LOOP SPACE HOMOLOGY OF A SMALL CATEGORY

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Abstract. In a 2009 paper, Dave Benson gave a description in purely algebraic terms of the mod p homology of $\Omega(BG^\wedge_p)$, when G is a finite group, $BG^\wedge_p$ is the p-completion of its classifying space, and $\Omega(BG^\wedge_p)$ is the loop space of $BG^\wedge_p$. The main purpose of this work is to shed new light on Benson's result by extending it to a more general setting. As a special case, we show that if $\mathcal{C}$ is a small category, $|\mathcal{C}|$ is the geometric realization of its nerve, R is a commutative ring, and $|\mathcal{C}|^+_R$ is a "plus construction" for $|\mathcal{C}|$ in the sense of Quillen (taken with respect to R-homology), then $H_*(\Omega(|\mathcal{C}|^+_R);R)$ can be described as the homology of a chain complex of projective $R\mathcal{C}$-modules satisfying a certain list of algebraic conditions that determine it uniquely up to chain homotopy. Benson's theorem is now the case where $\mathcal{C}$ is the category of a finite group G, $R=\mathbb{F}_p$ for some prime p, and $|\mathcal{C}|^+_R = BG^\wedge_p$.

Introduction

Let G be a finite group, and let $BG^\wedge_p$ denote its classifying space after p-completion in the sense of Bousfield and Kan [BK]. In general, the higher homotopy groups $\pi_i(BG^\wedge_p)$ for $i \geq 2$ can be nonvanishing, and hence the loop space $\Omega(BG^\wedge_p)$ is interesting in its own right. These spaces are the subject of several papers by the second author (e.g., [L, Theorem 1.1.4]). In particular, the homology of $\Omega(BG^\wedge_p)$ with p-local coefficients is known to have some very interesting properties, as described in [CL, § 2]. This helped to motivate the question of whether the homology of $\Omega(BG^\wedge_p)$ admits a purely algebraic definition (e.g., in [CL, § 2.6]).

In [Be2], Benson answered this question by showing that $H_*(\Omega(BG^\wedge_p);k)$, for a field k of characteristic p, is isomorphic to the homology of what he called a "left k-squeezed resolution for G": a chain complex of projective kG-modules satisfying certain axioms. He also showed that any two such complexes are chain homotopy equivalent, and hence have the same homology. The k-homology of $\Omega(BG^\wedge_p)$ is thus determined by the axioms of a squeezed resolution.

Our original aim in this paper was to check whether Benson's concept of a squeezed resolution can be formulated in a more categorical context. This was motivated in part by the problem of identifying p-compact groups in the sense of Dwyer and Wilkerson: loop spaces with finite mod p homology and p-complete classifying spaces (see Section 5 for more discussion). When doing this, we discovered that in fact, squeezed resolutions can be defined in a much more general setting, where we call them Ω-resolutions to emphasize the connection

1991 *Mathematics Subject Classification.* Primary 55R35. Secondary 55R40, 20D20.

Key words and phrases. Classifying space, Loop space, Small category, p-completion, Finite groups, Fusion.

C. Broto acknowledges financial support from the Spanish Ministry of Economy through the "María de Maeztu" Programme for Units of Excellence in R&D (MDM-2014-0445) and FEDER-MINECO Grant MTM2016-80439-P and from the Generalitat de Catalunya through AGAUR Grant 2017SGR1725.

B. Oliver is partially supported by UMR 7539 of the CNRS.

R. Levi and B. Oliver were partly supported by FEDER-MINECO Grant MTM2016-80439-P during several visits to the Universitat Autònoma de Barcelona.

The authors also thank the University of Aberdeen for its support during visits by two of us.

to loop spaces. In this setting, Benson's result can be generalized to a statement about plus constructions (in the sense of Quillen) on nerves of small categories.

When $\mathcal{C}$ is a small category, we let $|\mathcal{C}|$ denote the geometric realization of the nerve of $\mathcal{C}$. If R is a commutative ring, then an *$R\mathcal{C}$-module* is a (covariant) functor $\mathcal{C} \longrightarrow R\text{-}\mathtt{mod}$, and a morphism of $R\mathcal{C}$-modules is a natural transformation of functors. When π is a group, we let $\mathcal{B}(\pi)$ be the category with one object $\circ_\pi$ and $\mathrm{End}(\circ_\pi) = \pi$. If X is a connected CW complex and $H \trianglelefteq \pi_1(X)$, then a *plus construction* for X with respect to R and H means a space $|\mathcal{C}|^+_R$ together with a map $\kappa\colon |\mathcal{C}| \longrightarrow |\mathcal{C}|^+_R$ such that $\pi_1(\kappa)$ is surjective with kernel H and $H_*(\kappa; R)$ is an isomorphism; Quillen's construction shows that this exists whenever $H_1(H; R) = 0$ (see Lemma A.3). Finally, a group G is *R-perfect* if $H_1(G; R) = 0$.

