

A GEOMETRIC CONSTRUCTION OF SATURATED FUSION SYSTEMS

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ABSTRACT. A saturated fusion system consists of a finite p -group S , together with a category which encodes “conjugacy” relations among subgroups of S , and which satisfies certain axioms which are motivated by properties of the fusion in a Sylow p -subgroup of a finite group. We describe here new ways of constructing abstract saturated fusion systems, first as fusion systems of spaces with certain properties, and then via certain graphs.

A *saturated fusion system* consists of a finite p -group S , together with a category \mathcal{F} whose objects are the subgroups of S , whose morphisms are group monomorphisms between those subgroups, and which satisfies certain axioms modelled on the fusion category for the p -subgroups of a finite group. The precise definition of a saturated fusion system is due to Puig [Pu], and our version of that definition is given in Section 1. Saturated fusion systems mimic in several ways the structure of finite groups and their classifying spaces. Examples have been known for some time of “exotic” saturated fusion systems — systems which do not arise from the fusion in any finite group — but the construction of such examples is very complicated, and we are looking for simpler and more systematic ways to construct them. One consequence of the main result in this paper is a way of constructing a variety of examples of saturated fusion systems. Of the examples constructed using this technique, some are then shown by other means to be exotic.

The definition of a fusion system over a p -group S is simple, and in most cases it is clear whether or not a given category satisfies it. In contrast, it is much harder to check whether a given fusion system is saturated. For example, for any map $f: BS \longrightarrow X$, where S is a finite p -group and X is a topological space, the fusion system of X over (S, f) is a category $\mathcal{F}_{S,f}(X)$ whose objects are the subgroups of S , and where $\text{Mor}_{\mathcal{F}_{S,f}(X)}(P, Q)$ is the set of all monomorphisms $\varphi \in \text{Hom}(P, Q)$ such that $(f|_{BP}) \circ B\varphi \simeq (f|_{BQ})$. This is always a fusion system in the sense of Definition 1.1, but is not in general saturated.

The central result in this paper is Theorem 2.1, where we list some conditions on the map f which ensure that the fusion system $\mathcal{F}_{S,f}(X)$ is saturated. These conditions also ensure that $\mathcal{F}_{S,f}(X)$ has an associated *linking system* (see Definition 1.3), and hence that X , S , and f define a p -local finite group. Afterwards, we construct more concrete examples using that theorem, and show in many cases that they are “exotic” in the sense of not coming from any finite group. For example, in Theorem 4.2, in certain cases when G is an amalgamated free product of finite groups, we apply our theorem to BG_p^\wedge to show that the fusion system of G (taken over a maximal p -subgroup of G) is

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saturated. This result, which is stated in terms of trees of groups, was discovered and first proved as a special case of Theorem 2.1, but we also include a more elementary, purely graph theoretic proof here.

The paper is organized as follows. In Section 1, we give the definitions of abstract fusion and linking systems, as well as definitions of fusion and linking systems of groups and spaces and some background results about them. Our main theorem is proven in Section 2. In Section 3, we describe conditions under which the main theorem can be applied to the space BG_p^\wedge for an infinite discrete group G , to prove that the fusion system of G with respect to some finite p -subgroup is saturated (Theorem 3.3). A special case of this is then studied in Section 4 — the case where G acts on a tree with finite isotropy subgroups — and this in turn is applied in Section 5 to construct concrete examples of fusion systems, some of which are then shown to be “exotic”. We hope to find other applications of our main Theorem 2.1 in the future which allow us to construct a still wider variety of examples.

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1. A SURVEY OF FUSION SYSTEMS

We first recall some definitions, mostly from [BLO2].

Definition 1.1 ([Pu] and [BLO2, Definition 1.1]). *A fusion system over a finite p -group S is a category \mathcal{F} , where $\text{Ob}(\mathcal{F})$ is the set of all subgroups of S , and which satisfies the following two properties for all $P, Q \leq S$:*

- $\text{Hom}_S(P, Q) \subseteq \text{Hom}_{\mathcal{F}}(P, Q) \subseteq \text{Inj}(P, Q)$; and
- each $\varphi \in \text{Hom}_{\mathcal{F}}(P, Q)$ is the composite of an isomorphism in \mathcal{F} followed by an inclusion.

Fusion systems as defined above are too general for our purposes, and some additional definitions and conditions are needed so that they more closely model the fusion in finite groups. If \mathcal{F} is a fusion system over a finite p -subgroup S , then two subgroups $P, Q \leq S$ are said to be \mathcal{F} -conjugate if they are isomorphic as objects of the category \mathcal{F} .

Definition 1.2 ([Pu], see [BLO2, Definition 1.2]). *Let \mathcal{F} be a fusion system over a p -group S .*

- A subgroup $P \leq S$ is fully centralized in \mathcal{F} if $|C_S(P)| \geq |C_S(P')|$ for all $P' \leq S$ which is \mathcal{F} -conjugate to P .
- A subgroup $P \leq S$ is fully normalized in \mathcal{F} if $|N_S(P)| \geq |N_S(P')|$ for all $P' \leq S$ which is \mathcal{F} -conjugate to P .
- \mathcal{F} is a saturated fusion system if the following two conditions hold:
 - (I) For all $P \leq S$ which is fully normalized in \mathcal{F} , P is fully centralized in \mathcal{F} and $\text{Aut}_S(P) \in \text{Syl}_p(\text{Aut}_{\mathcal{F}}(P))$.
 - (II) If $P \leq S$ and $\varphi \in \text{Hom}_{\mathcal{F}}(P, S)$ are such that φP is fully centralized, and if we set

$$N_\varphi = \{g \in N_S(P) \mid \varphi c_g \varphi^{-1} \in \text{Aut}_S(\varphi P)\},$$

then there is $\bar{\varphi} \in \text{Hom}_{\mathcal{F}}(N_\varphi, S)$ such that $\bar{\varphi}|_P = \varphi$.

If G is a finite group and $S \in \text{Syl}_p(G)$, then the category $\mathcal{F}_S(G)$ defined in the introduction is a saturated fusion system (see [BLO2, Proposition 1.3]).

An alternative, simplified pair of axioms for a fusion system being saturated has been given by Radu Stancu [St].

We now turn to centric linking systems associated to abstract fusion systems. Whenever \mathcal{F} is a fusion system over a finite p -group S , a subgroup $P \leq S$ is called \mathcal{F} -centric if $C_S(P') = Z(P')$ for all $P' \leq S$ which are \mathcal{F} -conjugate to P . We let $\mathcal{F}^c \subseteq \mathcal{F}$ denote the full subcategory whose objects are the \mathcal{F} -centric subgroups of S . If $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group G , then $P \leq S$ is \mathcal{F} -centric if and only if P is p -centric in G ; i.e., if and only if $Z(P) \in \text{Syl}_p(C_G(P))$.

Definition 1.3 ([BLO2, Definition 1.7]). *Let \mathcal{F} be a fusion system over the p -group S . A centric linking system associated to \mathcal{F} is a category \mathcal{L} whose objects are the \mathcal{F} -centric subgroups of S , together with a functor $\pi: \mathcal{L} \longrightarrow \mathcal{F}^c$, and “distinguished” monomorphisms $P \xrightarrow{\delta_P} \text{Aut}_{\mathcal{L}}(P)$ for each \mathcal{F} -centric subgroup $P \leq S$, which satisfy the following conditions.*

(A) *π is the identity on objects. For each pair of objects $P, Q \in \text{Ob}(\mathcal{L})$, $Z(P)$ acts freely on $\text{Mor}_{\mathcal{L}}(P, Q)$ by composition (upon identifying $Z(P)$ with $\delta_P(Z(P)) \leq \text{Aut}_{\mathcal{L}}(P)$), and π induces a bijection*

$$\text{Mor}_{\mathcal{L}}(P, Q)/Z(P) \xrightarrow{\cong} \text{Hom}_{\mathcal{F}}(P, Q).$$

(B) *For each \mathcal{F} -centric subgroup $P \leq S$ and each $x \in P$, $\pi(\delta_P(x)) = c_x \in \text{Aut}_{\mathcal{F}}(P)$.*

(C) *For each $f \in \text{Mor}_{\mathcal{L}}(P, Q)$ and each $x \in P$, $f \circ \delta_P(x) = \delta_Q(\pi f(x)) \circ f$.*

A p -local finite group is defined to be a triple $(S, \mathcal{F}, \mathcal{L})$, where S is a finite p -group, \mathcal{F} is a saturated fusion system over S , and \mathcal{L} is a centric linking system associated to \mathcal{F} . The *classifying space* of the triple $(S, \mathcal{F}, \mathcal{L})$ is the p -completed nerve $|\mathcal{L}|_p^\wedge$.

In the following definition, recall that a (possibly infinite) group G is p -perfect if it has no normal subgroup of index p ; or equivalently, if $\text{Hom}(G, \mathbb{Z}/p)$ contains only the trivial homomorphism. Clearly, if G is generated by p -perfect subgroups, then it is itself p -perfect. Hence any group G contains a maximal p -perfect subgroup, which is normal.

Definition 1.4. *Fix any pair $S \leq G$, where G is a (possibly infinite) group and S is a finite p -subgroup.*

(a) *Define $\mathcal{F}_S(G)$ to be the category whose objects are the subgroups of S , and where*

$$\begin{aligned} \text{Mor}_{\mathcal{F}_S(G)}(P, Q) &= \text{Hom}_G(P, Q) \stackrel{\text{def}}{=} \{c_g \in \text{Hom}(P, Q) \mid g \in G, gPg^{-1} \leq Q\} \\ &\cong N_G(P, Q)/C_G(P). \end{aligned}$$

Here c_g denotes the homomorphism conjugation by g ($x \mapsto gxg^{-1}$), and $N_G(P, Q) = \{g \in G \mid gPg^{-1} \leq Q\}$ (the transporter set).

(b) *For each $P \leq S$, let $C'_G(P)$ be the maximal p -perfect subgroup of $C_G(P)$. Let $\mathcal{L}_S^c(G)$ be the category whose objects are the $\mathcal{F}_S(G)$ -centric subgroups of S , and where*

$$\text{Mor}_{\mathcal{L}_S^c(G)}(P, Q) = N_G(P, Q)/C'_G(P).$$

Let $\pi: \mathcal{L}_S^c(G) \longrightarrow \mathcal{F}_S(G)$ be the functor which is the inclusion on objects and sends the class of $g \in N_G(P, Q)$ to conjugation by g . For each $\mathcal{F}_S(G)$ -centric subgroup

$P \leq G$, let $\delta_P: P \longrightarrow \text{Aut}_{\mathcal{L}_S^c(G)}(P)$ be the monomorphism induced by the inclusion $P \leq N_G(P)$.

It is clear from the definitions that $\mathcal{F}_S(G)$ is a fusion system for any S and G , and just as clear that it is not always saturated. When G is finite and $S \in \text{Syl}_p(G)$, then $\mathcal{F}_S(G)$ is always saturated (see [Pu], or [BLO2, Proposition 1.3]), and $\mathcal{L}_S^c(G)$ is a centric linking system associated to $\mathcal{F}_S(G)$. Thus in this case, $(S, \mathcal{F}_S(G), \mathcal{L}_S^c(G))$ is a p -local finite group, with classifying space $|\mathcal{L}_S^c(G)|_p^\wedge \simeq BG_p^\wedge$ (see [BLO1, Proposition 1.1]).

When G is infinite, we note the following condition for $\mathcal{L}_S^c(G)$ to be a centric linking system.

Lemma 1.5. *Fix any pair $S \leq G$, where G is a (possibly infinite) group and S is a finite p -subgroup, and set $\mathcal{F} = \mathcal{F}_S(G)$. Assume, for each \mathcal{F} -centric subgroup $P \leq S$, that $H^i(C_G(P)/Z(P); \mathbb{F}_p) = 0$ for $i = 1, 2$. Then $\mathcal{L}_S^c(G)$ is a centric linking system associated to \mathcal{F} .*

Proof. Conditions (B) and (C) in Definition 1.3 hold by definition of $\mathcal{L}_S^c(G)$, the projection functor π , and the distinguished monomorphisms δ_P . Also, for each pair of objects P, Q , $C_G(P)$ acts freely on $N_G(P, Q)$ by right multiplication, so $C_G(P)/C'_G(P)$ acts freely on $\text{Mor}_{\mathcal{L}_S^c(G)}(P, Q)$ with orbit set $\text{Hom}_{\mathcal{F}}(P, Q)$. So to prove that $\mathcal{L}_S^c(G)$ is a centric linking system associated to \mathcal{F} , it remains only to show that for each \mathcal{F} -centric subgroup $P \leq S$, the inclusion $Z(P) \leq C_G(P)$ induces an isomorphism $Z(P) \cong C_G(P)/C'_G(P)$.

The assumption $H^1(C_G(P)/Z(P); \mathbb{F}_p) = 0$ implies that $C_G(P)/Z(P)$ is p -perfect. Since $H^2(C_G(P)/Z(P); \mathbb{F}_p) = 0$ and $Z(P)$ is a finite p -group, the exact sequences in group cohomology for extensions of modules show that $H^2(C_G(P)/Z(P); Z(P)) = 0$, and hence that $C_G(P)$ splits as a product $Z(P) \times H$ for a normal subgroup $H \triangleleft C_G(P)$. Thus $H \cong C_G(P)/Z(P)$ is the maximal p -perfect subgroup of $C_G(P)$, and so $C_G(P)/C'_G(P) = C_G(P)/H \cong Z(P)$. \square

Fusion systems and linking systems can also be defined for spaces. In the following definition, if $H: X \times I \longrightarrow Y$ is a homotopy (where $I = [0, 1]$), then $[H]$ denotes its homotopy class among maps $X \times I \longrightarrow Y$ whose restriction to $X \times \{0, 1\}$ is the same as that of H . In other words, if we regard H as a path in $\text{Map}(X, Y)$ by adjunction, then $[H]$ denotes the homotopy class of that path rel endpoints.

For any p -group P and any $g \in P$, let $H_g: BP \times I \longrightarrow BP$ be the homotopy from Id_{BP} to Bc_g induced by the natural transformation of functors $\mathcal{B}(G) \longrightarrow \mathcal{B}(G)$ which sends the unique object o_G in $\mathcal{B}(G)$ to the morphism \check{g} corresponding to $g \in G$.

Definition 1.6. *Fix a space X , a finite p -group S , and a map $f: BS \longrightarrow X$.*

(a) *Define $\mathcal{F}_{S,f}(X)$ to be the category whose objects are the subgroups of S , and whose morphisms are given by*

$$\text{Hom}_{\mathcal{F}_{S,f}(X)}(P, Q) = \{\varphi \in \text{Inj}(P, Q) \mid f|_{BP} \simeq f|_{BQ} \circ B\varphi\}$$

for each $P, Q \leq S$.

(b) *Define $\mathcal{F}'_{S,f}(X) \subseteq \mathcal{F}_{S,f}(X)$ to be the subcategory with the same objects as $\mathcal{F}_{S,f}(X)$, and where $\text{Mor}_{\mathcal{F}'_{S,f}(X)}(P, Q)$ (for $P, Q \leq S$) is the set of all composites of restrictions of morphisms in $\mathcal{F}_{S,f}(X)$ between $\mathcal{F}_{S,f}(X)$ -centric subgroups.*

(c) Define $\mathcal{L}_{S,f}^c(X)$ to be the category whose objects are the $\mathcal{F}_{S,f}(X)$ -centric subgroups of S , and whose morphisms are defined by

$$\text{Mor}_{\mathcal{L}_{S,f}^c(X)}(P, Q) = \{(\varphi, [H]) \mid \varphi \in \text{Inj}(P, Q), H: BP \times I \longrightarrow X, \\ H|_{BP \times 0} = f|_{BP}, H|_{BP \times 1} = f|_{BQ \circ B\varphi}\}.$$

The composite in $\mathcal{L}_{S,f}^c(X)$ of morphisms

$$P \xrightarrow{(\varphi, [H])} Q \xrightarrow{(\psi, [K])} R,$$

where $H: BP \times I \rightarrow X$ and $K: BQ \times I \rightarrow X$ are homotopies as described above, are defined by setting

$$(\psi, [K]) \circ (\varphi, [H]) = (\psi \circ \varphi, [(K \circ (B\varphi \times \text{Id})) \cdot H]),$$

where \cdot denotes composition (juxtaposition) of homotopies. Let

$$\pi: \mathcal{L}_{S,f}^c(X) \longrightarrow \mathcal{F}_{S,f}(X)$$

be the forgetful functor: it is the inclusion on objects, and sends a morphism $(\varphi, [H])$ to φ . For each $\mathcal{F}_{S,f}(X)$ -centric subgroup $P \leq S$, let

$$\delta_P: P \longrightarrow \text{Aut}_{\mathcal{L}_{S,f}^c(X)}(P)$$

be the “distinguished homomorphism” which sends $g \in P$ to $(c_g, [f|_{BP \circ H_g}])$.

Equivalently, via adjunction, a morphism from P to Q in $\mathcal{L}_{S,f}^c(X)$ can be thought of as a pair $(\varphi, [H])$, where $\varphi \in \text{Hom}(P, Q)$, H is a path in the mapping space $\text{Map}(BP, X)$ from $f|_{BP}$ to $f|_{BQ \circ B\varphi}$, and $[H]$ is the homotopy class of the path H rel endpoints.

The categories $\mathcal{F}'_{S,f}(X) \subseteq \mathcal{F}_{S,f}(X)$ are always fusion systems over S , but are not in general saturated. However, in certain situations we consider, $\mathcal{F}'_{S,f}(X)$ will be a saturated fusion system, even though $\mathcal{F}_{S,f}(X)$ might not be (see Example 3.4).

Theorem A of [5a1] says that if all morphisms in a fusion system are obtained as composites of restrictions of morphisms between centric subgroups, then it is saturated if the saturation conditions (I) and (II) hold on centric subgroups. Thus it makes sense, for a general abstract fusion system \mathcal{F} , to define the subsystem $\mathcal{F}' \subseteq \mathcal{F}$ over the same p -group S to be the subcategory with the same objects, but with only those morphisms which are obtained as composites of restrictions of morphisms in \mathcal{F} between \mathcal{F} -centric subgroups. One particularly well behaved situation is that in which \mathcal{F}' has no more centric subgroups than those already centric in \mathcal{F} . In this case, it clearly follows that the full subcategories of centric objects in \mathcal{F} and in \mathcal{F}' are equal, and hence that one can check conditions (I) and (II) in either subcategory.

These arguments are collected in the following proposition.

Proposition 1.7. *Fix a space X , a finite p -group S , and a map $f: BS \longrightarrow X$. If all $\mathcal{F}_{S,f}(X)$ -centric subgroups $P \leq S$ satisfy conditions (I) and (II) in Definition 1.2, and if all $\mathcal{F}'_{S,f}(X)$ -centric subgroups of S are $\mathcal{F}_{S,f}(X)$ -centric, then $\mathcal{F}'_{S,f}(X)$ is a saturated fusion system.*

In the situation of Proposition 1.7, there could possibly be a $\mathcal{F}'_{S,f}(X)$ -centric subgroup $P \leq S$ which is not $\mathcal{F}_{S,f}(X)$ -centric, because it is $\mathcal{F}_{S,f}(X)$ -conjugate to a subgroup which is not centric in S . When this is the case, [5a1, Theorem 2.2] cannot be applied to prove the above proposition, since we’ve changed the set of centric subgroups in question. This is why we need to assume that the two fusion systems have the same centric subgroups.

Our main theorem will give some conditions on a map $BS \xrightarrow{f} X$ which ensure that a triple $(S, \mathcal{F}'_{S,f}(X), \mathcal{L}^c_{S,f}(X))$ is a p -local finite group. More generally, however, without any extra hypotheses, the category $\mathcal{L}^c_{S,f}(X)$ does satisfy most of the axioms for being a centric linking system associated to $\mathcal{F}_{S,f}(X)$.

In the following lemma, for any $f: BS \longrightarrow X$ as above, and any $P \leq S$, we let

$$\omega_P: BZ(P) \longrightarrow \text{Map}(BP, X)_{f|_{BP}}$$

be the map which is adjoint to the composite

$$BZ(P) \times BP \xrightarrow{B\mu} BS \xrightarrow{f} X,$$

where $\mu: Z(P) \times P \longrightarrow S$ is multiplication.

Lemma 1.8. *Fix a space X , a finite p -group S , and a map $f: BS \longrightarrow X$. Then the category $\mathcal{L}^c_{S,f}(X)$, together with the functor*

$$\pi: \mathcal{L}^c_{S,f}(X) \longrightarrow \mathcal{F}_{S,f}(X)$$

and the distinguished homomorphisms $\delta_P: P \longrightarrow \text{Aut}_{\mathcal{L}^c_{S,f}(X)}(P)$, satisfy axioms (B) and (C) in Definition 1.3. If in addition,

$$Z(P) \xrightarrow{\pi_1(\omega_P)} \pi_1(\text{Map}(BP, X), f|_{BP}) \text{ is an isomorph. } \forall \mathcal{F}_{S,f}(X)\text{-centric } P \leq S, \quad (*)$$

then $\mathcal{L}^c_{S,f}(X)$ is a linking system associated to $\mathcal{F}_{S,f}(X)$.

Proof. Proving this means essentially repeating the proof of [BLO2, Theorem 7.5]. Set $\mathcal{F} = \mathcal{F}_{S,f}(X)$ and $\mathcal{L} = \mathcal{L}^c_{S,f}(X)$ for short. Condition (B) in Definition 1.3 clearly holds.

For $g \in P \leq S$, set $\widehat{H}_g = f|_{BP} \circ H_g$: a homotopy $BP \times I \rightarrow X$ from $f|_{BP}$ to $f|_{BP} \circ Bc_g$. Thus the distinguished homomorphism $P \xrightarrow{\delta_P} \text{Aut}_{\mathcal{L}}(P)$ is defined by sending $g \in P$ to $(c_g, [\widehat{H}_g])$.

Condition (C) means showing, for each $(\varphi, [H]) \in \text{Mor}_{\mathcal{L}}(P, Q)$ and each $g \in P$, that the following square commutes:

$$\begin{array}{ccc} P & \xrightarrow{(\varphi, [H])} & Q \\ (c_g, [\widehat{H}_g]) \downarrow & & \downarrow (c_{\varphi(g)}, [\widehat{H}_{\varphi(g)}]) \\ P & \xrightarrow{(\varphi, [H])} & Q. \end{array}$$

Here, $H: BP \times I \longrightarrow X$ is a homotopy from $f|_{BP}$ to $f|_{BQ} \circ B\varphi$. Clearly, $\varphi \circ c_g = c_{\varphi(g)} \circ \varphi$. It remains to check that the two juxtaposed homotopies described in the following diagram are homotopic among homotopies from $f|_{BP}$ to $f|_{BQ} \circ B(\varphi \circ c_g)$:

$$\begin{array}{ccc} f|_{BP} & \xrightarrow{H} & f|_{BQ} \circ B\varphi \\ \widehat{H}_g \downarrow & & \downarrow \widehat{H}_{\varphi(g)} \circ (B\varphi \times \text{Id}) \\ f|_{BP} \circ Bc_g & \xrightarrow{H \circ (Bc_g \times \text{Id})} & f|_{BQ} \circ B(\varphi \circ c_g). \end{array}$$

The map

$$F: BP \times I \times I \longrightarrow X \quad \text{defined by} \quad F(x, s, t) = H(H_g(x, t), s)$$

defines a homotopy between them, since

$$\begin{aligned} F(x, s, 0) &= H(x, s), \\ F(x, 1, t) &= (f|_{BQ} \circ B\varphi \circ H_g)(x, t) = \widehat{H}_{\varphi(g)}(B\varphi(x), t), \\ F(x, 0, t) &= f|_{BP} \circ H_g(x, t) = \widehat{H}_g(x, t), \quad \text{and} \\ F(x, s, 1) &= H(BC_g(x), s). \end{aligned}$$

It remains to prove (A) while assuming that $(*)$ holds. For any \mathcal{F} -centric subgroup $P \leq S$, we identify $\pi_1(\text{Map}(BP, X), f|_{BP})$ as a subgroup of $\text{Aut}_{\mathcal{L}}(P)$: the subgroup of elements of the form $(\text{Id}, [H])$ when H is a homotopy from $f|_{BP}$ to itself. Under this identification, δ_P restricts to the homomorphism from $Z(P)$ to $\pi_1(\text{Map}(BP, X), f|_{BP})$ which sends $g \in Z(P)$ to $[\widehat{H}_g]$, where \widehat{H}_g is now regarded as a loop in $\text{Map}(BP, X)$. By definition of \mathcal{F} and \mathcal{L} , for any other \mathcal{F} -centric subgroup $Q \leq S$, $\pi_1(\text{Map}(BP, X), f|_{BP})$ acts freely on $\text{Mor}_{\mathcal{L}}(P, Q)$ with orbit set $\text{Hom}_{\mathcal{F}}(P, Q)$. So to prove (A), we must show that the isomorphism $\pi_1(\omega_P)$ of $(*)$ sends $g \in Z(P)$ to $[\widehat{H}_g]$.

Let $[1]$ be the category with two objects $0, 1$, and one nonidentity morphism $0 \rightarrow 1$. Fix $g \in Z(P)$, and consider the composite functor

$$\Psi: \mathcal{B}(P) \times [1] \xrightarrow{\text{Id} \times \psi} \mathcal{B}(P) \times \mathcal{B}(Z(P)) \xrightarrow{\mathcal{B}(\mu)} \mathcal{B}(P),$$

where $\psi: [1] \rightarrow \mathcal{B}(Z(P))$ sends $0 \rightarrow 1$ to the morphism g , and where $\mathcal{B}(\mu)$ is induced by multiplication. Then

$$|\Psi|: BP \times I \longrightarrow BP$$

is induced by the natural homomorphism of functors from $\text{Id}_{\mathcal{B}(P)}$ to itself defined by sending the object o_P to the morphism corresponding to g , and is thus the homotopy H_g of Definition 1.6. By definition, $\pi_1(\omega_P)(g)$ is the homotopy class of $f \circ |\Psi|$, when regarded as a loop in $\text{Map}(BP, X)$, and is thus equal to $[f \circ H_g] = [\widehat{H}_g]$. This finishes the proof of (A), and hence of the lemma. \square

We will refer several times to the following classical result.

Proposition 1.9. *For any pair of discrete groups H and G , the natural map*

$$\text{Rep}(H, G) \stackrel{\text{def}}{=} \text{Hom}(H, G) / \text{Inn}(G) \longrightarrow [BH, BG]$$

is a bijection. For each $\rho \in \text{Hom}(H, G)$, the homomorphism $BC_G(\rho(H)) \times H \longrightarrow G$ is adjoint to a homotopy equivalence

$$BC_G(\rho(H)) \xrightarrow{\cong} \text{Map}(BH, BG)_{B\rho}.$$

Proof. See, for example, [BrK, Proposition 7.1]. \square

2. A NEW TOPOLOGICAL CHARACTERIZATION OF FUSION SYSTEMS

In this section, we show, for a p -complete space X , a p -group S , and a map $f: BS \rightarrow X$, that the triple $(S, \mathcal{F}'_{S,f}(X), \mathcal{L}_{S,f}^c(X))$ is a p -local finite group if X , S , and f satisfy certain conditions listed in Theorem 2.1 below.

When S is a p -group, a map $f: BS \rightarrow X$ will be called *Sylow* if every map $BP \rightarrow X$, for a p -group P , factors through f up to homotopy. A map $f: X \rightarrow Y$ between arbitrary spaces is called *centric* if the induced map

$$\text{Map}(X, X)_{\text{Id}} \xrightarrow{f \circ -} \text{Map}(X, Y)_f$$

is a homotopy equivalence.

In [BLO2, Theorem 7.5], we showed that a p -complete space X is the classifying space of some p -local finite group if and only if there is a pair (S, f) , where S is a p -group and $f: BS \rightarrow X$ is a map, such that (a) $\mathcal{F}_{S,f}(X)$ is saturated, (b) $X \simeq |\mathcal{L}_{S,f}^c(X)|_p^\wedge$, and (c) $f|_{BP}$ is a centric map for each $\mathcal{F}_{S,f}(X)$ -centric subgroup $P \leq S$. The following theorem is similar in nature, although aimed at finding conditions for $(S, \mathcal{F}_{S,f}(X), \mathcal{L}_{S,f}^c(X))$ to be a p -local finite group rather than for X to be the classifying space of a p -local finite group. The main new result here is the geometric condition for the fusion system $\mathcal{F}_{S,f}(X)$ to be saturated.

Theorem 2.1. *Fix a space X , a p -group S , and a map $f: BS \rightarrow X$. Assume that*

- (a) f is Sylow;
- (b) $f|_{BP}$ is a centric map for each $\mathcal{F}_{S,f}(X)$ -centric subgroup $P \leq S$; and
- (c) every $\mathcal{F}'_{S,f}(X)$ -centric subgroup of S is also $\mathcal{F}_{S,f}(X)$ -centric.

Then the triple $(S, \mathcal{F}'_{S,f}(X), \mathcal{L}_{S,f}^c(X))$ is a p -local finite group.

Proof. For each $\mathcal{F}_{S,f}(X)$ -centric subgroup $P \leq S$, (b) implies that composition with $f|_{BP}$ induces a homotopy equivalence $\text{Map}(BP, BP)_{\text{Id}} \rightarrow \text{Map}(BP, X)_f$. Also, by Proposition 1.8, $\text{Map}(BP, BP)_{\text{Id}} \simeq BZ(P)$, and the resulting homotopy equivalence $BZ(P) \xrightarrow[\simeq]{\eta} \text{Map}(BP, X)_f$ is adjoint to the composite

$$BZ(P) \times BP \xrightarrow{B\mu} BS \xrightarrow{f} X,$$

where $\mu: Z(P) \times P \rightarrow S$ is multiplication. Thus $\omega_P = \eta$, where ω_P is the map of Lemma 1.8, and hence is a homotopy equivalence.

Condition (*) of Lemma 1.8 thus holds, and so $\mathcal{L}_{S,f}^c(X)$ is a linking system associated to $\mathcal{F}_{S,f}(X)$ by that lemma. It remains only to prove that $\mathcal{F}'_{S,f}(X)$ is saturated. This proof is based on two lemmas which will be stated and proven later in this section.

Write $\mathcal{L} = \mathcal{L}_{S,f}^c(X)$, $\mathcal{F} = \mathcal{F}_{S,f}(X)$, and $\mathcal{F}' = \mathcal{F}'_{S,f}(X)$ for short. By Proposition 1.7 and (c), in order to prove that \mathcal{F}' is saturated, it suffices to show that conditions (I) and (II) in Definition 1.2 hold for all \mathcal{F}' -centric subgroups $P \leq S$. If $P \leq S$ is \mathcal{F}' -centric, then it is also \mathcal{F} -centric by (c), and hence $f|_{BP}$ is a centric map by (b).

We first prove condition (I). Assume $P \leq S$ is \mathcal{F}' -centric and fully normalized in \mathcal{F}' . Since P is \mathcal{F}' -centric, it is fully centralized, and it remains only to show that $\text{Aut}_S(P) \in \text{Syl}_p(\text{Aut}_{\mathcal{F}'}(P))$. We identify P with $\delta_P(P) \leq \text{Aut}_{\mathcal{L}}(P)$. Since \mathcal{L} is a centric linking system associated to \mathcal{F} or \mathcal{F}' , the homomorphism

$$\pi_{P,P}: \text{Aut}_{\mathcal{L}}(P) \longrightarrow \text{Aut}_{\mathcal{F}}(P) = \text{Aut}_{\mathcal{F}'}(P)$$

induced by the functor $\pi: \mathcal{L} \rightarrow \mathcal{F}$ is surjective with kernel $Z(P)$. Also, $\pi_{P,P}(P) = \text{Inn}(P)$ is normal in $\text{Aut}_{\mathcal{F}}(P)$, and thus $P \triangleleft \text{Aut}_{\mathcal{L}}(P)$. By axiom (B) for a linking system, $\pi_{P,P}$ sends $g \in \text{Aut}_{\mathcal{L}}(P)$ to $c_g \in \text{Aut}(P)$, and thus

$$C_{\text{Aut}_{\mathcal{L}}(P)}(P) = \text{Ker}(\pi_{P,P}) = Z(P). \quad (1)$$

By Lemma 2.2, $f|_{BP}$ extends up to homotopy to a map $\hat{f}: B\text{Aut}_{\mathcal{L}}(P) \rightarrow X$ (by definition, $\text{Aut}_{\mathcal{L}}(P) = \text{Aut}_{\mathcal{L}_{P,f}^c(X)}(P)$). If T is any Sylow p -subgroup of $\text{Aut}_{\mathcal{L}}(P)$, then $T \geq P$ since $P \triangleleft \text{Aut}_{\mathcal{L}}(P)$, $\hat{f}|_{BT}$ factors through BS by condition (a), and thus there is a homomorphism $\varphi: T \rightarrow S$ such that $\varphi|_P \in \text{Hom}_{\mathcal{F}'}(P, S)$. In particular,

$\varphi|_P$ is a monomorphism. Hence since $Z(T) \leq C_T(P) \leq P$ by (1), $\text{Ker}(\varphi) \cap Z(T) \leq \text{Ker}(\varphi) \cap P = 1$; and (since a nontrivial normal subgroup intersects nontrivially with the center) this implies that φ is a monomorphism. Hence $|N_S(\varphi(P))| \geq |T|$ since $N_S(\varphi(P)) \geq \varphi(T)$; and also $|N_S(P)| \geq |N_S(\varphi(P))|$ since P is fully normalized. Since P is centric in S ,

$$|\text{Aut}_S(P)| = |N_S(P)|/|Z(P)| \geq |N_S(\varphi(P))|/|Z(P)| \geq |T/Z(P)|;$$

and thus $\text{Aut}_S(P) \in \text{Syl}_p(\text{Aut}_{\mathcal{F}'}(P))$ since $\text{Aut}_{\mathcal{F}'}(P) \cong \text{Aut}_{\mathcal{L}}(P)/Z(P)$ and $T \in \text{Syl}_p(\text{Aut}_{\mathcal{L}}(P))$.

It remains to prove condition (II). Fix a morphism $\varphi \in \text{Hom}_{\mathcal{F}'}(P, S)$, and set

$$N = N_\varphi = \{g \in N_S(P) \mid \varphi c_g \varphi^{-1} \in \text{Aut}_S(\varphi(P))\}$$

as usual. Consider the diagram

$$\begin{array}{ccc} BP & \xrightarrow{B\varphi} & BS \\ \text{incl} \downarrow & \nearrow^{B\varphi'} & \downarrow f \\ BN & \xrightarrow{f|_{BN}} & X. \end{array}$$

The square commutes up to homotopy since φ is a morphism in \mathcal{F}' , and condition (1) in Lemma 2.3 holds by definition of N . Thus, by Lemma 2.3, there is a homomorphism $\varphi' \in \text{Hom}(N, S)$ such that $B\varphi'$ makes both triangles in the above diagram commute up to homotopy. The commutativity of the lower triangle means that $\varphi' \in \text{Hom}_{\mathcal{F}'}(N, S)$. The commutativity of the upper triangle implies that $\varphi'|_P = \varphi \circ c_g$ for some $g \in P$ (Proposition 1.9), and thus $\bar{\varphi} \stackrel{\text{def}}{=} \varphi' \circ c_g^{-1}$ is an extension of φ which lies in $\text{Hom}_{\mathcal{F}'}(N, S)$. This finishes the proof of (II). \square

It remains to state and prove the technical lemmas used in the proof of Theorem 2.1.

Lemma 2.2. *Fix a space X , a p -group P , and a centric map $f: BP \longrightarrow X$. Set $\mathcal{L} = \mathcal{L}_{P,f}^c(X)$ for short. Then f extends (up to homotopy) to a map*

$$\bar{f}: B\text{Aut}_{\mathcal{L}}(P) \longrightarrow X.$$

Proof. We first consider the following abstract situation. Fix a space Y , a basepoint $y_0 \in Y$, and a finite group G with a right action on Y . Consider the following commutative diagram

$$\begin{array}{ccccccc} \Omega(Y \times_G EG) & \longrightarrow & G & \xrightarrow{\iota_1} & Y \times EG & \longrightarrow & Y \times_G EG \\ \text{pr}_1 \downarrow \simeq & & \parallel & & \downarrow \simeq & & \\ F & \longrightarrow & G & \xrightarrow{\iota_2} & Y & & \end{array}$$

Here, ι_1 and ι_2 are defined by the action at the basepoints: $\iota_1(g) = (y_0g, g^{-1})$ and $\iota_2(g) = y_0g$. Also, F is the “standard” homotopy fiber of ι_2 :

$$F = \{(g, H) \mid g \in G, H: I \rightarrow Y, H(0) = y_0, H(1) = y_0g\}. \quad (I = [0, 1])$$

This is an H -space, via the product $(g', H')(g, H) = (g'g, (R_g \circ H') \cdot H)$, where R_g denotes the right action of g on Y and “ \cdot ” denotes composition of paths in Y . We can also regard $\Omega(Y \times_G EG)$ as the standard homotopy fiber of ι_1 . Then pr_1 is defined

by projecting a path in $Y \times EG$ to the first factor, and is a map of H -spaces and a homotopy equivalence. In particular, it induces an isomorphism of groups

$$\pi_1(Y \times_G EG) \cong \pi_0(\Omega(Y \times_G EG)) \xrightarrow{\pi_0(\text{pr}_1)} \pi_0(F). \quad (1)$$

Set $\mathcal{F} = \mathcal{F}_{P,f}(X)$ for short. We apply the above remarks to the space $Y = \text{Map}(BP, X)_f$, the point $y_0 = f$, and the group $G = \text{Aut}_{\mathcal{F}}(P)$, where $\alpha \in \text{Aut}_{\mathcal{F}}(P)$ acts on Y via right composition by $B\alpha$. Thus after replacing paths in Y by homotopies,

$F = \{(\varphi, H) \mid \varphi \in \text{Aut}_{\mathcal{F}}(P), H: BP \times I \longrightarrow X, H|_{BP \times 0} = f, H|_{BP \times 1} = f \circ B\varphi\}$, and $\text{Aut}_{\mathcal{L}}(P) = \pi_0(F)$ by definition. Also, since f is centric,

$$Y = \text{Map}(BP, X)_f \simeq \text{Map}(BP, BP)_{\text{Id}} \simeq BZ(P),$$

where the last equivalence follows from Proposition 1.9. Then $Y \times_G EG$ is also aspherical, and so $Y \times_G EG \simeq B \text{Aut}_{\mathcal{L}}(P)$ by (1).

Since $B\varphi$ fixes the base point of BP for all $\varphi \in \text{Aut}(P)$, the evaluation map

$$Y = \text{Map}(BP, X)_f \xrightarrow{\text{eval}} X$$

is $\text{Aut}_{\mathcal{F}}(P)$ -equivariant (with respect to the trivial action on X). It thus factors through the orbit space, or alternatively through the Borel construction:

$$\bar{f}: B \text{Aut}_{\mathcal{L}}(P) \simeq \text{Map}(BP, X)_f \times_{\text{Aut}_{\mathcal{F}}(P)} E \text{Aut}_{\mathcal{F}}(P) \xrightarrow{\text{eval}} X.$$

It remains to show that $\bar{f}|_{BP} \simeq f$, where BP is included into $B \text{Aut}_{\mathcal{L}}(P)$ via the distinguished monomorphism δ_P . By the naturality of these maps, it suffices to do this when $X = BP$ and $f = \text{Id}$. In this case, that means showing that $\pi_1(\bar{f}|_{BP}) = \text{Id}_P$. Fix $g \in P$, and let $H_g: BP \times I \longrightarrow BP$ be as in Definition 1.6. We also regard H_g as a path in $\text{Map}(BP, BP)_{\text{Id}}$ from Id_{BP} to Bc_g , whose restriction to the basepoint of BP is by definition the loop in BP representing g . By the above construction, $g \in \pi_1(BP)$ corresponds to the class

$$[H_g, \phi] \in \pi_1(\text{Map}(BP, X)_f \times_{\text{Aut}_{\mathcal{F}}(P)} E \text{Aut}_{\mathcal{F}}(P)),$$

where ϕ is any path in $EP \subseteq E \text{Aut}_{\mathcal{F}}(P)$ from the vertex Id to the vertex c_g^{-1} . Hence upon evaluating this at the basepoint of BP , we see that $\text{eval}([H_g, \phi])$ is the loop in BP representing g , and thus that $\pi_1(\bar{f})(g) = g$. \square

It remains to prove the existence of certain homotopy liftings.

Lemma 2.3. *Fix a finite group H , a normal p -subgroup $P \triangleleft H$, a p -group S , and a monomorphism $\varphi: P \longrightarrow S$ such that $C_S(\varphi(P)) = Z(\varphi(P))$. Let X be a space, and let $f: BS \longrightarrow X$ be such that $f \circ B\varphi$ is centric. Assume that*

$$\text{for each } x \in H, \varphi c_x \varphi^{-1} \in \text{Aut}_S(\varphi(P)). \quad (1)$$

Let $s: BH \longrightarrow X$ be such that the square in the following diagram commutes up to homotopy:

$$\begin{array}{ccc} BP & \xrightarrow{B\varphi} & BS \\ \text{incl} \downarrow & \nearrow B\varphi' & \downarrow f \\ BH & \xrightarrow{s} & X. \end{array} \quad (2)$$

Then there is a homomorphism $\varphi' \in \text{Hom}(H, S)$ such that the two triangles in diagram (2) commute up to homotopy.

Proof. We identify BP with EH/P and BH with

$$(E(H/P) \times EH)/H = E(H/P) \times_{H/P} EH/P.$$

Thus the inclusion $BP \subseteq BH$ is induced by the inclusion of an orbit $H/P \subseteq E(H/P)$. Let

$$\widehat{\Phi}: EH/P \longrightarrow BS \quad \text{and} \quad \widehat{s}: E(H/P) \times_{H/P} EH/P \longrightarrow X$$

be maps homotopic to $B\varphi$ and s under these identifications.

By (1), the connected component $\text{Map}(EH/P, BS)_{B\varphi}$ is invariant under the action of H/P induced by the action of the group on EH/P . We thus get the following square of equivariant maps between spaces with (H/P) -action

$$\begin{array}{ccc} H/P & \xrightarrow{v} & \text{Map}(EH/P, BS)_{\widehat{\Phi}} \\ \text{incl} \downarrow & \nearrow \widetilde{u} & \simeq \downarrow f \circ - \\ E(H/P) & \xrightarrow{u} & \text{Map}(EH/P, X)_{f \circ \widehat{\Phi}} \end{array} \quad (3)$$

Here, u is adjoint to \widehat{s} (when regarded as a map defined on $E(H/P) \times EH/P$); and v is defined by setting $v(gP)(xP) = \widehat{\Phi}(xgP)$ for $x \in EH$. The square in (3) commutes up to equivariant homotopy by the commutativity of the square in (2).

Now, $\text{Map}(EH/P, BS)_{B\varphi} \simeq BC_S(\varphi(P)) \simeq BZ(P)$ by Proposition 1.9, and since $\varphi(P)$ is centric in S by assumption. Also, $\text{Map}(EH/P, X)_{f \circ \widehat{\Phi}} \simeq BZ(P)$ since $f \circ B\varphi$ is a centric map by assumption. Thus the map $(f \circ -)$ is H/P -equivariant and a homotopy equivalence. Since the H/P -action on $E(H/P)$ is free, there is an equivariant lifting \widetilde{u} of u as in the above diagram which makes both triangles in (3) commute up to equivariant homotopy. This is adjoint to an H/P -equivariant map from $E(H/P) \times EH/P$ to BS , which (since H/P acts trivially on BS) factors through

$$\widetilde{s}: BH = E(H/P) \times_{H/P} EH/P \longrightarrow BS$$

which makes the two triangles in (2) commute up to homotopy. Finally, $\widetilde{s} \simeq B\varphi'$ for some $\varphi' \in \text{Hom}(H, S)$ by Proposition 1.9 again, and this finishes the proof. \square

3. FUSION SYSTEMS OF COMPLETED CLASSIFYING SPACES OF GROUPS

In order to apply Theorem 2.1 to a space X , we must have good control over the mapping spaces $\text{Map}(BP, X)$ for finite p -groups P . One interesting case where we can do this is when $X = BG_p^\wedge$ for certain infinite groups G . This is based on a theorem of Broto and Kitchloo [BrK].

When G is an infinite group, we say that a subgroup $S \leq G$ is a *Sylow p -subgroup* if S is a finite p -subgroup, and if all other finite p -subgroups of G are conjugate to subgroups of S .

Proposition 3.1. *Fix a prime p and a discrete group G . Assume there is an \mathbb{F}_p -acyclic G -complex X with finitely many orbits of cells and with finite isotropy subgroups. Let $S \leq G$ be any finite p -subgroup, and let $f: BS \longrightarrow BG_p^\wedge$ be the inclusion. Then the following hold.*

- (a) $\mathcal{F}_{S,f}(BG_p^\wedge) = \mathcal{F}_S(G)$.
- (b) If S is a Sylow p -subgroup of G , then the map f is Sylow.

- (c) For any $P \leq S$, $f|_{BP}$ is a centric map if and only if $H^i(C_G(P)/Z(P); \mathbb{F}_p) = 0$ for all $i > 0$.

Proof. In the notation of [BrK], $\mathcal{K}_1\mathcal{X}$ is a class of topological groups which includes all discrete groups which act on \mathbb{F}_p -acyclic complexes with finitely many orbits of cells and with finite isotropy subgroups. (The definition in [BrK] also requires that the fixed point set of any finite p -group be \mathbb{F}_p -acyclic, but this follows from Smith theory, since the complex is finite dimensional.) In particular, this class includes G . Hence by [BrK, Corollary 3.3], for any finite p -group P , the natural map

$$\text{Rep}(P, G) \stackrel{\text{def}}{=} \text{Hom}(P, G)/\text{Inn}(G) \xrightarrow{\cong} [BP, BG_p^\wedge] \quad (1)$$

is a bijection. Also, for each $\rho \in \text{Hom}(P, G)$, the homomorphism $P \times C_G(\rho(P)) \xrightarrow{(\rho, \text{incl})} G$ induces a homotopy equivalence

$$BC_G(P)_p^\wedge \xrightarrow{\simeq} \text{Map}(BP, BG_p^\wedge)_{B\rho}. \quad (2)$$

Point (a) follows immediately from (1).

Assume S is a Sylow p -subgroup of G . If P is any finite p -group and $s: BP \rightarrow BG_p^\wedge$ is a map, then $s \simeq B\varphi$ for some $\varphi \in \text{Hom}(P, G)$ by (1), $\varphi(P)$ is G -conjugate to some $Q \leq S$ since S is Sylow, and thus $B\varphi \simeq f \circ B\varphi'$ for some $\varphi' \in \text{Hom}(P, S)$. Thus the map f is Sylow, and this proves (b).

By (2), for any $P \leq S$, $f|_{BP}$ is a centric map if and only if the inclusion of $BZ(P)$ into $BC_G(P)_p^\wedge$ is a homotopy equivalence, or equivalently, if the inclusion of $BZ(P)$ into $BC_G(P)$ is an \mathbb{F}_p -homology isomorphism. Since $Z(P)$ is central in $C_G(P)$, this last condition is equivalent to requiring that $H^i(C_G(P)/Z(P); \mathbb{F}_p) = 0$ for all $i > 0$, and this proves (c). \square

Before stating our theorem, we need one more definition.

Definition 3.2. Fix a prime p .

- (a) If $H \leq G$ are finite groups, then H is strongly embedded in G at p if $p \mid |H|$, but $H \cap gHg^{-1}$ has order prime to p for all $g \in G \setminus N_G(H)$.
- (b) If G is a finite group and $S \in \text{Syl}_p(G)$, a subgroup $P \leq S$ is essential if either $P = S$, or P is p -centric in G and $\text{Out}_G(P)$ has a strongly embedded subgroup at p .

By Goldschmidt's version of Alperin's fusion theorem [Gd, Theorem 3.3], for any finite group G and any $S \in \text{Syl}_p(G)$, each morphism in $\mathcal{F}_S(G)$ is a composite of restrictions of morphisms between subgroups of S which are essential in G . Note that each essential subgroup is also radical — $\text{Out}_G(P)$ has no strongly embedded subgroup if $O_p(\text{Out}_G(P)) \neq 1$.

A finite group G has a strongly embedded subgroup H if and only if the poset of nontrivial p -subgroups of G is disconnected (cf. [As, 46.6]), in which case the stabilizer of a connected component is strongly embedded.

The following theorem is a first application of Theorem 2.1.

Theorem 3.3. Fix a prime p and a discrete group G . Let X be an \mathbb{F}_p -acyclic G -complex with finitely many orbits of cells and with finite isotropy subgroups. Fix a vertex $x_* \in X$, let G_* be the isotropy subgroup of x_* , choose $S \in \text{Syl}_p(G_*)$, and set $\mathcal{F} = \mathcal{F}_S(G)$ and $\mathcal{L} = \mathcal{L}_S^c(G)$. Assume the following hold:

- (a) For each finite p -subgroup $P \leq G$, X^P contains at least one point in the orbit Gx_* .
- (b) If $P \leq S$ is \mathcal{F} -centric, then $X^P/C_G(P)$ is \mathbb{F}_p -acyclic.
- (c) If $P \leq S$ is a Sylow p -subgroup of the isotropy subgroup of an edge of X , or an essential p -subgroup of the isotropy subgroup of a vertex, then P is \mathcal{F} -centric.

Then \mathcal{F} is a saturated fusion system over S , and \mathcal{L} is a centric linking system associated to \mathcal{F} .

Proof. If $f: BS \longrightarrow BG_p^\wedge$ is the map induced by the inclusion, then $\mathcal{F} = \mathcal{F}_{S,f}(BG_p^\wedge)$ by Proposition 3.1(a). For any finite p -subgroup $P \leq G$, there is $g \in G$ such that $gx_* \in X^P$ by point (a) above, and hence P is contained in the isotropy subgroup gG_*g^{-1} of gx_* . Since $gSg^{-1} \in \text{Syl}_p(gG_*g^{-1})$, this shows that P is G -conjugate to a subgroup of S . Thus S is a Sylow p -subgroup of G ; and hence by Proposition 3.1(b), the map f is Sylow.

By Theorem 2.1, to prove that $\mathcal{F} = \mathcal{F}_S(G)$ is saturated, it remains only to check condition 2.1(b), and to show that $\mathcal{F} = \mathcal{F}'_{S,f}(BG_p^\wedge)$. (This last claim also implies condition 2.1(c).) In Step 1, we prove condition 2.1(b), and also prove that $\mathcal{L} = \mathcal{L}_S^c(G)$ is a centric linking system associated to \mathcal{F} . In Step 2, we prove that $\mathcal{F} = \mathcal{F}'_{S,f}(BG_p^\wedge)$; i.e., that \mathcal{F} is generated by morphisms between \mathcal{F} -centric subgroups.

Step 1: Fix an \mathcal{F} -centric subgroup $P \leq S$. Thus $C_S(P) = Z(P)$ for each $P' \leq S$ which is G -conjugate to P . If $Q \leq C_G(P)$ is a finite p -subgroup, then PQ is a finite p -group, so $gPQg^{-1} \leq S$ for some $g \in G$, and $Q \leq Z(P)$ by the above remark applied to $P' = gPg^{-1}$. Thus $Z(P)$ is maximal among finite p -subgroups of $C_G(P)$.

Set $\overline{C}_G(P) = C_G(P)/Z(P)$ for short. For each $x \in X^P$, G_x is a finite group which contains P , so $C_{G_x}(P)/Z(P)$ is finite of order prime to p , and hence its classifying space is \mathbb{F}_p -acyclic. Consider the projection maps

$$B\overline{C}_G(P) \xleftarrow{\text{pr}_1} E\overline{C}_G(P) \times_{\overline{C}_G(P)} X^P \xrightarrow{\text{pr}_2} X^P/\overline{C}_G(P) = X^P/C_G(P)$$

associated to the Borel construction on X^P . All fibers (point inverses) of pr_1 are homeomorphic to X^P , and thus \mathbb{F}_p -acyclic by Smith theory (X is \mathbb{F}_p -acyclic and finite dimensional). For each $x \in X^P$ with orbit $\bar{x} \in X^P/C_G(P)$ and with stabilizer subgroup G_x ,

$$\text{pr}_2^{-1}(\bar{x}) \cong E\overline{C}_G(P)/C_{G_x}(P) \simeq B(C_{G_x}(P)/Z(P))$$

is \mathbb{F}_p -acyclic. Hence by a spectral sequence argument (or by an appropriate version of the Vietoris mapping theorem), pr_1 and pr_2 are both \mathbb{F}_p -homology equivalences. Since $X^P/C_G(P)$ is \mathbb{F}_p -acyclic by assumption, this implies that $B\overline{C}_G(P)$ is \mathbb{F}_p -acyclic.

Thus $H^i(\overline{C}_G(P); \mathbb{F}_p) = 0$ for all $i > 0$. Hence by Proposition 3.1(c), the map $f|_{BP}$ is centric, and this finishes the proof of condition 2.1(b). By Lemma 1.5, this also shows that $\mathcal{L} = \mathcal{L}_S^c(G)$ is a centric linking system associated to \mathcal{F} .

Step 2: Fix any $\varphi = c_g \in \text{Hom}_{\mathcal{F}}(P, Q)$ in \mathcal{F} . Then $P \leq S$ and $gPg^{-1} \leq S$, so P is contained in the isotropy subgroups of both x_* and $g^{-1}(x_*)$. Choose a path ϕ in the 1-skeleton of X^P from x_* to $g^{-1}(x_*)$. Let $v_0 = x_*, v_1, \dots, v_m = g^{-1}(x_*)$ be the successive vertices in the path ϕ , let e_i be the edge connecting v_{i-1} to v_i , and set $H_i = G_{e_i}$ and $K_i = G_{v_i}$. Thus by construction, $P \leq H_i$, and $K_{i-1} \geq H_i \leq K_i$ for all $1 \leq i \leq m$. Also, $S \in \text{Syl}_p(K_0)$ and $K_m = g^{-1}K_0g$.

Fix Sylow subgroups $P_i \in \text{Syl}_p(H_i)$ such that $P \leq P_i$. Choose $Q_i, Q'_i \in \text{Syl}_p(K_i)$ such that $Q'_{i-1} \geq P_i \leq Q_i$, and let $k_i \in K_i$ be such that $Q'_i = k_i Q_i k_i^{-1}$. We also assume that $Q_0 = S$ and $Q'_m = g^{-1} S g$. Finally, since S is Sylow in G , there are elements $g_i \in G$ such that $Q_i \leq g_i S g_i^{-1}$. In particular, when $i = m$, since

$$g^{-1} S g = Q'_m = k_m Q_m k_m^{-1},$$

we can choose $g_m = k_m^{-1} g^{-1}$.

Consider the following diagram, where all subgroups are contained in S :

$$\begin{array}{ccccccc} P & \xrightarrow{\psi_0} & P^{k_0} & \xrightarrow{\varphi_1} & P^{g_1} & \xrightarrow{\psi_1} & P^{k_1 g_1} & \xrightarrow{\varphi_2} & P^{g_2} & & P^{k_{m-1} g_{m-1}} & \xrightarrow{\varphi_m} & P^{g_m} & \xrightarrow{\psi_m} & P^{k_m g_m} \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \dots & \downarrow & & \downarrow & & \downarrow \\ & & P_1^{k_0} & \xrightarrow{\bar{\varphi}_1} & P_1^{g_1} & & P_2^{k_1 g_1} & \xrightarrow{\bar{\varphi}_2} & P_2^{g_2} & & P_m^{k_{m-1} g_{m-1}} & \xrightarrow{\bar{\varphi}_m} & P_m^{g_m} & & \end{array}$$

Here, we use the standard notation $H^g = g^{-1} H g$. Also, ψ_i is conjugation by $g_i^{-1} k_i^{-1} g_i \in K_i^{g_i}$ (where $g_0 = 1$), and φ_i and $\bar{\varphi}_i$ are conjugation by $g_i^{-1} k_{i-1} g_{i-1}$. All of these subgroups are contained in S by construction. Also, by the above choice of g_m , ψ_m is conjugation by $g g_m$. Thus the composite of these morphisms ψ_i and φ_i is conjugation by

$$(g g_m)(g_m^{-1} k_{m-1} g_{m-1})(g_{m-1}^{-1} k_{m-1}^{-1} g_{m-1}) \cdots (g_2^{-1} k_1 g_1)(g_1^{-1} k_1^{-1} g_1)(g_1^{-1} k_0 g_0)(g_0^{-1} k_0^{-1} g_0) = g.$$

(Recall that $g_0 = 1$.)

By (c), each subgroup of S which is conjugate to any P_i is \mathcal{F} -centric. Each ψ_i is a morphism in the fusion system of $K_i^{g_i}$, and hence a composite of restrictions of morphisms between essential p -subgroups of this group [Gd, Theorem 3.3]. Since all such subgroups are \mathcal{F} -centric by (c), this finishes the proof. \square

We finish the section with two very simple examples which illustrate why some of these assumptions are needed. The first example shows why condition (c) is needed in Theorem 3.3. It also shows why we cannot take $\mathcal{F}' = \mathcal{F}$ in Theorem 2.1.

Example 3.4. *Let S be an abelian p -group, $T \not\leq S$ a proper subgroup of order > 2 , and $H \leq \text{Aut}(T)$ a nontrivial subgroup of order prime to p . Set $G = S \underset{T}{*} (T \rtimes H)$. Then G acts on a tree with isotropy subgroups all G -conjugate to S , T , or $T \rtimes H$. This action satisfies conditions (a) and (b) in Theorem 3.3, but not condition (c); and the inclusion map $f: BS \longrightarrow BG_p^\wedge$ satisfies all of the hypotheses (a)–(c) in Theorem 2.1. The fusion system $\mathcal{F}_S(G) = \mathcal{F}_{S,f}(BG_p^\wedge)$ is not saturated. The fusion system $\mathcal{F}'_{S,f}(BG_p^\wedge)$ is equal to $\mathcal{F}_S(S)$, and is thus a proper subsystem of $\mathcal{F}_S(G)$ and is saturated.*

Proof. By [Se, Theorem I.9], G acts freely on a tree X with isotropy subgroups conjugate to S and $T \rtimes H$ on vertices, and to T on edges, and with fundamental domain an interval. Since S does not fix any edges (and X^S must be a tree), X^S is a point, and hence $X^S/C_G(S)$ is also a point. Since T has index prime to p in $T \rtimes H$, X^P contains elements in the orbit G/S for each finite p -subgroup $P \leq G$. Thus the action of G on X satisfies conditions 3.3(a) and 3.3(b). Conditions (a)–(c) in Theorem 2.1 then follow using Proposition 3.1.

The fusion system $\mathcal{F} = \mathcal{F}_S(G)$ is not saturated, since the automorphisms in the group $\text{Aut}_{\mathcal{F}}(T) \cong H$ do not extend to automorphisms in $\text{Aut}_{\mathcal{F}}(S)$. \square

The next example shows that condition (b) in Theorem 2.1 and condition (b) in Theorem 3.3 must be assumed (in each theorem) for all \mathcal{F} -centric subgroups: it does not suffice to assume them when $P = S$.

Example 3.5. *Set $p = 2$, $S = D_8 \times C_2^2$, $T = C_4 \times C_2^2$, and $H = C_4 \times A_4$, with the obvious inclusions of T in S and H . Set $G = S \underset{T}{*} H$. Then G acts on a tree with all isotropy subgroups G -conjugate to S , T , or H . This action satisfies conditions (a) and (c) in Theorem 3.3 as well as conditions (a) and (c) in Theorem 2.1, but does not satisfy condition (b) in either theorem. The inclusion map $f: BS \longrightarrow BG_p^\wedge$ is Sylow and centric, but the inclusion map $f|_{BT}: BT \longrightarrow BG_p^\wedge$ is not centric. Neither fusion system $\mathcal{F}_S(G) = \mathcal{F}_{S,f}(BG_p^\wedge)$ nor $\mathcal{F}'_{S,f}(BG_p^\wedge)$ is saturated.*

Proof. Set $\mathcal{F} = \mathcal{F}_{S,f}(BG_p^\wedge)$ and $\mathcal{F}' = \mathcal{F}'_{S,f}(BG_p^\wedge)$ for short. As in the last example, G acts on a tree X with isotropy subgroups as described, and with fundamental domain an interval, by [Se, Theorem I.9]. Since any action of a finite group on a tree has a fixed point, every finite subgroup of G is contained in an isotropy subgroup, and thus in a subgroup G -conjugate to S or H . This shows that S is a Sylow 2-subgroup of G , and hence (by Proposition 3.1(b)) that f is Sylow. Thus both conditions 2.1(a) and 3.3(a) hold. Also, f is centric by Proposition 3.1(c), since $C_G(S) = Z(S)$.

Since S is a 2-group and H has a normal Sylow 2-subgroup, their only radical 2-subgroups (hence their only essential 2-subgroups) are S and T , respectively. Since both are centric in S , condition 3.3(c) holds. As seen in Step 2 of the proof of Theorem 3.3, this implies that $\mathcal{F}' = \mathcal{F}$, and thus that condition 2.1(c) also holds.

Now, T is normal in G , since it is normal in S and H , and $G/T \cong (S/T) \underset{T}{*} (H/T) \cong C_2 \underset{T}{*} C_3$. Hence

$$C_G(T)/T \cong \text{Ker}[C_2 \underset{T}{*} C_3 \longrightarrow C_2 \times C_3],$$

and this is a free group (since it acts freely on a tree). In particular, $H^1(C_G(T)/T; \mathbb{F}_2) \neq 0$; and (since $C_G(T)/T$ acts freely on the tree X^T) $X^T/C_G(T) \simeq B(C_G(T)/T)$ is not \mathbb{F}_2 -acyclic. So by Proposition 3.1(c), $f|_{BT}$ is not a centric map; and this shows that conditions 2.1(b) and 3.3(b) both fail.

Now, $\text{Aut}_{\mathcal{F}}(C_4 \times C_2^2) \cong C_2 \times C_3$, but $\text{Out}_{\mathcal{F}}(D_8 \times C_2^2) \cong 1$ (and the same for \mathcal{F}'). The automorphism of $C_4 \times C_2^2$ of order 3 thus fails to extend to S , so axiom (II) fails, and \mathcal{F} is not saturated. \square

4. FUSION SYSTEMS OF TREES OF GROUPS

Let $(\mathcal{G}, \mathcal{T})$ be a tree of groups in the sense of [Se, §I.4.4]. Thus \mathcal{T} is a tree; and \mathcal{G} assigns groups $\mathcal{G}(v)$ and $\mathcal{G}(e)$ to each vertex $v \in \mathcal{T}^0$ and each edge $e \in \mathcal{T}^1$, and a monomorphism $\mathcal{G}(e) \rightarrow \mathcal{G}(v)$ for each pair (e, v) where v is an endpoint of e . For any such tree of groups $(\mathcal{G}, \mathcal{T})$, we let $\mathcal{G}_{\mathcal{T}}$ denote the amalgamated free product of the groups $\mathcal{G}(v)$ over the $\mathcal{G}(e)$, as described in [Se, §I.4.4]. Thus $\mathcal{G}_{\mathcal{T}}$ is the free product of the groups $\mathcal{G}(v)$ for all vertices $v \in \mathcal{T}^0$, modulo the relations given by the inclusions of groups $\mathcal{G}(e)$ for $e \in \mathcal{T}^1$ into the groups of the endpoints of e .

Alternatively, one can regard \mathcal{T} as a category whose set of objects is the disjoint union of \mathcal{T}^0 and \mathcal{T}^1 , and with a pair of morphisms $w \leftarrow e \rightarrow v$ for each edge $e \in \mathcal{T}^1$ with endpoints v, w . Then \mathcal{G} is a functor from \mathcal{T} to the category Gr^+ of groups and monomorphisms, and $\mathcal{G}_{\mathcal{T}} = \text{colim}_{\mathcal{T}}(\mathcal{G})$.

In this paper, we will be considering only *finite* trees of *finite* groups; i.e., pairs $(\mathcal{G}, \mathcal{T})$ where \mathcal{T} is a finite tree, and $\mathcal{G}(v)$ is a finite group for each $v \in \mathcal{T}^0$. Our goal in this section is to find some conditions on \mathcal{G} and \mathcal{T} which ensure that the group $\mathcal{G}_{\mathcal{T}}$ gives rise to a saturated fusion system and associated centric linking system.

If $(\mathcal{G}, \mathcal{T})$ is a tree of groups, and $G = \mathcal{G}_{\mathcal{T}}$, then we let $\tilde{\mathcal{T}}$ denote the graph with vertex and edge sets

$$\begin{aligned}\tilde{\mathcal{T}}^0 &= \{(g\mathcal{G}(v), v) \mid v \in \mathcal{T}^0, g \in G\} = \coprod_{v \in \mathcal{T}^0} (G/\mathcal{G}(v) \times \{v\}) \\ \tilde{\mathcal{T}}^1 &= \{(g\mathcal{G}(e), e) \mid e \in \mathcal{T}^1, g \in G\} = \coprod_{e \in \mathcal{T}^1} (G/\mathcal{G}(e) \times \{e\})\end{aligned}$$

(with the obvious choices of endpoints). Equivalently, $\tilde{\mathcal{T}} = \text{hocolim}_{\mathcal{T}} (G/-)$. By [Se, Theorem I.9, p. 38], $\tilde{\mathcal{T}}$ is a tree upon which G acts with orbit graph \mathcal{T} , with fundamental domain which can be identified with \mathcal{T} (the subtree spanned by vertices $(1\mathcal{G}(v), v)$), and with isotropy subgroups on the fundamental domain given by \mathcal{G} .

For any pair of groups H, G , let $\text{Rep}(H, G) = \text{Hom}(H, G)/\text{Inn}(G)$, and let $[\alpha] \in \text{Rep}(H, G)$ be the class of $\alpha \in \text{Hom}(H, G)$. If $(\mathcal{G}, \mathcal{T})$ is a tree of groups, and H is any finite group, we let $\text{Rep}(H, \mathcal{G})$ be the graph with vertex and edge sets

$$\begin{aligned}\text{Rep}(H, \mathcal{G})^0 &= \{(v, [\alpha]) \mid v \in \mathcal{T}^0, [\alpha] \in \text{Rep}(H, \mathcal{G}(v))\} \\ \text{Rep}(H, \mathcal{G})^1 &= \{(e, [\alpha]) \mid e \in \mathcal{T}^1, [\alpha] \in \text{Rep}(H, \mathcal{G}(e))\}.\end{aligned}$$

When $\alpha \in \text{Hom}(H, \mathcal{G}(x))$, where x is a vertex or edge in \mathcal{T} , we write $(x, [\alpha])$ for the pair $(x, [\alpha])$, where $[\alpha]$ is the class of α in $\text{Rep}(H, \mathcal{G}(x))$. Alternatively, if we regard \mathcal{T} as a category, then

$$\text{Rep}(H, \mathcal{G}) = \text{hocolim}_{\mathcal{T}} \text{Rep}(H, \mathcal{G}(-)).$$

Lemma 4.1. *Fix a finite tree of finite groups $(\mathcal{G}, \mathcal{T})$. Set $G = \mathcal{G}_{\mathcal{T}} = \text{colim}_{\mathcal{T}}(\mathcal{G})$, and let $\tilde{\mathcal{T}}$ be as above. Then the following hold for any vertex v_* of \mathcal{T} and any subgroup $H \leq \mathcal{G}(v_*)$:*

- (a) *The connected component of $\text{Rep}(H, \mathcal{G})$ which contains $(v_*, [\text{incl}_H^{\mathcal{G}(v_*)}])$ is isomorphic (as a graph) to $\tilde{\mathcal{T}}^H/C_G(H)$.*
- (b) *The natural map*

$$\Phi_H: \pi_0(\text{Rep}(H, \mathcal{G})) \xrightarrow{\cong} \text{Rep}(H, G)$$

is a bijection. In particular, for x a vertex or edge of \mathcal{T} and $\alpha \in \text{Hom}(H, \mathcal{G}(x))$, $(x, [\alpha])$ lies in the connected component of the vertex $(v_, [\text{incl}_H^{\mathcal{G}(v_*)}])$ if and only if $\alpha \in \text{Hom}_G(H, \mathcal{G}(x))$.*

Proof. When x is a vertex or edge of \mathcal{T} , we write $G_x = \mathcal{G}(x)$ for short: the isotropy subgroup at x of the G -action, when we regard \mathcal{T} as a subtree (a fundamental domain) of $\tilde{\mathcal{T}}$.

If $K \leq G$ and $gK \in (G/K)^H$, then $H \leq gKg^{-1}$, and so we can regard $c_g^{-1}: x \mapsto g^{-1}xg$ as a homomorphism from H to K whose class in $\text{Rep}(H, K)$ depends only on the coset gK . Hence it makes sense to define

$$f_H: \tilde{\mathcal{T}}^H \longrightarrow \text{Rep}(H, \mathcal{G})$$

by sending each vertex (gG_v, v) to the pair $(v, [c_g^{-1}])$ and each edge (gG_e, e) to the pair $(e, [c_g^{-1}])$. We claim the following hold:

- (i) $\text{Im}(f_H)$ is the connected component of $(v_*, [\text{incl}_H^{G_{v_*}}])$ in $\text{Rep}(H, \mathcal{G})$.
- (ii) A vertex $(v, [\alpha])$, for $\alpha \in \text{Hom}(H, G_v)$, lies in $\text{Im}(f_H)$ if and only if the composite $H \xrightarrow{\alpha} G_v \leq G$ is G -conjugate to the inclusion.
- (iii) f_H induces an isomorphism of graphs $(\tilde{\mathcal{T}}^H)/C_G(H) \cong \text{Im}(f_H)$.

Point (ii) is immediate.

If $(e, [\alpha])$ is an edge of $\text{Rep}(H, \mathcal{G})$ with endpoint $f_H(gG_v, v) = (v, [c_g^{-1}])$, then v is an endpoint of e , and $[\alpha] = [c_g^{-1}] \in \text{Rep}(H, G_v)$. Thus $\alpha = c_h^{-1}c_g^{-1} = c_{gh}^{-1}$ for some $h \in G_v$, and $(e, [\alpha]) = f_H(ghG_e, e)$. So an edge of $\text{Rep}(H, \mathcal{G})$ lies in $\text{Im}(f_H)$ if one of its endpoints lies in $\text{Im}(f_H)$, and thus $\text{Im}(f_H)$ is a union of connected components of $\text{Rep}(H, \mathcal{G})$. Since $\tilde{\mathcal{T}}^H$ is a tree by [Se, §I.6.1], $\text{Im}(f_H)$ is nonempty and connected, hence is a connected component of $\text{Rep}(H, \mathcal{G})$, and this finishes the proof of (i).

Two vertices (gG_v, v) and (hG_w, w) are sent to the same vertex of $\text{Rep}(H, \mathcal{G})$ if and only if $v = w$ and $[c_g^{-1}] = [c_h^{-1}]$ in $\text{Rep}(H, G_v)$. This last condition is equivalent to saying that $h \in C_G(H)gG_v$; i.e., that (gG_v, v) and (hG_w, w) are in the same $C_G(H)$ -orbit; and thus (iii) holds. Points (i) and (iii) together imply (a).

By (ii), for any $\alpha \in \text{Hom}(H, G)$, Φ_H sends the connected component of a vertex $(v, [\beta])$ in $\text{Rep}(H, \mathcal{G})$ to $[\alpha]$ if and only if $(v, [\beta\alpha^{-1}]) \in \text{Im}(f_{\alpha(H)})$. Hence by (i), $\Phi_H^{-1}([\alpha])$ contains exactly one connected component. This shows that Φ_H is a bijection, and proves (b). \square

We originally discovered the following theorem as a special case of Theorem 2.1 (and of Theorem 3.3), and it was certainly motivated by those results. However, since it also has a more elementary proof which does not use certain deep theorems in homotopy theory, we give both proofs here.

Theorem 4.2. *Fix a prime p and a finite tree of finite groups $(\mathcal{G}, \mathcal{T})$. Fix a vertex v_* of \mathcal{T} , set $G_* = \mathcal{G}(v_*)$ for short, and choose $S \in \text{Syl}_p(G_*)$. Set $G = \mathcal{G}_{\mathcal{T}} = \underline{\text{colim}}(\mathcal{G})$, $\mathcal{F} = \mathcal{F}_S(G)$, and $\mathcal{L} = \mathcal{L}_S^c(G)$. Assume the following hold:*

- (a) *For each vertex $v \neq v_*$ of \mathcal{T} , if e is the edge adjacent to v in the (unique) minimal path from v to v_* , then $[\mathcal{G}(v) : \mathcal{G}(e)]$ is prime to p .*
- (b) *If $P \leq S$ is \mathcal{F} -centric (equivalently, if $Z(P)$ is a maximal p -subgroup of $C_G(P)$), then the component of $(v_*, [\text{incl}_P^{G_*}])$ in the graph $\text{Rep}(P, \mathcal{G})$ is a tree.*
- (c) *If some $P \leq S$ is G -conjugate to an essential p -subgroup of $\mathcal{G}(v)$ for any vertex v , then P is \mathcal{F} -centric.*

Then \mathcal{F} is a saturated fusion system over S , and \mathcal{L} is a centric linking system associated to \mathcal{F} .

Proof. Let $\tilde{\mathcal{T}}$ be as defined above: the tree upon which G acts with orbit space and fundamental domain \mathcal{T} . When x is a vertex or edge of \mathcal{T} , we write $G_x = \mathcal{G}(x)$ for short.

We first show how the theorem follows as a special case of Theorem 3.3, applied to the action of G on $\tilde{\mathcal{T}}$. Condition 3.3(b) follows from condition (b) here, together with Lemma 4.1(a); while condition 3.3(c) follows from condition (c) here.

It remains to describe how condition 3.3(a) follows from condition (a) here. Let $P \leq G$ be any finite p -subgroup; we must show that $\tilde{\mathcal{T}}^P$ contains some vertex in the orbit of $(1G_*, v_*)$ in $\tilde{\mathcal{T}}$. Since $\tilde{\mathcal{T}}$ is a tree, the fixed point set of the P -action is also a tree, and hence its image in the orbit tree \mathcal{T} is nonempty and connected. Let v be the vertex in that image which is closest to v_* . If $v \neq v_*$, then there is some $g \in G$ such that $gG_v \in (G/G_v)^P$, and hence $g^{-1}Pg \leq G_v$. Let e be the edge adjacent to v on the minimal path from v to v_* ; then $[G_v : G_e]$ is prime to p by (a), and hence there is $g' \in G$ such that $g'^{-1}Pg' \leq G_e$. Then the edge $(g'G_e, e)$ is in $\tilde{\mathcal{T}}^P$, which contradicts the original assumption about v . This shows that $v = v_*$, and thus that some vertex of the form (gG_*, v_*) is in $\tilde{\mathcal{T}}^P$.

Since this theorem also has a more elementary algebraic proof, we give that here. We first note that the argument just given also shows:

(a') For each vertex v in \mathcal{T} and each p -subgroup $P \leq G_v$, P is G -conjugate to a subgroup of S .

By a proof identical to Step 2 in the proof of Theorem 3.3, we show (using (c)) that every morphism in \mathcal{F} is a composite of restrictions of morphisms between \mathcal{F} -centric subgroups. Hence by [5a1, Theorem 2.3], \mathcal{F} is saturated if it satisfies axioms (I) and (II) in Definition 1.2 for all \mathcal{F} -centric subgroups $P \leq S$. So it remains to prove (I) and (II) for \mathcal{F} -centric subgroups, and to prove that \mathcal{L} is a centric linking system associated to \mathcal{F} .

For any $H \leq G_*$, let $\text{Rep}(H, \mathcal{G})_*$ be the connected component of $(v_*, [\text{incl}_H^{G_*}])$ in $\text{Rep}(H, \mathcal{G})$. By Lemma 4.1(b), if x is any vertex or edge in \mathcal{T} , and $\varphi \in \text{Hom}(H, G_x)$, then $(x, [\varphi])$ lies in $\text{Rep}(H, \mathcal{G})_*$ if and only if $\varphi \in \text{Hom}_G(H, G_x)$.

Proof of (I) for \mathcal{F} -centric subgroups. Let $P \leq S$ be any subgroup which is \mathcal{F} -centric and fully normalized in \mathcal{F} . By Lemma 4.1, there is a bijection $\pi_0(\text{Rep}(P, \mathcal{G})) \cong \text{Rep}(P, G)$ which sends $\text{Rep}(P, \mathcal{G})_*$ to $[\text{incl}_P^G]$, and which is equivariant with respect to the $\text{Aut}(P)$ -action on both sets. Thus $\text{Aut}_{\mathcal{F}}(P) = \text{Aut}_G(P)$ is the isotropy subgroup of $\text{Rep}(P, \mathcal{G})_* \in \pi_0(\text{Rep}(P, \mathcal{G}))$ under the $\text{Aut}(P)$ -action. In particular, $\text{Aut}_{\mathcal{F}}(P)$ leaves $\text{Rep}(P, \mathcal{G})_*$ invariant. Since $\text{Rep}(P, \mathcal{G})_*$ is a tree by (b), and since every action of a finite group on a tree has a fixed point, there is a vertex $(v, [\alpha])$ in $\text{Rep}(P, \mathcal{G})_*$ which is fixed by $\text{Aut}_{\mathcal{F}}(P)$. Thus $\alpha \in \text{Hom}_G(P, G_v)$. Set $P' = \alpha(P) \leq G_v$, so that $\text{Aut}_{G_v}(P') = \alpha \text{Aut}_{\mathcal{F}}(P) \alpha^{-1}$. Fix $Q' \in \text{Syl}_p(N_{G_v}(P'))$. By (a'), there is $Q \leq S$ which is G -conjugate to Q' . Fix $\beta \in \text{Iso}_G(Q', Q)$, and set $P'' = \beta(P')$. Then

$$|\text{Aut}_S(P)| = \frac{|N_S(P)|}{|Z(P)|} \geq \frac{|N_S(P'')|}{|Z(P'')|} \geq \frac{|Q'|}{|Z(P')|} = |\text{Aut}_{Q'}(P')|;$$

where the first inequality holds since P'' is G -conjugate (hence \mathcal{F} -conjugate) to P and P is fully centralized in \mathcal{F} . Since $\text{Aut}_{Q'}(P') \in \text{Syl}_p(\text{Aut}_G(P'))$ and $\text{Aut}_G(P') \cong \text{Aut}_G(P)$, this proves that $\text{Aut}_S(P) \in \text{Syl}_p(\text{Aut}_G(P))$, and finishes the proof of (I).

Proof of (II) for \mathcal{F} -centric subgroups. Fix $\varphi \in \text{Hom}_{\mathcal{F}}(P, S)$ where P is \mathcal{F} -centric, and set

$$N_{\varphi} = \{g \in N_S(P) \mid \varphi c_g \varphi^{-1} \in \text{Aut}_S(\varphi(P))\} \quad \text{and} \quad K = \text{Aut}_{N_{\varphi}}(P).$$

We claim that

$$\mathrm{Im}[\mathrm{Rep}(N_\varphi, \mathcal{G})_* \xrightarrow{\mathrm{Res}} \mathrm{Rep}(P, \mathcal{G})_*] = \mathrm{Rep}(P, \mathcal{G})_*^K. \quad (1)$$

Clearly, if x is a vertex or edge in \mathcal{T} , then the restriction of any $\beta \in \mathrm{Rep}(N_\varphi, G_x)$ lies in $\mathrm{Rep}(P, G_x)^K$, so the problem is to prove that $\mathrm{Rep}(P, \mathcal{G})_*^K$ lies in the image. By (b), $\mathrm{Rep}(P, \mathcal{G})$ is a tree, and hence the fixed point set of the finite group K is also a tree. So to prove (1), it suffices to show, for any edge $(e, [\alpha])$ in $\mathrm{Rep}(P, \mathcal{G})_*^K$ and any vertex $(v, [\beta])$ in $\mathrm{Rep}(N_\varphi, \mathcal{G})_*$ such that $(v, [\beta|_P])$ is an endpoint of $(e, [\alpha])$, that $(e, [\alpha])$ also lies in the image of the restriction map.

In this situation, v is an endpoint of e , and we regard G_e as a subgroup of G_v as usual. Then $\beta|_P = c_g \circ \alpha$ for some $g \in G_v$. Set $P' = \alpha(P)$ and $K' = \alpha K \alpha^{-1} \leq \mathrm{Aut}(P')$ for short, and consider the subgroup

$$N_e = \{a \in N_{G_e}(P') \mid c_a \in K'\}.$$

Fix $Q \in \mathrm{Syl}_p(N_e)$. Then $\mathrm{Aut}_{N_e}(P') = K'$ since $(e, [\alpha])$ is fixed by K , and $\mathrm{Aut}_Q(P') = K'$ since K' is a p -group. Also, since $\beta|_P = c_g \circ \alpha$, $gQg^{-1} \leq \beta(N_\varphi) \cdot C_{G_v}(\beta(P))$. Since $\beta(P)$ is p -centric in G_v , gQg^{-1} and $\beta(N_\varphi)$ are both Sylow p -subgroups of this last group. Hence there is $h \in C_{G_v}(\beta(P))$ such that $hgQg^{-1}h^{-1} = \beta(N_\varphi)$. Set $\bar{\alpha} = c_{hg}^{-1} \circ \beta \in \mathrm{Iso}(N_\varphi, Q)$. Then $\bar{\alpha}|_P = c_g^{-1} \circ \beta|_P = \alpha$ since h centralizes $\beta(P)$, so $\mathrm{Res}((e, [\bar{\alpha}])) = (e, [\alpha])$. Also, $(v, [\bar{\alpha}]) = (v, [\beta])$ is an endpoint of $(e, [\bar{\alpha}])$ since $hg \in G_v$, so $(e, [\bar{\alpha}])$ is an edge in $\mathrm{Rep}(N_\varphi, \mathcal{G})$, and this finishes the proof of (1).

Now, $(v_*, [\varphi]) \in \mathrm{Rep}(P, \mathcal{G})_*$ by Lemma 4.1(b), and is fixed by the K action by definition of N_φ . So by (1), there is $\psi \in \mathrm{Hom}(N_\varphi, G_*)$ such that $(v_*, [\psi])$ is in $\mathrm{Rep}(N_\varphi, \mathcal{G})_*$ and $[\psi|_P] = [\varphi]$ in $\mathrm{Rep}(P, G_*)$. Thus $\psi \in \mathrm{Hom}_G(N_\varphi, G_*)$, and $\psi|_P = c_g \circ \varphi$ for some $g \in G_*$. By axiom (II) for the saturated fusion system $\mathcal{F}_{G_*}(S)$, $c_g^{-1} = \varphi \circ (\psi|_P)^{-1}$ extends to some $\chi \in \mathrm{Hom}_{G_*}(\psi(N_\varphi), S)$, and hence $\bar{\varphi} \stackrel{\mathrm{def}}{=} \chi \circ \psi \in \mathrm{Hom}_G(N_\varphi, S)$ extends φ . This finishes the proof of (II) for \mathcal{F} -centric subgroups.

\mathcal{L} is a centric linking system. If P is \mathcal{F} -centric, then by point (b) and Lemma 4.1, $C_G(P)/Z(P)$ acts on the tree $\tilde{\mathcal{T}}^P$ with orbit space a tree. Furthermore, since $Z(P)$ is maximal among finite p -subgroups of $C_G(P)$, all isotropy subgroups of this action are finite of order prime to p . Hence by [I.10, p.39]Serre, $C_G(P)/Z(P)$ is an amalgamated product of finite groups of order prime to p taken over a finite tree. Such a group is clearly p -perfect (it is generated by elements of order prime to p); and Mayer-Vietoris sequences for the homology of amalgamated products (cf. [Bw, §VII.9]) show that $H^i(C_G(P)/Z(P); \mathbb{F}_p) = 0$ for all $i > 0$. So by Lemma 1.5, $\mathcal{L} = \mathcal{L}_S^c(G)$ is a centric linking system associated to \mathcal{F} . \square

The most difficult hypothesis to check in the above theorem is (b). For this reason, we give here some equivalent formulations. The equivalence of the first three conditions is implicit in the above proof, but we make them more explicit here.

Lemma 4.3. *Fix a finite tree of finite groups $(\mathcal{G}, \mathcal{T})$, and set $G = \mathcal{G}_\mathcal{T} = \underline{\mathrm{colim}}(\mathcal{G})$. Choose a vertex v_* of \mathcal{T} , and a subgroup $H \leq \mathcal{G}(v_*)$. Then the following three conditions are equivalent:*

- (1) *The abelianization of $C_G(H)$ is finite.*
- (2) *$C_G(H)$ is a finite amalgamated product of finite groups.*
- (3) *The component of $(v_*, [\mathrm{incl}_H^{\mathcal{G}(v_*)}])$ in the graph $\mathrm{Rep}(H, \mathcal{G})$ is a tree.*

Furthermore, if (1)–(3) hold, then:

- (4) There is a vertex $v \in \mathcal{T}^0$ and an element $x \in G$, such that $xHx^{-1} \leq \mathcal{G}(v)$ and $\text{Aut}_G(xHx^{-1}) = \text{Aut}_{\mathcal{G}(v)}(xHx^{-1})$.

Proof. Let $\tilde{\mathcal{T}}$ be the tree upon which G acts with orbit space and fundamental domain \mathcal{T} , as in Lemma 4.1. Again, when x is a vertex or edge of \mathcal{T} , we write $G_x = \mathcal{G}(x)$ for short. By Lemma 4.1, the component of $\text{Rep}(H, \mathcal{G})$ which contains $(v_*, [\text{incl}_H^{G_{v_*}}])$ can be identified with $\tilde{\mathcal{T}}^H/C_G(H)$. If this orbit graph is a tree, then by [Se, Theorem I.10, p.39], $C_G(H)$ is an amalgamated product of finite groups taken over a finite tree; and in particular, its abelianization is finite. If the orbit graph $\tilde{\mathcal{T}}^H/C_G(H)$ is not a tree, then by [Se, Corollary 1, p. 55], there is a surjection of $C_G(H)$ onto its fundamental group, an infinite free group, and hence the abelianization of $C_G(H)$ is not finite and $C_G(H)$ is not a finite amalgamated product of free groups. This proves the equivalence of (1), (2), and (3).

Now assume that (3) holds, and thus (by Lemma 4.1) that $\tilde{\mathcal{T}}^H/C_G(H)$ is a tree. The finite group $\text{Aut}_G(H) \cong N_G(H)/C_G(H)$ acts on this tree, and hence fixes some vertex (cf. [Se, §I.6.1]). Assume the orbit of the vertex $(a^{-1}G_v, v)$ is fixed by $\text{Aut}_G(H)$; in particular, $aHa^{-1} \leq G_v$ since $aG_v \in (G/G_v)^H$. Also, each $\alpha \in \text{Aut}_G(H)$ is of the form $\alpha = c_g$ for some $g \in N_G(H)$ which fixes the vertex $(a^{-1}G_v, v)$ in $\tilde{\mathcal{T}}^H$, which implies that $aga^{-1} \in G_v$. This shows that $\text{Aut}_G(aHa^{-1}) = \text{Aut}_{G_v}(aHa^{-1})$, and thus that (4) holds. \square

As shown in [AC], the fusion systems $\mathcal{F}_{\text{Sol}}(q)$ constructed in [LO] by the second and third authors are the fusion systems of certain amalgamated products $\text{Spin}_7(q) \ast_B K$, where B is the normalizer in $\text{Spin}_7(q)$ of a certain elementary abelian 2-subgroup of rank 2, and K contains B with index 3. The proof in [LO] that these fusion systems are saturated is very long and technical, and so it is natural to wonder whether or not this could be shown as an application of Theorem 4.2. As seen in [LO] or [AC], when $\mathcal{F} = \mathcal{F}_{\text{Sol}}(q)$ and $S \in \text{Syl}_p(\text{Spin}_7(q))$, then there is an elementary abelian 2-subgroup $E \leq S$ of rank 4 such that $\text{Aut}_{\mathcal{F}}(E) = \text{Aut}(E) \cong GL_4(2)$. Hence if the saturation of \mathcal{F} could be proven using Theorem 4.2, then by Lemma 4.3(4), some vertex of the tree defining the amalgamated product would be fixed by an extension of E by $\text{Aut}(E)$, and this is not the case. More precisely, this shows that condition (b) in Theorem 4.2 fails to hold for this amalgamated product. So Theorem 4.2 cannot be applied in this case.

5. EXAMPLES

We now look at some applications of Theorem 4.2, to produce explicit exotic fusion systems. These examples will all be based on Proposition 5.1 below, which in turn is a special case of Theorem 4.2.

For any fusion system \mathcal{F}_0 over a p -group S , any collection of subgroups $Q_1, \dots, Q_m \leq S$, and outer automorphism groups $\Delta_i \leq \text{Out}(Q_i)$ containing $\text{Out}_{\mathcal{F}_0}(Q_i)$, let

$$\langle \mathcal{F}_0; \Delta_1, \dots, \Delta_m \rangle$$

denote the fusion system over S generated by \mathcal{F}_0 and restrictions of automorphisms in the Δ_i to subgroups of Q_i . In other words, $\mathcal{F} = \langle \mathcal{F}_0; \Delta_i, \dots, \Delta_m \rangle$ is the fusion system

over S such that for all $P, Q \leq S$, $\text{Hom}_{\mathcal{F}}(P, Q)$ is the set of composites

$$P = P_0 \xrightarrow{\varphi_1} P_1 \xrightarrow{\varphi_2} P_2 \longrightarrow \cdots \longrightarrow P_{k-2} \xrightarrow{\varphi_{k-1}} P_{k-1} \xrightarrow{\varphi_k} P_k = Q$$

such that each j , either φ_j lies in $\text{Hom}_{\mathcal{F}_0}(P_{j-1}, P_j)$, or for some $1 \leq i \leq m$, $P_{j-1}, P_j \leq Q_i$ and φ_j is the restriction of some $\alpha_i \in \text{Aut}(Q_i)$ such that $[\alpha_i] \in \Delta_i$.

The following proposition is also a generalization of [BLO2, Proposition 9.1].

Proposition 5.1. *Fix a finite group G , a Sylow p -subgroup $S \leq G$, and subgroups $Q_1, \dots, Q_m \leq S$ such that no Q_i is G -conjugate to a subgroup of Q_j for $i \neq j$. For each i , set $K_i = \text{Out}_G(Q_i)$, and fix subgroups $\Delta_i \leq \text{Out}(Q_i)$ which contain K_i . Set $\mathcal{F} = \langle \mathcal{F}_S(G); \Delta_1, \dots, \Delta_m \rangle$. Assume for each i that*

- (1) $p \nmid [\Delta_i; K_i]$;
- (2) Q_i is p -centric in G , but no proper subgroup $P \not\leq Q_i$ is \mathcal{F} -centric or an essential p -subgroup of G ; and
- (3) for all $\alpha \in \Delta_i \setminus K_i$, $K_i \cap \alpha K_i \alpha^{-1}$ has order prime to p .

Then \mathcal{F} is a saturated fusion system over S , and has an associated centric linking system.

Proof. For each i , set $H_i = N_G(Q_i)$ and $T_i = O^p(C_G(Q_i))$. Then T_i has order prime to p since Q_i is p -centric in H_i ; and thus $Q_i \cdot C_G(Q_i) = Q_i T_i \cong Q_i \times T_i$ and $K_i \cong H_i / Q_i T_i$.

We first construct a finite group $G_i \geq H_i$ such that H_i has index prime to p in G_i , $Q_i \triangleleft G_i$, and $\text{Out}_{G_i}(Q_i) = \Delta_i$. By (3), $K_i \cap \alpha K_i \alpha^{-1}$ has order prime to p for all $\alpha \in \Delta_i \setminus K_i$; and hence the restriction homomorphism

$$H^j(\Delta_i; Z(Q_i)) \cong H^j(K_i; Z(Q_i))$$

is an isomorphism for all $j > 0$ by the description in of the image in terms of stable (or G -invariant) elements (cf. [AM, Theorem II.6.6] or [Bw, Theorem III.10.3]). When $j = 3$, the injectivity of the restriction map tells us that the obstruction to the existence of an extension

$$1 \longrightarrow Q_i \longrightarrow G'_i \longrightarrow \Delta_i \longrightarrow 1$$

vanishes [McL, Theorem IV.8.7], since its restriction to $K_i \leq \Delta_i$ vanishes. When $j = 2$, the group $H^2(\Delta_i; Z(Q_i)) \cong H^2(K_i; Z(Q_i))$ acts freely and transitively on the sets of all such extensions of Q_i by Δ_i or by K_i [McL, Theorem IV.8.8], and thus G'_i can be chosen to contain the group H_i/T_i .

Now let $T_i \wr K_i$ and $T_i \wr \Delta_i$ be the ‘‘regular’’ wreath products: the semidirect products $T_i^{|K_i|} \rtimes K_i$ and $T_i^{|\Delta_i|} \rtimes \Delta_i$ where K_i and Δ_i permute the factors T_i freely and transitively. There is an obvious embedding of $T_i \wr K_i$ into $T_i \wr \Delta_i$; and by [Hu, I.15.9], there is an embedding of H_i/Q_i (as an extension of T_i by K_i) into $T_i \wr K_i$. We can thus regard H_i/Q_i as a subgroup of $T_i \wr \Delta_i$. So if we define G_i to be the pullback of the maps

$$G'_i \longrightarrow \Delta_i \longleftarrow T_i \wr \Delta_i,$$

then G_i sits in an extension

$$1 \longrightarrow Q_i \times T_i^{|\Delta_i|} \longrightarrow G_i \longrightarrow \Delta_i \longrightarrow 1;$$

and we can identify H_i (regarded as a pullback of H_i/T_i and H_i/Q_i over K_i) as a subgroup of G_i with index prime to p . Also, $Q_i \triangleleft G_i$ and $\Delta_i = \text{Out}_{G_i}(Q_i)$.

We will apply Theorem 4.2 to the tree which has $m + 1$ vertices v_*, v_1, \dots, v_m and edges e_i connecting v_* to v_i , and to the functor $\mathcal{G}(v_*) = G$, $\mathcal{G}(v_i) = G_i$, and $\mathcal{G}(e_i) = H_i$.

Let \widehat{G} denote the amalgamated product of this tree of groups. Then $\mathcal{F} = \mathcal{F}_S(\widehat{G})$, and it remains only to check that conditions (a), (b), and (c) in Theorem 4.2 hold. Condition (a) holds by (1).

We next check condition 4.2(c). By definition of \mathcal{F} , for any subgroup $P \leq S$ which p -centric in G , P is \mathcal{F} -centric unless there is some $P' \leq S$ which is \mathcal{F} -conjugate to P but not G -conjugate, in which case P must be G -conjugate to a proper subgroup $P' \subsetneq Q_i$ for some i . For each i , any Sylow p -subgroup of H_i , or any essential p -subgroup (hence radical p -subgroup) of G_i , must contain $O_p(H_i) \geq Q_i$; and the Q_i are all p -centric in G (hence \mathcal{F} -centric) by assumption. Any essential p -subgroup P of G is p -centric in G ; and hence is \mathcal{F} -centric since by (b) again, no essential p -subgroup of G is properly contained in any Q_i . This finishes the proof of (c) in Theorem 4.2.

It remains to check condition 4.2(b). Fix an \mathcal{F} -centric subgroup $P \leq S$; we must show that the component Γ of $(v_*, [\text{incl}_P^G])$ in $\text{Rep}(P, \mathcal{G})$ is a tree. The edges in $\text{Rep}(P, \mathcal{G})$ adjacent to $(v_*, [\text{incl}_P^G])$ are of the form $(e_i, [\alpha])$, for $\alpha \in \text{Hom}_G(P, H_i)$. For any such edge, its other vertex $(v_i, [\alpha])$ is the endpoint of a second edge $(e_i, [\beta])$ only if $\beta = c_g \circ \alpha$ for some $g \in G_i \setminus H_i$. In particular, $\alpha(P)$ and $g\alpha(P)g^{-1}$ are both contained in H_i , and thus $\alpha(P) \leq H_i \cap g^{-1}H_i g$. By (3), $(H_i \cap g^{-1}H_i g)/Q_i$ has order prime to p , and hence $\alpha(P) \leq Q_i$. Since Q_i is a minimal \mathcal{F} -centric subgroup, this implies that $\alpha(P) = Q_i$. Since no two of the Q_i are G -conjugate, this can occur for at most one i . We thus have two possibilities:

- P is not G -conjugate to any Q_i . In this case, every edge in Γ is adjacent to the vertex $(v_*, [\text{incl}_P^G])$, and Γ is a tree.
- P is G -conjugate to Q_j for some fixed $j \in \{1, \dots, m\}$. In this case, let $\Gamma_0 \subseteq \Gamma$ be the subgraph of all vertices sitting over v_* or v_j , and all edges sitting over e_j . Each vertex of Γ not in Γ_0 sits over v_i for some $i \neq j$, and by the above remarks is connected to Γ_0 by a unique edge. Thus Γ_0 is a deformation retract of Γ . Each edge in Γ_0 has the form $(e_j, [\alpha])$ for some $\alpha \in \text{Iso}(P, Q_i)$. If two edges $(e_j, [\alpha])$ and $(e_j, [\alpha'])$ have the same vertex $(v_*, [\alpha]) = (v_*, [\alpha'])$, then $\alpha' = c_g \circ \alpha$ for some $g \in G$, so $g \in N_G(Q_i) = H_i$, and the edges $(e_j, [\alpha])$ and $(e_j, [\alpha'])$ are equal. Thus no vertex in Γ_0 over v_* can be attached to two edges, and this proves that Γ_0 (and hence Γ) is a tree.

This finishes the proof of condition 4.2(b), and hence of the proposition. \square

Note that (3) implies (1) in the above proposition; condition (1) has been kept for emphasis.

Condition (3) means that the subgroup K_i is strongly embedded in Δ_i at the prime p (see Definition 3.2). This puts fairly restrictive conditions on K_i and Δ_i , especially when $p = 2$. By a theorem of Bender [Be], if Δ has a strongly embedded subgroup at $p = 2$, then either its Sylow 2-subgroups are cyclic or quaternion, or there is a normal series $A \triangleleft B \triangleleft \Delta$ where A and Δ/B have odd order, and B/A is isomorphic to $PSL_2(q)$, $Sz(q)$, or $PSU_3(q)$ for q some power of 2. The severe restrictions which this places on the groups involved when applying Proposition 5.1 with $p = 2$ help to explain why it seems unlikely that we could construct an exotic fusion system at the prime 2 using this proposition, although we are not yet able to completely exclude that possibility.

The following lemma is a refinement of [BLO2, Lemma 9.2], and will be used to show that certain fusion systems are not fusion systems of finite groups. When \mathcal{F} is a fusion

system over the p -group S , a subgroup $P \leq S$ is *strongly closed* if no element of P is \mathcal{F} -conjugate to an element of $S \setminus P$. The subgroup P is *normal in \mathcal{F}* if each morphism in \mathcal{F} extends to a morphism between subgroups containing P which sends P to itself. If P is normal in \mathcal{F} , then it is strongly closed, but not conversely.

As usual, a finite group G is *almost simple* if it contains a normal, nonabelian simple subgroup $L \triangleleft G$ such that $C_G(L) = 1$. In other words, G can be identified with a subgroup of $\text{Aut}(L)$, and G/L with a subgroup of $\text{Out}(L)$.

Lemma 5.2. *Let \mathcal{F} be a fusion system over a nonabelian p -group S . Assume, for each subgroup $1 \neq P \leq S$ which is strongly closed in \mathcal{F} , that*

- (a) P is centric in S (i.e., $C_S(P) = Z(P)$);
- (b) P is not normal in \mathcal{F} ; and
- (c) P does not factorize as a product of two or more subgroups which are permuted transitively by $\text{Aut}_{\mathcal{F}}(P)$.

Then if \mathcal{F} is the fusion system of a finite group, it is the fusion system of a finite almost simple group.

Proof. Assume that $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group G with $S \in \text{Syl}_p(G)$, and that G is a subgroup of minimal order with this property. Let $1 \neq L \triangleleft G$ be a minimal nontrivial normal subgroup. Set $P = L \cap S \in \text{Syl}_p(L)$; then P is strongly closed in \mathcal{F} . If $P = 1$ (i.e., L has order prime to p), then \mathcal{F} is also the fusion system of G/L , which contradicts the minimality assumption. If $L = P$ is an abelian p -group, then it is normal in \mathcal{F} , which contradicts (b). Thus, since L is minimal, it is a product of nonabelian simple groups isomorphic to each other (cf. [Go, Theorem 2.1.5]); and these must be permuted transitively by $N_G(L) = G$ since otherwise L is not minimal. Then L must be simple by (c). Also, $C_G(L) \cap S \leq C_S(P) \leq P$ by (a). Since $C_G(L) \triangleleft G$, this means it must have order prime to p (otherwise it would intersect every Sylow p -subgroup nontrivially); and this implies $C_G(L) = 1$ by the minimality assumption (again since $C_G(L) \triangleleft G$). Thus G is almost simple; i.e., $L \triangleleft G \leq \text{Aut}(L)$. \square

We next focus attention on cases where Proposition 5.1 can be applied with $Q_i \cong C_p^2$ (for p an odd prime). By [Hu, Satz III.14.23], a p -group S contains a centric subgroup of order p^2 if and only if it has maximal class; i.e., if and only if it has nilpotence class $n - 1$ when $|G| = p^n$. For odd p , the structure of p -groups of maximal class is described in detail in [Hu, §III.14], and include the following examples when $p = 3$.

Example 5.3. *Set $p = 3$, and let S be one of the following groups of order 3^4 :*

$$\begin{aligned} S' &= \langle a, b, x \mid a^9 = b^3 = x^3 = [a, b] = 1, \quad xax^{-1} = ab, \quad xbx^{-1} = ba^{-3} \rangle. \\ S'' &= \langle a, b, x \mid a^9 = b^3 = x^3 = [a, b] = 1, \quad xax^{-1} = ab, \quad xbx^{-1} = ba^3 \rangle. \end{aligned}$$

Let $\omega, \eta \in \text{Aut}(S)$ be the automorphisms

$$\omega(a) = a^{-1}, \quad \omega(b) = x^{-1}bx = \begin{cases} ba^3 & \text{if } S = S' \\ ba^{-3} & \text{if } S = S'', \end{cases} \quad \omega(x) = x^{-1}$$

$$\eta(a) = a^{-1}, \quad \eta(b) = b^{-1}, \quad \eta(x) = x.$$

For $i = 0, 1, 2$, set

$$R_i = \langle xa^i b^i, a^3 \rangle \quad \text{and} \quad Q_i = \langle R_i, b \rangle.$$

Then $R_i \cong C_3^2$ if $S = S'$ or if $i = 0$, and $R_i \cong C_9$ if $S = S''$ and $i = 1, 2$. Also, Q_i is extraspecial of order 27, and has exponent 3 if $S = S'$ or $i = 0$ and exponent 9 otherwise. All of these subgroups are invariant under ω , while η leaves R_0 and Q_0 invariant and switches R_1 and R_2 .

When $S = S''$, then the following fusion systems over S

$$\langle \mathcal{F}_S(S \rtimes \langle \omega \rangle); SL(R_0) \rangle \quad \text{and} \quad \langle \mathcal{F}_S(S \rtimes \langle \omega, \eta \rangle); GL(R_0) \rangle$$

are both saturated, and not fusion systems of any finite group. When $S = S'$, then the following table describes different fusion systems over S via the automorphism groups $\text{Out}_{\mathcal{F}}(P)$ for $P = S, R_i$, or Q_i , and where an asterisk marks those which are not fusion systems of any finite group:

$\text{Out}_{\mathcal{F}}(S)$	$\text{Aut}_{\mathcal{F}}(R_0)$	$\text{Out}_{\mathcal{F}}(Q_0)$	$\text{Aut}_{\mathcal{F}}(R_1)$	$\text{Aut}_{\mathcal{F}}(R_2)$	group
$\langle \omega \rangle$	$SL_2(3)$	—	$SL_2(3)$	$SL_2(3)$	$L_3^{\pm}(q)$ ($v_3(q \mp 1) = 2$)
$\langle \omega \rangle$	—	—	$SL_2(3)$	$SL_2(3)$	*
$\langle \omega \rangle$	$SL_2(3)$	—	—	—	*
$\langle \eta, \omega \rangle$	$GL_2(3)$	—	$SL_2(3)$		$L_3^{\pm}(q) \rtimes C_2$ ($v_3(q \mp 1) = 2$)
$\langle \eta, \omega \rangle$	—	$GL_2(3)$	$SL_2(3)$		${}^3D_4(q)$ ($v_3(q^2 - 1) = 1$)
$\langle \eta, \omega \rangle$	—	—	$SL_2(3)$		*
$\langle \eta, \omega \rangle$	$GL_2(3)$	—	—		*

Here, $L_n^+(q) = PSL_n(q)$ and $L_n^-(q) = PSU_n(q)$. Also, for any prime power q , ${}^3D_4(q)$ is the fixed subgroup of a certain “triatlity” graph automorphism of order 3 on $\text{Spin}_8(q^3)$.

Proof. That these fusion systems are all saturated is a special case of Proposition 5.1, applied with $G = S \rtimes \langle \omega \rangle$ or $G = S \rtimes \langle \eta, \omega \rangle$ as appropriate.

Let \mathcal{F} be any of these fusion systems, and assume $P \triangleleft S$ is a proper strongly closed subgroup. Then $P \geq \langle a^3 \rangle$ (any normal subgroup contains the center). If \mathcal{F} contains $SL(R_i)$ for some i , then $P \geq R_i$ since $\langle a^3 \rangle$ is \mathcal{F} -conjugate to the other subgroups of order 3 in R_i ; and hence $P \geq Q_i$ (the normal closure of R_i in S). By similar reasoning, if \mathcal{F} contains $SL(Q_i)$ for some i , then either $P = \langle a^3 \rangle$, or $P \geq Q_i$. Thus in all cases listed above, the only nontrivial subgroups strongly closed in \mathcal{F} are S , and possibly one of the Q_i . Hence by Lemma 5.2, if \mathcal{F} is the fusion system of a finite group, then it is the fusion system of a finite almost simple group G , which contains a normal simple group $L \triangleleft G$ with Sylow 3-subgroup S or Q_i . If L contains a Sylow 3-subgroup Q_i , then $3 \mid |G/L| \mid |\text{Out}(L)|$, and this is impossible by Lemma 5.4 below.

It remains to consider the case where $[G : L]$ is prime to 3, and thus where $S \in \text{Syl}_3(L)$. By [GLS, Tables 5.3 & 5.6.1], none of the sporadic simple groups has Sylow 3-subgroup of order 3^4 and rank 2. If $v_3(|A_n|) = 4$, then $n = 9, 10, 11$, and $\text{rk}_3(A_n) = 3$. By [GLS, Table 2.2], the only simple groups of Lie type in characteristic 3 whose Sylow 3-subgroups have order 3^4 are the groups $B_2(3)$, and these also have 3-rank equal to 3. Finally, using [GLS, Table 2.2] and [GL, 10-1 & 10-2], one checks that the only simple groups of Lie type whose Sylow 3-subgroups have order 3^4 and rank 2 are the groups $L_3(q)$ when $v_3(q - 1) = 2$, $U_3(q)$ when $v_3(q + 1) = 2$, and ${}^3D_4(q)$ when $v_3(q^2 - 1) = 1$. The precise fusion systems of these groups (and the fact that their Sylow subgroups are isomorphic to S') is determined directly, or with the help of the lists of maximal subgroups in [GLS, Theorem 6.5.3] (for $L_3^{\pm}(q)$) and [KI] (for ${}^3D_4(q)$). For example, by

[K1], there are subgroups

$$3^2 \cdot SL_2(3) \leq {}^3D_4(q) \quad \text{and} \quad 3_+^{1+2} \cdot GL_2(3) \leq SL_3^\pm(q) \cdot S_3 \leq {}^3D_4(q)$$

(when $q \equiv \pm 1 \pmod{3}$), and this determines the structure of the fusion system of ${}^3D_4(q)$. \square

It remains to prove the following lemma, which will also be used later.

Lemma 5.4. *There is no pair (L, p) , where L is a finite simple group, p is an odd prime, the Sylow p -subgroups of L are extraspecial of order p^3 , and $p \mid |\text{Out}(L)|$.*

Proof. If L is a sporadic or alternating group, then $|\text{Out}(L)|$ is a power of 2, so this is impossible. Thus L is of Lie type, and hence by [Ca, Theorem 12.5.1], $\text{Out}(L)$ is generated by field, graph, and diagonal automorphisms. We refer to [Ca, 9.4.10, 10.2.4–5, 14.3.2] for the orders of the simple groups of Lie type. The only simple groups with graph automorphisms of odd order are the groups $D_4(q)$ (with graph automorphisms of order 3), and $|D_4(q)| = q^{12}(q^8 + q^4 + 1)(q^6 - 1)(q^2 - 1)$ is a multiple of 3^4 for all q . If L has a field automorphism of order p , where p is an odd prime, then L is defined over a field of order q^p for some prime power q ; if $q^{pn} \pm 1$ is divisible by p then it is divisible by p^2 , and the list of orders of groups of Lie type makes it clear that this case is impossible. So if there is a pair (L, p) as above, then L must have a diagonal automorphism of order p .

The only simple groups of Lie type with diagonal automorphisms of order $p \geq 3$ are $PSL_n(q)$ (for $p \mid (n, q - 1)$), $PSU_n(q)$ (for $p \mid (n, q + 1)$), $E_6(q)$ (for $p = 3 \mid q - 1$), and ${}^2E_6(q)$ (for $p = 3 \mid q + 1$). Of these, the only cases where the simple group has p -rank ≤ 2 occur when $p = 3$, and $L = PSL_3(q)$ (where $3 \mid (q - 1)$ and $|L| = \frac{1}{3}q^3(q^2 - 1)(q^3 - 1)$) or $PSU_3(q)$ (where $3 \mid (q + 1)$ and $|L| = \frac{1}{3}q^3(q^2 - 1)(q^3 + 1)$). In both of these cases, $v_3(|L|) = 2v_3(q \pm 1) \neq 3$. \square

For the rest of the section, we let p be any odd prime, and consider the group

$$S = \langle a, b, c, x \mid a^p = b^p = c^p = x^p = [a, b] = [a, c] = [b, c] = 1, \\ xax^{-1} = a, \quad xbx^{-1} = ab, \quad xcx^{-1} = bc \rangle.$$

Set $A = \langle a, b, c \rangle$, $Q = \langle a, b, x \rangle$, and $R = \langle a, x \rangle$. Set

$$\Omega = (GL_2(p) \times \mathbb{F}_p^\times) / \{(uI, u^{-2}) \mid u \in \mathbb{F}_p^\times\},$$

and let $[B, u]$ denote the class of the pair (B, u) for $B \in GL_2(p)$ and $u \in \mathbb{F}_p^\times$. Define an action $\chi: \Omega \longrightarrow \text{Aut}(A)$ as follows. Identify A with the additive group $S_2(p)$ of symmetric 2×2 matrices by setting $a = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, and $c = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$; and let $[B, u] \in \Omega$ act by sending M to $u \cdot BMB^t$. These identifications are chosen so that the action of $X \stackrel{\text{def}}{=} [\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1]$ on A is precisely the action of $x \in S$ by conjugation. We can thus identify S as a Sylow p -subgroup of $\bar{G} \stackrel{\text{def}}{=} A \rtimes \Omega$. Then Ω is the normalizer in $\text{Aut}(A) \cong GL_3(p)$ of the orthogonal group $GO_3(p)$, for an appropriate choice of quadratic form on A .

For a given pair of fusion systems $\mathcal{F}' \subseteq \mathcal{F}$ over the same p -group S , we say that \mathcal{F}' has index prime to p in \mathcal{F} if $\text{Aut}_{\mathcal{F}'}(P) \geq O^{p'}(\text{Aut}_{\mathcal{F}}(P))$ for all $P \leq S$ [5a2, Definition 3.1]. In [5a2, §5], we prove that for any saturated fusion system \mathcal{F} , there is a unique minimal fusion subsystem $O^{p'}(\mathcal{F}) \subseteq \mathcal{F}$ of index prime to p . This terminology provides a convenient framework for describing the next result.

Example 5.5. Fix an odd prime p , and let S be the group of order p^4 defined above, with subgroups $A, Q, R \leq S$. Then the following hold.

(a) There are unique saturated fusion systems \mathcal{F}_Q and \mathcal{F}_R over S such that

$$\begin{aligned} \text{Out}_{\mathcal{F}_Q}(S) &\cong C_{p-1} \times C_{p-1}, & \text{Aut}_{\mathcal{F}_Q}(A) &= \Omega, & \text{Out}_{\mathcal{F}_Q}(Q) &= \text{Out}(Q) \cong GL_2(p) \\ \text{Out}_{\mathcal{F}_R}(S) &\cong C_{p-1} \times C_{p-1}, & \text{Aut}_{\mathcal{F}_R}(A) &= \Omega, & \text{Aut}_{\mathcal{F}_R}(R) &= \text{Aut}(R) \cong GL_2(p); \end{aligned}$$

and Q is not \mathcal{F}_R -radical.

(b) $O^{p'}(\mathcal{F}_Q)$ has index 2 in \mathcal{F}_Q , $\text{Out}_{O^{p'}(\mathcal{F}_Q)}(Q)$ is the unique subgroup of index 2 in $\text{Out}(Q) \cong GL_2(p)$, and

$$\text{Aut}_{O^{p'}(\mathcal{F}_Q)}(A) = \{[B, 1] \mid B \in GL_2(p)\}.$$

For all p , \mathcal{F}_Q is the fusion system of $\text{Aut}(PSp_4(p)) = PSp_4(p) \rtimes C_2$ (the extension by diagonal automorphisms), and $O^{p'}(\mathcal{F}_Q)$ is the fusion system of $PSp_4(p)$.

(c) $O^{p'}(\mathcal{F}_R)$ has index $(4, p-1)$ in \mathcal{F}_R , $\text{Aut}_{O^{p'}(\mathcal{F}_R)}(R)$ is the unique subgroup of index $(4, p-1)$ in $\text{Aut}(Q) \cong GL_2(p)$, and

$$\text{Aut}_{O^{p'}(\mathcal{F}_R)}(A) = \{[B, u] \in \Omega \mid \det(B) \cdot u^{-1} \in \mathbb{F}_p^{\times 4}\}.$$

When $p = 3$, \mathcal{F}_R is the fusion system of Σ_9 and $O^{3'}(\mathcal{F}_R)$ is the fusion system of A_9 . When $p = 5$, \mathcal{F}_R is the fusion system of $P\Sigma L_5(16) \cong PSL_5(16) \rtimes C_4$ (the extension by field automorphisms), and $O^{5'}(\mathcal{F}_R)$ is the fusion system of $PSL_5(16)$. When $p \geq 7$, no fusion subsystem of index prime to p in \mathcal{F}_R is the fusion system of a finite group.

Proof. Set $G = A \rtimes \Omega$, and identify S with $A \rtimes \langle X \rangle \leq G$. Since $N_\Omega(\langle X \rangle)$ is generated by $X = \left[\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right), 1 \right]$ together with elements $\left[\left(\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix} \right), w \right]$ for $u, v, w \in \mathbb{F}_p^\times$, and since $\left[\left(\begin{smallmatrix} u & 0 \\ 0 & u \end{smallmatrix} \right), u^{-2} \right] = 1$,

$$\text{Out}_{\mathcal{F}_Q}(S) = \text{Out}_{\mathcal{F}_R}(S) = \text{Out}_G(S) = \{[\eta_u \omega_v] \mid u, v \in \mathbb{F}_p^\times\} \cong C_{p-1} \times C_{p-1},$$

where $[\eta_u]$ and $[\omega_v]$ are the classes modulo $\text{Inn}(S)$ of the automorphisms

$$\begin{aligned} \eta_u = c\left(\left[\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), 1\right]\right) &: & a &\mapsto a & b &\mapsto b^u a^{-(u-1)/2} & c &\mapsto c^{u^2} & x &\mapsto x^{1/u} \\ \omega_v = c\left(\left[\left(\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right), v\right]\right) &: & a &\mapsto a^v & b &\mapsto b^v & c &\mapsto c^v & x &\mapsto x \end{aligned}$$

for all $u \in \mathbb{F}_p^\times$. Here, $c(g)$ denotes conjugation by g ($a \mapsto gag^{-1}$). Note also the relation

$$c\left(\left[\left(\begin{smallmatrix} u & 0 \\ 0 & u^{-1} \end{smallmatrix}\right), 1\right]\right) = c\left(\left[\left(\begin{smallmatrix} 1 & 0 \\ 0 & u^{-2} \end{smallmatrix}\right), u^2\right]\right) = \omega_{u^2} \eta_{1/u^2} = \omega_u^2 \eta_u^{-2}.$$

It follows that

$$\text{Out}_G(Q) = \langle \eta_u|_Q, \omega_u|_Q \rangle = N_{\text{Out}(Q)}(\text{Out}_G(Q)).$$

So we can apply Proposition 5.1 with $m = 1$ and $Q_1 = Q$ (and with G as above), to prove that the fusion system \mathcal{F}_Q is saturated. Similarly,

$$\text{Aut}_G(R) = \langle \eta_u|_R, \omega_u|_R \rangle = N_{\text{Aut}(R)}(\text{Aut}_G(R)),$$

and so \mathcal{F}_R is saturated by Proposition 5.1 again.

We next calculate $O^{p'}(\mathcal{F}_Q)$. Let $O_*^{p'}(\mathcal{F}_Q) \subseteq \mathcal{F}_Q$ be the fusion subsystem generated by the automorphism groups $O^{p'}(\text{Aut}_{\mathcal{F}}(P))$ for $P \leq S$. Consider the subgroup $\text{Out}_{\mathcal{F}_Q}^0(S) \triangleleft \text{Out}_{\mathcal{F}_Q}(S)$ as defined in [5a2, §5.1]:

$$\text{Out}_{\mathcal{F}_Q}^0(S) = \langle \alpha \in \text{Out}_{\mathcal{F}_Q}(S) \mid \alpha|_P \in \text{Mor}_{O_*^{p'}(\mathcal{F}_Q)}(P, S), \text{ some } \mathcal{F}_Q\text{-centric } P \leq S \rangle.$$

For $u, v \in \mathbb{F}_p^\times$, $(\eta_u \omega_v)|_Q \in O^{p'}(\text{Aut}_{\mathcal{F}_Q}(Q)) \cong SL_2(p)$ if and only if $v = 1$; while $(\eta_u \omega_v)|_A \in O^{p'}(\text{Aut}_{\mathcal{F}_Q}(A)) = \Omega_0$ if and only if $v = 1/u \in \mathbb{F}_p^{\times 2}$. Thus $\text{Out}_{\mathcal{F}_Q}^0(S) = \{\eta_u \omega_{v^2}\}$. This shows that $O^{p'}(\mathcal{F}_Q)$ has index 2 in \mathcal{F}_Q , and has the form described in (b).

A similar argument shows that $O^{p'}(\mathcal{F}_R)$ has index $(4, p-1)$ in \mathcal{F}_R , and has the form described in (c).

Thus for all \mathcal{F} -centric subgroups $P' \leq S$, $\text{Aut}_{\mathcal{F}}(P')$ contains $O^{p'}(\text{Aut}_{\mathcal{F}_P}(P'))$ ($P = Q$ or R). Hence \mathcal{F} has index prime to p in \mathcal{F}_P , and by [5a2, Theorem 5.4], $O^{p'}(\mathcal{F}_P) \subseteq \mathcal{F} \subseteq \mathcal{F}_P$.

It is straightforward to check that the finite groups listed in (b) and (c) have the automorphism groups as indicated, and we have seen that this determines their fusion system. So it remains to show that the fusion systems in (c) are not fusion systems of finite groups for $p \geq 7$.

Let \mathcal{F} be any of these fusion systems. If $1 \neq P \triangleleft S$ is strongly closed in \mathcal{F} , then it must contain $Z(S) = \langle a \rangle$ (any nontrivial normal subgroup intersects nontrivially with $Z(S)$); hence contains A (since the subgroups $\text{Aut}_{\mathcal{F}}(A)$ -conjugate to $\langle a \rangle$ generate A in all cases); and hence is equal to S since either Q or R is \mathcal{F} -radical. So by [BLO2, Lemma 9.2], if \mathcal{F} is the fusion system of a finite group, it must be the fusion system of a finite almost simple group. More precisely, $\mathcal{F} = \mathcal{F}_S(G)$ for some G with normal simple subgroup $L \triangleleft G$ of index prime to p such that $C_G(L) = 1$. By a direct check through the list of finite simple groups, one sees that the following are the only simple groups which have Sylow p -subgroup isomorphic to S :

- (any p) $PSp_4(p)$
- ($p = 3$) $PSL_4(q)$ ($q \equiv 4, 7 \pmod{9}$), $PSU_4(q)$ ($q \equiv 2, 5 \pmod{9}$), $PSp_6(q)$ ($q \equiv \pm 2, \pm 4 \pmod{9}$), $\Omega_7(q)$ ($q \equiv \pm 2, \pm 4 \pmod{9}$), A_n ($n = 9, 10, 11$).
- ($p = 5$) $PSL_5(q)$ ($q \equiv 6, 11, 16, 21 \pmod{5}$), $PSU_5(q)$ ($q \equiv 4, 9, 14, 19 \pmod{5}$), C_{O_1} .

By elimination, none of the $\mathcal{F}_{R,i}$ for $p \geq 7$ is the fusion system of a finite group. \square

In fact, in the above situation, if \mathcal{F} is any saturated fusion system over S such that A is \mathcal{F} -radical but not normal in \mathcal{F} , then \mathcal{F} is isomorphic to a fusion system \mathcal{F}' over S which has index prime to p in one of the fusion systems \mathcal{F}_Q or \mathcal{F}_R . To see this, set $\Gamma = \text{Aut}_{\mathcal{F}}(A) \leq GL_3(p)$ for short, and let Γ' be the image of $\Gamma \cap SL_3(p)$ in $PSL_3(p)$. If Γ' has a nontrivial normal subgroup of order prime to p , then either the action on A is decomposable (which is impossible since the action of $\text{Aut}_S(A)$ is indecomposable), or A splits as a sum of three subspaces which are permuted by Γ . The latter case implies that $\Gamma \leq C_{p-1} \wr \Sigma_3$, and thus is possible only if $p = 3$ and $\Gamma \triangleleft C_2 \wr \Sigma_3 = \Omega$. Otherwise, if Γ' has no nontrivial normal subgroups of prime power order, then by [Bl, Theorem 1.1], Γ' must be isomorphic to $PSL_2(p)$ or $PGL_2(p)$; and conjugate to the indecomposable representation of these groups described in [Bl, Lemma 6.3]. Hence up to conjugacy, $\Gamma \cap SL_3(p)$ contains Ω_0 as a normal subgroup, and hence that $\Gamma \leq N_{GL_3(p)}(PSL_2(p)) = \Omega$.

In particular, every proper subgroup $P \not\leq A$ not contained in A is either \mathcal{F} -conjugate to Q or R ; or else $p = 3$ and P is cyclic or extraspecial of exponent 9 (in which case P cannot be \mathcal{F} -radical). Hence since A is not normal in \mathcal{F} , one of the subgroups Q or R must be \mathcal{F} -radical. If $P = Q$ or R is \mathcal{F} -radical, then $\text{Out}_{\mathcal{F}}(P) \leq \text{Out}(P) \cong GL_2(p)$

contains at least two subgroups of order p ; any two such subgroups generate $SL_2(p)$; and thus $\text{Out}_{\mathcal{F}}(P) \geq SL_2(p)$.

When $p \geq 5$, there are also saturated fusion systems over S where A is not radical, but is not normal either. Fix any subset $I \subseteq \{0, \dots, p-1\}$ with $|I| \geq 2$, and choose $P_i = \langle a, b, c^i x \rangle$ or $\langle a, c^i x \rangle$ for each $i \in I$. Let \mathcal{F}_I be the fusion system over S generated by $\text{Out}_{\mathcal{F}_I}(S) = \langle \eta_u, \omega_u \rangle \cong C_{p-1}^2$ (where η_u and ω_u are defined as above), and $\text{Out}_{\mathcal{F}_I}(P_i) = \text{Out}(P_i)$ for all $i \in I$. (This depends not just on I but also on the choice of the P_i .) These are saturated fusion systems by Proposition 5.1, and have no proper strongly closed subgroups. So by the list of simple groups with Sylow subgroup S given above, these systems are all exotic.

We look more closely only at the case where $|I| = 1$ and $P_i \cong R$. In this case, there is a proper strongly closed subgroup.

Example 5.6. Fix an odd prime p , and let S be the group of order p^4 defined above. For $u = 1, \dots, p-1$, let $\omega_u, \eta_u \in \text{Aut}(S)$ be the automorphisms

$$\begin{aligned} \omega_u(a) &= a^u, & \omega_u(b) &= b^u, & \omega_u(c) &= c^u, & \omega_u(x) &= x; \\ \eta_u(a) &= a, & \eta_u(b) &= b^u a^{-(u-1)/2}, & \eta_u(c) &= c^{u^2}, & \eta_u(x) &= x^{1/u}. \end{aligned}$$

Set

$$\Gamma = \{\omega_u \eta_v \mid u, v = 1, \dots, p-1\} \quad \text{and} \quad \Gamma_0 = \{\omega_u \eta_u \mid u = 1, \dots, p-1\}.$$

Set $R = \langle x, a \rangle \cong C_p^2$; a subgroup invariant under each ω_i and η_i . Then the fusion systems

$$\mathcal{F} = \langle \mathcal{F}_S(S \rtimes \Gamma); GL(R) \rangle \quad \text{and} \quad \mathcal{F}_0 = \langle \mathcal{F}_S(S \rtimes \Gamma_0); SL(R) \rangle$$

are both saturated, and neither is the fusion system of a finite group.

Proof. The fusion systems \mathcal{F} and \mathcal{F}_0 are saturated by Proposition 5.1, applied with $G = S \rtimes \Gamma$ or $S \rtimes \Gamma_0$, respectively.

If $P \neq 1$ is strongly closed in \mathcal{F} or \mathcal{F}_0 , then $P \geq \langle a \rangle = Z(S)$ (since any nontrivial normal subgroup intersects nontrivially with the center), hence $P \geq R$, and hence $P \geq Q = \langle a, b, x \rangle$ since bx is S -conjugate to x . Thus $P = Q$ or S . Hence by Lemma 5.2, if \mathcal{F} or \mathcal{F}_0 is the fusion system of a finite group G , then we can assume that there is a normal simple subgroup $L \triangleleft G$ such that $C_G(L) = 1$ (so $G/L \leq \text{Out}(L)$), and such that $L \geq S$ or $L \cap S = Q$. If $L \cap S = Q$, then $p \mid |G/L| \mid |\text{Out}(L)|$, and this is impossible by Lemma 5.4. Thus $S \in \text{Syl}_p(L)$.

In the proof of Example 5.5, we listed all simple groups L with Sylow p -subgroup isomorphic to S for some odd p . In all cases, the (unique) abelian subgroup of index p in S is radical in L , and thus $\mathcal{F}_S(L)$ is not contained in \mathcal{F} . \square

REFERENCES

- [AM] A. Adem & J. Milgram, Cohomology of finite groups, Springer-Verlag (1994)
- [As] M. Aschbacher, Finite group theory, Cambridge Univ. Press (1986)
- [AC] M. Aschbacher & A. Chermak, A group-theoretic approach to a family of 2-local finite groups constructed by Levi and Oliver (in preparation)
- [Be] H. Bender, Transitive Gruppen gerader Ordnung, in denen jede Involution genau einen Punkt festlässt, J. Algebra 17 (1971), 527–554
- [Bl] D. Bloom, The subgroups of $PSL(3, q)$ for odd q , Trans. Amer. Math. Soc. 127 (1967), 150–178

- [5a1] C. Broto, N. Castellana, J. Grodal, R. Levi, & B. Oliver, Subgroup families controlling p -local finite groups, Proc. London Math. Soc. (to appear)
- [5a2] C. Broto, N. Castellana, J. Grodal, R. Levi, & B. Oliver, Extensions of p -local finite groups (preprint)
- [BrK] C. Broto & N. Kitchloo, Classifying spaces of Kač-Moody groups, Math. Z. 240 (2002), 621–649
- [BLO1] C. Broto, R. Levi, & B. Oliver, Homotopy equivalences of p -completed classifying spaces of finite groups, Invent. math. 151 (2003), 611–664
- [BLO2] C. Broto, R. Levi, & B. Oliver, The homotopy theory of fusion systems, Journal Amer. Math. Soc. 16 (2003), 779–856
- [Bw] K. Brown, Cohomology of groups, Springer-Verlag (1982)
- [Ca] R. Carter, Simple groups of Lie type, Wiley (1989)
- [Gd] D. Goldschmidt, A conjugation family for finite groups, J. Algebra 16 (1970), 138–142
- [Go] D. Gorenstein, Finite groups, Harper & Row (1968)
- [GL] D. Gorenstein & R. Lyons, The local structure of finite groups of characteristic 2 type, Memoirs Amer. Math. Soc. 276 (1983)
- [GLS] D. Gorenstein, R. Lyons, & R. Solomon, The classification of the finite simple groups, nr. 3, Amer. Math. Soc. (1998)
- [Hu] B. Huppert, Endliche Gruppen I, Springer-Verlag (1967)
- [Kl] P. Kleidman, The maximal subgroups of the Steinberg triality groups ${}^3D_4(q)$ and of their automorphism groups, J. Algebra 115 (1988), 182–199
- [LO] R. Levi, & B. Oliver, Construction of 2-local finite groups of a type studied by Solomon and Benson, Geometry & Topology 6 (2002), 917–990
- [McL] S. Mac Lane, Homology, Springer-Verlag (1963, 1975)
- [Pu] L. Puig, Unpublished notes (ca. 1990)
- [Se] J.-P. Serre, Trees, Springer-Verlag (1980)
- [St] R. Stancu, Equivalent definitions of fusion systems, preprint

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