3)^3$ acts on $V \cong \mathbb{F}_2^6$, there is a decomposition $V = V_1 \times V_2 \times V_3$ such that each element of $N_{GL_6(2)}(K) \cong \Sigma_3 \wr \Sigma_3$ permutes the subgroups $V_i \cong \mathbb{F}_2^2$.

Proof. Since $O_{2'}(G) \neq 1$ is solvable by the odd order theorem [FT], it contains an elementary abelian p-subgroup K which is characteristic (for some odd p). Since $|GL_6(2)| = 2^{15}{\cdot}3^4{\cdot}5{\cdot}7^2{\cdot}31$, K must be isomorphic to one of the subgroups in the above list.

The list of normalizers now follows from elementary representation theoretic considerations. When $K \in \mathrm{Syl}_p(GL_6(2))$, then there is only one possibility for K up to conjugacy, and its normalizer is as described above. In all other cases, $N_{GL_6(2)}(K)$ depends on the description of $\mathbb{F}_2^6$ as an $\mathbb{F}_2[K]$-module.

For example, when $K \cong (C_3)^2$, the module could be a sum of two irreducible modules of dimension 2 and a 2-dimensional module with trivial action, or three irreducibles of dimension 2 of which two are isomorphic, or three pairwise nonisomorphic irreducible modules of dimension 2. The three possible normalizers listed above correspond to these three choices. □

Lemma A.7. *Let P be a finite 2-group such that $\mathrm{rk}(P/\Phi(P)) \leq 6$. Assume $G \leq \mathrm{Out}(P)$ has a strongly 2-embedded subgroup, where $S \in \mathrm{Syl}_2(G)$ and $S \cong C_2 \times C_2$. Then $G \cong A_5 \times (C_3)^s$ for some $s \leq 2$.*

Proof. Set $H = O_{2'}(G)$. Since G has a strongly 2-embedded subgroup, $O^{2'}(G/H) \cong A_5$ by Bender's theorem [Be, Satz 1]. Since $|\mathrm{Out}(A_5)| = 2$ (see [Sz1, (3.2.17)]), $G/H \cong A_5$. In particular, $O_2(G) = 1$, and by Lemma A.1, G acts faithfully on $P/\Phi(P) \cong (C_2)^r$ $(r \leq 6)$. We can thus identify G as a subgroup of $GL_6(2)$.

If $H = O_{2'}(G) \neq 1$, then let $K \trianglelefteq G \leq GL_6(2)$ be as in Lemma A.6. Then $N_{GL_6(2)}(K)$ is nonsolvable and $5 \,\big|\, |N_{GL_6(2)}(K)/K|$, and so by the same lemma, $K \cong C_3$ or $(C_3)^2$. From the list of possibilities for $N_{GL_6(2)}((C_3)^s)$ for $s = 1, 2$, we see that $G \cong H \times (G/H) \cong H \times A_5$, where $H = K \cong C_3$ or $(C_3)^2$. (Note that $GL_2(4) \cong C_3 \times A_5$.) $\square$

There are four distinct irreducible $\bar{\mathbb{F}}_2[A_5]$-modules: the trivial module, the natural 2-dimensional $\bar{\mathbb{F}}_2[SL_2(4)]$-module and its Galois conjugate, and the natural 4-dimensional module for A_5 (cf. [Se, § 18.6]). The first and last are realizable over $\mathbb{F}_2$, while the 2-dimensional modules are realizable over $\mathbb{F}_4$ and hence induce a 4-dimensional irreducible $\mathbb{F}_2[A_5]$-module by restriction of scalars. Thus there are three irreducible $\mathbb{F}_2[A_5]$-modules: the trivial module $\mathbb{F}_2$, and two 4-dimensional modules W_1 and W_2 described as follows:

- W_1 is generated by a G-orbit of five elements whose sum is zero; and
- $W_2 \cong \mathbb{F}_4^2$ with the canonical action of $G \cong SL_2(4)$.

We keep this notation for the irreducible modules in the statement of the following lemma.

Lemma A.8. *Fix $G \cong A_5$. Let V be a finitely generated $\mathbb{F}_2[G]$-module such that for each involution $x \in G$, $\dim([x, V]) = 2$. Then the composition factors of V include exactly one irreducible 4-dimensional module. Also, the following hold for any $S \in \mathrm{Syl}_2(G)$.*

(a) *If $[S, V] \nleq C_V(S)$, then $V \cong \mathbb{F}_2^k \oplus W_1$ ($k = \dim(V) - 4$).*

(b) *Assume $[S, V] \le C_V(S)$. Then there is a composition factor of V isomorphic to W_2, and V is indecomposable if and only if $[S, V] = C_V(S)$. Also, $C_V(G) = C_V(N_G(S))$ and $\dim(C_V(G)) = \dim(C_V(S)) - 2$. If V is indecomposable and $C_V(G) \ne 0$, then $\bigcap_{1 \ne x \in S}[x, V] = 0$.*

Proof. As noted above, each irreducible $\mathbb{F}_2[G]$-module is isomorphic to $\mathbb{F}_2$, W_1, or W_2. If each composition factor of V is 1-dimensional, then for an appropriate choice of basis, G would act on V via upper triangular matrices, and thus via the identity (Lemma A.1).

By a direct check, $\dim([x, W_j]) = 2$ for $1 \ne x \in S$ and $j = 1, 2$. If the composition factors of V include more than one 4-dimensional module, then this would mean $\dim([x, V]) \ge 4$. Since $\dim([x, V]) = 2$ for $1 \ne x \in S$, there is exactly one such composition factor.

(a) Let $V_1 \le V_2 \le V$ be submodules such that V_2/V_1 is 4-dimensional and irreducible. We just saw that G acts trivially on V_1 and on V/V_2. In particular, we can assume they were chosen so that $V_1 = C_V(G)$. Also, $\dim([x, V_2]) = 2$ for $1 \ne x \in S$, and thus $[x, V_2] = [x, V]$ and $[S, V_2] = [S, V]$. Likewise, $\dim([x, V_2]) = \dim([x, V_2/V_1])$ for $1 \ne x \in S$ implies $V_1 \cap [x, V_2] = 0$, and hence $C_{V_2}(x)$ surjects onto $C_{V_2/V_1}(x)$. Since G acts trivially on V_1, this shows that $C_{V_2/V_1}(S) = C_{V_2}(S)/V_1$.

Upon combining these observations, we see that

$$\begin{aligned} [S, V] \le C_V(S) \quad &\iff \quad [S, V_2] = [S, V] \le C_V(S) \cap V_2 = C_{V_2}(S) \\ &\iff \quad [S, V_2] + V_1 \le C_{V_2}(S) \\ &\iff \quad [S, V_2/V_1] \le C_{V_2}(S)/V_1 = C_{V_2/V_1}(S)\,. \end{aligned}$$

Since $[S, W_2] = C_{W_2}(S)$ while $[S, W_1] \nleq C_{W_1}(S)$, we conclude that $[S, V] \le C_V(S)$ if and only if $V_2/V_1 \cong W_2$.

Since W_1 is free (hence projective) as an $\mathbb{F}_2[S]$-module, it is also projective as an $\mathbb{F}_2[G]$-module (see [Bs, Corollary 3.6.10]). Hence W_1 is also injective by, e.g., [Bs, Proposition 3.1.2]. So if $V_2/V_1 \cong W_1$, then $V \cong W_1 \oplus \mathbb{F}_2^k$ where $k = \dim(V) - 4$.

(b) Now assume that $[S,V]\le C_V(S)$, and thus as shown above that $V_2/V_1\cong W_2$. If V is decomposable, then it contains a nonzero direct summand with trivial action, in which case $[S,V]<C_V(S)$. In other words, V is indecomposable if $[S,V]=C_V(S)$.

Conversely, assume $[S,V]<C_V(S)$. Fix a set $X\subseteq G$ of representatives for the left cosets $gS\subseteq G$. For each $v\in C_V(S)$, set $v^*=\sum_{g\in X}gv$. Then $v^*\in C_V(G)=V_1$ and $v^*\equiv|G/S|{\cdot}v \pmod{[G,V]}$, so $v\in V_1+[G,V]\le V_2$. Hence $C_V(S)\le V_2$, and $C_V(S)/V_1=C_{V_2/V_1}(S)=[S,V_2/V_1]\le([S,V]+V_1)/V_1$ by the above remarks. Thus $[S,V]\not\ge V_1$ since $[S,V]<C_V(S)$. So there are $v\in V_1$ and $\varphi\in \mathrm{Hom}_S(V,\mathbb{F}_2)$ such that $\varphi(v)\ne 0$. Define $\psi\in \mathrm{Hom}_G(V,\mathbb{F}_2)$ by $\psi(x)=\sum_{g\in X}\varphi(g^{-1}x)$; then $\psi(v)=|G/S|\varphi(v)\ne 0$. Thus the inclusion $\langle v\rangle<V$ is split by the map $(x\mapsto\psi(x)v)$, and V is decomposable.

It remains to prove the claims in the last two sentences in (b). Since V is the direct sum of an indecomposable module and a module with trivial action, it suffices to show them when V is indecomposable, and thus when $[S,V]=C_V(S)$. Recall that $V_1=C_V(G)$. Also, $C_V(S)=[S,V]\le[G,V]\le V_2$, and thus $C_V(S)/V_1=C_{V_2}(S)/V_1=C_{V_2/V_1}(S)\cong C_{W_2}(S)$ is 2-dimensional. So $\dim(C_V(G))=\dim(C_V(S))-2$, and $C_V(N_G(S))=C_V(G)$ since $C_{W_2}(N_G(S))=0$.

Recall that $\dim([x,V])=2$ and $[x,V]\cap C_V(G)=0$ for $1\ne x\in S$. Set $V_0=\bigcap_{1\ne x\in S}[x,V]$. Either $V_0=0$; or $\dim(V_0)=1$ and $V_0\le C_V(N_G(S))=C_V(G)$, which is impossible; or V_0 is 2-dimensional, hence is equal to $[S,V]=C_V(S)$, and so $C_V(G)=0$. □

In fact, up to isomorphism, there are two distinct indecomposable $\mathbb{F}_2[A_5]$-modules of dimension 5 and three of dimension 6 which satisfy the hypotheses of Lemma A.8(b). The three of dimension 6 are the permutation module for the A_5-action on A_5/D_{10}, the group of symmetric (2×2) matrices over $\mathbb{F}_4$ with the canonical action of $A_5\cong SL_2(4)$, and the dual of this last module.

References

[Alp] J. Alperin, Sylow 2-subgroups of 2-rank three, Finite groups '72, Proc. Gainesville conf. 1972, North-Holland (1973), 3–5

[AOV1] K. Andersen, B. Oliver, & J. Ventura, Reduced, tame, and exotic fusion systems, Proc. London Math. Soc. 105 (2012), 87–152

[AOV2] K. Andersen, B. Oliver, & J. Ventura, Fusion systems and amalgams, Math. Z. 274 (2013), 1119–1154

[A1] M. Aschbacher, Finite Group Theory, Cambridge Univ. Press (1986)

[A2] M. Aschbacher, The generalized Fitting subsystem of a fusion system, Memoirs Amer. Math. Soc. 209 (2011), nr. 986

[AKO] M. Aschbacher, R. Kessar, & B. Oliver, Fusion systems in algebra and topology, Cambridge Univ. Press (2011)

[Be] H. Bender, Transitive Gruppen gerader Ordnung, in denen jede Involution genau einen Punkt festlässt, J. Algebra 17 (1971), 527–554

[Bs] D. Benson, Representations and cohomology I: Basic representation theory of finite groups and associative algebras, Cambridge Univ. Press (1995)

[Bg] M. Berger, Géométrie 1, Nathan (1990)

[BCGLO] C. Broto, N. Castellana, J. Grodal, R. Levi, & B. Oliver, Extensions of p-local finite groups, Trans. Amer. Math. Soc. 359 (2007), 3791–3858

[BLO1] C. Broto, R. Levi, & B. Oliver, Homotopy equivalences of p-completed classifying spaces of finite groups, Invent. math. 151 (2003), 611–664

[BLO2] C. Broto, R. Levi, & B. Oliver, The homotopy theory of fusion systems, J. Amer. Math. Soc. 16 (2003), 779–856

[BMO1] C. Broto, J. Møller, & B. Oliver, Equivalences between fusion systems of finite groups of Lie type, Journal Amer. Math. Soc. 25 (2012), 1–20

[BMO2] C. Broto, J. Møller, & B. Oliver, Automorphisms of fusion systems of finite simple groups of Lie type, `arXiv:1601.04566`

[Ca] R. Carter, Simple groups of Lie type, Wiley (1972)

[Cr] D. Craven, The theory of fusion systems, an algebraic approach, Cambridge Studies in Advanced Mathematics 131, Cambridge Univ. Press (2011)

[CR] C. Curtis & I. Reiner, Methods of representation theory, vol. I, Wiley (1981)

[FT] W. Feit & J. Thompson, Solvability of groups of odd order, Pacific J. Math. 13 (1963), 775–1029

[FF] R. Flores & R. Foote, Strongly closed subgroups of finite groups, Adv. in Math. 222 (2009), 453–484

[Ft] R. Foote, A characterization of finite groups containing a strongly closed 2-subgroup, Comm. algebra 25 (1997), 593–6062

[Fr] J. Frame, Computation of characters of the Higman-Sims group and its automorphism group, J. Algebra 20 (1972), 320–349

[Gd1] D. Goldschmidt, 2-Fusion in finite groups, Annals of Math. 99 (1974), 70–117

[Gd2] D. Goldschmidt, Strongly closed 2-subgroups of finite groups, Annals of Math. 102 (1975), 475–489

[G] D. Gorenstein, Finite groups, Harper & Row (1968)

[GLS3] D. Gorenstein, R. Lyons, & R. Solomon, The classification of the finite simple groups, nr. 3, Amer. Math. Soc. surveys and monogr. 40 #3 (1997)

[GW] D. Gorenstein & J. Walter, The characterization of finite groups with dihedral Sylow 2-subgroups: I, II, III, J. Algebra 2 (1965), 85–151, 218–270, 354–393

[HS] D. Higman & C. Sims, A simple group of order 44,352,000, Math. Z. 105 (1968), 110–113

[Is] I. M. Isaacs, Character theory of finite groups, Academic Press (1976)

[JW] C. Jansen & R. Wilson, Two new constructions of the O'Nan group, J. London Math. Soc. 56 (1997), 579–583

[LP] K. Lux & H. Pahlings, Representations of groups: a computational approach, Cambridge Univ. Press (2010)

[Ly1] R. Lyons, Evidence for a new finite simple group, J. Algebra 20 (1972), 540–569; Erratum, J. Algebra 34 (1975), 188–189

[Ly2] R. Lyons, Automorphism groups of sporadic groups, `arXiv:1106.3760`

[O1] B. Oliver, Splitting fusion systems over 2-groups, Proc. Edinburgh Math. Soc., 56 (2013), 263-301

[O2] B. Oliver, Reduced fusion systems over 2-groups of sectional rank at most four, Memoirs Amer. Math. Soc. 239 (2016), nr. 1131

[OV] B. Oliver & J. Ventura, Saturated fusion systems over 2-groups, Trans. Amer. Math. Soc. 361 (2009), 6661–6728

[O'N] M. O'Nan, Some evidence for the existence of a new simple group, Proc. London Math. Soc. 32 (1976), 421–479

[Pg] L. Puig, Frobenius categories, J. Algebra 303 (2006), 309–357

[Sch] J. Schur, Untersuchungen über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen, J. reine angew. Math. 132 (1907), 85–137

[Se] J.-P. Serre, Linear representations of finite groups, Springer-Verlag (1977)

[So1] R. Solomon, Finite groups with Sylow 2-subgroups of type $\mathfrak{A}_{12}$, J. Algebra 24 (1973), 346–378

[So2] R. Solomon, Finite groups with Sylow 2-subgroups of type $\Omega(7,q)$, $q \equiv \pm 3 \pmod 8$, J. Algebra 28 (1974), 174–181

[Sz1] M. Suzuki, Group theory I, Springer-Verlag (1982)

[Sz2] M. Suzuki, Group theory II, Springer-Verlag (1986)

[Ta] D. Taylor, The geometry of the classical groups, Heldermann Verlag (1992)

[Wg] A. Wagner, The subgroups of $PSL(5,2^a)$, Resultate Math. 1 (1978), 207–226

[Mg] W. Bosma, J. Cannon, & C. Playoust, The Magma algebra system I: The user language, J. Symbolic Comput. 24 (1997), 235–265, `http://dx.doi.org/10.1006/jsco.1996.0125`

[Gp] The GAP Group, GAP — Groups, Algorithms, and Programming, Version 4.7.9 (2015), `http://www.gap-system.org`

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