Theorem A. *Fix a commutative ring R, a small connected category $\mathcal{C}$, a group π, and a functor $\theta\colon \mathcal{C} \longrightarrow \mathcal{B}(\pi)$ such that $\pi_1(|\theta|)\colon \pi_1(|\mathcal{C}|) \longrightarrow \pi$ is surjective. Set $H = \mathrm{Ker}(\pi_1(|\theta|))$, and assume that H is R-perfect. Then there is an Ω-resolution*

$$\cdots \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} \theta^*(R\pi) \longrightarrow 0$$

(a chain complex of $R\mathcal{C}$-modules satisfying conditions listed in Definition 1.5 or Lemma 3.10), and $H_*(C_*, \partial_*) \cong H_*(\Omega(|\mathcal{C}|^+_R); R)$ for each such (C_*, ∂_*) and each plus construction $|\mathcal{C}|^+_R$ for $|\mathcal{C}|$ with respect to R and H.

Theorem A will be stated in a more precise form as Theorem 4.4. Upon restricting to the case where $R = \mathbb{F}_p$ for a prime p, $\mathcal{C} = \mathcal{B}(G)$ for some finite group G, and $\pi = G/O^p(G) \cong \pi_1(BG^\wedge_p)$ (the largest p-group quotient of G), we recover Benson's theorem, since $BG^\wedge_p$ is a plus construction on $BG = |\mathcal{B}(G)|$ with respect to the ring $\mathbb{F}_p$ and the subgroup $O^p(G)$.

As another special case of Theorem A, let $(S, \mathcal{F}, \mathcal{L})$ be a p-local compact group in the sense of [BLO, Definition 4.2]. Thus S is a discrete p-toral group (an extension of $(\mathbb{Z}/p^\infty)^r$ by a finite p-group), $\mathcal{F}$ is a saturated fusion system over S, and $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$. Set $\pi = \pi_1(|\mathcal{L}|^\wedge_p)$: a finite p-group by [BLO, Proposition 4.4]. By Theorem 4.7 or 4.4 applied with $\mathcal{L}$ in the role of $\mathcal{C}$, $H_*(\Omega(|\mathcal{L}|^\wedge_p); \mathbb{F}_p)$ can be described in terms of Ω-resolutions. As noted above, our original motivation for this work was the search for new conditions sufficient to guarantee that $\Omega(|\mathcal{L}|^\wedge_p)$ has finite homology, and hence that $|\mathcal{L}|^\wedge_p$ is a p-compact group in the sense of Dwyer and Wilkerson [DW]. We did not succeed in doing this, but our attempt to do so is what led to this more general setting. Also, we do construct some examples in Propositions 5.5, 5.6, and 5.11 of explicit Ω-resolutions of finite length (in fact, of minimal length) for certain p-compact groups.

It turns out that Ω-resolutions can be defined in much greater generality than that needed in Theorem A. Let $\bigl(\mathcal{A} \underset{\theta^*}{\overset{\theta_*}{\rightleftarrows}} \mathcal{B}\bigr)$ be an *Ω-system*: a pair of abelian categories and additive functors such that θ_* is left adjoint to θ^*, $\theta_*\theta^* \cong \mathrm{Id}_{\mathcal{B}}$, and θ^* is exact (Definition 1.1). In this situation, for a projective object X in $\mathcal{B}$, an *Ω-resolution* of X is a chain complex of projective objects in $\mathcal{A}$ augmented by a morphism to $\theta^*(X)$ which satisfies certain axioms listed in Definition 1.5. In particular, these axioms are minimal conditions needed to ensure the uniqueness of Ω-resolutions up to chain homotopy equivalence (Proposition 1.6). However, while each such X has at most one Ω-resolution up to homotopy, we have examples that show that it need not have any in this very general situation. The examples in Theorem A are the special case where $\mathcal{A} = R\mathcal{C}\text{-}\mathtt{mod}$, $\mathcal{B} = R\pi\text{-}\mathtt{mod}$, and θ_* is left Kan extension with respect to the functor θ. Another large family of examples, where we show that Ω-resolutions exist but haven't yet found a geometric interpretation of their homology, is described in the following proposition (and in more detail in Proposition 3.5).

Proposition B. *Let $\theta\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a functor between small categories that is bijective on objects and surjective on morphism sets, and which has the following property:*

> *for each $c, c' \in \mathrm{Ob}(\mathcal{C})$ and each $\varphi, \varphi' \in \mathrm{Mor}_{\mathcal{C}}(c, c')$ such that $\theta_{c,c'}(\varphi) = \theta_{c,c'}(\varphi')$, there is $\alpha \in \mathrm{Aut}_{\mathcal{C}}(c')$ such that $\theta_c(\alpha) = \mathrm{Id}_{\theta(c')}$ and $\varphi = \alpha\varphi'$.*

Then for each commutative ring R, $\Bigl(R\mathcal{C}\text{-}\mathsf{mod} \underset{\theta^*}{\overset{\theta_*}{\rightleftarrows}} R\mathcal{D}\text{-}\mathsf{mod}\Bigr)$ is an Ω-system, where θ_* is defined by left Kan extension. Furthermore, projective objects in $R\mathcal{D}\text{-}\mathsf{mod}$ all have Ω-resolutions if and only if $\mathrm{Ker}[\theta_c\colon \mathrm{Aut}_{\mathcal{C}}(c) \longrightarrow \mathrm{Aut}_{\mathcal{D}}(\theta(c))]$ *is R-perfect for each $c \in \mathrm{Ob}(\mathcal{C})$.*

We begin in Section 1 by defining Ω-resolutions in our most general setting and proving their uniqueness. In Section 2, we find some necessary conditions, and some sufficient conditions, for their existence. We then restrict in Section 3 to the special case of $R\mathcal{C}$-modules, and construct examples where Ω-resolutions do or do not exist (Propositions 3.5 and 3.12). Our results connecting the homology of certain Ω-resolutions to the homology of loop spaces are shown in Section 4, where Theorem A is stated and proved in a slightly more precise form as Theorem 4.4. Afterwards, we look in Section 5 at some detailed examples of Ω-resolutions arising from p-local compact groups in which the maximal torus is normal.

Notation: For each small category $\mathcal{C}$, $|\mathcal{C}|$ denotes its geometric realization. When $\mathcal{C}$ is a small category and R is a commutative ring, we let $R\mathcal{C}\text{-}\mathsf{mod}$ denote the category of "$R\mathcal{C}$-modules": covariant functors from $\mathcal{C}$ to $R\text{-}\mathsf{mod}$. When $\mathcal{C}$ is an abelian category, we write $\mathscr{P}(\mathcal{C})$ to denote the class of projective objects in $\mathcal{C}$. For a group G, we write $G^{\mathrm{ab}} = G/[G,G]$ for the abelianization, and let $\mathcal{B}(G)$ denote the category with one object $\circ_G$ and $\mathrm{End}_{\mathcal{B}(G)}(\circ_G) \cong G$.

1. Ω-systems and Ω-resolutions

We begin by defining Ω-resolutions and proving their uniqueness in a very general setting. We do not prove any results about the existence of Ω-resolutions in this section, but leave that for Sections 2 and 4.

Definition 1.1. *An* Ω-system $(\mathcal{A}, \mathcal{B}; \theta_*, \theta^*)$ *consists of a pair of abelian categories $\mathcal{A}$ and $\mathcal{B}$, together with additive functors*

$$\mathcal{A} \underset{\theta^*}{\overset{\theta_*}{\rightleftarrows}} \mathcal{B}$$

such that

(OP1) *θ_* is left adjoint to θ^*;*

(OP2) *θ_* is a retraction in the sense that the counit of the adjunction $\mathfrak{b}\colon \theta_* \circ \theta^* \longrightarrow \mathrm{Id}_{\mathcal{B}}$ is an isomorphism; and*

(OP3) *θ^* sends epimorphisms in $\mathcal{B}$ to epimorphisms in $\mathcal{A}$.*

It will be important, in the situation of Definition 1.1, to know that $\theta^*(\mathcal{B})$ is a full subcategory of $\mathcal{A}$. In fact, this holds without assuming condition (OP3).

Lemma 1.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be a pair of categories, together with functors*

$$\mathcal{A} \underset{\theta^*}{\overset{\theta_*}{\rightleftarrows}} \mathcal{B}$$

such that θ_* is left adjoint to θ^*, and such that the counit $\mathfrak{b}\colon \theta_* \circ \theta^* \longrightarrow \mathrm{Id}_{\mathcal{B}}$ of the adjunction is an isomorphism of functors. Then the image $\theta^*(\mathcal{B})$ is a full subcategory of $\mathcal{A}$.

Proof. Fix $B, B' \in \mathrm{Ob}(\mathcal{B})$ and $\alpha \in \mathrm{Mor}_{\mathcal{A}}(\theta^*(B), \theta^*(B'))$. Let $\mathfrak{a}\colon \mathrm{Id}_{\mathcal{A}} \longrightarrow \theta^* \circ \theta_*$ be the unit of the adjunction. Upon applying $\mathfrak{a}$ to α, one obtains the commutative square in the following diagram in $\mathcal{A}$:

$$\begin{tikzcd}
& \theta^*(B) \arrow[rr, "\alpha"] \arrow[dl, "\mathrm{Id}_{\theta^*(B)}"'] \arrow[d, "\mathfrak{a}_{\theta^*(B)}"] & & \theta^*(B') \arrow[d, "\mathfrak{a}_{\theta^*(B')}"'] \arrow[dr, "\mathrm{Id}_{\theta^*(B')}"] & \\
\theta^*(B) & \theta^*\theta_*\theta^*(B) \arrow[l, "\theta^*(\mathfrak{b}_B)", "\cong"'] \arrow[rr, "\theta^*\theta_*(\alpha)"] & & \theta^*\theta_*\theta^*(B') \arrow[r, "\theta^*(\mathfrak{b}_{B'})", "\cong"'] & \theta^*(B')\,.
\end{tikzcd}$$

The two triangles commute by, e.g., [Mac1, § V.5], and $\mathfrak{b}_B$ and $\mathfrak{b}_{B'}$ are isomorphisms in $\mathcal{B}$ by assumption. Hence $\alpha = \theta^*(\mathfrak{b}_{B'} \circ \theta_*(\alpha) \circ \mathfrak{b}_B^{-1})$ is a morphism in $\theta^*(\mathcal{B})$, and so $\theta^*(\mathcal{B})$ is a full subcategory of $\mathcal{A}$. □

The following is one family of Ω-systems to which we will frequently refer. More examples will be given in Section 3.

Example 1.3. Fix a commutative ring R, a pair of groups G and π, and a surjective homomorphism $\theta\colon G \longrightarrow \pi$. Let $RG\text{-}\mathsf{mod}$ and $R\pi\text{-}\mathsf{mod}$ be the categories of (left) RG- and $R\pi$-modules, respectively, and let

$$RG\text{-}\mathsf{mod} \underset{\theta^*}{\overset{\theta_*}{\rightleftarrows}} R\pi\text{-}\mathsf{mod}$$

be the functors defined as follows. For each RG-module M, set $\theta_*(M) = R\pi \otimes_{RG} M$ where $R\pi$ is regarded as a right RG-module via θ. For each $R\pi$-module N, set $\theta^*(N) = N$ regarded as an RG-module via θ. Then $(RG\text{-}\mathsf{mod}, R\pi\text{-}\mathsf{mod}; \theta_*, \theta^*)$ is an Ω-system.

Proof. If M and N are RG- and $R\pi$-modules, respectively, then there is an obvious natural bijection $\mathrm{Hom}_{RG}(M, \theta^*(N)) \cong \mathrm{Hom}_{R\pi}(\theta_*(M), N)$. Thus (OP1) holds: θ_* is left adjoint to θ^*. Conditions (OP2) and (OP3) are clear. □

The following properties of Ω-systems follow easily from the basic properties of abelian categories and adjoint functors (See, e.g., [Mac1, § IV-V].).

Lemma 1.4. *The following hold for each Ω-system $(\mathcal{A}, \mathcal{B}; \theta_*, \theta^*)$.*

(a) *The functor θ^* is exact, and θ_* is right exact.*

(b) *The functor θ_* sends projectives to projectives.*

(c) *A sequence in $\mathcal{B}$ is exact if and only if its image under θ^* is exact in $\mathcal{A}$. A morphism in $\mathcal{B}$ is an isomorphism, an epimorphism, or a monomorphism if and only if the same is true in $\mathcal{A}$ of its image under θ^*.*

Proof. **(a,b)** Since θ_* is left adjoint to θ^*, θ_* is right exact and θ^* is left exact. By (OP3), θ^* also preserves epimorphisms, and hence is exact. Since θ_* has a right adjoint that is exact, it sends projectives to projectives.

(c) The exactness of θ^* implies that it sends the kernel, cokernel, and image of each morphism ψ in $\mathcal{B}$ to the kernel, cokernel, and image in $\mathcal{A}$ of $\theta^*(\psi)$. Also, if $\varphi \in \mathrm{Iso}_{\mathcal{A}}(\theta^*(N), \theta^*(N'))$ for N, N' in $\mathcal{B}$, then since $\theta^*(\mathcal{B})$ is a full subcategory of $\mathcal{A}$ and $\theta_*\theta^*$ is naturally isomorphic to the identity, $\varphi = \theta^*(\psi)$ for some $\psi \in \mathrm{Iso}_{\mathcal{B}}(N, N')$. Hence a sequence in $\mathcal{B}$ whose image under θ^* is exact in $\mathcal{A}$ is also exact in $\mathcal{B}$, and if $\theta^*(\varphi)$ is a monomorphism or epimorphism in $\mathcal{A}$, then φ is a monomorphism or epimorphism, respectively, in $\mathcal{B}$. This proves the "if" statements, and the converse in all cases holds by the exactness of θ^*. □

We are now ready to define Ω-resolutions.

Definition 1.5. *Let $(\mathcal{A},\mathcal{B};\theta_*,\theta^*)$ be an Ω-system. For a projective object X in $\mathcal{B}$, an* Ω-resolution *of X with respect to $(\mathcal{A},\mathcal{B};\theta_*,\theta^*)$ is a chain complex*

$$\boldsymbol{R} = \Bigl(\cdots \xrightarrow{\ \partial_3\ } P_2 \xrightarrow{\ \partial_2\ } P_1 \xrightarrow{\ \partial_1\ } P_0 \xrightarrow{\ \varepsilon\ } \theta^*(X) \longrightarrow 0\Bigr) \tag{1}$$

in $\mathcal{A}$ such that

(Ω-1) *P_n is projective in $\mathcal{A}$ for all $n \geq 0$;*

(Ω-2) *$\theta_*(\boldsymbol{R})$ is exact; and*

(Ω-3) *$H_n(P_*,\partial_*)$ is isomorphic to an object in $\theta^*(\mathcal{B})$ for each $n \geq 0$, and ε induces an isomorphism $H_0(P_*,\partial_*) \cong \theta^*(X)$.*

If an Ω-resolution $\boldsymbol{R}$ exists as above, then we set $H_*^{\Omega}(\mathcal{A},\mathcal{B};X) = \theta_*\bigl(H_*(P_*,\partial_*)\bigr)$: the image under θ_* of the homology of the complex (P_*,∂_*).

There are, in fact, many Ω-systems for which Ω-resolutions do not exist. In the situation of Example 1.3, when R is a field and $\theta\colon G \longrightarrow \pi$ is a surjection of groups, we will see in Example 2.9 that a nonzero projective object in $\mathcal{B}$ has an Ω-resolution if and only if $H_1(\mathrm{Ker}(\theta);R) = 0$. However, whenever X does have at least one Ω-resolution, the next proposition implies that $H_*^{\Omega}(\mathcal{A},\mathcal{B};X)$ is unique up to natural isomorphism.

Proposition 1.6. *Let $(\mathcal{A},\mathcal{B};\theta_*,\theta^*)$ be an Ω-system. Let X and Y be projective objects in $\mathcal{B}$, and let $f \in \mathrm{Mor}_{\mathcal{B}}(X,Y)$ be a morphism. Let*

$$\cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} \theta^*(X) \longrightarrow 0 \quad \text{and} \quad \cdots \xrightarrow{\partial_2'} P_1' \xrightarrow{\partial_1'} P_0' \xrightarrow{\varepsilon'} \theta^*(Y) \longrightarrow 0$$

be chain complexes in $\mathcal{A}$, where the first satisfies conditions (Ω-1) *and* (Ω-2) *in Definition 1.5 and the second satisfies condition* (Ω-3)*. Then there are morphisms $f_n \in \mathrm{Mor}_{\mathcal{A}}(P_n,P_n')$ which make the following diagram commute:*

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_3"] & P_2 \arrow[r, "\partial_2"] \arrow[d, "f_2"'] & P_1 \arrow[r, "\partial_1"] \arrow[d, "f_1"'] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "f_0"'] & \theta^*(X) \arrow[r] \arrow[d, "\theta^*(f)"'] & 0 \\
\cdots \arrow[r, "\partial_3'"] & P_2' \arrow[r, "\partial_2'"] & P_1' \arrow[r, "\partial_1'"] & P_0' \arrow[r, "\varepsilon'"] & \theta^*(Y) \arrow[r] & 0
\end{tikzcd}$$

Furthermore, $\{f_n\}_{n\in\mathbb{N}}$ is unique up to chain homotopy.

Proof. For each $i \geq 0$, $\theta_*(P_i)$ is projective in $\mathcal{B}$ by Lemma 1.4(b) and since P_i is projective. Also, $\theta_*\theta^*(X) \cong X$ by (OP2), and X is projective in $\mathcal{B}$ by assumption. So by (Ω-2), $\theta_*(P_*) \longrightarrow \theta_*\theta^*(X) \to 0$ is an exact sequence of objects projective in $\mathcal{B}$, and hence splits in each degree.

Existence of f_*: Since P_0 is projective, and the augmentation $\varepsilon'\colon P_0' \longrightarrow \theta^*(Y)$ is onto by (Ω-3), $\theta^*(f)\circ\varepsilon$ lifts to a homomorphism $f_0\colon P_0 \longrightarrow P_0'$.

Assume, for some $n \geq 0$, that f_i has been constructed for all $0 \leq i \leq n$, where $f_{n-1}\circ \partial_n = \partial_n' \circ f_n$. Then $f_n \circ \partial_{n+1}$ sends P_{n+1} into $\mathrm{Ker}(\partial_n')$, and hence induces a homomorphism $\chi\colon P_{n+1} \longrightarrow H_n(P_*',\partial_*')$. By assumption, $H_n(P_*',\partial_*') \cong \theta^*(B)$ for some B in $\mathcal{B}$, and hence there are natural bijections

$$\mathrm{Mor}_{\mathcal{A}}(P_i, H_n(P_*',\partial_*')) \cong \mathrm{Mor}_{\mathcal{A}}(P_i,\theta^*(B)) \cong \mathrm{Mor}_{\mathcal{B}}(\theta_*(P_i),B)$$

for $i = n, n+1, n+2$. Since the complex $(\theta_*(P_*),\theta_*(\partial_*)) \longrightarrow \theta_*(\theta^*X) \to 0$ is exact and split and $\chi\circ\partial_{n+2} = 0$ by construction, $\theta_*(\chi)$ factors through $\mathrm{Im}(\theta_*(\partial_{n+1}))$ and extends to $\theta_*(P_n)$. By adjointness, there is $\varphi\colon P_n \longrightarrow H_n(P_*',\partial_*')$ such that $\chi = \varphi\circ\partial_{n+1}$.

Since P_n is projective, φ lifts to a morphism $\widetilde{\varphi}\colon P_n \longrightarrow \operatorname{Ker}(\partial'_n)$. An easy diagram chase now shows that $\operatorname{Im}((f_n-\widetilde{\varphi})\circ\partial_{n+1}) \le \operatorname{Im}(\partial'_{n+1})$. Hence, upon replacing f_n by $f_n-\widetilde{\varphi}$, $f_{n-1}\circ\partial_n = \partial'_n \circ f_n$ still holds (where $f_{-1} = f$ if $n = 0$) and $\operatorname{Im}(f_n \circ \partial_{n+1}) \le \operatorname{Im}(\partial'_{n+1})$. Upon using the projectivity of P_{n+1} again, one can lift $f_n \circ \partial_{n+1}$ to a homomorphism $f_{n+1}\colon P_{n+1} \longrightarrow P'_{n+1}$ such that $f_n \circ \partial_{n+1} = \partial'_{n+1} \circ f_{n+1}$. We now continue inductively.

Uniqueness of f_*: Let f'_* and f''_* be two homomorphisms covering f, and set $t_* = f'_* - f''_*$. Thus $t_*\colon (P_*, \partial_*) \longrightarrow (P'_*, \partial'_*)$ is a homomorphism covering $X \xrightarrow{0} Y$, and we must construct a chain homotopy $D\colon P_* \longrightarrow P'_*$ of degree $+1$ such that $D \circ \partial + \partial' \circ D = t_*$.

Set $D_{-1} = 0\colon \theta^*(X) \longrightarrow P'_0$. Since $\varepsilon' \circ t_0 = 0$, and the sequence

$$P'_1 \xrightarrow{\partial'_1} P'_0 \xrightarrow{\varepsilon'} \theta^*(Y) \longrightarrow 0$$

is exact by condition (Ω-3), t_0 lifts to a homomorphism $D_0\colon P_0 \longrightarrow P'_1$. The rest of the proof is carried out using arguments similar to those used to show existence. □

When $(\mathcal{A}, \mathcal{B}; \theta_*, \theta^*)$ is an Ω-system and $X \in \mathscr{P}(\mathcal{B})$ has an Ω-resolution, there is a spectral sequence that links the Ω-homology of X to higher derived functors of θ_*.

Proposition 1.7. *Let $(\mathcal{A}, \mathcal{B}; \theta_*, \theta^*)$ be an Ω-system, and assume $\mathcal{A}$ has enough projectives. Let X be a projective object in $\mathcal{B}$ that has an Ω-resolution. Then there is a first quadrant spectral sequence $E^r_{*,*}$ in $\mathcal{B}$ such that*

$$E^2_{i,j} \cong (L_i\theta_*)\bigl(\theta^*(H^\Omega_j(\mathcal{A}, \mathcal{B}; X))\bigr) \qquad \text{and} \qquad E^\infty_{i,j} \cong \begin{cases} X & \text{if } (i,j) = (0,0) \\ 0 & \text{if } (i,j) \ne (0,0). \end{cases}$$

Proof. Let (P_*, ∂_*) be an Ω-resolution of X. Let $\bigl\{Q_{ij}\bigr\}_{i,j\ge 0}$ be a *proper projective resolution* of (P_*, ∂_*); i.e., a double complex of projective objects in $\mathcal{A}$, where for each j, the sequences

(i) $0 \longleftarrow P_j \longleftarrow Q_{0,j} \longleftarrow Q_{1,j} \longleftarrow \cdots,$

(ii) $0 \longleftarrow H_j(P_*) \longleftarrow H_j(Q_{0,*}) \longleftarrow H_j(Q_{1,*}) \longleftarrow \cdots$, and

(iii) $0 \longleftarrow Z_j(P_*) \longleftarrow Z_j(Q_{0,*}) \longleftarrow Z_j(Q_{1,*}) \longleftarrow \cdots$

are all projective resolutions (see [Be1, Definition 3.6.1]). Proper projective resolutions exist by [Mac2, Proposition XII.11.6] (see also, [Be1, Lemma 3.6.2]).

Consider the two spectral sequences associated to the double complex $\theta_*(Q_{*,*})$. Since each row $Q_{*,j}$ is a projective resolution of the projective object P_j, $\theta_*(Q_{*,j})$ is a resolution of $\theta_*(P_j)$, and thus $\bar{E}^1_{0,j} \cong \theta_*(P_j)$, while $\bar{E}^1_{i,j} = 0$ if $i \ge 1$. Since P_* is an Ω resolution of X, we now obtain $\bar{E}^2_{0,0} \cong X$, while $\bar{E}^2_{i,j} = 0$ if $(i,j) \ne (0,0)$.

Now consider the other spectral sequence $E^r_{i,j}$, where we first take homology of the columns. By (ii), $H_j(Q_{i,*})$ and $Z_j(Q_{i,*})$ are projective for each $i,j \ge 0$, so $B_j(Q_{i,*})$ is also projective, and all sequences involved in the homology of $Q_{i,*}$ split. In other words,

$$E^1_{i,j} = H_j(\theta_*(Q_{i,*})) \cong \theta_*(H_j(Q_{i,*}))$$

for all $i,j \ge 0$. By (ii) again, the j-th row in the E^1-page is obtained by applying θ_* to a projective resolution of $H_j(P_*)$, and so

$$E^2_{i,j} \cong (L_i\theta_*)(H_j(P_*, \partial_*)) \cong (L_i\theta_*)\bigl(\theta^*(H^\Omega_j(\mathcal{A}, \mathcal{B}; X))\bigr).$$

Since $\bar{E}^\infty_{i,j} = 0$ for all $(i,j) \ne (0,0)$, the two spectral sequences have isomorphic E^∞-pages, and this proves the proposition. □

We finish the section with the following observation.

Remark 1.8. If $(\mathcal{A},\mathcal{B};\theta_*,\theta^*)$ and $(\mathcal{B},\mathcal{C};\eta_*,\eta^*)$ are two Ω-systems, then their composite $(\mathcal{A},\mathcal{C};\eta_*\theta_*,\theta^*\eta^*)$ is easily seen to be an Ω-system. In other words, there is a category whose objects are the small abelian categories and whose morphisms are isomorphism classes of Ω-systems. One obvious question is whether there is a natural way to construct Ω-resolutions for the composite Ω-system from Ω-resolutions for the two factors, and if so, what connection there is (if any) between the homology groups of these three complexes.

2. The existence of Ω-resolutions

We saw in the last section that Ω-resolutions, when they exist, are unique up to chain homotopy. The question of when they do exist is more complicated, and in this section, we give some necessary conditions and some sufficient conditions for that to happen. When doing this, the following slightly more general form of Definition 1.5 will be needed: one that allows for truncated resolutions.

Definition 2.1. *Let $(\mathcal{A},\mathcal{B};\theta_*,\theta^*)$ be an Ω-system. For $X\in\mathscr{P}(\mathcal{B})$ and $1\le n\le\infty$, an* Ω_n-resolution *of X is a chain complex*

$$\boldsymbol{R}_n=\begin{cases}\bigl(P_n\xrightarrow{\ \partial_n\ }\cdots\xrightarrow{\ \partial_3\ }P_2\xrightarrow{\ \partial_2\ }P_1\xrightarrow{\ \partial_1\ }P_0\xrightarrow{\ \varepsilon\ }\theta^*(X)\longrightarrow 0\bigr) & \text{if } n<\infty\\ \bigl(\cdots\xrightarrow{\ \partial_3\ }P_2\xrightarrow{\ \partial_2\ }P_1\xrightarrow{\ \partial_1\ }P_0\xrightarrow{\ \varepsilon\ }\theta^*(X)\longrightarrow 0\bigr) & \text{if } n=\infty\end{cases}\tag{1}$$

in $\mathcal{A}$ such that

(Ω_n-1) *$P_i\in\mathscr{P}(\mathcal{A})$ for all $0\le i\le n$ (for all $i\ge 0$ if $n=\infty$);*

(Ω_n-2) *$\theta_*(\boldsymbol{R}_n)$ is exact;*

(Ω_n-3) *$H_i(P_*,\partial_*)$ is isomorphic to an object in $\theta^*(\mathcal{B})$ for each $0\le i<n$, and ε induces an isomorphism $H_0(P_*,\partial_*)\cong\theta^*(X)$; and*

(Ω_n-4) *if $n<\infty$, the inclusion $\mathrm{Im}(\partial_n)\le P_{n-1}$ induces a monomorphism $\theta_*(\mathrm{Im}(\partial_n))\longrightarrow\theta_*(P_{n-1})$.*

In particular, an Ω_∞-resolution is the same as an Ω-resolution (Definition 1.5).

Lemma 2.2. *Condition (Ω_n-4) can be replaced by the following equivalent condition:*

(Ω_n-4′) *If $n<\infty$, the sequence $\theta_*(\mathrm{Ker}(\partial_n))\xrightarrow{\theta_*(\mathrm{incl})}\theta_*(P_n)\xrightarrow{\theta_*(\partial_n)}\theta_*(P_{n-1})$ is exact.*

Proof. The sequence $\theta_*(\mathrm{Ker}(\partial_n))\longrightarrow\theta_*(P_n)\longrightarrow\theta_*(\mathrm{Im}(\partial_n))\to 0$ is exact since θ_* is right exact by Lemma 1.4(a). Hence the sequence in (Ω_n-4′) is exact if and only if the inclusion of $\mathrm{Im}(\partial_n)$ in P_{n-1} induces a monomorphism $\theta_*(\mathrm{Im}(\partial_n))\longrightarrow\theta_*(P_{n-1})$. □

Lemma 2.3. *Fix an Ω-system $(\mathcal{A},\mathcal{B};\theta_*,\theta^*)$, and a projective object $X\in\mathscr{P}(\mathcal{B})$. Let $P_m\longrightarrow\cdots\longrightarrow P_0\longrightarrow X\to 0$ be an Ω_m-resolution of X for some $1<m\le\infty$. Then for each $1\le n<m$, the truncation $P_n\longrightarrow\cdots\longrightarrow P_0\longrightarrow X\to 0$ is an Ω_n-resolution of X.*

Proof. Conditions (Ω_n-1)–(Ω_n-3) in Definition 2.1 follow immediately from (Ω_m-1)–(Ω_m-3), so we need only prove that (Ω_n-4) holds. Consider the following commutative diagram:

$$\begin{tikzcd}
\theta_*(P_{n+1}) \arrow[r, "\theta_*(\partial_{n+1})"] \arrow[drr, bend right, "0"'] & \theta_*(P_n) \arrow[rr, "\theta_*(\partial_n)"] \arrow[dr, two heads, "\theta_*(\partial_n^*)"'] & & \theta_*(P_{n-1}) \\
& & \theta_*(\mathrm{Im}(\partial_n)) \arrow[ur, "\theta_*(\mathrm{incl})"'] &
\end{tikzcd}\tag{2}$$

where $\partial_n^*\colon P_n \longrightarrow \mathrm{Im}(\partial_n)$ is the restriction of ∂_n. The row in (2) is exact by (Ω_m-2) and since $m > n$, and $\theta_*(\partial_n^*)$ is an epimorphism since θ_* is right exact. Hence $\mathrm{Ker}(\theta_*(\partial_n^*)) = \mathrm{Ker}(\theta_*(\partial_n))$ and $\theta_*(\mathrm{incl})$ is a monomorphism, and (Ω_n-4) holds. □

Definition 2.4. *When $(\mathcal{A}, \mathcal{B}; \theta_*, \theta^*)$ is an Ω-system, we say that $\theta^*(\mathcal{B})$* closed under subobjects in $\mathcal{A}$ *if for each monomorphism $A_1 \longrightarrow A_2$ in $\mathcal{A}$, A_1 is isomorphic to an object of $\theta^*(\mathcal{B})$ if A_2 is. Similarly, we say that $\theta^*(\mathcal{B})$* is closed under extensions in $\mathcal{A}$ *if for each short exact sequence $0 \to M' \longrightarrow M \longrightarrow M'' \to 0$ in $\mathcal{A}$, M is isomorphic to an object in $\theta^*(\mathcal{B})$ if M' and M'' are isomorphic to objects in $\theta^*(\mathcal{B})$.*

We will show that for each Ω-system $(\mathcal{A}, \mathcal{B}; \theta_*, \theta^*)$ in which $\theta^*(\mathcal{B})$ is closed under subobjects and extensions, all projectives in $\mathcal{B}$ have Ω-resolutions (Proposition 2.8).

Lemma 2.5. *Let $(\mathcal{A}, \mathcal{B}; \theta_*, \theta^*)$ be an Ω-system, where $\mathcal{A}$ has enough projectives.*

(a) *The following are equivalent:*
   (a.i) *$\theta^*(\mathcal{B})$ is closed under subobjects in $\mathcal{A}$.*
   (a.ii) *For each M in $\mathcal{A}$, the unit morphism $\mathfrak{a}_M\colon M \longrightarrow \theta^*\theta_*(M)$ is an epimorphism.*

(b) *If either condition* (a.i) *or* (a.ii) *holds, then the following two conditions are equivalent:*
   (b.i) *$\theta^*(\mathcal{B})$ is closed under extensions in $\mathcal{A}$.*
   (b.ii) *For each N in $\mathcal{B}$, $(L_1\theta_*)(\theta^*(N)) = 0$.*

(c) *If $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is an extension in $\mathcal{A}$, where M' and M'' are in $\theta^*(\mathcal{B})$ but M is not isomorphic to an object in $\theta^*(\mathcal{B})$, then $(L_1\theta_*)(M'') \neq 0$.*

Proof. **(a.i $\Longrightarrow$ a.ii)** