HOMOTOPY EQUIVALENCES OF p-COMPLETED CLASSIFYING SPACES OF FINITE GROUPS

CARLES BROTO, RAN LEVI, AND BOB OLIVER

Abstract. We study homotopy equivalences of p-completions of classifying spaces of finite groups. To each finite group G and each prime p, we associate a finite category $\mathcal{L}_p^c(G)$ with the following properties. Two p-completed classifying spaces $BG_p^\wedge$ and $BG'^\wedge_p$ have the same homotopy type if and only if the associated categories $\mathcal{L}_p^c(G)$ and $\mathcal{L}_p^c(G')$ are equivalent. Furthermore, the topological monoid $\mathrm{Aut}(BG_p^\wedge)$ of self equivalences is determined by the self equivalences of the associated category $\mathcal{L}_p^c(G)$.

Throughout this paper, let p denote a fixed prime. For any finite group G, we write $BG_p^\wedge$ to denote the Bousfield-Kan p-completion of BG. The main results in this paper are algebraic conditions for the existence of homotopy equivalences between $BG_p^\wedge$ and $BG'^\wedge_p$, for any pair of finite groups G and G', and a description of the space of self homotopy equivalences of $BG_p^\wedge$.

Spaces of the form $BG_p^\wedge$ for G finite are not generally aspherical. In general, if $G = \pi_1(X)$ is finite, then $\pi_1(X_p^\wedge) \cong G/O^p(G)$, where $O^p(G) \lhd G$ is the smallest normal subgroup of p-power index in G (see Proposition A.2). Thus if G is p-perfect (if $G = O^p(G)$), then $BG_p^\wedge$ is simply connected, but $\widetilde{H}^*(BG_p^\wedge;\mathbb{F}_p) \cong \widetilde{H}^*(BG;\mathbb{F}_p)$ vanishes only if G has order prime to p, and so $BG_p^\wedge$ is contractible only in this case. In fact, $BG_p^\wedge$ has non-vanishing higher homotopy groups in arbitrarily high dimensions whenever $O^p(G)$ has order a multiple of p. For a general discussion of $BG_p^\wedge$ for G finite we refer to [Lev].

This work originated as a study by the first two authors of the space of self equivalences of $BG_p^\wedge$, which was motivated by the problem of classifying fibrations with $BG_p^\wedge$ as fiber. Some of their work on this subject appears in [BL]. A possibly more general motivation for the project is the analogy with compact Lie groups. For compact connected simple Lie groups G the space of self maps of BG was studied by Jackowski, McClure and Oliver [JMO]. In their study homology decompositions for classifying spaces were used to provide a description of the corresponding mapping spaces. Homology decompositions are one of the most important tools in the study of

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classifying spaces and maps between them. Our approach however is different. Central to our study are certain *centric linking categories* $\mathcal{L}_p^c(G)$, defined for any finite group G and any prime p, which we proceed by describing.

A p-subgroup $P \le G$ is *p-centric* if its center $Z(P)$ is the Sylow p-subgroup of its centralizer $C_G(P)$. Equivalently, the centralizer of such a subgroup splits naturally as a cartesian product $C_G(P) \cong Z(P) \times O^p(C_G(P))$, where the second factor has order prime to p (see Lemma A.4). The objects of $\mathcal{L}_p^c(G)$ are the p-centric subgroups $P \le G$. For any pair of such subgroups $P, Q \le G$,

$$\mathrm{Mor}_{\mathcal{L}_p^c(G)}(P,Q) = N_G(P,Q)/O^p(C_G(P)),$$

where $N_G(P,Q) = \{x \in G \,|\, xPx^{-1} \le Q\}$.

Our first theorem is the following:

Theorem A. *For any prime p and any pair G, G' of finite groups, the p-completed classifying spaces $BG^\wedge_p$ and $BG'^\wedge_p$ are homotopy equivalent if and only if the centric linking categories $\mathcal{L}_p^c(G)$ and $\mathcal{L}_p^c(G')$ are equivalent.*

To prove Theorem A, we first show that there is an equivalence $|\mathcal{L}_p^c(G)|^\wedge_p \simeq BG^\wedge_p$ which is natural in G. This proves the "if" part of the theorem. The converse is then proven by constructing a certain category $\mathcal{L}_p^c(X)$ associated to any space X, and then showing that the categories $\mathcal{L}_p^c(BG^\wedge_p)$ and $\mathcal{L}_p^c(G)$ are equivalent. The objects in this *linking category* $\mathcal{L}_p^c(X)$ consist of pairs (P, α), where P is a p-group and $BP \xrightarrow{\alpha} X$ is a "centric monomorphism" in a certain sense defined later. Morphisms between objects consist of monomorphisms between the p-groups, together with paths in certain mapping spaces. The precise definition of $\mathcal{L}_p^c(X)$ is given in Section 2 (Definition 2.5).

The work in [BL] was motivated in part by the paper [MP], where the authors claim an algebraic condition for determining whether or not two distinct p-completed classifying spaces have the same homotopy type. In the course of trying to compare this claim to their own work, the authors of this article were led to the discovery of an error in [MP], which invalidates the proof of the main theorem there (stated below as Conjecture D). Our attempts to fix this error led to Theorem A, which gives a different algebraic criterion for the existence of an equivalence.

We next consider the self homotopy equivalences of $BG^\wedge_p$. For any space X, let $\mathrm{Aut}(X)$ denote the topological monoid of (unpointed) self homotopy equivalences of X, and let $\mathrm{Out}(X)$ denote the group of components $\pi_0(\mathrm{Aut}(X))$. For any small category $\mathcal{C}$, let $\mathcal{A}ut(\mathcal{C})$ denote the strict monoidal category (with strictly associative operation given by composition) whose objects are self equivalences $F : \mathcal{C} \longrightarrow \mathcal{C}$ and whose morphisms are natural isomorphisms of functors. Note that every morphism in $\mathcal{A}ut(\mathcal{C})$ is invertible, making it into a groupoid. Let $\mathrm{Out}(\mathcal{C})$ denote the group of components of $\mathcal{A}ut(\mathcal{C})$, namely the set of equivalence classes obtained by identifying two functors $F', F'' \in \mathcal{A}ut(\mathcal{C})$ if they are naturally isomorphic. These uses of the notation $\mathrm{Out}(-)$ are motivated by the observation that $\mathrm{Out}(BG) \cong \mathrm{Out}(\mathcal{B}(G)) \cong$

$\mathrm{Out}(G)$ for any discrete group G (where $\mathcal{B}(G)$ is the category with one object and morphism group G).

The linking categories $\mathcal{L}_p^c(G)$ come equipped with natural forgetful functors λ_G from $\mathcal{L}_p^c(G)$ to the category of groups. A self equivalence ψ of the category $\mathcal{L}_p^c(G)$ is called *isotypical* if the functor λ_G and the composite functor $\lambda_G \circ \psi$ are naturally isomorphic. The full subcategory $\mathcal{A}ut_{\mathrm{typ}}(\mathcal{L}_p^c(G))$ of $\mathcal{A}ut(\mathcal{L}_p^c(G))$ whose objects are all isotypical self equivalences of $\mathcal{L}_p^c(G)$ inherits a monoidal structure. We let $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_p^c(G)) \leq \mathrm{Out}(\mathcal{L}_p^c(G))$ denote the subgroup of classes of isotypical self equivalences. Our next theorem gives a description of $\mathrm{Out}(BG^\wedge_p)$.

Theorem B. *For any prime p and any finite group G, there is an isomorphism of groups*

$$\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_p^c(G)) \xrightarrow{\ \cong\ } \mathrm{Out}(BG^\wedge_p).$$

The homotopy groups of a small category $\mathcal{C}$ can be defined purely algebraically or equivalently as the homotopy groups of the geometric realization of its nerve. Theorem B claims in fact that $\mathrm{Out}(BG^\wedge_p) = \pi_0(\mathrm{Aut}(BG^\wedge_p))$ is isomorphic as a group to $\pi_0(\mathcal{A}ut_{\mathrm{typ}}(\mathcal{L}_p^c(G)))$. The next theorem generalizes Theorem B by showing that in fact the groupoid $\mathcal{A}ut_{\mathrm{typ}}(\mathcal{L}_p^c(G))$ completely describes the homotopy type of $\mathrm{Aut}(BG^\wedge_p)$. The proof makes use of the fact, proven in [BL], that each component of $\mathrm{Aut}(BG^\wedge_p)$ is aspherical. For a small category $\mathcal{C}$ let $|\mathcal{C}|$ denote the geometric realization of the nerve of $\mathcal{C}$.

Theorem C. *For any prime p and any finite group G, there is a weak homotopy equivalence*

$$\mathrm{Aut}(BG^\wedge_p) \simeq |\mathcal{A}ut_{\mathrm{typ}}(\mathcal{L}_p^c(G))|.$$

Moreover, these spaces are weakly equivalent as topological monoids, in the sense that their classifying spaces are also weakly homotopy equivalent.

Theorem C is stated in terms of weak equivalences rather than homotopy equivalence because of the usual problem of whether or not a mapping space between two CW complexes has itself the homotopy type of a CW complex. If one is willing to work simplicially, i.e. replace spaces by their singular simplicial sets and mapping spaces by the respective simplicial mapping spaces, then one may replace "weak equivalence" by "equivalence" in the statement of the theorem. Throughout the paper we shall not distinguish between homotopy equivalences and weak homotopy equivalences, but the above comment should be kept in mind.

The identity component of $\mathrm{Aut}(BG^\wedge_p)$ was shown in [BL] to have the homotopy type of $BZ(\bar{G})$, where $\bar{G} = G/O_{p'}G$, the quotient of G by its maximal normal subgroup of order prime to p.

Theorem A will be proven in a more precise form as Theorem 2.9 below. Theorems B and C will be proven together as Theorem 4.5.

Let G and G' be finite groups, with Sylow p-subgroups $S \leq G$ and $S' \leq G'$. An isomorphism $\varphi : S \xrightarrow{\cong} S'$ is called *fusion preserving* if for any $P, Q \leq S$, a group isomorphism $\alpha : P \xrightarrow{\cong} Q$ is induced by conjugation in G if and only if the corresponding isomorphism $\varphi\alpha\varphi^{-1} : \varphi(P) \to \varphi(Q)$ is induced by conjugation in G'.

The claim of the main theorem in [MP] can be reformulated in these terms. However, an error in its proof occurs on [MP, p.129], where a certain functor fails to be well defined on morphisms. Recently, the third author has found a proof of this result, but one which is far beyond the scope of this paper. Hence we restate it here as the following:

Conjecture D (Martino-Priddy [MP])**.** *For any prime p and any pair G, G' of finite groups, $BG^\wedge_p \simeq BG'^\wedge_p$ if and only if there is a fusion preserving isomorphism between their Sylow p-subgroups.*

The proof of Conjecture D when p is odd is given in [Ol]; the proof for $p = 2$ is still in preparation.

Theorem A gives a different purely algebraic condition for the spaces $BG^\wedge_p$ and $BG'^\wedge_p$ to be homotopy equivalent. However, the condition in the Martino-Priddy Conjecture is simpler to check in practice. It is easier in general to construct an isomorphism between Sylow subgroups and show that it preserves fusion, than to construct an equivalence between centric linking categories.

In Section 6, we study the relation between Conjecture D and our Theorem A. Let $\mathcal{O}^c_p(G)$ be the *centric orbit category* of G: the category whose objects are the p-centric subgroups of G, and for which

$$\mathrm{Mor}_{\mathcal{O}^c_p(G)}(P, Q) = Q\backslash N_G(P, Q) \cong \mathrm{Map}_G(G/P, G/Q).$$

Let $\mathcal{Z}_G$ be the contravariant functor from $\mathcal{O}^c_p(G)$ to finite abelian p-groups which sends P to its center $Z(P)$. Let G be a finite group with a Sylow p-subgroup S. Let $\mathrm{Out}_{\mathrm{fus}}(S)$ denote the quotient of the group of fusion preserving automorphisms of S by the normal subgroup of all automorphisms induced by conjugation in G. The following theorem is an algebraic analogue of [BL, Theorem 1.6].

Theorem E. *For any prime p and any finite group G, there is an exact sequence*

$$0 \longrightarrow \lim_{\mathcal{O}^c_p(G)}{}^1(\mathcal{Z}_G) \xrightarrow{\lambda_G} \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}^c_p(G)) \xrightarrow{\mu_G} \mathrm{Out}_{\mathrm{fus}}(S) \xrightarrow{\omega_G} \lim_{\mathcal{O}^c_p(G)}{}^2(\mathcal{Z}_G),$$

where λ_G and μ_G are group homomorphisms, and where exactness at $\mathrm{Out}_{\mathrm{fus}}(S)$ means that $\mathrm{Im}(\mu_G) = \omega_G^{-1}(0)$.

The $\lim^1$ term in the theorem is known not to vanish in general. The $\lim^2$ term is the group containing the obstructions for lifting fusion preserving outer automorphisms of S to isotypical outer automorphisms of $\mathcal{L}^c_p(G)$.

Recently, this $\lim^2$ obstruction group has been shown to vanish for all finite groups G, and it is this vanishing result by the third author which implies Conjecture D. The proof of this result is, however, still very long and complicated, and uses the classification theorem for finite simple groups. For this reason, it is still useful to have more elementary proofs of this result in certain specialized cases. In this paper, we show (in Corollary 6.4) that $\lim^2(\mathcal{Z}_G) = 0$ whenever $\mathrm{rk}_p(G) < p^2$. In particular, this implies the following theorem (see Theorem 6.5 below):

Theorem F. *For any prime p and any pair of finite groups G and G', Conjecture D is true if* $\mathrm{rk}_p(G) < p^2$.

The paper is organized as follows. In Section 1, we define the centric linking categories and discuss some of their basic properties. In Section 2, we construct the categories $\mathcal{L}_p^c(X)$ for an arbitrary space X and prove Theorem A. Sections 3 and 4 are devoted to the proof of Theorems B and C. In Sections 5 and 6, we discuss fusion preserving isomorphisms between Sylow subgroups, and higher limits obstructions to the existence and uniqueness of a lift of a given fusion preserving isomorphism to a homotopy equivalence between the respective classifying spaces. In particular, Theorems E and F are proven in Section 6. Finally, in Section 7, we give some examples to demonstrate the use of these methods to describe $\mathrm{Out}(BG^\wedge_p)$ in certain cases.

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1. From groups to categories

Fix a finite group G and a prime p. We begin by defining the *transporter category* $\widetilde{\mathcal{L}}_p(G)$ and the *linking category* $\mathcal{L}_p(G)$ which will play a central role throughout this paper. These are both special cases of the "localité" categories defined by Puig [Pu].

An object in $\widetilde{\mathcal{L}}_p(G)$ is a p-subgroup $P \le G$. For each pair of p-subgroups $P, Q \le G$, we define

$$\mathrm{Mor}_{\widetilde{\mathcal{L}}_p(G)}(P,Q) = N_G(P,Q) \overset{\text{def}}{=} \{x \in G \,|\, xPx^{-1} \le Q\}$$

(the transporter). The morphism from P to Q corresponding to $x \in N_G(P,Q)$ will be denoted $\widehat{x}$. Composition is given by $\widehat{x} \circ \widehat{y} = \widehat{xy}$.

As usual, for any finite group H, $O^p(H)$ denotes the maximal normal p-perfect subgroup of H; equivalently the minimal normal subgroup of p-power index (see the appendix). We define the *linking category* $\mathcal{L}_p(G)$ to be the category with the same objects as $\widetilde{\mathcal{L}}_p(G)$, and with morphism sets

$$\mathrm{Mor}_{\mathcal{L}_p(G)}(P,Q) = N_G(P,Q)/O^p(C_G(P)).$$

In other words, we have divided out by the action of $O^p(C_G(P))$ on $N_G(P,Q)$ given by right multiplication. The morphism $P \longrightarrow Q$ corresponding to the class of $x \in N_G(P,Q)$ will also be denoted $\widehat{x}$.

To see that composition in $\mathcal{L}_p(G)$ is well defined, note first that for all $P,Q\le S$ and $g\in N_G(P,Q)$, $g^{-1}C_G(Q)g\le C_G(P)$ since $gPg^{-1}\le Q$, and hence $g^{-1}O^p(C_G(Q))g\le O^p(C_G(P))$ by Lemma A.1. Thus, for any $R\le S$, and any $h\in N_G(Q,R)$, $x\in O^p(C_G(P))$, and $y\in O^p(C_G(Q))$,

$$(hy)(gx)=hg{\cdot}(g^{-1}yg)x\in hg{\cdot}O^p(C_G(P)),$$

and hence $\widehat{hy}\circ\widehat{gx}=\widehat{h}\circ\widehat{g}$.

Our goal here is to construct a category, depending on G and p, which reflects the structure of $BG^\wedge_p$ in a way which will be made clear later. In order to do this, we must further restrict the categories $\widetilde{\mathcal{L}}_p(G)$ and $\mathcal{L}_p(G)$. Following the terminology of Dwyer, we call a subgroup $H\le G$ *centric* if $C_G(H)\le H$ (i.e., if $C_G(H)=Z(H)$). A p-subgroup $P\le G$ is *p-centric* if $Z(P)\in\mathrm{Syl}_p(C_G(P))$; or equivalently (Lemma A.4) if

$$C_G(P)=Z(P)\times O^p(C_G(P))$$

and $O^p(C_G(P))$ has order prime to p. For such subgroups, we write $C'_G(P)\overset{\text{def}}{=}O^p(C_G(P))$ for short. We let $\widetilde{\mathcal{L}}^c_p(G)\subseteq\widetilde{\mathcal{L}}_p(G)$ and $\mathcal{L}^c_p(G)\subseteq\mathcal{L}_p(G)$ denote the full subcategories whose objects are the p-centric subgroups of G. In particular, for any p-centric $P,Q\le G$,

$$\mathrm{Mor}_{\mathcal{L}^c_p(G)}(P,Q)=N_G(P,Q)/C'_G(P).$$

For any finite group G, let $\mathcal{B}(G)$ denote the category with one object o_G and morphism group G. The morphism in $\mathcal{B}(G)$ corresponding to $x\in G$ will be denoted $\widetilde{x}$, and $\widetilde{x}\circ\widetilde{y}=\widetilde{xy}$. We identify the classifying space BG with the (realization of the) nerve of $\mathcal{B}(G)$. Let

$$\widetilde{\alpha}_G\colon|\widetilde{\mathcal{L}}_p(G)|\longrightarrow|\mathcal{B}(G)|=BG$$

be the nerve of the functor $\widetilde{\mathcal{L}}_p(G)\longrightarrow\mathcal{B}(G)$ which sends each object of $\widetilde{\mathcal{L}}_p(G)$ to o_G, and which sends a morphism $P\xrightarrow{\widehat{x}}Q$ (for $x\in N_G(P,Q)$) to $\widetilde{x}\in\mathrm{Aut}_{\mathcal{B}(G)}(o_G)$. The main result of this section is the following:

Proposition 1.1. *For any finite group G, the maps*

$$\widetilde{\alpha}_G\colon|\widetilde{\mathcal{L}}^c_p(G)|\longrightarrow BG\qquad\text{and}\qquad|\pi|\colon|\widetilde{\mathcal{L}}^c_p(G)|\longrightarrow|\mathcal{L}^c_p(G)|$$

are both $\mathbb{F}_p$-homology equivalences. Hence there is a homotopy equivalence

$$\alpha_G\colon|\mathcal{L}^c_p(G)|^\wedge_p\xrightarrow{\ \simeq\ }BG^\wedge_p,$$

unique up to homotopy, such that the following triangle of homotopy equivalences commutes up to homotopy:

$$\begin{array}{ccc}|\widetilde{\mathcal{L}}^c_p(G)|^\wedge_p & \xrightarrow[\simeq]{|\pi|^\wedge_p} & |\mathcal{L}^c_p(G)|^\wedge_p\\ & {\scriptstyle(\widetilde{\alpha}_G)^\wedge_p}\searrow\simeq \quad \simeq\swarrow{\scriptstyle\alpha_G} & \\ & BG^\wedge_p\,. & \end{array}$$

Proof. This follows from Lemmas 1.2 and 1.3 below. Lemma 1.2 says that $\widetilde{\alpha}_G$ is an $\mathbb{F}_p$-homology equivalence, and hence induces a homotopy equivalence $(\widetilde{\alpha}_G)^\wedge_p$ after p-completion. Lemma 1.3 implies as a special case that the projection functor $\pi\colon \widetilde{\mathcal{L}}^c_p(G) \longrightarrow \mathcal{L}^c_p(G)$ induces an $\mathbb{F}_p$-homology equivalence between the nerves, and hence a homotopy equivalence between their p-completions. The other statements are then clear. □

The rest of the section is devoted to the proofs of Lemmas 1.2 and 1.3. For any set $\mathcal{C}$ of subgroups of G closed under conjugation, let $K_{\mathcal{C}}$ denote the category whose set of objects is $\mathcal{C}$, and with a unique morphism $P \longrightarrow Q$ whenever $P \leq Q$. Recall [Dw] that a family $\mathcal{C}$ of p-subgroups of G (closed under conjugation) is called *ample* if the projection $EG \times_G |K_{\mathcal{C}}| \longrightarrow BG$ is an $\mathbb{F}_p$-homology equivalence. Here, $|K_{\mathcal{C}}|$ denotes the realization of the nerve of the poset $\mathcal{C}$, and the action of G is induced by conjugation. In particular, the family of p-centric subgroups of G is ample by [Dw, §8]. In general we will denote by $\widetilde{\mathcal{L}}^{\mathcal{C}}_p(G) \subseteq \widetilde{\mathcal{L}}_p(G)$ and $\mathcal{L}^{\mathcal{C}}_p(G) \subseteq \mathcal{L}_p(G)$ the full subcategories whose objects are in $\mathcal{C}$.

Lemma 1.2. *For any finite group G, any prime p, and any ample family $\mathcal{C}$ of p-subgroups of G, $\widetilde{\alpha}_G$ restricts to an $\mathbb{F}_p$-homology equivalence*

$$\widetilde{\alpha}^{\mathcal{C}}_G\colon |\widetilde{\mathcal{L}}^{\mathcal{C}}_p(G)| \longrightarrow BG.$$

Proof. Let $\mathcal{E}(G)$ be the category whose objects are the elements of G, and with a unique morphism between each pair of objects (i.e., the category whose nerve is the universal free G-space EG). Let

$$\gamma\colon \mathcal{E}(G) \times_G K_{\mathcal{C}} \longrightarrow \widetilde{\mathcal{L}}^{\mathcal{C}}_p(G)$$

be the functor which sends a pair of objects (g,P) to ${}^gP = gPg^{-1}$, and which sends a pair of morphisms $(g \to xg, P{\leq}Q)$ to $\widehat{x}\colon {}^gP \to {}^{xg}Q$. This clearly induces bijections on objects and on morphisms, and hence a homeomorphism

$$|\gamma|\colon EG \times_G |K_{\mathcal{C}}| \xrightarrow{\ \cong\ } |\widetilde{\mathcal{L}}^{\mathcal{C}}_p(G)|.$$

Let $\delta\colon \mathcal{E}(G) \times_G K_{\mathcal{C}} \longrightarrow \mathcal{B}(G)$ be the functor which sends all objects to o_G, and which sends a morphism $(g \to xg, P{\leq}Q)$ to $\widetilde{x} \in \operatorname{Aut}_{\mathcal{B}(G)}(o_G)$. The following triangle of categories and functors

$$\begin{array}{ccc} \mathcal{E}(G) \times_G K_{\mathcal{C}} & \xrightarrow{\ \gamma\ } & \widetilde{\mathcal{L}}^{\mathcal{C}}_p(G) \\ & {\scriptstyle\delta}\searrow \quad \swarrow{\scriptstyle\widetilde{\alpha}^{\mathcal{C}}_G} & \\ & \mathcal{B}(G) & \end{array}$$

commutes by definition, and hence induces a commutative triangle of spaces

$$\begin{array}{ccc} EG\times_G |K_{\mathcal{C}}| & \xrightarrow[\cong]{|\gamma|} & |\widetilde{\mathcal{L}}_p^{\mathcal{C}}(G)| \\ & {\scriptstyle |\delta|}\searrow \quad \swarrow{\scriptstyle |\widetilde{\alpha}_G^{\mathcal{C}}|} & \\ & BG\,. & \end{array}$$

Since $|\delta|$ is an $\mathbb{F}_p$-homology equivalence (by definition of "ample"), $|\widetilde{\alpha}_G^{\mathcal{C}}|$ is also an $\mathbb{F}_p$-homology equivalence. □

The following lemma implies as a special case that the projection from $|\widetilde{\mathcal{L}}_p^c(G)|$ to $|\mathcal{L}_p^c(G)|$ is an $\mathbb{F}_p$-homology equivalence.

Lemma 1.3. *Fix a prime p. Let $\psi\colon \mathcal{C} \longrightarrow \mathcal{C}'$ be any functor between small categories which has the following properties:*

(i) *ψ is bijective on isomorphism classes of objects and is surjective on morphism sets.*

(ii) *For each object c in $\mathcal{C}$, the subgroup*

$$K(c) \stackrel{\text{def}}{=} \operatorname{Ker}\bigl[\operatorname{Aut}_{\mathcal{C}}(c) \longrightarrow \operatorname{Aut}_{\mathcal{C}'}(\psi(c))\bigr]$$

is finite of order prime to p.

(iii) *For each pair of objects c and d, and each $f,g\colon c\to d$ in $\mathcal{C}$, $\psi(f)=\psi(g)$ if and only if there is some $\sigma\in K(c)$ such that $g=f\circ\sigma$ (i.e., $\operatorname{Mor}_{\mathcal{C}'}(\psi(c),\psi(d)) \cong \operatorname{Mor}_{\mathcal{C}}(c,d)/K(c)$).*

Then for any functor $F\colon \mathcal{C}' \longrightarrow \mathsf{Spaces}$, the induced map

$$\operatorname*{hocolim}_{\mathcal{C}}(F) \longrightarrow \operatorname*{hocolim}_{\mathcal{C}'}(F\circ\psi)$$

is an $\mathbb{F}_p$-homology equivalence, and hence induces a homotopy equivalence between the p-completions. Also, for any functor $T\colon \mathcal{C}'^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{-}\mathsf{mod}$,

$$\lim_{\mathcal{C}}{}^*(T) \cong \lim_{\mathcal{C}'}{}^*(T\circ\psi).$$

Proof. We can assume that ψ is bijective on objects (not just on isomorphism classes). If not, then replace $\mathcal{C}'$ by the equivalent category $\mathcal{C}''$ whose objects are those of $\mathcal{C}$, and where $\operatorname{Mor}_{\mathcal{C}''}(c_1,c_2) = \operatorname{Mor}_{\mathcal{C}'}(\psi(c_1),\psi(c_2))$.

Let $\mathcal{C}$-mod and $\mathcal{C}'$-mod denote the categories of functors $\mathcal{C}^{\mathrm{op}}\to\mathsf{Ab}$ and $\mathcal{C}'^{\mathrm{op}}\to\mathsf{Ab}$. We proceed by showing that the functor $\psi^*: \mathcal{C}'\text{-}\mathsf{mod}\to\mathcal{C}\text{-}\mathsf{mod}$ given by composition on the right with ψ has a left adjoint, which is a useful observation in proving our claim. Define $(-)_K : \mathcal{C}\text{-}\mathsf{mod} \longrightarrow \mathcal{C}'\text{-}\mathsf{mod}$ by

$$D_K(c') = D(c)_{K(c)} \stackrel{\text{def}}{=} \mathbb{Z}\otimes_{\mathbb{Z}K(c)} D(c) = H_0(K(c);D(c))$$

for any D in $\mathcal{C}$-mod, and any $c'\in\mathcal{C}'$ and $c=\psi^{-1}(c')$. Note that c is uniquely defined by the assumption that ψ is bijective on objects. To see that D_K is well defined as a (contravariant) functor on $\mathcal{C}'$, let $f' : \psi(c) \to \psi(d)$ be any morphism and let $f\colon c \to d$ be a lift to $\mathcal{C}$. Then for any $\tau \in K(d)$ there exists, by (iii) above (applied with $g = \tau\circ f$), some $\sigma \in K(c)$ such that $f\circ\sigma = \tau\circ f$, and any other lift of f' differs from f by an element of $K(c)$.

By construction, for any D in $\mathcal{C}$-mod and D' in $\mathcal{C}'$-mod,

$$\mathrm{Hom}_{\mathcal{C}'\text{-}\mathsf{mod}}(D_K, D') \cong \mathrm{Hom}_{\mathcal{C}\text{-}\mathsf{mod}}(D, D'\circ\psi), \tag{1}$$

and we have thus shown that $(-)_K$ is left adjoint to ψ^*. In particular, this shows that D_K is $\mathcal{C}'$-mod-projective if D is $\mathcal{C}$-mod-projective. So if

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow 0$$

is any $\mathcal{C}$-mod projective resolution of the constant functor $\mathbb{Z}$, then

$$\cdots \longrightarrow (P_2)_K \longrightarrow (P_1)_K \longrightarrow (P_0)_K \longrightarrow \underline{\mathbb{Z}} \longrightarrow 0$$

is a chain complex of $\mathcal{C}'$-mod-projectives which is exact (a resolution) after localization at p (the groups $K(c)$ all being finite of order prime to p). Since

$$\lim_{\mathcal{C}}(T\circ\psi) \cong \mathrm{Hom}_{\mathcal{C}\text{-}\mathsf{mod}}(\underline{\mathbb{Z}}, T\circ\psi),$$

we have

$$\lim_{\mathcal{C}}{}^*(T\circ\psi) \cong \mathrm{Ext}^*_{\mathcal{C}\text{-}\mathsf{mod}}(\underline{\mathbb{Z}}, T\circ\psi) \cong H^*\bigl(\mathrm{Hom}_{\mathcal{C}\text{-}\mathsf{mod}}(P_*, T\circ\psi)\bigr).$$

Similarly, since $T(c)$ is a $\mathbb{Z}_{(p)}$-module for all c in $\mathcal{C}'$,

$$\lim_{\mathcal{C}'}{}^*(T) \cong \mathrm{Ext}^*_{\mathcal{C}'\text{-}\mathsf{mod}}(\underline{\mathbb{Z}}, T) \cong H^*\bigl(\mathrm{Hom}_{\mathcal{C}\text{-}\mathsf{mod}}((P_*)_K, T)\bigr).$$

So $\lim_{\mathcal{C}}{}^*(T\circ\psi) \cong \lim_{\mathcal{C}'}{}^*(T)$ by (1).

The $\mathbb{F}_p$-homology equivalence between homotopy colimits now follows from the isomorphisms

$$\lim_{\mathcal{C}}{}^*\, H^*(F\circ\psi(-);\mathbb{F}_p) \cong \lim_{\mathcal{C}'}{}^*\, H^*(F(-);\mathbb{F}_p),$$

together with the spectral sequence for the cohomology of a homotopy colimit. □

Remark 1.4. According to [Dw], any ample family $\mathcal{C}$ of p-subgroups of a finite group G gives rise to an $\mathbb{F}_p$-homology equivalence

$$\operatorname*{hocolim}_{\mathcal{F}_p^{\mathcal{C}}(G)} EG/C_G(-) \xrightarrow{\quad} BG$$

where $\mathcal{F}_p^{\mathcal{C}}(G)$ is the fusion category whose objects are the subgroups in $\mathcal{C}$ and whose morphisms are homomorphisms induced by conjugation in G. That is,

$$\mathrm{Mor}_{\mathcal{F}_p^{\mathcal{C}}(G)}(P,Q) \cong N_G(P,Q)/C_G(P)\,.$$

By the same methods, we can obtain another $\mathbb{F}_p$-homology equivalence

$$\operatorname*{hocolim}_{\mathcal{L}_p^{\mathcal{C}}(G)} EG/O^p(C_G(-)) \longrightarrow BG.$$

The relationship between these two decompositions can be described as moving information from the functor in the first decomposition to the indexing category in the second. When $\mathcal{C}$ is the family of p-centric subgroups of G, then $EG/O^p(C_G(P)) \simeq BO^p(C_G(P))$ is $\mathbb{F}_p$-acyclic for all $P \in \mathcal{C}$, which explains the existence of the homotopy equivalence $\alpha_G\colon |\mathcal{L}_p^c(G)|_p^\wedge \xrightarrow{\simeq} BG_p^\wedge$ of Proposition 1.1 in terms of the above $\mathbb{F}_p$-homology equivalences. This shows that we get a similar result if we replace the family of p-centric subgroups by that of all p-subgroups $P \le G$ for which $O^p(C_G(P))$ has order prime to p, and explains why we do not expect a similar equivalence to be true for families of subgroups which are larger than that.

2. From spaces to categories

In this section we introduce linking categories for spaces: homotopy theoretic analogues of the linking categories $\mathcal{L}_p(G)$ for finite groups G. The remarkable feature of this construction is that $\mathcal{L}_p(BG) \simeq \widetilde{\mathcal{L}}_p(G)$ and $\mathcal{L}_p(BG_p^\wedge) \simeq \mathcal{L}_p(G)$. The following proposition suggests a way of doing this. For any pair of groups G, H, we set

$$\operatorname{Rep}(G,H) = \operatorname{Hom}(G,H)/\operatorname{Inn}(H) :$$

the set of homomorphisms $G \longrightarrow H$ modulo conjugacy in H.

Proposition 2.1. *For any finite p-group P and any finite group G, the p-completion map $BG \longrightarrow BG_p^\wedge$ induces a (weak) homotopy equivalence*

$$\operatorname{Map}(BP,BG)_p^\wedge \xrightarrow{\ \simeq\ } \operatorname{Map}(BP,BG_p^\wedge).$$

In particular, the maps

$$\operatorname{Rep}(P,G) \xrightarrow[\cong]{B} [BP,BG] \xrightarrow[\cong]{\kappa_*} [BP,BG_p^\wedge],$$

where B sends a homomorphism $\rho\colon P \longrightarrow G$ to $B\rho$, are bijections. Also, for each $\rho\colon P \longrightarrow G$, the induced product map $C_G(\rho) \times P \longrightarrow G$ induces (after taking adjoints) a homotopy equivalence

$$BC_G(\rho) \longrightarrow \operatorname{Map}(BP,BG)_{B\rho}$$

and a weak homotopy equivalence

$$BC_G(\rho)_p^\wedge \longrightarrow \operatorname{Map}(BP,BG_p^\wedge)_{B\rho}\,.$$

Proof. The description of $\operatorname{Map}(BP,BG)$ is classical, and in fact holds for any pair of discrete groups. The fact that $\operatorname{Map}(BP,BG_p^\wedge)$ is the p-completion of $\operatorname{Map}(BP,BG)$ (when P is a p-group and G is finite) is shown in [BL, Proposition 2.1]. □

Proposition 2.1 suggests replacing a homomorphism $P \longrightarrow G$, for a p-group P, by a map $BP \longrightarrow X$. We next state a homotopy theoretic condition which corresponds to a homomorphism being injective. There could be several natural choices for such a condition. The following definition of a *homotopy monomorphism* seems to be the most useful for our purposes here.

Definition 2.2. *A map $f\colon X \longrightarrow Y$ is said to be a* homotopy monomorphism *at p if $H^*(X;\mathbb{F}_p)$ is a finitely generated module over $H^*(Y;\mathbb{F}_p)$ via the induced map f^*. For any space X and any prime p, a* subgroup *of X at p (a p*-subgroup *of X) is a pair (G,α) where G is a group (a p-group) and $\alpha\colon BG \longrightarrow X$ is a homotopy monomorphism.*

This definition of homotopy monomorphisms at p is motivated by the following lemma.

Lemma 2.3. *Let G and G' be two finite groups and $f\colon G \longrightarrow G'$ a homomorphism. Then $Bf\colon BG \longrightarrow BG'^{\wedge}_p$ is a homotopy monomorphism at p if and only if* $\mathrm{Ker}(f)$ *has order prime to p.*

Proof. This statement is implicit in [Sw]. Assume first that $f\colon G \longrightarrow G'$ is a monomorphism, and fix an embedding $G' \subseteq U(n)$ for some n. The composite $G \longrightarrow G' \longrightarrow U(n)$ induces a fibration

$$U(n)/G \longrightarrow BG \longrightarrow BU(n),$$

and the E^2-term of the spectral sequence for this fibration is finitely generated as a module over $H^*(BU(n);\mathbb{F}_p)$. Hence, since $H^*(BU(n);\mathbb{F}_p)$ is noetherian, the E^∞-term is also finitely generated. The composite $BG \longrightarrow BG' \longrightarrow BU(n)$ thus makes $H^*(G;\mathbb{F}_p)$ into a finitely generated $H^*(BU(n);\mathbb{F}_p)$-module; and hence a finitely generated $H^*(G';\mathbb{F}_p)$-module.

If $f\colon G \longrightarrow G'$ is such that $K = \mathrm{Ker}(f)$ has order prime to p, then $H^*(G;\mathbb{F}_p) \cong H^*(G/K;\mathbb{F}_p)$ by the Serre spectral sequence for the fibration $BK \to BG \to B(G/K)$. So $H^*(G;\mathbb{F}_p)$ is finitely generated as an $H^*(G';\mathbb{F}_p)$-module in this case also.

Conversely, assume that $H^*(G;\mathbb{F}_p)$ is finitely generated over $H^*(G';\mathbb{F}_p)$, and again set $K = \mathrm{Ker}(f)$. Since $K \le G$, $H^*(K;\mathbb{F}_p)$ is a finitely generated $H^*(G;\mathbb{F}_p)$-module, and thus a finitely generated $H^*(G';\mathbb{F}_p)$-module. Since $f|_K\colon K \longrightarrow G'$ is trivial, $\mathrm{Im}[H^*(G';\mathbb{F}_p) \longrightarrow H^*(K;\mathbb{F}_p)] = \mathbb{F}_p$; and hence $\dim_{\mathbb{F}_p}(H^*(K;\mathbb{F}_p)) < \infty$. By [Sw, Corollary 1], this implies that $|K|$ is prime to p. □

We also need to define *p-centric subgroups* of a space. Recall (Proposition 2.1) that for any finite group G and any p-subgroup $P \le G$, $BC_G(P) \simeq \mathrm{Map}(BP,BG)_{\mathrm{incl}}$ and $BC_G(P)^{\wedge}_p \simeq \mathrm{Map}(BP,BG^{\wedge}_p)_{\mathrm{incl}}$. This motivates the following definition, originally due to Dwyer and Kan [DK].

Definition 2.4. *A map $f\colon X \longrightarrow Y$ is* centric *if the induced map*

$$(f\circ -)\colon\ \mathrm{Map}(X,X)_{id} \longrightarrow \mathrm{Map}(X,Y)_f$$

is a homotopy equivalence, and is p-centric *if $(f\circ -)$ is an $\mathbb{F}_p$-homology equivalence. A p-subgroup (P,α) of a space X is p-centric if the map $\alpha\colon BP \longrightarrow X$ is p-centric.*

We are now ready to define the linking category, and centric linking category, of a space.

Definition 2.5. *Fix a space X. Define $\mathcal{L}_p(X)$ to be the category whose objects are the p-subgroups (P,α) of X, and whose morphisms from (P,α) to (Q,β) are the pairs (φ,η), where $\varphi\colon P \longrightarrow Q$ is a homomorphism and η is a homotopy class of paths in $\mathrm{Map}(BP,X)$ from α to $\beta\circ B\varphi$. The identity morphism of an object (P,α) is $(\mathrm{Id}_P, C_\alpha)$, where C_α is the homotopy class of the constant path at α. The composite of two morphisms*

$$(P,\alpha) \xrightarrow{(\varphi,\eta)} (Q,\beta) \xrightarrow{(\psi,\theta)} (R,\gamma)$$

is the morphism $\big(\psi\circ\varphi, (\theta\circ B\varphi)\cdot\eta\big)$, where $(-)\cdot(-)$ denotes the composite (from right to left) of two homotopy classes of paths. Let $\mathcal{L}_p^c(X)$ be the full subcategory of $\mathcal{L}_p(X)$ whose objects are the p-centric subgroups of X.

If $f\colon X \longrightarrow Y$ is a homotopy monomorphism at p, then there is an induced functor $\mathcal{L}_p(f)\colon \mathcal{L}_p(X) \longrightarrow \mathcal{L}_p(Y)$, and this correspondence satisfies the usual functorial properties. In order to obtain similar functoriality properties for the construction $\mathcal{L}_p^c$ we need maps to preserve p-centricity; that is, for (P,j) a p-centric subgroup of X we need that $(P, f\circ j)$ is also p-centric in Y. This happens only under very restrictive conditions. One relevant case of interest to us is that of the p-completion map $BG \longrightarrow BG^\wedge_p$, for a finite group G. This is clearly a homotopy monomorphism at p and by Proposition 2.1, preserves p-centricity. It therefore induces functors

$$\mathcal{L}_p(\kappa)\colon \mathcal{L}_p(BG) \longrightarrow \mathcal{L}_p(BG^\wedge_p) \qquad \text{and} \qquad \mathcal{L}_p^c(\kappa)\colon \mathcal{L}_p^c(BG) \longrightarrow \mathcal{L}_p^c(BG^\wedge_p)\,.$$

For any finite group G, any p-subgroups $P,Q \le G$, and any $x \in N_G(P,Q)$, we let $c_x\colon P \longrightarrow Q$ denote the homomorphism $c_x(g) = xgx^{-1}$. Also, $\eta_x\colon BP\times I \longrightarrow BG$ denotes the homotopy (i.e., path in $\mathrm{Map}(BP,BG)$) induced by the natural transformation $o_P \mapsto \widetilde{x} \in \mathrm{Aut}_{\mathcal{B}(G)}(o_G)$ between the functors $\mathcal{B}(\mathrm{incl}), \mathcal{B}c_x\colon \mathcal{B}(P) \longrightarrow \mathcal{B}(G)$.

Proposition 2.6. *For any finite group G and any prime p, let*

$$\widetilde{\beta}_G\colon \widetilde{\mathcal{L}}_p(G) \longrightarrow \mathcal{L}_p(BG)$$

be the functor defined as follows. For any p-subgroup $P \le G$, set $\widetilde{\beta}_G(P) = (P, i_P)$, where $i_P\colon BP \longrightarrow BG$ denotes the inclusion. For any pair of p-subgroups $P,Q \le G$, and any $x \in N_G(P,Q)$, set

$$\widetilde{\beta}_G(\widehat{x}) = (c_x, \eta_x)\colon (P,i_P) \longrightarrow (Q,i_Q).$$

Then $\widetilde{\beta}_G$ is an equivalence of categories, and restricts to an equivalence

$$\beta_G^c\colon \widetilde{\mathcal{L}}_p^c(G) \longrightarrow \mathcal{L}_p^c(BG).$$

Proof. By Proposition 2.1, $[BP,BG] \cong \mathrm{Rep}(P,G)$ for any finite group P. By Lemma 2.3, a homomorphism $\rho\colon P \longrightarrow G$ is injective if and only if $B\rho\colon BP \longrightarrow BG$

is a homotopy monomorphism. This shows that $\widetilde{\beta}_G$ induces a bijection between isomorphism classes of objects in $\widetilde{\mathcal{L}}_p(G)$ and in $\mathcal{L}_p(BG)$.

By Proposition 2.1 again, for any p-subgroup $P \le G$, (P, i_P) is p-centric in the sense of Definition 2.4 if and only if the inclusion $BZ(P) \longrightarrow BC_G(P)$ is an $\mathbb{F}_p$-homology equivalence. Since $Z(P)$ is central in $C_G(P)$, the spectral sequence of the fibration

$$BZ(P) \longrightarrow BC_G(P) \longrightarrow B(C_G(P)/Z(P))$$

shows that this inclusion is an $\mathbb{F}_p$-homology equivalence if and only if $B(C_G(P)/Z(P))$ is $\mathbb{F}_p$-acyclic. Finally, by Lemma 2.3 (applied with $G' = 1$), $B(C_G(P)/Z(P))$ is $\mathbb{F}_p$-acyclic if and only if $C_G(P)/Z(P)$ has order prime to p; i.e., P is p-centric in G. This shows that $\widetilde{\beta}^c_G$ also induces a bijection between isomorphism classes of objects.

It remains to show, for all $P, Q \le G$, that $\widetilde{\beta}_G$ induces a bijection

$$\mathrm{Mor}_{\widetilde{\mathcal{L}}_p(G)}(P, Q) \longrightarrow \mathrm{Mor}_{\mathcal{L}_p(BG)}((P, i_P), (Q, i_Q)).$$

For any $x \in N_G(P, Q)$ (i.e., any morphism $\widehat{x}\colon P \longrightarrow Q$ in $\widetilde{\mathcal{L}}_p(G)$), $\widetilde{\beta}_G(\widehat{x}) = (c_x, \eta_x)$, and by construction the restriction of $\eta_x\colon BP \times I \longrightarrow BG$ to the basepoint of BP is a loop which represents the element $x \in \pi_1(BG) \cong G$. This proves that the map from $\mathrm{Mor}_{\widetilde{\mathcal{L}}_p(G)}(P, Q)$ to $\mathrm{Mor}_{\mathcal{L}_p(BG)}((P, i_P), (Q, i_Q))$ is injective. So it remains only to show that these morphism sets have the same order.

By definition, $\mathrm{Mor}_{\mathcal{L}_p(BG)}((P, i_P), (Q, i_Q))$ consists of pairs (ρ, η) where $\rho\colon P \longrightarrow Q$ is a homomorphism and η represents a homotopy class of paths in $\mathrm{Map}(BP, BG)$ from i_P to $i_Q \circ B\rho$. For any fixed ρ, $\pi_1(\mathrm{Map}(BP, BG)_{i_P}) \cong C_G(P)$ acts freely and transitively on the set of homotopy classes of paths of $\mathrm{Map}(BP, BG)$ from i_P to $i_Q \circ B\rho$. Hence $\pi_1(\mathrm{Map}(BP, BG)_{i_P}) \cong C_G(P)$ acts freely on $\mathrm{Mor}_{\mathcal{L}_p(BG)}((P, i_P), (Q, i_Q))$ and orbits are in one to one correspondence with all homomorphism $P \longrightarrow Q$ induced by conjugation; that is, $N_G(P, Q)/C_G(P)$. In particular,

$$\begin{aligned} |\mathrm{Mor}_{\mathcal{L}_p(BG)}((P, i_P), (Q, i_Q))| &= \big|N_G(P, Q)/C_G(P)\big| \cdot |C_G(P)| = \\ &= |N_G(P, Q)| = |\mathrm{Mor}_{\widetilde{\mathcal{L}}_p(G)}(P, Q)|\,. \end{aligned}$$ □

We next show that the categories $\mathcal{L}^c_p(G)$ and $\mathcal{L}^c_p(BG^\wedge_p)$ are equivalent for any finite group G. This uses the description in Proposition 2.1 of maps $BP \longrightarrow BG^\wedge_p$ for p-groups P.

Proposition 2.7. *The composite functor*

$$\widetilde{\mathcal{L}}_p(G) \xrightarrow{\ \widetilde{\beta}_G\ } \mathcal{L}_p(BG) \xrightarrow{\ \mathcal{L}_p(\kappa)\ } \mathcal{L}_p(BG^\wedge_p)$$

factors through an equivalence of categories

$$\beta_G\colon \mathcal{L}_p(G) \longrightarrow \mathcal{L}_p(BG^\wedge_p),$$

which restricts to an equivalence of subcategories

$$\beta^c_G\colon \mathcal{L}^c_p(G) \longrightarrow \mathcal{L}^c_p(BG^\wedge_p).$$

Proof. Define β_G on objects by setting $\beta_G(P) = (P, i_P)$ for any p-subgroup $P \le G$, where i_P now denotes the p-completion $BP \to BG^\wedge_p$ of the map induced by inclusion. Since $[BP, BG^\wedge_p] \cong [BP, BG]$ for any p-group P (Proposition 2.1), this induces a bijection between isomorphism classes of objects in $\mathcal{L}_p(G)$ and those in $\mathcal{L}_p(BG^\wedge_p)$. Also, i_P is p-centric as a map to $BG^\wedge_p$ if and only if it is p-centric as a map to BG (Proposition 2.1 again), and this holds if and only if P is p-centric by Proposition 2.6.

It remains to show, for any pair of p-subgroups $P, Q \le G$, that there is a (unique) bijection $\beta_G(P,Q)$ which makes the following square commute:

$$\begin{array}{ccc} N_G(P,Q) = \mathrm{Mor}_{\widetilde{\mathcal{L}}_p(G)}(P,Q) & \xrightarrow[\cong]{\widetilde{\beta}_G(P,Q)} & \mathrm{Mor}_{\mathcal{L}_p(BG)}((P,i_P),(Q,i_Q)) \\ \big\downarrow & & \big\downarrow{\scriptstyle \kappa(P,Q)} \\ N_G(P,Q)/O^p(C_G(P)) = \mathrm{Mor}_{\mathcal{L}_p(G)}(P,Q) & \xrightarrow{\beta_G(P,Q)} & \mathrm{Mor}_{\mathcal{L}_p(BG^\wedge_p)}((P,i_P),(Q,i_Q)). \end{array} \tag{1}$$

The same argument as that used in the proof of Proposition 2.6 shows that the group $\pi_1(\mathrm{Map}(BP, BG^\wedge_p)_{Bi_P})$ acts freely on $\mathrm{Mor}_{\mathcal{L}_p(BG^\wedge_p)}((P,i_P),(Q,i_Q))$ with orbit set $N_G(P,Q)/C_G(P)$, the same as the orbit set of the action of $\pi_1(\mathrm{Map}(BP, BG)_{Bi_P})$ on $\mathrm{Mor}_{\mathcal{L}_p(BG)}((P,i_P),(Q,i_Q))$. Furthermore, the map $\kappa(P,Q)$ is equivariant via the induced map of fundamental groups

$$\pi_1(\mathrm{Map}(BP, BG)_{Bi_P}) \longrightarrow \pi_1(\mathrm{Map}(BP, BG^\wedge_p)_{Bi_P}). \tag{2}$$

By Propositions 2.1 and A.2,

$$\pi_1\big(\mathrm{Map}(BP, BG^\wedge_p)_{i_P}\big) \cong \pi_1(BC_G(P)^\wedge_p) \cong C_G(P)/O^p(C_G(P)), \tag{3}$$

and homomorphism (2) is just the projection of $C_G(P)$ onto $C_G(P)/O^p(C_G(P))$. Thus both vertical maps in diagram (1) involve dividing out by a free action of $O^p(C_G(P))$.

More precisely, for any $x, y \in N_G(P,Q)$ such that $c_x = c_y \in \mathrm{Hom}(P,Q)$, η_x and η_y are homotopic as paths in $\mathrm{Map}(BP, BG^\wedge_p)$ from i_P to $i_Q \circ Bc_x$ if and only if $\eta_x\eta_y^{-1}$ is trivial in $\pi_1\big(\mathrm{Map}(BP, BG^\wedge_p)_{i_P}\big)$. By (3), this is the case if and only if $xy^{-1} \in O^p(C_G(P))$, or equivalently $\widehat{x} = \widehat{y}$ in $\mathrm{Mor}_{\mathcal{L}_p(G)}(P,Q)$. This shows that $\beta_G(P,Q)$ is well defined and a bijection. Since β_G takes p-centric objects to p-centric objects, its restriction to $\mathcal{L}^c_p(G)$ is also an equivalence of categories. □

Upon combining Propositions 2.6 and 2.7 with Lemma 1.3, we get the following:

Corollary 2.8. *For any finite group G, the functor $\mathcal{L}^c_p(\kappa)\colon \mathcal{L}^c_p(BG) \longrightarrow \mathcal{L}^c_p(BG^\wedge_p)$ induces an $\mathbb{F}_p$-homology equivalence on nerves.*

We are now ready to prove Theorem A, which we restate as

Theorem 2.9. *For any pair G, G' of finite groups, $BG^\wedge_p$ and $BG'^\wedge_p$ are homotopy equivalent if and only if the categories $\mathcal{L}^c_p(G)$ and $\mathcal{L}^c_p(G')$ are equivalent.*

Proof. A homotopy equivalence $BG^\wedge_p \longrightarrow BG'^\wedge_p$ induces an equivalence of categories $\mathcal{L}^c_p(BG^\wedge_p) \longrightarrow \mathcal{L}^c_p(BG'^\wedge_p)$, and hence an equivalence $\mathcal{L}^c_p(G) \xrightarrow{\simeq} \mathcal{L}^c_p(G')$ by Proposition 2.7. Conversely, if the categories $\mathcal{L}^c_p(G)$ and $\mathcal{L}^c_p(G')$ are equivalent, then their nerves are homotopy equivalent, and hence $BG^\wedge_p \simeq BG'^\wedge_p$ by Proposition 1.1. □

3. Spaces of equivalences

We now turn to the proofs of Theorems B and C. They will first be reduced, in Lemma 3.1, to a problem of comparing two "categories of automorphisms": one of the space $BG^\wedge_p$, and the other of the category $\mathcal{L}^c_p(G)$. We begin by defining these categories of automorphisms.

For any category $\mathcal{C}$, we denote by $\mathcal{A}ut(\mathcal{C})$ the category whose objects are the equivalences of categories $\psi : \mathcal{C} \to \mathcal{C}$, and whose morphisms are the natural isomorphisms $\Psi : \psi \longrightarrow \psi'$ between equivalences. We think of an equivalence between categories either as a functor which is invertible modulo natural isomorphisms of functors, or equivalently as a functor which induces bijections on isomorphism classes of objects and on all morphism sets. Analogously, for any CW complex X, we denote by $\mathcal{A}ut(X)$ the fundamental groupoid of $\mathrm{Aut}(X)$; i.e., the category whose objects are the self homotopy equivalences $\varphi : X \xrightarrow{\simeq} X$, and where $\mathrm{Mor}_{\mathcal{A}ut(X)}(\varphi, \varphi')$ is the set of homotopy classes of paths from φ to φ' in the mapping space. All of these categories $\mathcal{A}ut(-)$ are given the discrete topology. In both cases, we write $\mathrm{Out}(-) = \pi_0(\mathcal{A}ut(-))$. Thus, $\mathrm{Out}(\mathcal{C})$ is the group of self equivalences of $\mathcal{C}$ up to natural isomorphisms of functors, and $\mathrm{Out}(X)$ is the group of homotopy classes of self homotopy equivalences of X.

In general, we write $\mathcal{C} \xrightarrow{\simeq} \mathcal{D}$ to indicate that a functor is an equivalence of categories, and $\psi \cong \psi'$ to denote that two functors ψ and ψ' are naturally isomorphic. When functors $\psi, \psi' \colon \mathcal{C} \longrightarrow \mathcal{D}$ are already defined, a natural isomorphism $\psi \xrightarrow{\Phi} \psi'$ will be described as a function which sends each object $c \in \mathrm{Ob}(\mathcal{C})$ to an isomorphism Ψ_c in $\mathcal{D}$, satisfying the obvious naturality properties.

Recall that a natural morphism $\psi \xrightarrow{\Phi} \psi'$ of functors $\mathcal{C} \longrightarrow \mathcal{D}$ is equivalent to a functor $\widehat{\Phi} \colon \mathcal{C} \times [1] \longrightarrow \mathcal{D}$, where $[1]$ is the category with to objects $0, 1$ and one non-identity morphism $(0 \to 1)$. For each $c \in \mathrm{Ob}(\mathcal{C})$, $\widehat{\Phi}$ sends $(c, 0)$ to $\psi(c)$ and $(c, 1)$ to $\psi'(c)$, while for each morphism $c \xrightarrow{f} d$,

$$\widehat{\Phi}(f, \mathrm{Id}_0) = \psi(f), \quad \widehat{\Phi}(f, \mathrm{Id}_1) = \psi'(f), \quad \text{and} \quad \widehat{\Phi}(f, 0 \to 1) = \Phi_d \circ \psi(f) = \psi'(f) \circ \Phi_c.$$

Each of these categories $\mathcal{A}ut(-)$ is a strict monoidal category, in the sense that composition defines a strictly associative functor

$$\mathcal{A}ut(-) \times \mathcal{A}ut(-) \longrightarrow \mathcal{A}ut(-)$$

with strict identity. The nerve of each $\mathcal{A}ut(-)$ is thus a simplicial monoid, and $|\mathcal{A}ut(-)|$ is a topological monoid.

For any space X, let $S_\bullet X$ denote its singular simplicial set, and let $\pi(X)$ denote its fundamental groupoid. Let

$$X \xleftarrow{\ \mathrm{ev}_X\ } |S_\bullet X| \xrightarrow{\ \sigma_X\ } |\pi(X)|$$

denote the evaluation map, and the map which takes each simplex to its homotopy class, respectively. More precisely, σ_X is the realization of the map of simplicial sets which is the identity on vertices (the elements of X in both cases), and sends a singular simplex $\Delta^n \xrightarrow{\ \phi\ } X$ to the image under ϕ of the sequence of edges in Δ^n which connects the successive vertices.

The following lemma shows that $\mathcal{A}ut(BG^\wedge_p) = \pi(\mathrm{Aut}(BG^\wedge_p))$ and $\mathrm{Aut}(BG^\wedge_p)$ have the same homotopy type, and hence that it suffices from now on to work with the former.

Lemma 3.1. *For any finite group G,*

$$\mathrm{Aut}(BG^\wedge_p) \xleftarrow[\simeq]{\ \mathrm{ev}\ } |S_\bullet \mathrm{Aut}(BG^\wedge_p)| \xrightarrow[\simeq]{\ \sigma\ } |\mathcal{A}ut(BG^\wedge_p)|$$

are homotopy equivalences of topological monoids.

Proof. Both maps are clearly morphisms of topological monoids. The first is a (weak) homotopy equivalence (cf. [GJ, Theorem I.11.4]), and the second is a homotopy equivalence since by [BL, Theorem 1.1], $\mathrm{Aut}(BG^\wedge_p)$ is aspherical. □

It now remains to compare $\mathcal{A}ut(BG^\wedge_p)$ with $\mathcal{A}ut(\mathcal{L}^c_p(G))$. But in fact, we do not deal with all self equivalences of the category $\mathcal{L}^c_p(G)$, but only certain "isotypical" equivalences.

Definition 3.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories equipped with functors $\mathcal{C} \xrightarrow{\ \gamma\ } \mathsf{Gr}$ and $\mathcal{D} \xrightarrow{\ \delta\ } \mathsf{Gr}$ to the category of groups. A functor $\psi : \mathcal{C} \longrightarrow \mathcal{D}$ is* isotypical *if γ is naturally isomorphic to $\delta \circ \psi$. When γ is understood, $\mathcal{A}ut_{\mathrm{typ}}(\mathcal{C})$ denotes the strict monoidal category of isotypical self-equivalences and natural isomorphisms between them, and $\mathrm{Out}_{\mathrm{typ}}(\mathcal{C})$ denotes the monoid of isotypical self-equivalences of $\mathcal{C}$ modulo natural isomorphisms of functors.*

We emphasize that the definition of an isotypical functor does not include the natural isomorphism $\gamma \xrightarrow{\ \cong\ } \delta \circ \psi$ as part of the data, but only requires that such an isomorphism exists. Thus, if we think of a category together with a functor to groups as a "diagram of groups", then an equivalence between diagrams of groups defines an isotypical equivalence, but the isotypical equivalence contains less information than the equivalence between diagrams.

The following lemma is immediate, and shows for example that $\mathrm{Out}_{\mathrm{typ}}(\mathcal{C})$ is always a group.

Lemma 3.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories equipped with functors $\mathcal{C} \xrightarrow{\ \gamma\ } \mathsf{Gr}$ and $\mathcal{D} \xrightarrow{\ \delta\ } \mathsf{Gr}$, and let $\psi : \mathcal{C} \longrightarrow \mathcal{D}$ be an isotypical functor. Then any functor*

$\psi' : \mathcal{C} \longrightarrow \mathcal{D}$ naturally isomorphic to ψ is isotypical, and any right inverse to ψ up to natural isomorphism (i.e., any $\mathcal{D} \xrightarrow{\psi^*} \mathcal{C}$ such that $\psi \circ \psi^* \cong \mathrm{Id}_{\mathcal{D}}$) is isotypical. In particular, $\mathrm{Out}_{\mathrm{typ}}(\mathcal{C})$ is a group. □

The categories $\widetilde{\mathcal{L}}_p(G)$, $\mathcal{L}_p(G)$, $\mathcal{L}_p(X)$ and all their variations are examples of such diagrams of groups, where the functors

$$\lambda_G : \mathcal{L}_p(G) \longrightarrow \mathsf{Gr} \qquad \text{and} \qquad \lambda_X : \mathcal{L}_p(X) \longrightarrow \mathsf{Gr},$$

are the obvious forgetful functors

$$\lambda_G\big(P \xrightarrow{\widehat{x}} Q\big) = \big(P \xrightarrow{c_x} Q\big) \qquad \text{and} \qquad \lambda_X\big((P,\alpha) \xrightarrow{(\varphi,\eta)} (Q,\beta)\big) = \big(P \xrightarrow{\varphi} Q\big).$$

(Recall that c_x is the homomorphism $g \mapsto xgx^{-1}$.)

For the categories $\mathcal{L}_p^c(G)$, there is an alternative criterion for a functor being isotypical, which is more useful in concrete situations. For each p-centric subgroup $P \le G$, let $D_P(G)$ denote the "distinguished subgroup" of $\mathrm{Aut}_{\mathcal{L}_p^c(G)}(P)$ given by

$$D_G(P) = \{\widehat{g} \,|\, g \in P\} \subseteq \mathrm{Aut}_{\mathcal{L}_p^c(G)}(P) \cong N_G(P)/C'_G(P).$$

Since $C'_G(P)$ has order prime to p, $D_G(P) \cong P$, and we can identify these two groups. The next lemma says that an equivalence $\psi\colon \mathcal{L}_p^c(G) \xrightarrow{\simeq} \mathcal{L}_p^c(G')$ is isotypical if and only if it sends distinguished subgroups isomorphically to distinguished subgroups.

Lemma 3.4. *Fix finite groups G and G', and an equivalence $\psi\colon \mathcal{L}_p^c(G) \longrightarrow \mathcal{L}_p^c(G')$. Then ψ is isotypical if and only if*

$$\psi_{P,P}\colon\ \mathrm{Aut}_{\mathcal{L}_p^c(G)}(P) \xrightarrow{\ \cong\ } \mathrm{Aut}_{\mathcal{L}_p^c(G')}(\psi(P))$$

sends $D_G(P)$ isomorphically to $D_{G'}(\psi(P))$ for each p-centric $P \le G$. In this case, if $P \xrightarrow{\psi_P} \psi(P)$ denotes the restriction of $\psi_{P,P}$ under the identifications $P = D_G(P)$ and $\psi(P) = D_{G'}(\psi(P))$, then $(P \mapsto \psi_P)$ is a natural isomorphism of functors $\lambda_G \xrightarrow{\cong} \lambda_{G'} \circ \psi$.

Proof. To simplify notation, we write $P' = \psi(P)$ for any P in $\mathcal{L}_p^c(G)$. Assume first that ψ is isotypical, and let $\Lambda\colon \lambda_G \xrightarrow{\cong} \lambda_{G'} \circ \psi$ be a natural isomorphism. Fix P, let $g \in P$ be any element, and set $\widehat{x} = \psi_{P,P}(\widehat{g})$, where $x \in N_{G'}(P')$. Then Λ sends P to an isomorphism $\Lambda_P \in \mathrm{Iso}(P,P')$ of groups, and

$$c_{\Lambda_P(g)} \circ \Lambda_P = \Lambda_P \circ c_g = c_x \circ \Lambda_P :$$

the first equality holds when Λ_P is replaced by any homomorphism $P \longrightarrow P'$, and the second holds by the naturality of Λ with respect to $(P \xrightarrow{\widehat{g}} P)$. Thus $c_{\Lambda_P(g)} = c_x$, so $x^{-1}\Lambda_P(g) \in C_{G'}(P')$, and we can assume that $x^{-1}\Lambda_P(g) \in Z(P')$ without changing the class $\widehat{x} = x{\cdot}C'_{G'}(P')$. It follows that $x \in P'$, i.e. that $\psi_{P,P}(\widehat{g}) \in D_{G'}(P')$, and thus $\psi_{P,P}(D_G(P)) \subseteq D_{G'}(P')$. Finally, these two sets are equal, since the distinguished subgroups are abstractly isomorphic (and $\psi_{P,P}$ is an isomorphism).

Now assume that $\psi_{P,P}(D_G(P)) = D_{G'}(P')$ for each P, and let $\psi_P\colon P \xrightarrow{\cong} P'$ be the restriction of $\psi_{P,P}$ under our identifications. We must show that $(P \mapsto \psi_P)$ is natural as an isomorphism of functors $\lambda_G \longrightarrow \lambda_{G'} \circ \psi$; i.e., that

$$\psi_Q \circ c_x = c_y \circ \psi_P \in \mathrm{Hom}(P, Q') \tag{1}$$

for any morphism $P \xrightarrow{\widehat{x}} Q$ in $\mathcal{L}_p^c(G)$, where $\widehat{y} = \psi_{P,Q}(\widehat{x})$. For any $g \in P$, ψ sends $\widehat{x} \circ \widehat{g} = \widehat{{}^x g} \circ \widehat{x}$ in $\mathcal{L}_p^c(G)$ to

$$\widehat{y} \circ \widehat{\psi_P(g)} = \widehat{\psi_Q({}^x g)} \circ \widehat{y} \in \mathrm{Mor}_{\mathcal{L}_p^c(G')}(P', Q') = N_{G'}(P', Q')/C'_{G'}(P'),$$

where $C_{G'}(P') = Z(P') \times C'_{G'}(P')$ and the second factor has order prime to p. Hence there is $h \in C'_{G'}(P')$ such that $y\psi_P(g) = \psi_Q(xgx^{-1})yh$,

$$yhy^{-1} = \psi_Q(xgx^{-1})^{-1} y\psi_P(g)y^{-1} \in Q'$$

(since $yP'y^{-1} \le Q'$), and $h = 1$ since it has order prime to p. Thus, $\psi_Q(xgx^{-1}) = y\psi_P(g)y^{-1}$ for all $g \in P$, and this proves (1). $\square$

From now on, for any isotypical equivalence $\mathcal{L}_p^c(G) \underset{\simeq}{\xrightarrow{\psi}} \mathcal{L}_p^c(G')$, and any p-centric $P \le G$, we let $\psi_P\colon P \xrightarrow{\cong} \psi(P)$ denote the isomorphism obtained by restricting $\psi_{P,P}$.

We next define functors

$$\mathcal{A}ut(BG^\wedge_p) \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{A}ut_{\mathrm{typ}}(\mathcal{L}_p^c(G)),$$

which will later be seen to be inverses up to natural isomorphism. Very roughly, R is defined to be the realization functor from $\mathcal{A}ut_{\mathrm{typ}}(\mathcal{L}_p^c(G))$ to $\mathcal{A}ut(|\mathcal{L}_p^c(G)|^\wedge_p)$ followed by an equivalence of categories induced by the homotopy equivalence $|\mathcal{L}_p^c(G)|^\wedge_p \simeq BG^\wedge_p$; while L is induced by the functor $\mathcal{L}(-)$ to $\mathcal{A}ut_{\mathrm{typ}}(\mathcal{L}_p^c(BG^\wedge_p))$ followed by an equivalence induced by $\mathcal{L}_p^c(BG^\wedge_p) \simeq \mathcal{L}_p^c(G)$.

More precisely, for any category $\mathcal{C}$, let

$$\mathcal{A}ut(\mathcal{C}) \xrightarrow{|-|} \mathcal{A}ut(|\mathcal{C}|) \qquad \text{and} \qquad \mathcal{A}ut(\mathcal{C}) \xrightarrow{|-|^\wedge_p} \mathcal{A}ut(|\mathcal{C}|^\wedge_p)$$

denote the functors which take a self equivalence of $\mathcal{C}$ to its geometric realization (before or after p-completion), and which take a natural isomorphism, interpreted as a functor from $\mathcal{C} \times [1]$ to $\mathcal{C}$, to the homotopy class of its geometric realization. For any finite group G, let $|\mathcal{L}_p^c(G)|^\wedge_p \xrightarrow{\alpha_G} BG^\wedge_p$ be the homotopy equivalence of Proposition 1.1, and fix a homotopy inverse

$$\alpha^* : BG^\wedge_p \longrightarrow |\mathcal{L}_p^c(G)|^\wedge_p.$$

Let

$$c_\alpha : \mathcal{A}ut(|\mathcal{L}_p^c(G)|^\wedge_p) \longrightarrow \mathcal{A}ut(BG^\wedge_p)$$

denote the functor "conjugation by α", defined on objects and morphisms by setting

$$c_\alpha(f) = \alpha^* \circ f \circ \alpha_G \qquad \text{and} \qquad c_\alpha(F) = \alpha^* \circ F \circ (\alpha_G \times I)$$

for any $f \in \mathrm{Aut}(|\mathcal{L}_p^c(G)|_p^\wedge)$ and any homotopy F. We now define the realization functor

$$R = R_G\colon \mathcal{A}ut_{\mathrm{typ}}(\mathcal{L}_p^c(G)) \xrightarrow{\ |-|_p^\wedge\ } \mathcal{A}ut(|\mathcal{L}_p^c(G)|_p^\wedge) \xrightarrow[\simeq]{c_\alpha} \mathcal{A}ut(BG_p^\wedge).$$

The first functor in this composite preserves the monoidal structures of the categories, but c_α does not in general.

For any space X, define a functor

$$\mathcal{L}_X : \mathcal{A}ut(X) \longrightarrow \mathcal{A}ut_{\mathrm{typ}}(\mathcal{L}_p^c(X))$$

as follows. On objects, $\mathcal{L}_X$ sends a self equivalence $X \xrightarrow{f} X$ to $\mathcal{L}_p^c(f)$. If $F : X \times I \longrightarrow X$ is a homotopy, representing a morphism in $\mathcal{A}ut(X)$ from f to f', then $\mathcal{L}_X(F)$ is defined to be the natural isomorphism of functors which sends an object (P, α) to the morphism $(\mathrm{Id}_P, [F \circ (\alpha \times I)])$. (Note that this only depends on the homotopy class of F, as a path in $\mathrm{Aut}(X)$ from f to f'.) One easily checks that $\mathcal{L}_X$ preserves compositions of homotopies and of homotopy equivalences, and is thus a a well defined functor of monoidal categories.

Since $\beta_G^c : \mathcal{L}_p^c(G) \longrightarrow \mathcal{L}_p^c(BG_p^\wedge)$ is an inclusion and an equivalence of categories (Proposition 2.7), it has a left inverse β^*, defined by sending any object (P, α) in $\mathcal{L}_p^c(BG_p^\wedge)$ not in the image of β_G^c to some $Q \le G$ such that $(Q, i_Q) = \beta_G^c(Q)$ is isomorphic to (P, α) in $\mathcal{L}_p^c(BG_p^\wedge)$. Also, β_G^c is isotypical since $\lambda_G = \lambda_{BG_p^\wedge} \circ \beta_G^c$, and hence β^* is isotypical by Lemma 3.3. Let

$$\mathcal{A}ut_{\mathrm{typ}}(\mathcal{L}_p^c(BG_p^\wedge)) \underset{c_\beta^*}{\overset{c_\beta}{\rightleftarrows}} \mathcal{A}ut_{\mathrm{typ}}(\mathcal{L}_p^c(G))$$

denote the equivalences induced by composition with β_G^c and β^*. Thus

$$c_\beta\big(\mathcal{L}_p^c(BG_p^\wedge) \xrightarrow{\rho} \mathcal{L}_p^c(BG_p^\wedge)\big) = \beta^* \circ \rho \circ \beta_G^c \quad \text{and} \quad c_\beta^*\big(\mathcal{L}_p^c(G) \xrightarrow{\psi} \mathcal{L}_p^c(G)\big) = \beta_G^c \circ \psi \circ \beta^*,$$

and similarly for morphisms. Note that c_β^* preserves the monoidal structures of these categories (since $\beta^* \circ \beta_G^c = \mathrm{Id}$ by assumption), while c_β does not in general preserve them. Let $L = L_G$ to be the composite

$$L = L_G\colon \mathcal{A}ut(BG_p^\wedge) \xrightarrow{\ \mathcal{L}_{BG_p^\wedge}\ } \mathcal{A}ut_{\mathrm{typ}}(\mathcal{L}_p^c(BG_p^\wedge)) \xrightarrow[\simeq]{c_\beta} \mathcal{A}ut_{\mathrm{typ}}(\mathcal{L}_p^c(G)).$$

It remains to show that the maps induced by R and L on nerves are mutual inverses up to natural isomorphism of functors, thus proving Theorems B and C. We first show that $L \circ R \cong \mathrm{Id}$.

For each p-centric subgroup $P \le G$, let $t_P\colon \mathcal{B}(P) \longrightarrow \mathcal{L}_p^c(G)$ denote the functor which sends the unique object o_P to P, and sends each morphism $\widetilde{x}$ in $\mathcal{B}(P)$ to $\widehat{x} \in \mathrm{Aut}_{\mathcal{L}_p^c(G)}(P)$. For each pair of p-centric subgroups $P, Q \le G$ and each $x \in N_G(P, Q)$, let

$$H(x)\colon t_P \longrightarrow t_Q \circ \mathcal{B}(c_x)$$

be the natural morphism of functors $\mathcal{B}(P) \longrightarrow \mathcal{L}_p^c(G)$ which sends the object o_P to the morphism $P \xrightarrow{\widehat{x}} Q$. This induces a homotopy

$$|H(x)|\colon BP \times I = |\mathcal{B}(P) \times [1]| \longrightarrow |\mathcal{L}_p^c(G)|;$$

i.e., a path in the mapping space $\mathrm{Map}(BP, |\mathcal{L}_p^c(G)|)$. Define

$$\Gamma_G\colon \mathcal{L}_p^c(G) \longrightarrow \mathcal{L}_p^c(|\mathcal{L}_p^c(G)|)$$

to be the functor which sends an object P to the pair $(P, |t_P|)$, and which sends a morphism $P \xrightarrow{\widehat{x}} Q$ in $\mathcal{L}_p^c(G)$ (for $x \in N_G(P,Q)$) to $\Gamma_G(\widehat{x}) = (c_x, |H(x)|)$.

The next two lemmas explore some of the features of Γ_G.

Lemma 3.5. *For any finite group G, the following square*

$$\begin{array}{ccc}
\mathcal{L}_p^c(G) & \xrightarrow{\beta_G^c} & \mathcal{L}_p^c(BG^\wedge_p) \\
\Big\downarrow{\scriptstyle \Gamma_G} & & \Big\uparrow{\scriptstyle \mathcal{L}_p^c(BG^\wedge_p)} \\
\mathcal{L}_p^c(|\mathcal{L}_p^c(G)|) & \xrightarrow{\mathcal{L}_p^c(\kappa)} & \mathcal{L}_p^c(|\mathcal{L}_p^c(G)|^\wedge_p)
\end{array}$$

commutes up to natural isomorphism of functors.

Proof. Consider the following commutative diagram

$$\begin{array}{ccccc}
 & & \mathcal{L}_p^c(BG) & \xrightarrow{\mathcal{L}_p^c(\kappa)} & \mathcal{L}_p^c(BG^\wedge_p) \\
 & \nearrow{\scriptstyle \widetilde{\beta}_G} & \Big\uparrow{\scriptstyle \mathcal{L}(\widetilde{\alpha}_G)} & & \simeq\Big\uparrow{\scriptstyle \mathcal{L}((\widetilde{\alpha}_G)^\wedge_p)} \\
\widetilde{\mathcal{L}}_p^c(G) & \xrightarrow{\widetilde{\Gamma}_G} & \mathcal{L}_p^c(|\widetilde{\mathcal{L}}_p^c(G)|) & \xrightarrow{\mathcal{L}_p^c(\kappa)} & \mathcal{L}_p^c(|\widetilde{\mathcal{L}}_p^c(G)|^\wedge_p) \\
\Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \mathcal{L}(|\pi|)} & & \simeq\Big\downarrow{\scriptstyle \mathcal{L}(|\pi|^\wedge_p)} \\
\mathcal{L}_p^c(G) & \xrightarrow{\Gamma_G} & \mathcal{L}_p^c(|\mathcal{L}_p^c(G)|) & \xrightarrow{\mathcal{L}_p^c(\kappa)} & \mathcal{L}_p^c(|\mathcal{L}_p^c(G)|^\wedge_p)
\end{array}$$

of functors between categories. Here $\widetilde{\Gamma}_G$ is defined in a way analogous to Γ_G. Since

$$\mathcal{L}((\widetilde{\alpha}_G)^\wedge_p) \cong \mathcal{L}(\alpha_G) \circ \mathcal{L}(|\pi|^\wedge_p)$$

by Proposition 1.1, this shows that the two composites

$$\widetilde{\mathcal{L}}_p^c(G) \xrightarrow{\ \pi\ } \mathcal{L}_p^c(G) \overset{\mathcal{L}_p^c(\alpha_G)\circ\mathcal{L}_p^c(\kappa)\circ\Gamma_G}{\underset{\beta_G^c}{\rightrightarrows}} \mathcal{L}_p^c(BG^\wedge_p)$$

are naturally isomorphic. Finally, since π induces a bijection on objects and a surjection on morphisms, any natural isomorphism of functors after composition with π (when regarded as a map from $\mathrm{Ob}(\widetilde{\mathcal{L}}_p^c(G)) = \mathrm{Ob}(\mathcal{L}_p^c(G))$ to $\mathrm{Mor}(\mathcal{L}_p^c(BG^\wedge_p))$) is also a natural isomorphism before composition. □

The second property we need of the functors Γ_G is their naturality with respect to isotypical equivalences.

Lemma 3.6. *For any isotypical equivalence $\psi\colon \mathcal{L}_p^c(G) \to \mathcal{L}_p^c(G)$, the following square*

$$\begin{array}{ccc} \mathcal{L}_p^c(G) & \xrightarrow{\Gamma_G} & \mathcal{L}_p^c(|\mathcal{L}_p^c(G)|) \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\mathcal{L}_p^c(|\psi|)} \\ \mathcal{L}_p^c(G) & \xrightarrow{\Gamma_G} & \mathcal{L}_p^c(|\mathcal{L}_p^c(G)|); \end{array} \tag{1}$$

commutes up to a natural isomorphism of functors

$$W(\psi)\colon \mathcal{L}_p^c(|\psi|)\circ\Gamma_G \xrightarrow{\ \cong\ } \Gamma_G\circ\psi,$$

which is itself natural in the sense that the following square commutes for any natural isomorphism $\Psi\colon \psi \xrightarrow{\cong} \psi'$:

$$\begin{array}{ccc} \mathcal{L}_p^c(|\psi|)\circ\Gamma_G & \xrightarrow{W(\psi)} & \Gamma_G\circ\psi \\ {\scriptstyle\mathcal{L}_p^c(|\Psi|)\circ\Gamma_G}\downarrow & & \downarrow{\scriptstyle\Gamma_G\circ\Psi} \\ \mathcal{L}_p^c(|\psi'|)\circ\Gamma_G & \xrightarrow{W(\psi')} & \Gamma_G\circ\psi'. \end{array} \tag{2}$$

Proof. We first claim that

$$W(\psi) : \mathcal{L}_p^c(|\psi|)\circ\Gamma_G \longrightarrow \Gamma_G\circ\psi,$$

defined by sending an object P in $\mathcal{L}_p^c(G)$ to the morphism

$$(\psi_P, C_P) \in \mathrm{Mor}_{\mathcal{L}_p^c(|\mathcal{L}_p^c(G)|)}((P, |\psi|\circ|t_P|), (P', |t_{P'}|)),$$

is a natural isomorphism of functors. Here, $P' = \psi(P)$ for short, and C_P denotes the constant homotopy. Note first that the objects are correct: $\mathcal{L}_p^c(|\psi|)(\Gamma_G(P)) = (P, |\psi|\circ|t_P|)$ and $\Gamma_G(\psi(P)) = (P', |t_{P'}|)$ by definition. Also, (ψ_P, C_P) is a morphism between these objects, since

$$|\psi|\circ|t_P| = |\psi\circ t_P| = |t_{P'}\circ\mathcal{B}(\psi_P)| = |t_{P'}|\circ B\psi_P$$

by definition of ψ_P. To show that $W(\psi)$ is natural, we must check, for each morphism $\widehat{x}\colon P \longrightarrow Q$ in $\mathcal{L}_p^c(G)$, with $\widehat{y} = \psi(\widehat{x})$ and $Q' = \psi(Q)$, that the following square commutes:

$$\begin{array}{ccc} (P, |\psi|\circ|t_P|) & \xrightarrow{(\psi_P, C_P)} & (P', |t_{P'}|) \\ {\scriptstyle(c_x, |\psi|\circ|H(x)|)}\downarrow & & \downarrow{\scriptstyle(c_y, |H(y)|)} \\ (Q, |\psi|\circ|t_Q|) & \xrightarrow{(\psi_Q, C_Q)} & (Q', |t_{Q'}|) \end{array}$$

in $\mathcal{L}_p^c(|\mathcal{L}_p^c(G)|)$. By Lemma 3.4, $(P \mapsto \psi_P)$ is a natural isomorphism of functors $\lambda_G \xrightarrow{\cong} \lambda_G\circ\psi$, and thus $c_y\circ\psi_P = \psi_Q\circ c_x$. So it remains to show that $|H(y)|\circ(B\psi_P\times I) = |\psi|\circ|H(x)|$; and this follows since both are induced by the natural isomorphism $(o_P \mapsto \widehat{y})$ of functors $\mathcal{B}(P) \longrightarrow \mathcal{L}_p^c(G)$.

It remains to check that square (2) commutes for all $\Psi : \psi \xrightarrow{\cong} \psi'$. This means showing, for each p-centric $P \le G$, and z such that $\widehat{z} = \Psi_P \in \mathrm{Iso}_{\mathcal{L}_p^c(G)}(\psi(P), \psi'(P))$, that the following square commutes in $\mathcal{L}_p^c(|\mathcal{L}_p^c(G)|)$:

$$\begin{array}{ccc}
(P, |\psi| \circ |t_P|) & \xrightarrow{(\psi_P, C_P)} & (\psi(P), |t_{\psi(P)}|) \\
\Big\downarrow {\scriptstyle (\mathrm{Id}_P, |\Psi| \circ |t_P|)} & & {\scriptstyle \Gamma_G(\Psi_P)} \Big\downarrow {\scriptstyle =(c_z, |H(z)|)} \\
(P, |\psi'| \circ |t_P|) & \xrightarrow{(\psi'_P, C_P)} & (\psi'(P), |t_{\psi'(P)}|).
\end{array}$$

The square of group homomorphisms commutes ($c_z \circ \psi_P = \psi'_P$) since for each $g \in P$, the square

$$\begin{array}{ccc}
\psi(P) & \xrightarrow{\widehat{\psi_P(g)} = \psi_{P,P}(\widehat{g})} & \psi(P) \\
{\scriptstyle \Psi_P = \widehat{z}} \Big\downarrow \cong & & \cong \Big\downarrow {\scriptstyle \Psi_P = \widehat{z}} \\
\psi'(P) & \xrightarrow{\widehat{\psi'_P(g)} = \psi'_{P,P}(\widehat{g})} & \psi'(P)
\end{array}$$

commutes by naturality of Ψ. Since the C_P are constant homotopies, it remains to check that $|\Psi| \circ |t_P|$ and $|H(\Psi_P)| \circ (B\psi_P \times I)$ are homotopic as paths in the space $\mathrm{Map}(BP, |\mathcal{L}_p^c(G)|)$ from $|\psi| \circ |t_P| = |t_{\psi(P)}| \circ B\psi_P$ to $|\psi'| \circ |t_P| = |t_{\psi'(P)}| \circ B\psi'_P$ — and this holds since both are induced by the natural isomorphism of functors $\mathcal{B}(P) \longrightarrow \mathcal{L}_p^c(G)$ which sends the object o_P to the morphism Ψ_P. $\square$

We are now ready to show:

Proposition 3.7. *For any finite group G, the composite*

$$\mathcal{A}ut_{\mathrm{typ}}(\mathcal{L}_p^c(G)) \xrightarrow{\;R\;} \mathcal{A}ut(BG_p^\wedge) \xrightarrow{\;L\;} \mathcal{A}ut_{\mathrm{typ}}(\mathcal{L}_p^c(G))$$

is naturally isomorphic to the identity.

Proof. Fix an isotypical equivalence $\psi : \mathcal{L}_p^c(G) \longrightarrow \mathcal{L}_p^c(G)$, and consider the following diagram:

$$\begin{array}{ccccccc}
 & \mathcal{L}_p^c(G) & & \xrightarrow{\qquad\qquad\psi\qquad\qquad} & & \mathcal{L}_p^c(G) & \\
{}^{\Gamma_G}\swarrow & & \searrow^{\beta_G^c}_{\simeq} & & {}^{\beta^*}_{\simeq}\nearrow & & \searrow^{\Gamma_G} \\
\mathcal{L}_p^c(|\mathcal{L}_p^c(G)|) & & \mathcal{L}_p^c(BG_p^\wedge) & \xrightarrow{\mathcal{L}_p^c(R(\psi))} & \mathcal{L}_p^c(BG_p^\wedge) & & \mathcal{L}_p^c(|\mathcal{L}_p^c(G)|) \\
{}_{\mathcal{L}_p^c(\kappa)}\searrow & & {}^{\simeq}_{\mathcal{L}_p^c(\alpha^*)}\swarrow & & \nwarrow^{\simeq}_{\mathcal{L}_p^c(\alpha_G)} & & \swarrow_{\mathcal{L}_p^c(\kappa)} \\
 & \mathcal{L}_p^c(|\mathcal{L}_p^c(G)|_p^\wedge) & & \xrightarrow{\qquad\mathcal{L}_p^c(|\psi|_p^\wedge)\qquad} & & \mathcal{L}_p^c(|\mathcal{L}_p^c(G)|_p^\wedge). &
\end{array}$$

Here, α^* and β^* are the inverses (up to homotopy or natural isomorphism) of α_G and β_G^c used to define R and L. In particular, $L \circ R(\psi) = \beta^* \circ \mathcal{L}_p^c(R(\psi)) \circ \beta_G^c$, and proving the proposition means showing that the upper trapezoid commutes up to natural isomorphism of functors. This follows since the two squares in the diagram commute up to natural isomorphism by Lemma 3.5, the large hexagon by Lemma 3.6, and the lower trapezoid (commutes precisely) by definition of R.

To make this more precise, fix natural isomorphisms of functors

$$U\colon \mathcal{L}_p^c(\alpha^*)\circ\beta_G^c \longrightarrow \mathcal{L}_p^c(\kappa)\circ\Gamma_G$$

and

$$V\colon \beta^*\circ\mathcal{L}_p^c(\alpha_G)\circ\mathcal{L}_p^c(\kappa)\circ\Gamma_G \longrightarrow \mathrm{Id}$$

(chosen independently of ψ), and let

$$W_p^\wedge(\psi) : \mathcal{L}_p^c(|\psi|_p^\wedge)\circ\mathcal{L}_p^c(\kappa)\circ\Gamma_G \longrightarrow \mathcal{L}_p^c(\kappa)\circ\Gamma_G\circ\psi$$

be the natural isomorphism of Lemma 3.6 after composing with completion at p. Define $A(\psi) : L\circ R(\psi) \longrightarrow \psi$ to be the composite of natural isomorphisms

$$\begin{aligned} L\circ R(\psi) \overset{\text{def}}{=} \beta^*\circ\mathcal{L}_p^c(R\psi)\circ\beta_G^c &= \beta^*\circ\mathcal{L}_p^c(\alpha_G)\circ\mathcal{L}_p^c(|\psi|_p^\wedge)\circ\mathcal{L}_p^c(\alpha^*)\circ\beta_G^c \\ &\xrightarrow{(-)\circ U} \beta^*\circ\mathcal{L}_p^c(\alpha_G)\circ\mathcal{L}_p^c(|\psi|_p^\wedge)\circ\mathcal{L}_p^c(\kappa)\circ\Gamma_G \\ &\xrightarrow{(-)\circ W_p^\wedge(\psi)} \beta^*\circ\mathcal{L}_p^c(\alpha_G)\circ\mathcal{L}_p^c(\kappa)\circ\Gamma_G\circ\psi \xrightarrow{V\circ\psi} \psi. \end{aligned}$$

To see that $A : L\circ R \longrightarrow \mathrm{Id}$ is an isomorphism of functors, it remains only to check its naturality with respect to morphisms in $\mathcal{A}ut_{\mathrm{typ}}(\mathcal{L}_p^c(G))$. Since U and V are independent of ψ, the naturality of A follows from the naturality of $W_p^\wedge(\psi)$ with respect to isomorphisms $\Psi : \psi \longrightarrow \psi'$, as shown in Lemma 3.6. □

It remains to show that $R\circ L$ is naturally equivalent to the identity functor. Notice that the map L_G is defined as the composite $c_\beta\circ\mathcal{L}_{BG_p^\wedge}$. Thus, to prove this claim, it suffices to show that the functor $|\mathcal{L}_{BG_p^\wedge}|$ induces a monomorphism on homotopy groups. One way to do this is via a homology decomposition of BG, using standard techniques to construct maps on a homotopy colimit. However, we give a different argument in the next section: one which uses the techniques already developed in this paper.

4. Proof of Theorems B and C

We now finish the proof that $\mathrm{Aut}(BG_p^\wedge)$ has the homotopy type of $|\mathcal{A}ut_{\mathrm{typ}}(\mathcal{L}_p^c(G))|$, by showing that $\mathcal{A}ut(BG_p^\wedge) \xrightarrow{L} \mathcal{A}ut_{\mathrm{typ}}(\mathcal{L}_p^c(G))$ is homotopy split injective. This could be done more directly if there were natural "evaluation" maps $\mathcal{L}_p^c(X) \longrightarrow X$, induced by evaluating each $BP \longrightarrow X$ at the basepoint of BP. Since there are no such natural maps, we instead define a simplicial space $\mathbf{M}_\bullet^c(X)$, whose topological realization $|\mathbf{M}_\bullet^c(X)|$ has the mod-p homotopy type of $|\mathcal{L}_p^c(X)|$ when $X = BG$ or $BG_p^\wedge$, and which is equipped with a natural evaluation map to X. Note that we use $|-|$ to denote both the topological realization of a simplicial space and the topological realization of the nerve of a category.

To help motivate this construction, note the following analogy between $\mathcal{L}_p(X)$ and the fundamental groupoid $\pi(X)$, which can be thought of as the full subcategory of $\mathcal{L}_p(X)$ whose objects are the trivial subgroups $(\mathrm{pt} \to X)$. The realization $|\pi(X)|$ does not have a (natural) evaluation map to X, but there is an obvious map $|S_\bullet(X)| \xrightarrow{\mathrm{ev}} X$ defined on the realization of the singular simplicial set. There is also a natural map $|S_\bullet(X)| \xrightarrow{\sigma_X} |\pi(X)|$ (defined at the beginning of Section 3), which sends a singular simplex $(\Delta^n \to X)$ to the sequence of paths obtained by restriction to the edges $[v_i, v_{i+1}]$; and this map is a homotopy equivalence if X is aspherical. By analogy, $\mathbf{M}^c_\bullet(X)$ should be a simplicial space whose n-simplices are maps on Δ^n which send each point to some mapping space $\mathrm{Map}(BP, X)$ with appropriate continuity conditions. In practice, it is simpler to define $\mathbf{M}^c_\bullet(X)$ using certain spaces $\Delta(\mathbf{P})$ defined as follows.

For any sequence

$$\mathbf{P} = \big(P_0 \xrightarrow{\varphi_1} P_1 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_n} P_n\big)$$

of finite p-groups and monomorphisms, define the space $\Delta(\mathbf{P})$ inductively as follows. Set $\Delta(P_0) = BP_0$, let $\Delta(P_0 \xrightarrow{\varphi_1} P_1)$ be the mapping cylinder of the map $BP_0 \xrightarrow{B\varphi_1} BP_1$, and in general let $\Delta(\mathbf{P})$ be the mapping cylinder of the map from $\Delta(P_0 \to \cdots \to P_{n-1})$ to BP_n induced by the $B\varphi_i$. Upon identifying the n-simplex Δ^n with the mapping cylinder of $(\Delta^{n-1} \to \mathrm{pt})$, the inclusions of basepoints into the BP_i and the projections to points induce maps

$$\Delta^n \xrightarrow{\ \iota_{\mathbf{P}}\ } \Delta(\mathbf{P}) \xrightarrow{\ \omega_{\mathbf{P}}\ } \Delta^n.$$

Let $\{v_0, \dots, v_n\}$ denote the vertices of Δ^n, and write $BP_i \times v_i$ ($\cong BP_i$) to denote $\omega_{\mathbf{P}}^{-1}(v_i)$.

Equivalently, for $\mathbf{P}$ as above, we can define

$$\Delta(\mathbf{P}) = \Big(\coprod_{i=0}^{n} (BP_i \times \Delta^{n-i})\Big)\Big/\sim,$$

where the equivalence relation is defined by

$$(x, d_0(t)) \sim (B\varphi_i(x), t) \quad \text{for all} \quad x \in BP_{i-1},\ t \in \Delta^{n-i}.$$

Here, $d_0\colon \Delta^{n-i} \longrightarrow \Delta^{n-i+1}$ is the face map which sends the vertex v_j (for $0 \le j \le i-1$) to v_{j+1}. Under this identification, $\Delta^n \xrightarrow{\iota_{\mathbf{P}}} \Delta(\mathbf{P})$ is induced by the inclusion of the basepoint in BP_0, and $\Delta(\mathbf{P}) \xrightarrow{\omega_{\mathbf{P}}} \Delta^n$ is induced by projection to the second factor followed by the face maps $\Delta^{n-i} \longrightarrow \Delta^n$ which send v_j to v_{j+i}.

Alternatively, $\Delta(\mathbf{P})$ can be regarded as the homotopy colimit of the sequence

$$BP_0 \longrightarrow BP_1 \longrightarrow \cdots \longrightarrow BP_n,$$

regarded as a functor on the poset category $[n] = \{0 < 1 < \cdots < n\}$. Or it can be thought of as the realization of the simplicial space $\Delta_\bullet(\mathbf{P})$, where

$$\Delta_k(\mathbf{P}) = \coprod_{0 \le i_0 \le \cdots \le i_k \le n} BP_{i_0},$$

with the obvious face and degeneracy maps.

Definition 4.1. *For any space X, let $\mathbf{M}^c_\bullet(X)$ be the simplicial space defined as follows. The vertices in $\mathbf{M}^c_\bullet(X)$ are the p-centric subgroups (P,α) in X. An n-simplex $\eta \in \mathbf{M}^c_n(X)$ is a map*

$$\eta : \Delta(\mathbf{P}) \longrightarrow X,$$

for some sequence $\mathbf{P} = \big(P_0 \to \cdots \to P_n\big)$ of finite p-groups and monomorphisms, such that $\eta|_{(BP_i \times v_i)}$ is a p-centric subgroup of X for each i. This n-simplex spans the vertices $(P_i, \eta|_{(BP_i\times v_i)})$ for $i = 0,\dots,n$. The set of vertices $\mathbf{M}^c_0(X)$ is given the discrete topology. The set of n-simplices spanning a given set of vertices (P_i,α_i) and based on a given sequence of monomorphisms $\mathbf{P}$ has the compact-open topology, and is open in the space of all n-simplices.

The face and degeneracy maps in $\mathbf{M}^c_\bullet(X)$ are described as follows. For any $\mathbf{P} = \big(P_0 \to \cdots \to P_n\big)$ and any morphism $\underline{m} \xrightarrow{\gamma} \underline{n}$ in the simplicial category, let $\gamma^*\mathbf{P}$ denote the sequence $P_{\gamma(0)} \to \cdots \to P_{\gamma(m)}$, and let

$$\Delta(\gamma,\mathbf{P})\colon \Delta(\gamma^*\mathbf{P}) \longrightarrow \Delta(\mathbf{P})$$

denote the map defined by sending $(BP_{\gamma(i)} \times v_i)$ to $(BP_{\gamma(i)} \times v_{\gamma(i)})$ (for $i = 0,\dots,m$) and extending linearly. Then $\mathbf{M}^c_n(X) \xrightarrow{\gamma^*} \mathbf{M}^c_m(X)$ sends an n-simplex $\Delta(\mathbf{P}) \xrightarrow{\eta} X$ to the m-simplex $\eta \circ \Delta(\gamma,\mathbf{P})$.

In fact, wherever $\mathbf{M}^c_\bullet(X)$ is used throughout this section, it will be necessary to replace it by its "levelwise" singular simplicial set $S_\bullet\mathbf{M}^c_\bullet(X)$. In other words, this is the bisimplicial set where each space $\mathbf{M}^c_n(X)$ is replaced by its singular simplicial set $S_\bullet\mathbf{M}^c_n(X)$. In general, this replacement goes through without problems; we make a few comments at key points in the arguments where this can make a difference in statements or definitions.

For each n-simplex $\Delta(\mathbf{P}) = \Delta(P_0 \to \cdots \to P_n) \xrightarrow{\eta} X$ in $\mathbf{M}^c_\bullet(X)$, the restriction of η to each mapping cylinder $\omega_{\mathbf{P}}^{-1}(\langle v_i, v_{i+1}\rangle) \cong \Delta(P_i \to P_{i+1})$ determines a morphism in $\mathcal{L}^c_p(X)$. The n-simplex η thus determines an n-simplex in the nerve of $\mathcal{L}^c_p(X)$, regarded as a simplicial space. This induces a map of simplicial spaces

$$\tau_X\colon \mathbf{M}^c_\bullet(X) \longrightarrow N_\bullet(\mathcal{L}^c_p(X)),$$

where $N_\bullet(-)$ denotes the nerve of the category (as opposed to its topological realization).

In the following lemma, the simplicial space $\mathbf{M}^c_\bullet(-)$ must in fact be replaced by $S_\bullet\mathbf{M}^c_\bullet(-)$ to get a "genuine" (not just weak) homotopy equivalence. This will be important in the proof of Proposition 4.4 (see the explanation after diagram (1) in that proof).

Lemma 4.2. *If $X = BG$ or $BG^\wedge_p$ for a finite group G, then $|\mathbf{M}^c_\bullet(X)| \xrightarrow{|\tau_X|} |\mathcal{L}^c_p(X)|$ is a homotopy equivalence.*

Proof. When $X = BG$ or $BG^\wedge_p$, then for any p-centric subgroup $BP \xrightarrow{\alpha} X$, $\mathrm{Map}(BP, X)_\alpha$ is aspherical by Proposition 2.1. We claim that $|\tau_X|$ is a homotopy equivalence for any X with this property.

Fix a set of vertices (P_i, α_i), and a sequence of monomorphisms $\mathbf{P} = \bigl(P_0 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_n} P_n\bigr)$. Let $\Delta_0(\mathbf{P})$ be the union of the mapping cylinders of the $B\varphi_i$, regarded as a subspace of $\Delta(\mathbf{P})$ (i.e., $\Delta_0(\mathbf{P}) = \bigcup_{i=1}^n (\omega_{\mathbf{P}}^{-1}[v_{i-1}, v_i])$). Let Y denote the space of maps $\Delta(\mathbf{P}) \xrightarrow{\eta} X$ such that $\eta|_{BP_i\times v_i} = \alpha_i$ for each i, and let Y_0 denote the space of maps $\Delta_0(\mathbf{P}) \longrightarrow X$ with the same property. Since $\Delta_0(\mathbf{P})$ is a deformation retract of $\Delta(\mathbf{P})$, the restriction map $Y \longrightarrow Y_0$ is a homotopy equivalence. Also, $\pi_0(Y_0)$ is the set of sequences of morphisms in $\mathcal{L}^c_p(X)$ of the form

$$(P_0, \alpha_0) \xrightarrow{(\varphi_1,\eta_1)} (P_1, \alpha_1) \xrightarrow{(\varphi_2,\eta_2)} \cdots \xrightarrow{(\varphi_n,\eta_n)} (P_n, \alpha_n).$$

In other words, the set of n-simplices in the nerve of $\mathcal{L}^c_p(X)$ is the set of connected components of the space of n-simplices in $\mathbf{M}^c_\bullet(X)$ (this holds for any X), and it remains only to show that each component in Y_0 is contractible. But Y_0 is the product over all i of the spaces of paths in $\mathrm{Map}(BP_i, X)$ from α_i to $\alpha_{i+1}\circ B\varphi_{i+1}$; and the connected components of these spaces are contractible since $\mathrm{Map}(BP_i, X)_{\alpha_i}$ is aspherical by assumption.

This shows that τ_X, as a map of simplicial spaces, is a homotopy equivalence in each degree. So $|\tau_X|$ is a homotopy equivalence by [GJ, Proposition IV.1.7]. □

For any X, define maps

$$\mathrm{ev}_{X,n} \colon \mathbf{M}^c_n(X) \times \Delta^n \longrightarrow X,$$

for each $n \geq 0$, by setting

$$\mathrm{ev}_{X,n}(\eta, t) = \eta(\iota_{\mathbf{P}}(t))$$

for each $\Delta(\mathbf{P}) \xrightarrow{\eta} X$ in $\mathbf{M}^c_n(X)$ and each $t \in \Delta^n$. In other words, this is the map defined by restriction to basepoints in the BP_i (recall that $\Delta^n \xrightarrow{\iota_{\mathbf{P}}} \Delta(\mathbf{P})$ is induced by inclusions of basepoints). The $\mathrm{ev}_{X,n}$ are continuous and commute with all face and degeneracy maps, and hence combine to define an evaluation map

$$\mathrm{ev}_X \colon |\mathbf{M}^c_\bullet(X)| \longrightarrow X.$$

Lemma 4.3. *If $X = BG$ or $BG^\wedge_p$ for some finite group G, then $|\mathbf{M}^c_\bullet(X)| \xrightarrow{\mathrm{ev}_X} X$ is an $\mathbb{F}_p$-homology equivalence.*

Proof. Assume first that $X = BG$, and consider the following diagram:

$$\begin{array}{ccccccc} |\widetilde{\mathcal{L}}_p^c(G)| & \xrightarrow[\simeq]{\beta_G^c} & |\mathcal{L}_p^c(BG)| & \xleftarrow[\simeq]{|\tau_{BG}|} & |\mathbf{M}_\bullet^c(BG)| & \xrightarrow{\mathrm{ev}_{BG}} & \\ {\scriptstyle \widetilde{\alpha}_G}\downarrow\simeq & & \downarrow{\scriptstyle \epsilon_1} & & \downarrow{\scriptstyle \epsilon_2} & & BG. \\ |\mathcal{B}(G)| & \xrightarrow[\simeq]{\gamma_G} & |\pi(BG)| & \xleftarrow[\simeq]{|\tau'|} & |S_\bullet^{\mathrm{top}}(BG)| & \xrightarrow[\simeq]{\epsilon_3} & \end{array} \tag{1}$$

Here, $S_\bullet^{\mathrm{top}}(BG)$ is the singular simplicial set for BG, but topologized so that the set of vertices is discrete, but the set of n-simplices $\Delta^n \longrightarrow BG$ with any given set of vertices is given the compact-open topology. The maps ϵ_1 and ϵ_2 are both defined by restricting maps $BP \longrightarrow BG$ to the basepoint (in the case of ϵ_2 via composition with $\iota_{\mathbf{P}}$), while ϵ_3 is the evaluation map for singular simplices in BG. Also, τ' is defined by sending each vertex in $S_\bullet^{\mathrm{top}}(BG)$ (i.e., each point of BG) to itself considered as an object in $\pi(BG)$, and by sending each singular simplex $\sigma : \Delta^n \longrightarrow BG$ to its homotopy class regarded as a sequence of paths in BG (i.e., an n-simplex in the nerve of $\pi(BG)$). Finally, γ_G is induced by regarding $\mathcal{B}(G)$ as a subcategory of $\pi(BG)$ — the full subcategory with object the basepoint $*$ of BG. The triangle in (1) commutes by definition of ev_{BG}, and both squares are induced by commutative squares of categories and functors or of simplicial spaces.

Now, $|\tau'|$ is a homotopy equivalence since τ' is a map of simplicial spaces which is a homotopy equivalence at each level (the components of the space of n-simplices in $S_\bullet^{\mathrm{top}}(BG)$ are contractible since BG is aspherical). The evaluation map ϵ_3 is also a homotopy equivalence, since $S_n^{\mathrm{top}}(BG)$ is homotopy equivalent to BG for each n. Also, $|\tau_{BG}|$ is a homotopy equivalence by Lemma 4.2, β_G^c and $\widetilde{\alpha}_G$ are homotopy equivalences by Propositions 2.6 and 1.1, and γ_G is an equivalence since it is induced by an equivalence of categories. The commutativity of the diagram now proves that ev_{BG} is a homotopy equivalence.

To see that $\mathrm{ev}_{BG^\wedge_p}$ is an $\mathbb{F}_p$-homology equivalence, consider the following diagram, where the vertical maps are all induced by p-completion:

$$\begin{array}{ccccc} |\mathcal{L}_p^c(BG)| & \xleftarrow{|\tau_{BG}|} & |\mathbf{M}_\bullet^c(BG)| & \xrightarrow{\mathrm{ev}_{BG}} & BG \\ {\scriptstyle \mathcal{L}_p^c(\kappa)}\downarrow & & {\scriptstyle \mathbf{M}_\bullet^c(\kappa)}\downarrow & & \downarrow{\scriptstyle \kappa} \\ |\mathcal{L}_p^c(BG^\wedge_p)| & \xleftarrow{|\tau_{BG^\wedge_p}|} & |\mathbf{M}_\bullet^c(BG^\wedge_p)| & \xrightarrow{\mathrm{ev}_{BG^\wedge_p}} & BG^\wedge_p. \end{array}$$

The squares commute by naturality of τ and ev. Also, $|\mathcal{L}_p^c(\kappa)|$ is an $\mathbb{F}_p$-homology equivalence by Corollary 2.8, and κ is clearly an equivalence. We have just shown that ev_{BG} is a homotopy equivalence, and $|\tau_{BG}|$ and $|\tau_{BG^\wedge_p}|$ are homotopy equivalences by Lemma 4.2. So $\mathrm{ev}_{BG^\wedge_p}$ is also an $\mathbb{F}_p$-homology equivalence. □

All of the definitions and results so far in this section were developed to prove the following result:

Proposition 4.4. *For any finite group G, the map*

$$|\mathcal{L}_{BG^\wedge_p}| : |\mathcal{A}ut(BG^\wedge_p)| \longrightarrow |\mathcal{A}ut_{\mathrm{typ}}(\mathcal{L}^c_p(BG^\wedge_p))|$$

induces a split monomorphism on π_i for all $i \geq 0$.

Proof. For any pair of spaces X and Y, let $\mathcal{E}qv(X,Y)$ denote the groupoid whose objects are the mod-p-equivalences $X \longrightarrow Y$, and whose morphisms are homotopy classes of homotopies. Clearly, composition with homotopy equivalences induces homotopy equivalences between the nerves of such categories.

To shorten the notation, we write $\mathbf{M}^c_\bullet = \mathbf{M}^c_\bullet(BG^\wedge_p)$ and $\mathcal{L}^c_p = \mathcal{L}^c_p(BG^\wedge_p)$. We will construct a functor $\xi : \mathcal{A}ut(BG^\wedge_p) \longrightarrow \mathcal{A}ut(\mathbf{M}^c_\bullet)$ which makes the following diagram commute (precisely, not just up to natural isomorphism):

$$\begin{array}{ccccc}
 & \mathcal{A}ut(BG^\wedge_p) & \xrightarrow{\ \mathcal{L}_{BG^\wedge_p}\ } & \mathcal{A}ut(\mathcal{L}^c_p) & \\
 {\scriptstyle -\circ\mathrm{ev}}\ \swarrow\ {\scriptstyle \simeq} & & & & \searrow\ {\scriptstyle |-|} \\
\mathcal{E}qv\big(|\mathbf{M}^c_\bullet|^\wedge_p, BG^\wedge_p\big) & \Big\downarrow{\scriptstyle \xi} & & & \mathcal{A}ut(|\mathcal{L}^c_p|^\wedge_p) \\
 {\scriptstyle \mathrm{ev}\circ-}\ \nwarrow\ {\scriptstyle \simeq} & & & & \swarrow\ {\scriptstyle -\circ|\tau|}\ {\scriptstyle \simeq} \\
 & \mathcal{A}ut(|\mathbf{M}^c_\bullet|^\wedge_p) & \xrightarrow[\simeq]{\ |\tau|\circ-\ } & \mathcal{E}qv\big(|\mathbf{M}^c_\bullet|^\wedge_p, |\mathcal{L}^c_p|^\wedge_p\big) &
\end{array} \qquad (1)$$

The lemma then follows immediately, since the maps labeled "$\simeq$" are all induced by composition with homotopy equivalences (and hence themselves induce homotopy equivalences on nerves).

In fact, when carrying out this argument, it is essential to replace $|\mathbf{M}^c_\bullet(BG^\wedge_p)|$ by $|S_\bullet\mathbf{M}^c_\bullet(BG^\wedge_p)|$ in the above diagram. This is necessary to guarantee that the maps

$$BG^\wedge_p \xleftarrow{\ |\tau|\ } |S_\bullet\mathbf{M}^c_\bullet(BG^\wedge_p)|^\wedge_p \xrightarrow{\ |\mathrm{ev}|\ } |\mathcal{L}^c_p(BG^\wedge_p)|^\wedge_p$$

are homotopy equivalences (not just weakly), and hence that composition with $|\tau|$ or $|\mathrm{ev}|$ is a homotopy equivalence.

We define ξ as follows. On objects, ξ sends a homotopy equivalence $BG^\wedge_p \xrightarrow{f} BG^\wedge_p$ to the induced functor $|\mathbf{M}^c_\bullet(BG^\wedge_p)| \xrightarrow[=|\mathbf{M}^c_\bullet(f)|]{\xi(f)} |\mathbf{M}^c_\bullet(BG^\wedge_p)|$, defined via composition with f. So it remains to define $\xi(F) : |\mathbf{M}^c_\bullet| \times I \longrightarrow |\mathbf{M}^c_\bullet|$ for each homotopy $F : BG^\wedge_p \times I \longrightarrow BG^\wedge_p$. To do this, we regard I as the realization of the simplicial set $[1]$, where $[1]_n$ is the set of sequences $\sigma = (\sigma(0) \leq \cdots \leq \sigma(n))$ taking values in $\{0,1\}$. For any such σ, we let $\Delta^n \xrightarrow{|\sigma|} I$ denote the affine map which sends v_i to $\sigma(i)$. Let

$$\xi_\bullet(F)\colon \mathbf{M}^c_\bullet \times [1] \longrightarrow \mathbf{M}^c_\bullet$$

be the map of simplicial spaces which sends the n-simplex $(\Delta(\mathbf{P}) \xrightarrow{\eta} BG^\wedge_p, \sigma)$ in $\mathbf{M}^c_\bullet \times [1]$ to the n-simplex

$$F(\eta,\sigma)\colon \Delta(\mathbf{P}) \xrightarrow{(\mathrm{Id},|\sigma|\circ\omega_{\mathbf{P}})} \Delta(\mathbf{P}) \times I \xrightarrow{\eta\times I} BG^\wedge_p \times I \xrightarrow{F} BG^\wedge_p,$$

where $\omega_{\mathbf{P}}\colon \Delta(\mathbf{P}) \longrightarrow \Delta^n$ is the map defined at the beginning of the section. The $F(\eta,\sigma)$ are easily seen to be continuous and to commute with face and degeneracy maps, and hence define a homotopy $\xi(F) = |\xi_\bullet(F)|$.

Note that an obvious modification in the above definition of ξ gives a functor which takes values in the automorphism category $\mathcal{A}ut(|S_\bullet \mathbf{M}^c_\bullet(BG^\wedge_p)|^\wedge_p)$.

The commutativity of (1) follows from the commutativity of the following squares, for any homotopy $F : BG^\wedge_p \times I \longrightarrow BG^\wedge_p$:

$$\begin{array}{ccccc}
BG^\wedge_p \times I & \xleftarrow{\mathrm{ev}\times I} & |\mathbf{M}^c_\bullet(BG^\wedge_p)| \times I & \xrightarrow{\tau\times I} & |\mathcal{L}^c_p(BG^\wedge_p)| \times I \\
\Big\downarrow{\scriptstyle F} & & \Big\downarrow{\scriptstyle \xi_\bullet(F)} & & \Big\downarrow{\scriptstyle \mathcal{L}_{BG^\wedge_p}(F)} \\
BG^\wedge_p & \xleftarrow{\mathrm{ev}} & |\mathbf{M}^c_\bullet(BG^\wedge_p)| & \xrightarrow{\tau} & |\mathcal{L}^c_p(BG^\wedge_p)|\,,
\end{array}$$

and this is easily checked. □

We are now ready to prove Theorems B and C. In particular, we want to show that $\mathrm{Aut}(BG^\wedge_p)$ and $|\mathcal{A}ut(\mathcal{L}^c_p(G))|$ are homotopy equivalent, and moreover equivalent as monoids. There is no obvious way to construct a map between these two spaces which is both a homotopy equivalence and a morphism of monoids, so instead we connect them via a sequence of maps. Here $S_\bullet X$, for any space X, denotes the (discrete) singular simplicial set of X.

Theorem 4.5. *For any finite group G and any prime p, $\mathrm{Aut}(BG^\wedge_p)$ has the homotopy type of $|\mathcal{A}ut_{\mathrm{typ}}(\mathcal{L}^c_p(G))|$. More precisely:*

(a) *R and L induce isomorphisms*

$$\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}^c_p(G)) \underset{\pi_0 R}{\overset{\pi_0 L}{\rightleftarrows}} \mathrm{Out}(BG^\wedge_p)$$

between the groups of components.

(b) *The following maps are morphisms of topological monoids and homotopy equivalences:*

$$\mathrm{Aut}(BG^\wedge_p) \underset{\simeq}{\xleftarrow{\mathrm{ev}}} |S_\bullet \mathrm{Aut}(BG^\wedge_p)| \underset{\simeq}{\xrightarrow{\sigma}} |\mathcal{A}ut(BG^\wedge_p)|$$
$$\underset{\simeq}{\xrightarrow{|\mathcal{L}_{BG^\wedge_p}|}} |\mathcal{A}ut_{\mathrm{typ}}(\mathcal{L}^c_p(BG^\wedge_p))| \underset{\simeq}{\xleftarrow{|c^*_\beta|}} |\mathcal{A}ut_{\mathrm{typ}}(\mathcal{L}^c_p(G))|,$$

where σ is the map defined in Section 3. In particular,

$$B\mathrm{Aut}(BG^\wedge_p) \simeq B(|\mathcal{A}ut_{\mathrm{typ}}(\mathcal{L}^c_p(G))|).$$

Proof. Consider the maps

$$|\mathcal{A}ut_{\text{typ}}(\mathcal{L}_p^c(G))| \underset{|R|}{\overset{|L|}{\rightleftarrows}} |\mathcal{A}ut(BG_p^\wedge)|$$

defined in Section 4. By Proposition 3.7, $L \circ R$ is naturally isomorphic to the identity on $\mathcal{A}ut_{\text{typ}}(\mathcal{L}_p^c(G))$; while $|L|$ (equivalently $|\mathcal{L}_{BG_p^\wedge}|$) is split injective on homotopy groups by Proposition 4.4. So R and L induce homotopy equivalences between these nerves; and in particular an isomorphism of groups between $\text{Out}(BG_p^\wedge) = \pi_0(|\mathcal{A}ut(BG_p^\wedge)|)$ and $\text{Out}_{\text{typ}}(\mathcal{L}_p^c(G)) = \pi_0(|\mathcal{A}ut_{\text{typ}}(\mathcal{L}_p^c(G))|)$.

All four of the above maps in (b) are morphisms of monoids by construction, and ev and σ are homotopy equivalences by Lemma 3.1. Also, $|c_\beta^*|$ is a homotopy equivalence since it is induced by composition with equivalences between the two categories involved. Finally, $|\mathcal{L}_{BG_p^\wedge}|$ induces surjections on all homotopy groups by Proposition 3.7, and injections by Proposition 4.4.

The last statement now follows since a morphism of monoids which is a homotopy equivalence induces a homotopy equivalence between the classifying spaces, using [GJ, Proposition IV.1.7]. □

5. Fusion preserving isomorphisms

Let G and G' be finite groups with Sylow p-subgroups S and S' respectively, and assume $S \cong S'$. Recall that an isomorphism $S \xrightarrow{f} S'$ is called *fusion preserving* if given subgroups $P, Q \leq S$ and an isomorphism $P \xrightarrow{\alpha} Q$ between them, α is induced by conjugation in G if and only if $f(P) \xrightarrow{f\alpha f^{-1}} f(Q)$ is induced by conjugation in G'.

Let $\mathcal{F}_p(G)$ denote the *fusion category* of G: the category whose objects are the p-centric subgroups of G, and with $\text{Mor}_{\mathcal{F}_p(G)}(P,Q) = N_G(P,Q)/C_G(P)$ (regarded as a subset of $\text{Hom}(P,Q)$). Martino and Priddy in [MP] defined an *isotypical equivalence of fusion categories* to be an equivalence $\psi : \mathcal{F}_p(G) \longrightarrow \mathcal{F}_p(G')$ together with isomorphisms $P \xrightarrow{\cong} \psi(P)$ which are natural with respect to all morphisms in $\mathcal{F}_p(G)$. This is clearly the same as an isotypical equivalence in the sense of Definition 3.2. Martino and Priddy then showed [MP, Corollary 1.2] that there is an isotypical equivalence between $\mathcal{F}_p(G)$ and $\mathcal{F}_p(G')$ if and only if there is a fusion preserving isomorphism between the Sylow p-subgroups of G and G'.

What we now want is to understand the relationship between, on the one hand fusion preserving isomorphisms between Sylow subgroups, and on the other hand isotypical equivalences between linking categories. The first lemma describes how to do this in one direction.

Throughout this section and the next, for any functor $\psi : \mathcal{C} \to \mathcal{C}'$ and any objects c, d in $\mathcal{C}$, we let $\psi_{c,d}$ denote the induced map from $\text{Mor}_{\mathcal{C}}(c,d)$ to $\text{Mor}_{\mathcal{C}'}(\psi(c), \psi(d))$.

Lemma 5.1. *Fix finite groups G and G', and an isotypical equivalence*

$$\psi : \mathcal{L}_p^c(G) \longrightarrow \mathcal{L}_p^c(G')$$

of categories. Let $S \le G$ be a Sylow p-subgroup. Then $S' \overset{\text{def}}{=} \psi(S)$ is a Sylow p-subgroup of G', and

$$N_G(S)/C'_G(S) = \operatorname{Aut}_{\mathcal{L}_p^c(G)}(S) \xrightarrow[\cong]{\psi_{S,S}} \operatorname{Aut}_{\mathcal{L}_p^c(G')}(S') = N_{G'}(S')/C'_{G'}(S')$$

restricts to a fusion preserving isomorphism $\psi_S : S \xrightarrow{\cong} S'$. Furthermore:

(a) *ψ is naturally isomorphic to an equivalence ψ' which agrees with ψ on $\operatorname{Aut}_{\mathcal{L}_p^c(G)}(S)$ and hence on S (thus ${\psi'}_S = \psi_S$), and which sends inclusions of subgroups of S to inclusions of subgroups of S'. In other words, for any $P \le S$, $\psi'(P \xrightarrow{\widehat{1}} S) = (\psi'(P) \xrightarrow{\widehat{1}} S')$.*

(b) *Assume that ψ sends inclusions of subgroups of S to inclusions of subgroups of S'. Then for any p-centric $P \le S$, $\psi(P) = \psi_S(P)$, and $\psi_{P,P}$ sends $\widehat{g} \in \operatorname{Aut}_{\mathcal{L}_p^c(G)}(P)$ to $\widehat{\psi_S(g)} \in \operatorname{Aut}_{\mathcal{L}_p^c(G')}(\psi(P))$ for all $g \in P$.*

Proof. Since $S \lhd N_G(S)/C'_G(S)$ and $S' \lhd N_{G'}(S')/C'_{G'}(S')$ are (unique) Sylow p-subgroups, $\psi_{S,S}$ restricts to an isomorphism $\psi_S : S \xrightarrow{\cong} S'$. We prove that ψ_S is fusion preserving as point (c) below.

(a) For each p-centric $P \le G$, define a subgroup $\psi'(P) \le S'$ and an isomorphism $\alpha_P : \psi(P) \xrightarrow{\cong} \psi'(P)$ in $\mathcal{L}_p^c(G')$ as follows. If $P \lneqq S$, let $y \in N_{G'}(\psi(P), S')$ be such that $\psi_{P,S}(P \xrightarrow{\widehat{1}} S) = \widehat{y}$, and set $\psi'(P) = y(\psi(P))y^{-1}$ (the "image" of $\psi_{P,S}(\widehat{1})$) and $\alpha_P = \widehat{y}$. Otherwise, set $\psi'(P) = \psi(P)$ and $\alpha_P = \mathrm{Id}$.

Now let ψ' be the functor which sends P to $\psi'(P)$, and where

$$\psi'_{P,Q}\Big(P \xrightarrow{\widehat{x}} Q\Big) = \Big(\psi'(P) \xrightarrow{\alpha(Q)\circ\psi(\widehat{x})\circ\alpha(P)^{-1}} \psi'(Q)\Big).$$

Then ψ' is well defined as a functor, α is a natural isomorphism from ψ to ψ', and ψ' sends inclusions into S to inclusions into S'. Notice that ψ' is automatically an isotypical equivalence.

(b) Now assume ψ sends inclusions to inclusions. Fix a p-centric subgroup $P \le S$, and set $P' = \psi(P) \le S'$ for short. For any element $g \in P$, the following square commutes in $\mathcal{L}_p^c(G')$

$$\begin{array}{ccc}
P' & \xrightarrow{\psi(\widehat{g})=\widehat{h}} & P' \\
\Big\downarrow{\scriptstyle \widehat{1}} & & \Big\downarrow{\scriptstyle \widehat{1}} \\
S' & \xrightarrow{\psi(\widehat{g})=\widehat{\psi_S(g)}} & S'
\end{array}$$

(for some $h \in N_{G'}(P')$) since it is the image under ψ of a commutative square in $\mathcal{L}_p^c(G)$. Thus $h \equiv \psi_S(g) \pmod{C'_{G'}(P')}$, so $\psi_S(g) \in N_{G'}(P')$, and $\psi_{P,P}(\widehat{g}) = \widehat{\psi_S(g)}$. In particular, since ψ is isotypical, this shows that $P' = \psi_S(P)$.

(c) It remains to show that ψ_S is fusion preserving. This is an easy consequence of Alperin's fusion theorem in the form [Al, Theorem 5.1]; but since our result is much more elementary we give a direct proof.

For each P, let $P \xrightarrow[\cong]{\theta_P} P'$ be the restriction of $\psi_{P,P}$ to $P \le N_G(P)/C'_G(P)$. By Lemma 3.4, this defines a natural isomorphism of functors $\theta\colon \lambda_G \xrightarrow{\cong} \lambda_G \circ \psi$. Thus ψ_S may be identified with $\theta_S : S \longrightarrow S'$. By (a), we can assume that ψ sends inclusions of subgroups in S to inclusions of subgroups in S'. It follows that $\theta_P : P \longrightarrow P'$ is given by the restriction of θ_S to P. If $P, Q \le S$ are p-centric in G and $\alpha = c_x : P \longrightarrow Q$ is a group isomorphism with $x \in N_G(P, Q)$ then

$$\psi_S \alpha \psi_S^{-1} = \theta_Q \alpha \theta_P^{-1} = \psi(\alpha) = c_{x'}$$

for some $x' \in N_{G'}(P', Q')$. Hence fusion is preserved among p-centric subgroups of S and S'.

Fix a subgroup $P \le S$ which is not p-centric. We may assume inductively that ψ_S preserves fusion among all subgroups of strictly larger order. Let $\mathcal{P}$ be the set of subgroups of S which are G-conjugate to P. We may assume that $N_S(P)$ is a Sylow p-subgroup of $N_G(P)$ (replace P by some other subgroup in $\mathcal{P}$ if necessary).

Let $\overline{P} \le S$ be a Sylow p-subgroup of $C_G(P){\cdot}P$. Then

$$P \lneqq \overline{P} \le N_G(P) \tag{1}$$

since P is not p-centric, and we claim that

$$N_G(P) = \bigl(N_G(P) \cap N_G(\overline{P})\bigr){\cdot}C_G(P). \tag{2}$$

To see this, notice first that the right hand side is clearly a subgroup of $N_G(P)$. The other inclusion follows because if $x \in N_G(P)$, then $xc \in N_G(\bar{P})$ for some $c \in C_G(P)$. For any $x \in N_G(P)$, $c_x = c_y \in \mathrm{Aut}(P)$ for some $y \in N_G(P) \cap N_G(\overline{P})$ by (2); ψ_S sends $c_y \in \mathrm{Aut}(\overline{P})$ to $c_{y'} \in \mathrm{Aut}(\psi_S(\overline{P}))$ for some y' by (1) and the induction hypothesis; and hence sends $c_x \in \mathrm{Aut}(P)$ to $c_{y'} \in \mathrm{Aut}(\psi_S(P))$. So fusion is preserved for automorphisms of P.

Now let Q be any other subgroup in $\mathcal{P}$, and set $R = N_S(Q)$ ($\gneqq Q$) for short. Since $N_S(P)$ is a Sylow p-subgroup of $N_G(P)$, there is $x \in N_G(Q, P)$ such that $xRx^{-1} \le N_S(P)$. By the induction hypothesis, there is $x' \in G'$ such that ψ_S sends $R \xrightarrow[\cong]{c_x} xRx^{-1}$ to $\psi_S(R) \xrightarrow[\cong]{c_{x'}} \psi_S(xRx^{-1})$. But then $x' \in N_G(\psi_S(Q), \psi_S(P))$, and so $\psi_S(Q)$ and $\psi_S(P)$ are conjugate in G'.

This shows that ψ_S sends fusion among subgroups of S to fusion among subgroups of S'. The same argument applied to ψ_S^{-1} shows the converse, and thus that ψ_S is fusion preserving. □

The isomorphism $\psi_S : S \longrightarrow S'$ of Lemma 5.1 will be referred to as the underlying fusion preserving isomorphism of the equivalence $\psi : \mathcal{L}_p^c(G) \longrightarrow \mathcal{L}_p^c(G')$.

Martino and Priddy [MP, Corollary 1.2] showed that if $BG^\wedge_p \simeq BG'^\wedge_p$, then there is a fusion preserving isomorphism between Sylow p-subgroups of G and of G'; this is a consequence of the description of $[BP, BG^\wedge_p]$ (shown here as Proposition 2.1). By Theorem A, an equivalence $BG^\wedge_p \simeq BG'^\wedge_p$ determines an equivalence of categories $\mathcal{L}_p^c(G) \simeq \mathcal{L}_p^c(G')$, and Lemma 5.1 gives an algebraic explanation of why this determines a fusion preserving isomorphism.

In the next section, we will describe the obstructions to constructing an isotypical equivalence of categories $\mathcal{L}_p^c(G) \longrightarrow \mathcal{L}_p^c(G')$ which has a given isomorphism between Sylow p-subgroups as its underlying fusion preserving isomorphism. The following lemma plays a key role in that construction.

Lemma 5.2. *Let G and G' be finite groups with Sylow p-subgroup S and S' respectively. Let $f : S \longrightarrow S'$ be a fusion preserving isomorphism. Then for any $P, Q \leq S$ p-centric in G, there is a map*

$$\Phi_{P,Q} \colon N_G(P,Q) \longrightarrow N_{G'}(f(P), f(Q))$$

which satisfies the following three conditions for all $x \in N_G(P,Q)$ (all x such that $xPx^{-1} \leq Q$):

(a) *The following square commutes (in the category of groups):*

$$\begin{array}{ccc} P & \xrightarrow{\;c_x\;} & Q \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ f(P) & \xrightarrow{\;c_{x'}\;} & f(Q), \end{array}$$

where $x' = \Phi_{P,Q}(x)$.

(b) $\Phi_{P,Q}(gx) = f(g)\Phi_{P,Q}(x)$ *for all $g \in Q$.*

(c) $\Phi_{P,Q}$ *induces a bijection*

$$\widehat{\Phi}_{P,Q} \colon N_G(P,Q)/C'_G(P) \xrightarrow{\;\cong\;} N_{G'}(f(P), f(Q))/C'_{G'}(f(P)).$$

Proof. To simplify notation, for any $P \leq S$ we write $P' = f(P) \leq S'$. Conditions (a) and (c) mean that there is a commutative diagram of the following form

$$\begin{array}{ccccccc} N_G(P,Q) & \twoheadrightarrow & N_G(P,Q)/C'_G(P) & \twoheadrightarrow & N_G(P,Q)/C_G(P) & \subseteq & \mathrm{Hom}(P,Q) \\ {\scriptstyle \Phi_{P,Q}}\downarrow & & {\scriptstyle \widehat{\Phi}_{P,Q}}\downarrow\cong & & \downarrow\cong & & {\scriptstyle f_*}\downarrow\cong \\ N_{G'}(P',Q') & \twoheadrightarrow & N_{G'}(P',Q')/C'_{G'}(P') & \twoheadrightarrow & N_{G'}(P',Q')/C_{G'}(P') & \subseteq & \mathrm{Hom}(P',Q'), \end{array}$$

where f_* is the bijection between homomorphism sets induced by the isomorphisms $P \cong P'$ and $Q \cong Q'$. The restriction of f_* to $N_G(P,Q)/C_G(P)$ is defined and bijective by the assumption that f is fusion preserving.

Let $X \subseteq N_G(P,Q)$ be a set of orbit representatives for the left action of Q and the right action of $C'_G(P)$. For each $x \in X$, choose an arbitrary element $x' = \Phi_{P,Q}(x)$ such that $c_{x'} = f_*(c_x)$. Extend this to all of $N_G(P,Q)$ by setting $\Phi_{P,Q}(gxa) = f(g)\Phi_{P,Q}(x)$ for all $x \in X$, $g \in Q$, and $a \in C'_G(P)$. Then $\Phi_{P,Q}$ satisfies (a) and (b) above, and induces a map $\widehat{\Phi}_{P,Q}$ which makes the above diagram commute. Furthermore, $\widehat{\Phi}_{P,Q}$ is bijective, since the third column of the diagram is obtained by dividing out by the free action of $Z(P) \cong Z(P')$ on the second column. □

6. Higher limit obstructions

We now look more closely at ways of constructing isotypical equivalences between linking categories $\mathcal{L}_p^c(G)$ and $\mathcal{L}_p^c(G')$ of finite groups. We construct obstructions to the existence and uniqueness of such equivalences which "extend" given fusion preserving isomorphisms between Sylow p-subgroups. This is then used to prove Conjecture D whenever G has p-rank less than p^2.

For any small category $\mathcal{C}$ and any contravariant functor $\mathcal{C} \to \mathsf{Ab}$, let $C^*(\mathcal{C};F)$ denote the chain complex

$$C^n(\mathcal{C};F) = \prod_{c_0 \to \cdots \to c_n} F(c_0),$$

with differentials $\phi \in C^{n-1}(\mathcal{C};F) \xrightarrow{d} C^n(\mathcal{C};F)$ defined by

$$d\phi(c_0 \xrightarrow{f} c_1 \to \cdots \to c_n) = f^*\phi(c_1 \to \cdots \to c_n) + \sum_{k=1}^{n} (-1)^k \phi(c_0 \to \cdots \widehat{c_k} \cdots \to c_n).$$

Then $\lim\limits_{\mathcal{C}}{}^*(F) \cong H^*(C^*(\mathcal{C};F),d)$ (cf. [GZ, Appendix II, Proposition 3.3], applied with $\mathcal{M} = \mathsf{Ab}^{\mathrm{op}}$).

Recall the notation used in Section 4 for objects and maps in orbit categories. For any finite G, $\mathcal{O}_p^c(G)$ is the category whose objects are the p-centric subgroups $P \le G$, and where $\mathrm{Mor}_{\mathcal{O}_p^c(G)}(P,Q) = Q\backslash N_G(P,Q)$ for any objects $P,Q \le G$. The morphism corresponding to $x \in N_G(P,Q)$ is denoted $\mathring{x}$, and corresponds to the G-map $G/P \to G/Q$ which sends gP to $gx^{-1}Q$ in the usual definition of $\mathcal{O}_p^c(G)$.

Define

$$\mathcal{Z}_G \colon \mathcal{O}_p^c(G) \longrightarrow \mathbb{Z}_{(p)}\text{-}\mathsf{mod}$$

to be the functor defined by setting $\mathcal{Z}_G(P) = Z(P)$. If $P,Q \le G$ are p-centric and $x \in N_G(P,Q)$, then we let $\mathcal{Z}_G(\mathring{x})$ be the restriction to Sylow p-subgroups of the homomorphism

$$C_G(Q) \xrightarrow{c_x^{-1}} C_G(P)$$

induced by $g \mapsto x^{-1}gx$. Since this clearly depends only on the coset Qx, $\mathcal{Z}_G$ is well defined as a functor on the orbit category.

For any element g in a finite group G, we let $g_p \in G$ denote its "p-part". This is the unique element of p-power order in the cyclic subgroup $\langle g\rangle$ such that $|g^{-1}g_p|$ is prime to p. Equivalently, it is the unique element of p-power order in G for which there is an element $g' \in G$ of order prime to p such that $[g_p, g'] = 1$ and $g = g_p{\cdot}g'$. In particular, for any p-centric subgroup $P \le G$, the projection of $C_G(P) = Z(P) \times C'_G(P)$ to $Z(P)$ sends each $g \in C_G(P)$ to $g_p \in Z(P)$.

Proposition 6.1. *Fix finite groups G and G', and Sylow p-subgroups $S \le G$ and $S' \le G'$. Let $f : S \to S'$ be any fusion preserving isomorphism. Then there is an element $\omega_G(f) \in \lim\limits_{\mathcal{O}_p^c(G)}{}^2(\mathcal{Z}_G)$ with the following property: there is an isotypical equivalence of categories*

$$\psi : \mathcal{L}_p^c(G) \xrightarrow{\;\cong\;} \mathcal{L}_p^c(G')$$

with $\psi_S = f$, if and only if $\omega_G(f) = 0$.

Proof. Again, for any $P \le S$ we write $P' = f(P) \le S'$. Let $\mathcal{L}_S^c(G) \subseteq \mathcal{L}_p^c(G)$ and $\mathcal{O}_S^c(G) \subseteq \mathcal{O}_p^c(G)$ denote the full subcategories whose objects are the p-centric subgroups of G contained in S, and similarly for $\mathcal{L}_{S'}^c(G') \subseteq \mathcal{L}_p^c(G')$. All three of these inclusions are equivalences of categories, since every p-centric subgroup of G is conjugate to a subgroup of S. It follows that $\lim\limits_{\mathcal{O}_S^c(G)}{}^*(\mathcal{Z}_G) \cong \lim\limits_{\mathcal{O}_p^c(G)}{}^*(\mathcal{Z}_G)$. Also, any equivalence $\psi : \mathcal{L}_S^c(G) \to \mathcal{L}_{S'}^c(G')$ extends to an equivalence $\mathcal{L}_p^c(G) \to \mathcal{L}_p^c(G')$. So we can restrict our attention to these subcategories.

We define ψ on objects by setting $\psi(P) = P'$ for all $P \le S$ p-centric in G. By Lemma A.5, any $P \le S$ is p-centric in G if and only if each $Q \le S$ which is G-conjugate to P contains its centralizer in S. Hence P' is p-centric in G' if and only if P is p-centric in G; and so ψ defines a bijection between objects in $\mathcal{L}_S^c(G)$ and $\mathcal{L}_{S'}^c(G')$. The problem is to define ψ on morphisms. Thus, our goal is to construct the obstruction class $\omega_G(f)$, and show that ψ can be well defined on morphisms if and only if $\omega_G(f) = 0$.

Given a choice of the $\Phi_{P,Q}$, for all p-centric $P, Q \le S$, satisfying conditions (a), (b), and (c) of Lemma 5.2, we now define an element

$$\omega_\Phi \in C^2(\mathcal{O}_S^c(G); \mathcal{Z}_G) = \prod_{P\to Q\to R} \mathcal{Z}_G(P)$$

as follows. For each $x \in N_G(P,Q)$ and $y \in N_G(Q,R)$,

$$f_*(c_y)f_*(c_x) = f_*(c_{yx}),$$

where f_* is as in Lemma 5.2. Hence $\Phi_{P,R}(yx)^{-1}\Phi_{Q,R}(y)\Phi_{P,Q}(x) \in C_{G'}(P')$. Define

$$\omega'_\Phi(y,x) \overset{\text{def}}{=} \bigl(\Phi_{P,R}(yx)^{-1}\Phi_{Q,R}(y)\Phi_{P,Q}(x)\bigr)_p \in Z(P').$$

For each $P \xrightarrow{\mathring{x}} Q \xrightarrow{\mathring{y}} R$ in $\mathcal{O}_S^c(G)$, set

$$\omega_\Phi(y,x) = \omega_\Phi\bigl(P \xrightarrow{\mathring{x}} Q \xrightarrow{\mathring{y}} R\bigr) \overset{\text{def}}{=} f^{-1}\bigl(\omega'_\Phi(y,x)\bigr) \in Z(P).$$

To show that this is well defined we must show that $\omega'_\Phi(hy,gx)=\omega'_\Phi(y,x)$ for any $g\in Q$ and any $h\in R$. Indeed

$$\Phi_{P,R}(hygx)=f(hygy^{-1})\Phi_{P,R}(yx),\quad \Phi_{Q,R}(hy)\Phi_{P,Q}(gx)=f(h)\Phi_{Q,R}(y)f(g)\Phi_{P,Q}(x),$$

and hence

$$\begin{aligned}\Phi_{P,R}(hygx)^{-1}&\Phi_{Q,R}(hy)\Phi_{P,Q}(gx)\\ &=\Phi_{P,R}(yx)^{-1}f(yg^{-1}y^{-1})\Phi_{Q,R}(y)f(g)\Phi_{P,Q}(x)\\ &=\Phi_{P,R}(yx)^{-1}(f(yg^{-1}y^{-1})\Phi_{Q,R}(y)f(g)\Phi_{Q,R}(y)^{-1})\Phi_{Q,R}(y)\Phi_{P,Q}(x).\end{aligned}$$

But $f(ygy^{-1})=\Phi_{Q,R}(y)f(g)\Phi_{Q,R}(y)^{-1}$ by (a). Hence

$$\Phi_{P,R}(hygx)^{-1}\Phi_{Q,R}(hy)\Phi_{P,Q}(gx)=\Phi_{P,R}(yx)^{-1}\Phi_{Q,R}(y)\Phi_{P,Q}(x),$$

and $\omega'_\Phi(hy,gx)=\omega'_\Phi(y,x)$ as claimed.

To see that ω_Φ is a 2-cocycle, we must show, for each sequence

$$P\xrightarrow{\mathring{x}}Q\xrightarrow{\mathring{y}}R\xrightarrow{\mathring{z}}T$$

in $\mathcal{O}^c_S(G)$, that

$$d(\omega_\Phi)(P\overset{\mathring{x}}{\to}Q\overset{\mathring{y}}{\to}R\overset{\mathring{z}}{\to}T)\overset{\text{def}}{=}\big(x^{-1}\omega(z,y)x\big)\cdot\omega(z,yx)^{-1}\omega(zy,x)\omega(y,x)^{-1}=1.$$

For any $g\in Z(Q)$,

$$f(x^{-1}gx)\equiv\Phi_{P,R}(yx)^{-1}\Phi_{Q,R}(y)f(g)\Phi_{Q,R}(y)^{-1}\Phi_{P,Q}(yx)$$

by condition (a) (applied to the squares for y and yx); so it suffices to show that

$$\Phi_{P,R}(yx)^{-1}\Phi_{Q,R}(y)f(\omega(z,y))\Phi_{Q,R}(y)^{-1}\Phi_{P,R}(yx)\cdot f\Big(\omega(z,yx)^{-1}\omega(zy,x)\omega(y,x)^{-1}\Big)=1.$$

This follows immediately by substitution and cancellation.

Now set $\omega_G(f)=[\omega_\Phi]\in\lim\limits_{\mathcal{O}^c_S(G)}{}^2(\mathcal{Z}_G)$. We claim that $\omega_G(f)$ depends only on the isomorphism $f:S\to S'$, and that f extends to an isotypical equivalence $\psi:\mathcal{L}^c_S(G)\to\mathcal{L}^c_{S'}(G')$ of categories (i.e., $\psi_S=f$) if and only if $\omega_G(f)=0$. Clearly, the $\Phi_{P,Q}$ induce a functor $\psi:\mathcal{L}^c_S(G)\to\mathcal{L}^c_{S'}(G')$ by taking the corresponding bijections $\widehat{\Phi}_{P,Q}$ if and only if $\omega_\Phi=0$. Also, for any isotypical equivalence $\psi:\mathcal{L}^c_S(G)\longrightarrow\mathcal{L}^c_{S'}(G')$ with $\psi_S=f$ which sends inclusions in S to inclusions in S', the maps $\Phi_{P,Q}$ defined by ψ between morphism sets satisfy conditions (a) and (b) above by Lemma 5.1(b). So it remains to show that

(1) for any two choices of maps $\Phi_{P,Q}$ and $\Phi'_{P,Q}$, the corresponding 2-cycles ω_Φ and $\omega_{\Phi'}$ differ by a coboundary, and

(2) all coboundaries can be realized in this way.

To prove point (1), fix maps $\Phi_{P,Q}$ and $\Phi'_{P,Q}$ satisfying conditions (a), (b), and (c) of Lemma 5.2. For each morphism $P \xrightarrow{x} Q$ in $\mathcal{L}_S^c(G)$, one has $\Phi_{P,Q}(x)^{-1}\Phi'_{P,Q}(x) \in C_{G'}(P')$ by condition (a). Set

$$A(x) \stackrel{\mathrm{def}}{=} \Phi_{P,Q}(x)^{-1}\Phi'_{P,Q}(x); \quad \text{and} \quad \alpha(x) \stackrel{\mathrm{def}}{=} f^{-1}((A(x))_p) \in Z(P). \tag{1}$$

Then $\alpha(x)$ does not depend of the choice of the lifts $\Phi_{P,Q}$ and $\Phi'_{P,Q}$. Also, $\alpha(x)$ depends only on the class $\mathring{x} \in \mathrm{Mor}_{\mathcal{O}_S^c(G)}(P,Q)$ by condition (b). So

$$\alpha \in C^1(\mathcal{O}_S^c(G); \mathcal{Z}_G) = \prod_{P \to Q} Z(P).$$

Write $a_x = \alpha(x)$ for short. For each sequence $P \xrightarrow{\mathring{x}} Q \xrightarrow{\mathring{y}} R$ in $\mathcal{O}_S^c(G)$, repeated application of condition (a) gives

$$\begin{aligned}
f\bigl(\omega_{\Phi'}(y,x)\bigr)& \\
&= \bigl(\Phi'(yx)^{-1}\Phi'(y)\Phi'(x)\bigr)_p \\
&= \bigl(A(yx)^{-1}(\Phi(yx)^{-1}\Phi(y)\Phi(x))(\Phi(x)^{-1}A(y)\Phi(x))A(x)\bigr)_p \\
&= f(a_{yx})^{-1}\omega'_\Phi(y,x)\bigl(\Phi(x)^{-1}A(y)\Phi(x)\bigr)_p f(a_x) \\
&= \omega'_\Phi(y,x)[\Phi(x)^{-1}f(a_y)\Phi(x)]f(a_{yx})^{-1}f(a_x) \\
&= \omega'_\Phi(y,x)f((x^{-1}a_yx)a_{yx}^{-1}a_x) \\
&= f\bigl(\omega_\Phi(y,x)\bigr)\cdot f\bigl(d\alpha(P \xrightarrow{\mathring{x}} Q \xrightarrow{\mathring{y}} R)\bigr).
\end{aligned}$$

Thus $\omega_{\Phi'} = \omega_\Phi + d\alpha$.

Finally to see point (2), notice that any 1-cochain α can be realized as the difference between Φ and some Φ', since one can use (1) to define Φ'. This completes the proof of the proposition. □

Given a fusion preserving isomorphism f between Sylow p-subgroups of G and G', one can try directly to construct a homotopy equivalence

$$BG_p^\wedge \simeq \Bigl(\operatorname*{hocolim}_{P \in \mathcal{O}_p^c(G)}(EG/P)\Bigr)_p^\wedge \longrightarrow BG'^\wedge_p.$$

Maps $EG/P \longrightarrow BG'^\wedge_p$ are determined up to homotopy by f, and the obstruction to extending them to a map defined on the homotopy colimit lies precisely in $\lim\limits_{\mathcal{O}_p^c(G)}{}^2(\mathcal{Z}_G)$. It was this observation — that one finds the same obstruction group to constructing an equivalence of categories $\mathcal{L}_p^c(-)$ and to constructing an equivalence of spaces — which first suggested Theorem A to us.

Just as the obstruction to realizing a fusion preserving isomorphism as the underlying isomorphism of an equivalence of categories $\mathcal{L}_p^c(-)$ lies in $\lim^2(\mathcal{Z}_G)$, the obstruction to the uniqueness of such a realization lies in $\lim^1(\mathcal{Z}_G)$. To simplify this discussion, we restrict attention to automorphisms of $\mathcal{L}_p^c(G)$. Recall that $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_p^c(G))$ denotes the group of isotypical self equivalences modulo natural isomorphism. By analogy, we let $\mathrm{Aut}_{\mathrm{fus}}(S)$, for any Sylow p-subgroup $S \le G$, denote the group of fusion preserving automorphisms of S, and set

$$\mathrm{Out}_{\mathrm{fus}}(S) = \mathrm{Aut}_{\mathrm{fus}}(S)/\{c_x \,|\, x \in N_G(S)\}.$$

We are now ready to restate and prove Theorem E. As pointed out in the introduction, this is an algebraic version of the exact sequence of [BL, Theorem 1.6], where $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_p^c(G))$ is replaced by $\mathrm{Out}(BG^\wedge_p)$.

Theorem 6.2. *For any finite group G, there is an exact sequence*

$$0 \longrightarrow \lim_{\mathcal{O}_p^c(G)}{}^1(\mathcal{Z}_G) \xrightarrow{\lambda_G} \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_p^c(G)) \xrightarrow{\mu_G} \mathrm{Out}_{\mathrm{fus}}(S) \xrightarrow{\omega_G} \lim_{\mathcal{O}_p^c(G)}{}^2(\mathcal{Z}_G),$$

Here, λ_G and μ_G are group homomorphisms, μ_G sends the class of an isotypical equivalence to the class of its underlying fusion preserving isomorphism, and ω_G sends the class of a fusion preserving isomorphism f to the obstruction $\omega_G(f)$ of Proposition 6.1. Exactness at $\mathrm{Out}_{\mathrm{fus}}(S)$ means that $\mathrm{Im}(\mu_G) = \omega_G^{-1}(0)$.

Proof. As in the proof of Proposition 6.1, we replace $\mathcal{L}_p^c(G)$ and $\mathcal{O}_p^c(G)$ by their equivalent subcategories $\mathcal{L}_S^c(G)$ and $\mathcal{O}_S^c(G)$: the full subcategories whose objects are the subgroups of S which are p-centric in G.

Let ω_G be the obstruction map of Proposition 6.1, and let μ_G be the map which sends $\psi \in \mathrm{Aut}_{\mathrm{typ}}(\mathcal{L}_p^c(G))$ to ψ_S. If ψ and ψ' are naturally isomorphic, by a natural isomorphism which sends S to $\widehat{x} \in \mathrm{Aut}_{\mathcal{L}_S^c(G)}(S)$, then ψ_S and $\psi'{}_S$ differ by conjugation by $x \in N_G(S)$. So μ_G is well defined, and the sequence is exact at $\mathrm{Out}_{\mathrm{fus}}(S)$ by Proposition 6.1.

A 1-cocycle $\alpha \in Z^1(\mathcal{O}_S^c(G); \mathcal{Z}_G)$ is a collection of maps $\mathrm{Mor}_{\mathcal{O}_S^c(G)}(P,Q) \xrightarrow{\alpha_{P,Q}} Z(P)$ such that

$$\big(x^{-1}\alpha(y)x\big)\cdot\alpha(yx)^{-1}\cdot\alpha(x) = 1 \tag{1}$$

for all $\big(P \xrightarrow{\mathring{x}} Q \xrightarrow{\mathring{y}} R\big)$. In particular, $\alpha(\mathring{g}) = \alpha(\mathrm{Id}_P) = 1$ for all $g \in P$. For any such α, set $\lambda_G(\alpha) = [\psi_\alpha]$, where $\psi_\alpha(P) = P$ for all objects P, and $\psi_\alpha(\widehat{x}) = \widehat{x\alpha(x)}$. Then ψ_α is a functor by (1). If β is a 0-cochain, then the elements $\beta(P) \in Z(P)$ define a natural isomorphism $\mathrm{Id} \longrightarrow \psi_{d\beta}$. One easily checks that $\psi_{\alpha+\alpha'} = \psi_\alpha \circ \psi'_\alpha$; and this shows that λ_G is a well defined homomorphism.

It is straightforward to show that $\mathrm{Im}(\lambda_G) = \mathrm{Ker}(\mu_G)$. To see that λ_G is injective, fix $[\alpha] \in \mathrm{Ker}(\lambda_G)$, and let $\beta : \mathrm{Id} \longrightarrow \psi_\alpha$ be a natural isomorphism. This consists of a choice of $\widehat{\beta(P)} \in \mathrm{Aut}_{\mathcal{L}_p^c(G)}(P)$ for each P, where $\beta(P) \in N(P)$. Naturality with respect to automorphisms $P \xrightarrow{\widehat{g}} P$ for $g \in P$ implies that $\beta(P) \in Z(P)$ for each P (since $\psi_\alpha(\widehat{g}) = \widehat{g}$). Thus $\beta \in C^0(\mathcal{O}_S^c(G); \mathcal{Z}_G)$, and $\alpha = d\beta$. □

A priori, the obstruction map ω_G defined above, from $\mathrm{Out}_{\mathrm{fus}}(S)$ to $\lim^2(\mathcal{Z}_G)$, need not be a homomorphism. Rather, one can show directly that its behavior with respect to composition of fusion preserving automorphisms is twisted by an action of the group $\mathrm{Out_{fus}}(S)$ on $\lim^2(\mathcal{Z}_G)$. However, recent results have made this observation irrelevant, since the third author has recently shown that $\lim\limits_{\mathcal{O}_p^c(G)}{}^2(\mathcal{Z}_G) = 0$ for all primes p and all finite groups G.

However, as noted in the introduction, the proof of this result is sufficiently long and complicated that we still feel it useful to have more elementary proofs of special cases. Recall that the *p-rank* $\mathrm{rk}_p(G)$ of a finite group G is the rank of the largest elementary abelian p-subgroup of G. We will show that $\lim\limits_{\mathcal{O}_p^c(G)}{}^r(\mathcal{Z}_G) = 0$ whenever $\mathrm{rk}_p(G) < p^r$. This, together with Proposition 6.1, will then imply Theorem F as special case.

We first discuss more generally the vanishing of higher limits groups over orbit categories. One way to compute higher limits of functors over orbit categories is to filter the functor in a way so that each of the quotient functors vanishes except on one isomorphism class of objects. In particular, if higher limits of the original functor are non-vanishing, then they must also be non-vanishing for at least one of the quotient functors. This motivated the definition of graded abelian groups $\Lambda^*(G;M)$ in [JMO]. For any prime p, any finite group G, and any $\mathbb{Z}_{(p)}[G]$-module M,

$$\Lambda^*(G;M) \overset{\text{def}}{=} \lim_{\mathcal{O}_p(G)}{}^*(F_M), \qquad \text{where} \qquad F_M(P) = \begin{cases} M & \text{if } P = 1 \\ 0 & \text{otherwise.} \end{cases}$$

These groups have the property [JMO, Lemma 5.4] that if $F : \mathcal{O}_p(G)^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{-}\mathsf{mod}$ is *any* functor which vanishes except on subgroups conjugate to P, then $\lim^*(F) \cong \Lambda^*(N(P)/P; F(P))$.

The proof of the following proposition is based on the description by Grodal [Gr] of these groups, in terms of the cohomology of certain posets, and the part about $\mathrm{rk}_p(G)$ was shown there by Grodal.

Proposition 6.3. *For any finite group G, any finitely generated $\mathbb{F}_p[G]$-module M, and any $k \geq 1$ such that $\Lambda^k(G;M) \neq 0$, there is an elementary abelian p-subgroup $A \leq G$ of rank k such that $M|_A$ contains the free module $\mathbb{F}_p[A]$ as a direct summand. In particular,*

$$\mathrm{rk}_p(G) \geq k \qquad \textit{and} \qquad \dim_{\mathbb{F}_p}(M) \geq p^k.$$

Proof. Let $\mathcal{S}_p(G)$ denote the nerve of the poset of nontrivial p-subgroups of G, and let $\widetilde{C}_*(\mathcal{S}_p(G))$ denote its reduced chain complex. Thus, $\widetilde{C}_{-1}(\mathcal{S}_p(G)) = \mathbb{Z}$, and $\widetilde{C}_k(\mathcal{S}_p(G))$ (for $k \geq 0$) is the free $\mathbb{Z}$-module with basis the set of sequences $1 \neq P_0 \lneqq P_1 \lneqq \cdots \lneqq P_k$. The conjugation action of G on the basis makes this into a chain complex of $\mathbb{Z}[G]$-modules. By [Gr, Theorem 1.2], for any $\mathbb{Z}_{(p)}[G]$-module M,

$$\Lambda^*(G;M) \cong H^{*-1}\big(\mathrm{Hom}_G(\widetilde{C}_*(\mathcal{S}_p(G)), M)\big).$$

Similarly, let $\mathcal{A}_p(G)$ denote the nerve of the poset of nontrivial elementary abelian p-subgroups of G, and set $C_* = \widetilde{C}_*(\mathcal{A}_p(G); \mathbb{F}_p)$. By [Qu], $\mathcal{A}_p(G)$ and $\mathcal{S}_p(G)$ are G-homotopy equivalent, and so $\Lambda^*(G; M) \cong H^{*-1}(\mathrm{Hom}_G(C_*, M))$ for any $\mathbb{F}_p[G]$-module M. By [Wb2, Theorem 2.7.1], C_* splits as a sum $C_* = D_* \oplus P_*$ of complexes of $\mathbb{F}_p[G]$-modules, where D_* is exact (in fact, split exact), and where each P_k is projective. In particular, $\Lambda^*(G; M)$ is the cohomology of the complex $\mathrm{Hom}_G(P_*, M)$.

Assume that $\Lambda^k(G; M) \neq 0$. Then $\mathrm{Hom}_G(P_{k-1}, M) \neq 0$ by the above remarks. Since P_{k-1} is $\mathbb{F}_p[G]$-projective, the module $\mathrm{Hom}_{\mathbb{F}_p}(P_{k-1}, M)$ is also $\mathbb{F}_p[G]$-projective. Notice that for any projective G-module P, the submodule of G-invariants is given by

$$P^G = \{\nu_G \cdot x \mid x \in P\},$$

where ν_G is the norm element $\sum_{g \in G} g \in \mathbb{F}_p[G]$. The group G acts on $\mathrm{Hom}_{\mathbb{F}_p}(P_{k-1}, M)$ by $(g\phi)(x) = g \cdot \phi(g^{-1}x)$, and

$$0 \neq \mathrm{Hom}_G(P_{k-1}, M) \cong \big(\mathrm{Hom}_{\mathbb{F}_p}(P_{k-1}, M)\big)^G.$$

Hence there is some $\phi : P_{k-1} \to M$ such that $\sum_{g \in G} g\phi \neq 0$. By extension, we may assume that ϕ with this property is defined on $C_{k-1} = D_{k-1} \oplus P_{k-1}$.

The complex C_* is generated as an $\mathbb{F}_p[G]$-module in dimension $k-1$ by the $(k-1)$-simplices of $\mathcal{A}_p(G)$, namely, by sequences of the form $\tau = (E_1 \gneqq \cdots \gneqq E_k)$. Hence there exists such a simplex τ such that

$$\sum_{g \in G} (g\phi)(\tau) = \sum_{g \in G} g \cdot \phi(g^{-1}\tau) \neq 0,$$

where ϕ is as above.

Let $b_1, \ldots, b_r$ be a choice of right coset representatives for E_k in G. Notice that E_k is contained in the stabilizer group of τ. Thus

$$0 \neq \sum_{g \in G} (g\phi)(\tau) = \sum_{g \in E_k} \sum_{i=1}^{r} g b_i \phi(b_i^{-1}\tau).$$

Set $x = \sum_{i=1}^r b_i \phi(b_i^{-1}\tau) \in M$. Then $\sum_{g \in E_k} gx \neq 0$. Notice that $\mathrm{rk}_p(G) \geq \mathrm{rk}_p(E_k) \geq k$.

Let $\iota_x : \mathbb{F}_p[E_k] \longrightarrow M$ denote the E_k-map sending 1 to x. Then $\mathrm{Ker}(\iota_x)$ does not contain the norm element ν_{E_k} and hence is the zero ideal in the group ring (cf. [Se, §8.3, Proposition 26]). Thus ι_x is a monomorphism. Furthermore, since $\mathbb{F}_p[E_k]$ is an injective E_k-module, the module $M|_{E_k}$ contains $\mathbb{F}_p[E_k]$ as a split summand. In particular, $\dim_{\mathbb{F}_p}(M) \geq |E_k| \geq p^k$. This completes the proof. □

The idea of the last part of the above proof was suggested to us by the proof of [Wb1, 5.3].

The inequalities of Proposition 6.3 are in fact the best possible, as is shown by the following examples. Let V denote the $\mathbb{F}_p[\Sigma_{p+1}]$-module $V = (\mathbb{F}_p)^{p+1}/(\mathrm{diag})$. Then

$\Lambda^1(\Sigma_{p+1};V)\cong\mathbb{F}_p$ by [JMO, Proposition 6.2(i)]. So for any $k>0$,

$$\Lambda^k\big((\Sigma_{p+1})^k,V^{\otimes k}\big)\neq 0$$

by the Künneth formula in [JMO, Proposition 6.1(v)]; and

$$\Lambda^k\big(\Sigma_{p+1}\underbrace{\wr C_p\wr\cdots\wr C_p}_{(k-1)\text{ times}},\,V^{p^{k-1}}\big)\neq 0$$

by a general formula $\Lambda^i(G\wr C_p;M^p)\cong\Lambda^{i-1}(G;M)$ when $p\big||G|$ (unpublished). These are thus two examples of pairs (G,M) for which $\Lambda^k(G;M)\neq 0$, $\mathrm{rk}_p(G)=k$, and $\dim_{\mathbb{F}_p}(M)=p^k$.

When $k=1$, a similar argument gives a much stronger result. If M is an $\mathbb{F}_p[G]$-module with $\Lambda^1(G;M)\neq 0$, then for any Sylow p-subgroup $S\le G$, $M|_S$ contains a summand isomorphic to $\mathbb{F}_p[S]$, and in particular $\dim_{\mathbb{F}_p}(M)\ge|S|$.

As an easy consequence of Proposition 6.3, we get:

Corollary 6.4. *Fix a finite group G, and a functor $F:\mathcal{O}_p^c(G)^{\mathrm{op}}\to\mathbb{Z}_{(p)}\text{-}\mathsf{mod}$ which takes values in finite abelian p-groups. Assume, for some $k>0$, that $\lim\limits_{\mathcal{O}_p^c(G)}{}^k(F)\neq 0$. Then there is a p-centric subgroup $P\le G$ such that $\mathrm{rk}_p(F(P))\ge p^k$.*

Proof. Extend F to a functor on $\mathcal{O}_p(G)$ by setting $F(P)=0$ whenever P is not p-centric. Since any p-subgroup of G which contains a p-centric subgroup is itself p-centric, the chain complexes $C^*(\mathcal{O}_p(G);F)$ and $C^*(\mathcal{O}_p^c(G);F)$ are isomorphic. So higher limits of F are the same whether taken over $\mathcal{O}_p(G)$ or over $\mathcal{O}_p^c(G)$.

Now let $P_1,P_2,\dots,P_k$ be conjugacy class representatives for p-centric subgroups of G, arranged such that $|P_1|\le|P_2|\le\cdots$. For $0\le i\le k$, let F_i be the functor

$$F_i(P)=\begin{cases}0 & \text{if } P\ge xP_jx^{-1}\text{, some } x\in G \text{ and } j>i\\ F(P) & \text{otherwise.}\end{cases}$$

Thus, $F_0=0$, $F_k=F$, and F_i is a subfunctor of F_{i+1} for each i. By the exact sequences of higher limits for extensions of functors, there is some $1\le i\le k$ such that $\lim^k(F_i/F_{i-1})\neq 0$. The functor F_i/F_{i-1} vanishes except on subgroups conjugate to P_i, and hence

$$\Lambda^k(N(P_i)/P_i;F(P_i))\cong\lim_{\mathcal{O}_p(G)}{}^k(F_i/F_{i-1})\neq 0$$

by [JMO, Lemma 5.4]. Then $\Lambda^k(N(P)/P;p^jF(P)/p^{j+1}F(P))\neq 0$ for some $j\ge 0$ (where $P=P_i$), so this module has rank at least p^k by Proposition 6.3, and hence $\mathrm{rk}_p(F(P))\ge p^k$. □

We are now ready to prove Theorem F.

Theorem 6.5. *Fix a prime p, and a finite group G such that $\mathrm{rk}_p(G)<p^2$. Then for any other finite group G', $BG^\wedge_p\simeq BG'^\wedge_p$ if and only if there is a fusion preserving isomorphism between Sylow p-subgroups of G and of G'.*

Proof. If $BG^\wedge_p \simeq BG'^\wedge_p$, then there is an equivalence of categories

$$\psi : \mathcal{L}_p^c(G) \xrightarrow{\ \simeq\ } \mathcal{L}_p^c(G')$$

by Theorem A, and hence a fusion preserving isomorphism between the Sylow p-subgroups by Lemma 5.1.

Conversely, if $\mathrm{rk}_p(G) < p^2$ then $\mathrm{rk}_p(Z(P)) < p^2$ for each p-centric $P \le G$, and Corollary 6.4 implies that $\lim^2(\mathcal{Z}_G) = 0$. Here, as usual, $\mathcal{Z}_G : \mathcal{O}_p^c(G) \to \mathbb{Z}_{(p)}\text{-}\mathtt{mod}$ is the functor which sends P to $Z(P)$. So if there is a fusion preserving isomorphism between Sylow p-subgroups of G and G', then the categories $\mathcal{L}_p^c(G)$ and $\mathcal{L}_p^c(G')$ are equivalent by Proposition 6.1, and so $BG^\wedge_p \simeq BG'^\wedge_p$ by Theorem A. □

7. An example: $BG^\wedge_2$ when $G = PSL_2(q)$ and q is odd

Throughout this section, we fix $G = PSL_2(q)$, where $q = p^e$ and p is an odd prime. Set $k = \nu_2(|G|)$, where in general $2^{\nu_2(n)}$ is the highest power of 2 dividing n. Then $k \ge 2$, and $q \equiv 2^k \pm 1 \pmod{2^{k+1}}$. Fix a Sylow 2-subgroup $S \le G$, and let $\mathcal{L}_S^c(G) \subseteq \mathcal{L}_2^c(G)$ denote the full subcategory whose objects are the 2-centric subgroups of G which are contained in S. As usual, this inclusion is an equivalence of categories.

For a positive integer n, C_n denotes a cyclic group of order n; and (if n is even) D_n denotes the dihedral group of order n.

Lemma 7.1. *The Sylow 2-subgroup S in $PSL_2(q)$ is dihedral of order 2^k. If $k = 2$ then $N_G(S) \cong A_4$, while if $k \ge 3$ then $N_G(S) = S$. If $k \ge 3$, and if T_1 and T_2 are representatives for the two conjugacy classes of subgroups $T_i \cong C_2^2$ in S, then T_1 and T_2 also represent the two conjugacy classes of such subgroups in G, and $N_G(T_i) \cong \Sigma_4$. If $k \ge 3$, and $z \in Z(S)$ is a generator, then $C_G(z) \cong D_{q-1}$ if $q \equiv 1$ (mod 8), and $C_G(z) \cong D_{q+1}$ if $q \equiv -1$ (mod 8).*

Proof. See, for example, [Sz, Theorems 3.6.25–26] or [Hu, Satz 8.27] for a complete description of the subgroups of $PSL_2(q)$. □

The next lemma provides an explicit description of the morphism sets in $\mathcal{L}_S^c(G)$.

Lemma 7.2. *Assume $|S| \ge 8$, and let $P, Q \le S$ be any pair of subgroups which are 2-centric in G. If $P \not\cong C_2^2$, then the map*

$$N_S(P,Q) \xrightarrow{\ \cong\ } \mathrm{Mor}_{\mathcal{L}_S^c(G)}(P,Q)$$

sending $x \in N_S(P,Q)$ to $\widehat{x}$ is a bijection; while if $P \cong C_2^2$, then there is a bijection

$$N_S(P,Q) \times_{N_S(P)} N_G(P) \xrightarrow{\ \cong\ } \mathrm{Mor}_{\mathcal{L}_S^c(G)}(P,Q)$$

which sends a pair (x,g) to the composite $\widehat{x} \circ \widehat{g}$.

Proof. Since S is dihedral of order $2^k \geq 8$, its centric subgroups consist of S itself, the (unique) cyclic subgroup of index 2, and two S-conjugacy classes of dihedral subgroups of each order 2^m for $2 \leq m < k$. Here, by a dihedral subgroup of order 4 is meant a group isomorphic to C_2^2.

Let $z \in Z(S)$ be the generator, and set $D = C_G(z) \cong D_{q\pm1}$ (Lemma 7.1). If P, P' are two dihedral subgroups of the same order $2^m \geq 4$, and $gPg^{-1} = P'$ for some $g \in G$, then either $m = 2$ and P, P' are S-conjugate by Lemma 7.1; or $m \geq 3$, $Z(P) = Z(P') = \langle z \rangle$, and $g \in C_G(z) \cong D_{q\pm1}$. Thus P and P' are D-conjugate, and we leave it as an exercise for the reader to check that this implies they are S-conjugate.

Hence for any 2-centric $P, Q \leq S$, every morphism in $\mathrm{Mor}_{\mathcal{L}_S^c(G)}(P, Q)$ has the form $\widehat{x} \circ \widehat{g}$ for some $g \in N_G(P)$ and some $x \in N_S(P, Q)$. Also, $\widehat{x} \circ \widehat{g} = \widehat{x'} \circ \widehat{g'}$ if and only if there is some $a \in N_S(P)$ such that $x' = xa$ and $g' = a^{-1}g$. Furthermore, since $z \in P$, $C_G(P) \leq C_G(z) = D$, and thus (since each noncyclic subgroup of a dihedral group contains its centralizer) $C'_G(P) = 1$ if P is noncyclic. Also, if P is cyclic of index 2 in S, then $N_G(P) = D$, and hence $N_G(P)/C'_G(P) \cong N_S(P) = S$. It follows that for all 2-centric subgroups $P, Q \leq S$,

$$\begin{aligned}\mathrm{Mor}_{\mathcal{L}_p^c(G)}(P, Q) &\cong N_S(P, Q) \times_{N_S(P)} \big(N_G(P)/C'_G(P)\big) \\ &\cong \begin{cases} N_S(P, Q) \times_{N_S(P)} N_S(P) \cong N_S(P, Q) & \text{if } P \text{ is cyclic} \\ N_S(P, Q) \times_{N_S(P)} N_G(P) & \text{otherwise.} \end{cases}\end{aligned}$$

If P is dihedral of order ≥ 8, then $z \in P$ is the unique central element of order 2, so $N_G(P) \leq C_G(z) = D$. We again leave it to the reader to check that $N_D(P) = N_S(P)$; and thus that $N_G(P) = N_S(P)$ in this case. Hence $N_S(P, Q) \times_{N_S(P)} N_G(P) \cong N_S(P, Q)$, and this finishes the proof of the lemma. □

We also need the following elementary lemma about isomorphisms between groups isomorphic to Σ_4.

Lemma 7.3. *Assume $G \cong G' \cong \Sigma_4$, fix Sylow 2-subgroups $S \leq G$ and $S' \leq G'$, and let $T \lhd G$ and $T' \lhd G'$ be the normal subgroups of order 4. Then any isomorphism $\theta\colon S \longrightarrow S'$ such that $\theta(T) = T'$ extends to an isomorphism $\widehat{\theta}\colon G \longrightarrow G'$. If $\widehat{\theta}$ and $\widehat{\theta}'$ are two such extensions, then there is $x \in Z(S)$ such that $\widehat{\theta}' = \widehat{\theta} \circ c_x$.*

Proof. Fix an isomorphism $\psi\colon G' \xrightarrow{\cong} G$ such that $\psi(S') = S$. Clearly, $\psi(T') = T$. Then $\psi \circ \theta \in \mathrm{Aut}(S)$ and $\psi \circ \theta(T) = T$. This implies that $\psi \circ \theta$ is the identity on $S/Z(S)$, and all such automorphisms are inner. Hence $\psi \circ \theta$ extends to an inner automorphism $\varphi \in \mathrm{Inn}(G)$, and $\psi^{-1} \circ \varphi \in \mathrm{Iso}(G, G')$ extends θ.

By [Sz, 3.2.17], all automorphisms of Σ_4 are inner. So if $\widehat{\theta} \in \mathrm{Aut}(G)$ is the identity on S, then it must be conjugation by an element of $C_G(S) = Z(S)$. The last statement now follows. □

As a first consequence of Lemmas 7.1, 7.2 and 7.3, we determine which of the spaces $BPSL_2(q)^\wedge_2$ are homotopy equivalent to each other.

Proposition 7.4. *Let q and q' be odd prime powers, and set $G = PSL_2(q)$ and $G' = PSL_2(q')$. Set $k = \nu_2(|G|)$ and $k' = \nu_2(|G'|)$. Then $BG^\wedge_2 \simeq BG'^\wedge_2$ if and only if $k = k'$.*

Proof. If $BG^\wedge_2 \simeq BG'^\wedge_2$, then their Sylow 2-subgroups must have the same order (see Proposition 2.1), and hence $k = k'$.

Conversely, assume $k = k'$, let $S \le G$ and $S' \le G'$ be Sylow 2-subgroups, and fix an isomorphism $\theta : S \to S'$ (both are dihedral of order $2^k = 2^{k'}$). If $k = 2$, then $S \cong C_2^2$ is the only object in $\mathcal{L}_S^c(G)$, and θ extends to an isomorphism $N_G(S) \xrightarrow{\cong} N_{G'}(S') \cong A_4$, and hence to an equivalence of categories $\mathcal{L}_S^c(G) \xrightarrow{\cong} \mathcal{L}_{S'}^c(G')$.

Now assume $k \ge 3$. To simplify the notation in the following argument, we write $P' = \theta(P)$ for any $P \le S$, and $g' = \theta(g)$ for any $g \in S$. Let $T_1, T_2 \le S$ be representatives for the two conjugacy classes of subgroups $T_i \cong C_2^2$ in S. For $i = 1, 2$, $R_i \overset{\text{def}}{=} N_S(T_i) \cong D_8$ is a Sylow 2-subgroup in $N_G(T_i) \cong \Sigma_4$. By Lemma 7.3, $\theta|_{R_i} \colon R_i \longrightarrow R_i'$ extends to an isomorphism $\theta_i : N_G(T_i) \longrightarrow N_{G'}(T_i')$. Now, for each subgroup $T \le S$ such that $T \cong C_2^2$, define

$$\theta_T \colon N_G(T) \longrightarrow N_{G'}(T')$$

by setting, for some $x \in N_S(T, T_i)$ (where $i = 1, 2$ is the unique choice such that T_i is conjugate to T),

$$\theta_T(g) = x'^{-1} \cdot \theta_i(xgx^{-1}) \cdot x' .$$

To see that this is independent of the choice of x, let $y \in N_S(T, T_i)$ is another element, and set $a = yx^{-1} \in N_S(T_i)$ (so $y = ax$). Then

$$y'^{-1} \cdot \theta_i(ygy^{-1}) \cdot y' = x'^{-1} \cdot (a'^{-1}\theta_i(a)) \cdot \theta_i(xgx^{-1}) \cdot (\theta_i(a)^{-1}a') \cdot x' = \theta_T(g)$$

since $\theta_i(a) = \theta(a) = a'$ by assumption.

For each pair of subgroups $P, Q \le S$ which are 2-centric in G, define maps

$$\Theta_{P,Q} \colon \mathrm{Mor}_{\mathcal{L}_S^c(G)}(P, Q) \longrightarrow \mathrm{Mor}_{\mathcal{L}_{S'}^c(G')}(P', Q')$$

as follows. If $P \not\cong C_2^2$, then $\Theta_{P,Q}(\widehat{g}) = \widehat{g'}$ for all $g \in N_S(P, Q)$. If $P \cong C_2^2$, then for all $g \in N_S(P, Q)$ and all $x \in N_G(P)$, set $\Theta_{P,Q}(\widehat{g} \circ \widehat{x}) = \widehat{g'} \circ \widehat{\theta_P(x)}$. Since $\theta_P|_{N_S(P)} = \theta|_{N_S(P)}$, these maps are well defined by Proposition 7.2.

Now define $\Theta \colon \mathcal{L}_S^c(G) \longrightarrow \mathcal{L}_{S'}^c(G')$ by setting $\Theta(P) = P'$ for all $P \le S$ p-centric in G, and by letting $\Theta_{P,Q}$ be defined as above. This is clearly a functor, and an isomorphism of categories. Thus

$$\mathcal{L}_p^c(G) \simeq \mathcal{L}_S^c(G) \cong \mathcal{L}_{S'}^c(G') \simeq \mathcal{L}_p^c(G'),$$

and so $BG^\wedge_2 \simeq BG'^\wedge_2$ by Theorem A. $\square$

We now want to study the group $\mathrm{Out}(BG_2^\wedge) \cong \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_S^c(G))$, using methods similar to those of Proposition 7.4 to construct equivalences of categories. To do this, we first define a group $X(G)$ which is motivated by the proof of Proposition 7.4 above.

Set

$$\mathcal{T} = \{T \le S \,|\, T \cong C_2^2\}\,.$$

Let $X(G)$ be the set of all $(\theta; \{\theta_T\}_{T\in\mathcal{T}})$, where $\theta \in \mathrm{Aut}(S)$, $\theta_T \in \mathrm{Iso}(N_G(T), N_G(\theta T))$ for each $T \in \mathcal{T}$, and such that the relations

$$\theta_T|_{N_S(T)} = \theta|_{N_S(T)} \qquad \text{and} \qquad \theta_{T'} \circ c_g = c_{\theta(g)} \circ \theta_T \tag{7.5}$$

hold for all $T, T' \in \mathcal{T}$ and all $g \in N_S(T, T')$. We define the group structure on $X(G)$ by setting

$$(\psi; \{\psi_T\}_{T\in\mathcal{T}}) \cdot (\theta; \{\theta_T\}_{T\in\mathcal{T}}) = (\psi \circ \theta; \{\psi_{\theta T} \circ \theta_T\}_{T\in\mathcal{T}}).$$

This clearly defines a group structure, with identity $(\mathrm{Id}_S, \{\mathrm{Id}_{N_G(T)}\}_{T\in\mathcal{T}})$, and with inverses

$$(\theta; \{\theta_T\}_{T\in\mathcal{T}})^{-1} = (\theta^{-1}; \{\theta_{\theta^{-1}T}{}^{-1}\}_{T\in\mathcal{T}}).$$

Let $X_S(G) \le X(G)$ be the subgroup

$$X_S(G) = \big\{C_x \overset{\text{def}}{=} (c_x; \{c_x\}_{T\in\mathcal{T}}) \,\big|\, x \in S\big\}.$$

It is not hard to see that this is a normal subgroup. More precisely, using relations (7.5), one shows that conjugation by $(\theta; \{\theta_T\}_{T\in\mathcal{T}})$ sends C_x to $C_{\theta x}$.

Proposition 7.6. *If $k \ge 3$, then there is an isomorphism of groups*

$$\Lambda\colon\ \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_S^c(G)) \xrightarrow{\ \cong\ } X(G)/X_S(G)$$

with the following property: for any $\Theta \in \mathrm{Aut}_{\mathrm{typ}}(\mathcal{L}_S^c(G))$ which sends inclusions to inclusions,

$$\Lambda(\Theta) = (\Theta_{S,S}; \{\Theta_{T,T}\}_{T\in\mathcal{T}}).$$

Proof. Let $\mathrm{Aut}^1_{\mathrm{typ}}(\mathcal{L}_S^c(G))$ denote the monoid of isotypical equivalences of $\mathcal{L}_S^c(G)$ to itself which send inclusions to inclusions. This is, in fact, a group: any isotypical equivalence of $\mathcal{L}_S^c(G)$ which sends inclusions to inclusions must be bijective on objects (see Lemma 5.1(b)), and hence is an isomorphism of categories.

By Lemma 5.1(a), each isotypical equivalence of $\mathcal{L}_S^c(G)$ is naturally isomorphic to an element of $\mathrm{Aut}^1_{\mathrm{typ}}(\mathcal{L}_S^c(G))$. Thus,

$$\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_S^c(G)) \cong \mathrm{Aut}^1_{\mathrm{typ}}(\mathcal{L}_S^c(G))/\,\mathrm{Aut}^1_0(\mathcal{L}_S^c(G)),$$

where $\mathrm{Aut}^1_0(\mathcal{L}_S^c(G))$ denotes the subgroup of those equivalences which are naturally isomorphic to the identity. We must show Λ is well defined on this quotient.

The above formula clearly defines a homomorphism from $\mathrm{Aut}^1_{\mathrm{typ}}(\mathcal{L}_S^c(G))$ to $X(G)$. If $\Theta \in \mathrm{Aut}^1_0(\mathcal{L}_S^c(G))$ — if Θ is naturally isomorphic to the identity by a natural isomorphism $\mathrm{Id} \xrightarrow{\ \alpha\ } \Theta$ — then set $\alpha(S) = \widehat{x}$ and $\alpha(T) = \widehat{x_T}$ for $T \in \mathcal{T}$. Here,

$x \in N_G(S) = S$, and $x_T \in N_G(T, \Theta(T))$. The naturality of α implies that the following squares commute for each $T \in \mathcal{T}$ and each $g \in N_G(T)$:

$$\begin{array}{ccc} T & \xrightarrow{\widehat{1}} & S \\ {\scriptstyle\widehat{x_T}}\downarrow & & \downarrow{\scriptstyle\widehat{x}} \\ \Theta(T) & \xrightarrow{\widehat{1}} & S \end{array} \qquad\qquad \begin{array}{ccc} T & \xrightarrow{\widehat{g}} & T \\ {\scriptstyle\widehat{x_T}}\downarrow & & \downarrow{\scriptstyle\widehat{x_T}} \\ \Theta(T) & \xrightarrow{\Theta_{T,T}(\widehat{g})} & \Theta(T) \end{array}$$

This shows that $x_T = x$ and $\Theta_{T,T}(\widehat{g}) = c_x(\widehat{g})$ for each $T \in \mathcal{T}$, and hence that $\Lambda(\Theta) = C_x \in X_S(G)$. Thus, Λ is well defined as a homomorphism on $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_S^c(G))$.

If $\Theta \in \mathrm{Aut}_{\mathrm{typ}}^1(\mathcal{L}_S^c(G))$, and if $\Theta_{S,S}$, Θ_{T_1,T_1}, and Θ_{T_2,T_2} are all conjugation by some fixed element $x \in S$, then Θ is naturally isomorphic to the identity by the natural isomorphism $\mathrm{Id} \xrightarrow{\cong} \Theta$ which sends each object to the morphism $\widehat{x}$. Thus $[\Theta] = 1$ in $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_S^c(G))$, and this shows that Λ is injective.

It remains to show that Λ is surjective, and this follows by exactly the same argument as that used to construct an equivalence in the proof of Proposition 7.4. □

We now look at the groups

$$\mathrm{Out}(BG_2^\wedge) \cong \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_p^c(G)) \cong X(G)/X_S(G).$$

The first equivalence is a consequence of Theorem B, and the second follows from Proposition 7.6. We want to compare these groups with the group $\mathrm{Out}(G)$ of outer automorphisms. We first recall the well known description of this group.

Recall that $q = p^e$ where p is prime. We regard $\mathrm{Aut}(\mathbb{F}_q) \cong C_e$ as a subgroup of $\mathrm{Aut}(G)$: each $\varphi \in \mathrm{Aut}(\mathbb{F}_q)$ acts on G by sending a matrix $A = (a_{i,j})$ to the matrix $\varphi(A) = (\varphi(a_{i,j}))$.

Lemma 7.7. *Let $\omega \in \mathrm{Aut}(G)$ be any automorphism induced by conjugation by an element $\gamma \in PGL_2(q) \smallsetminus PSL_2(q)$; i.e., $\gamma = [A]$ for some $A \in GL_2(q)$ whose determinant is not a square in $(\mathbb{F}_q)^*$. Then*

$$\mathrm{Out}(G) = \langle \omega \rangle \times \mathrm{Aut}(\mathbb{F}_q) \cong C_2 \times C_e.$$

Proof. See [Ctr, Theorem 12.5.1]. Note that ω is called the "diagonal automorphism", and the elements coming from $\mathrm{Aut}(\mathbb{F}_q)$ the "field automorphisms". □

We next compare the group $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_S^c(G))$ to a certain quotient of $\mathrm{Aut}(S)$. The statement of the following lemma should be compared with [BL, Theorem 1.7(3)].

In order to simplify the notation for elements of $X(G)$, we fix subgroups $T_1, T_2 \in \mathcal{T}$ which represent the two (S- or G-) conjugacy classes, and write elements of $X(G)$ as triples $(\theta; \theta_1, \theta_2)$ for $\theta \in \mathrm{Aut}(S)$ and $\theta_i \in \mathrm{Iso}(N_G(T_i), N_G(\theta(T_i)))$ such that $\theta_i|_{N_S(T_i)} = \theta|_{N_S(T_i)}$. Any such triple extends to a unique element $(\theta; \{\theta_T\}_{T\in\mathcal{T}})$ via the defining relations (7.5) for $X(G)$. Also, for any such triple, we write $[\theta; \theta_1, \theta_2]$ to denote its class in $X(G)/X_S(G)$.

Lemma 7.8. *Let* $\mathrm{Aut}_G(S)$ *denote the group of automorphisms of* S *induced by conjugation in* G. *Let*

$$\sigma\colon \mathrm{Out}(BG^\wedge_2) \cong \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}^c_S(G)) \longrightarrow \mathrm{Aut}(S)/\mathrm{Aut}_G(S)$$

be the homomorphism defined by restricting an automorphism of the category to its action on $\mathrm{Aut}_{\mathcal{L}^c_S(G)}(S) = N_G(S) = S$.

(a) *If* $k = 2$ *(so* $S \cong C_2^2$*), then* σ *is an isomorphism, and* $\mathrm{Aut}(S)/\mathrm{Aut}_G(S) \cong \Sigma_3/C_3 \cong C_2$.

(b) *Assume* $k \geq 3$ *(so* $S \cong D_{2^k}$*), and let* $z \in S$ *be the central element of order* 2. *Then* σ *is surjective, and*

$$\mathrm{Aut}(S)/\mathrm{Aut}_G(S) = \mathrm{Out}(S) \cong C_2 \times C_{2^{k-3}}.$$

Also, $\mathrm{Ker}(\sigma)$ *has order* 2, *and under the identification of Proposition 7.6 is generated by the element* $[\mathrm{Id}_S; \mathrm{Id}_{T_1}, c_z]$.

Proof. The surjectivity of σ (for arbitrary k) follows as in the proof of Proposition 7.4, by using Lemma 7.3 to show that each $\theta \in \mathrm{Aut}(S)$ extends to a triple $(\theta; \theta_1, \theta_2)$.

If $k = 2$ (if $q \equiv \pm 3 \pmod 8$), then $S \cong C_2^2$, and $N_G(S) \cong A_4$ by Lemma 7.1. Then σ is injective since S is the only object in $\mathcal{L}^c_S(G)$.

If $k \geq 3$, then fix generators $\alpha, \beta \in S \cong D_{2^k}$, where $|\alpha| = 2^{k-1}$ and $|\beta| = 2$. Then

$$\mathrm{Aut}(D_{2^k}) \cong C_{2^{k-1}} \rtimes \mathrm{Aut}(C_{2^{k-1}}),$$

where the normal cyclic subgroup is generated by the automorphism $\alpha \mapsto \alpha$ and $\beta \mapsto \alpha\beta$; and where the second factor is the subgroup of automorphisms fixing β. Also, $\mathrm{Aut}_G(S) = \mathrm{Inn}(S)$ since $N_G(S) = S$. It now follows easily that

$$\mathrm{Aut}(S)/\mathrm{Aut}_G(S) = \mathrm{Out}(D_{2^k}) \cong C_2 \times \mathrm{Aut}(C_{2^{k-1}})/\{\pm 1\} \cong C_2 \times C_{2^{k-3}},$$

and it remains to calculate the kernel of σ.

Under the identification of $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}^c_p(G))$ with $X(G)/X_S(G)$ given in Proposition 7.6, an element in $\mathrm{Ker}(\sigma)$ is of the form $[Id_S; \theta_1, \theta_2]$, where $\theta_i \in \mathrm{Aut}(N_G(T_i))$ for $i = 1, 2$ are such that $\theta_i|_{N_S(T_i)} = \mathrm{Id}$. By Lemma 7.3, the only nontrivial automorphism of $N_G(T_i) \cong \Sigma_4$ which is the identity on a Sylow 2-subgroup D_8 is conjugation by the central element of that subgroup. Thus, each θ_i is the identity or conjugation by $z \in Z(S)$. The element $[\mathrm{Id}_S; c_z, c_z]$ is equal to $C_z \in X_S(G)$; while the element $[\mathrm{Id}_S; \mathrm{Id}, c_z]$ is not in $X_S(G)$. It follows that $|\mathrm{Ker}(\sigma)| = 2$. □

By Lemma 7.7, $\mathrm{Out}(G)$ is a finite abelian group. We let $\mathrm{Out}(G)_{(2)}$ denote its Sylow 2-subgroup We are now ready to determine the group $\mathrm{Out}(BG^\wedge_2)$ and compare it to $\mathrm{Out}(G)_{(2)}$.

Proposition 7.9. *Fix* $q = p^e$, *where* p *is an odd prime and* $e \geq 1$, *and set* $G = PSL_2(q)$. *Then the natural homomorphism*

$$B : \mathrm{Out}(G)_{(2)} \longrightarrow \mathrm{Out}(BG^\wedge_2) \cong C_2 \times C_{2^{k-2}}$$

is always injective, and is an isomorphism if and only if $p \equiv \pm 3$ *(mod 8).*

Proof. We first fix Sylow 2-subgroups $S \le G = PSL_2(q)$ and $\widehat{S} \le PGL_2(q)$, such that $S \le \widehat{S}$ has index 2, as follows. If $q \equiv 1 \pmod 4$, then let S and $\widehat{S}$ be the subgroups generated by diagonal matrices, together with the matrix $\beta \overset{\text{def}}{=} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. If $q \equiv 3 \pmod 4$, then identify $(\mathbb{F}_q)^2$ with $\mathbb{F}_{q^2}$, regarded as a 2-dimensional $\mathbb{F}_q$-vector space, and set $H = \langle (\mathbb{F}_{q^2})^*, \beta \rangle$, where $\beta \in \mathrm{Aut}(\mathbb{F}_{q^2})$ is the automorphism of order 2. Let $\widehat{H}$ be the image of H in $PGL_2(q)$, let $\widehat{S}$ be the Sylow 2-subgroup of $\widehat{H}$ which contains β, and set $S = \widehat{S} \cap PSL_2(q)$.

In either case, $\widehat{S} \cong D_{2^{k+1}}$ and $S \cong D_{2^k}$ are both dihedral groups. Fix $\gamma \in \widehat{S}$ which generates the cyclic subgroup of index 2. By Lemma 7.7, conjugation by γ generates the first factor in $\mathrm{Out}(G) \cong C_2 \times \mathrm{Aut}(\mathbb{F}_q)$.

We must compare the following maps,

$$\begin{array}{ccc} \mathrm{Out}(G)_{(2)} & & \\ {\scriptstyle B}\downarrow & \overset{\rho}{\searrow} & \\ \mathrm{Out}(BG^\wedge_2) & \xrightarrow{\ \sigma\ } & \mathrm{Out}(S), \end{array} \qquad (1)$$

where σ is surjective by Lemma 7.8.

Case 1: Assume first that $q \not\equiv 1 \pmod 8$. Then e is odd (q is not a square), and so $\mathrm{Out}(G)_{(2)} = \langle c_\gamma \rangle$ has order 2. Also, $c_\gamma \notin \mathrm{Ker}(\rho)$ ($(c_\gamma)|_S \notin \mathrm{Inn}(S)$) by the above description of $\gamma \in \widehat{S}$. So B is injective since ρ is, and B is an isomorphism if and only if $|\,\mathrm{Out}(BG^\wedge_2)| = 2$, if and only if $k = 2$ ($q \equiv \pm 3 \pmod 8$) by Lemma 7.8.

Case 2: Now assume that $q \equiv 1 \pmod 8$. Set $\alpha = \gamma^2 = [\mathrm{diag}(v, v^{-1})]$, where $v \in (\mathbb{F}_q)^*$ has order 2^k. Then $S = \langle \alpha, \beta \rangle \cong D_{2^k}$, and conjugation by γ sends α to α and β to $\alpha\beta$. So

$$\mathrm{Out}(S) = \langle \rho(c_\gamma) \rangle \times (\mathrm{Aut}(C_{2^{k-1}})/\{\pm 1\}),$$

where the second factor operates on $\langle \alpha \rangle$ while fixing β. If q is not a square, then $|\,\mathrm{Aut}(\mathbb{F}_q)|$ is odd, $\mathrm{Out}(G)_{(2)}$ is generated by conjugation by γ, and so ρ (and hence B) are injective.

Assume now that $q = (q_1)^2$ is a square. Since $q \equiv 2^k + 1 \pmod{2^{k+1}}$, we have $q_1 \equiv 2^{k-1} \pm 1 \pmod{2^k}$. Let $\varphi_1 \in \mathrm{Aut}(\mathbb{F}_q) \le \mathrm{Aut}(G)$ denote the automorphism of order 2: $\varphi_1(a) = a^{q_1}$ for $a \in \mathbb{F}_q$. Since v has order 2^k in $(\mathbb{F}_q)^*$, this shows that $\varphi_1(v) = -v$ or $-v^{-1}$. In particular, $\varphi_1(\alpha) = \alpha$ or α^{-1} (where $\alpha = \mathrm{diag}(v, v^{-1})$), so $\varphi_1|_S \in \mathrm{Inn}(S)$, and $\varphi_1 \in \mathrm{Ker}(\rho)$. Furthermore, $\mathrm{Ker}(\rho)$ is generated by φ_1, since any automorphism $\varphi_2 \in \mathrm{Aut}(\mathbb{F}_q)$ of order 4 is of the form $(a \mapsto a^{q_2})$ where $q_2 \equiv 2^{k-2} \pm 1 \pmod{2^{k-1}}$.

Let $G_1 \le G = PSL_2(q)$ be the subgroup of elements fixed by φ_1; i.e., the group of classes (mod $\pm I$) of matrices $A \in SL_2(q)$ such that $\varphi_1(A) = \pm A$. This subgroup clearly contains $PSL_2(q_1)$, and $[G_1{:}PSL_2(q_1)] = 2$. From the formulas

$$|PSL_2(q)| = \tfrac{1}{2}q(q^2-1) = \tfrac{1}{2}q_1^2(q_1^2-1)(q_1^2+1) \quad \text{and} \quad |PSL_2(q_1)| = \tfrac{1}{2}q_1(q_1^2-1),$$

we see that G_1 has odd index in G, and hence that we can assume $S \le G_1$. Also, this shows that $S \cap PSL_2(q_1)$ is a dihedral subgroup of index 2 in S, and hence that it contains the subgroups in just one of the conjugacy classes in $\mathcal{T}$. Assume they are labelled so that $T_1 \le PSL_2(q_1)$ and $T_2 \not\le PSL_2(q_1)$.

In particular, $N_G(T_2) \cong \Sigma_4$ is not contained in G_1, since it has no subgroup of index 2 containd in $PSL_2(q_1)$. We claim that $N_G(T_1) \le G_1$. If $k \ge 4$ ($|S| \ge 16$), then $N_G(T_1)$ is contained in $PSL_2(q_1)$ by Lemma 7.1, and hence is contained in G_1. If $k = 3$ ($|S| = 8$), then $N_{PSL_2(q_1)}(T_1) \cong A_4$ by Lemma 7.1 again, and $T_1 \in \mathrm{Syl}_2(PSL_2(q_1))$. For any $g \in G_1 \smallsetminus PSL_2(q_1)$, gT_1g^{-1} is another Sylow subgroup, so there is $a \in PSL_2(q_1)$ such that $ag \in N_{G_1}(T_1)$, and thus $N_{G_1}(T_1) = N_G(T_1)$.

Since G_1 is by definition the fixed subgroup of φ_1, it now follows that φ_1 is the identity on S and on $N_G(T_1)$, but not on $N_G(T_2)$. So by Lemma 7.8(b), $B\varphi_1$ is the nontrivial element in $\mathrm{Ker}(\sigma)$. We have already seen that the class of φ_1 generates $\mathrm{Ker}(\rho)$, and hence this finishes the proof that B is injective.

To see when B is an isomorphism, note first that

$$|\,\mathrm{Out}(BG^\wedge_2)| = 2{\cdot}|\,\mathrm{Out}(S)| = 2^{k-1} \qquad \text{and} \qquad |\,\mathrm{Out}(G)_{(2)}| = 2^{m+1} \quad (m = \nu_2(e))$$

by Lemmas 7.7 and 7.8. So B is an isomorphism if and only if p has order $2^{k-1} = 2^{m+1}$ in $(\mathbb{Z}/2^{k+1})^*$; and this is the case if and only if $p \equiv \pm 3 \pmod 8$. In particular, since $BG^\wedge_2 \simeq BPSL_2(5^{2^{k-2}})^\wedge_2$ by Proposition 7.4,

$$\mathrm{Out}(BG^\wedge_2) \cong \mathrm{Out}(BPSL_2(5^{2^{k-2}})^\wedge_2) \cong \mathrm{Out}(PSL_2(5^{2^{k-2}})) \cong C_2 \times C_{2^{k-2}}. \qquad \square$$

Appendix A. $O^p(G)$ and p-centric subgroups

For any finite group G, $O^p(G)$ is defined to be the smallest normal subgroup of p-power index in G. Equivalently, $O^p(G)$ is the largest p-perfect subgroup of G, where a group K is p-perfect if it is generated by commutators and p-th powers, or equivalently if $H_1(K;\mathbb{F}_p) = 0$. We first note that this subgroup is natural with respect to group homomorphisms.

Lemma A.1. *For any prime p and any homomorphism $\varphi\colon G \longrightarrow H$ of finite groups, $f(O^p(G)) \le O^p(H)$.*

Proof. Since $O^p(G)$ is p-perfect and $H/O^p(H)$ is a p-group, the composite homomorphism from $O^p(G)$ to $H/O^p(H)$ is trivial. $\square$

One way in which $O^p(-)$ occurs in this paper is in the formula for the fundamental group of a p-completed space.

Proposition A.2. *For any connected complex X such that $\pi_1(X)$ is finite,*

$$\pi_1(X^\wedge_p) \cong \pi_1(X)/O^p(\pi_1(X)).$$

Proof. This is well known, but doesn't seem to be explicitly stated in [BK]. Set $G = \pi_1(X)$ and $\pi = G/O^p(G)$ for short. Let $\widehat{X}$ be the covering space of X with fundamental group $O^p(G)$. Since $\pi_1(\widehat{X})$ is p-perfect, $\widehat{X}^\wedge_p$ is simply connected by [BK, VII.3.2]. Also, $B\pi$ is p-complete since π is a p-group. Hence the sequence

$$\widehat{X}^\wedge_p \longrightarrow X^\wedge_p \longrightarrow B\pi$$

is a fibration sequence by [BK, II.5.2(iv)], and so $\pi_1(X^\wedge_p) \cong \pi$. □

The other way in which the subgroups $O^p(-)$ occur in this paper is in the definition of the linking categories $\mathcal{L}_p(G)$, and when working with p-centric subgroups of a finite group.

Definition A.3. *A p-subgroup P of a finite group G is* p-centric *if $Z(P)$ is a Sylow p-subgroup of $C_G(P)$.*

The following lemma is important when defining centric linking categories.

Lemma A.4. *Fix a finite group G and a p-subgroup $P \le G$. Then P is p-centric in G if and only if $C_G(P) \cong Z(P) \times O^p(C_G(P))$ and $O^p(C_G(P))$ has order prime to p.*

Proof. Assume P is p-centric in G. Then $Z(P)$ is a normal Sylow p-subgroup of $C_G(P)$, and so $C_G(P)/Z(P)$ has order prime to p. Thus $H^2(C_G(P)/Z(P); Z(P)) = 0$, and so $C_G(P)$ splits accordingly as a semidirect product. Furthermore, $Z(P)$ is central in $C_G(P)$, and hence must be a direct factor. So $O^p(C_G(P))$ has trivial intersection with $Z(P)$, hence has order prime to p, and $C_G(P) \cong Z(P) \times O^p(C_G(P))$.

The converse is clear. □

When P is p-centric in G, we write $C'_G(P) \stackrel{\text{def}}{=} O^p(C_G(P))$ for short; thus $C_G(P) \cong Z(P) \times C'_G(P)$ in this case.

We will also need the following characterization of p-centric subgroups.

Lemma A.5. *Let G be a finite group, and fix $S \in \mathrm{Syl}_p(G)$. Then a p-subgroup $P \le S$ is p-centric in G if and only if for all $Q \le S$ which is G-conjugate to P, $C_S(Q) \le Q$.*

Proof. If $P \le S$ is p-centric in G, then so is each $Q \le S$ which is G-conjugate to P. Hence $Z(Q)$ is a normal Sylow p-subgroup of $C_G(Q)$, and hence contains all p-subgroups of $C_G(Q)$. In particular, $C_S(Q) \le Z(Q) \le Q$.

If P is not p-centric in G, then choose $x \in G$ such that $x^{-1}Sx$ contains a Sylow p-subgroup of $P{\cdot}C_G(P)$. Set $Q = xPx^{-1}$. Then $Q \le S$, and $C_S(Q)$ is a Sylow p-subgroup of $C_G(Q)$. Since P is not p-centric in G, neither is Q, and so $C_S(Q) \not\le Q$. □

Lemma A.5 makes it clear that the Sylow p-subgroups of a finite group G are all p-centric in G, and that if P is p-centric in G then so are all other p-subgroups which contain P.

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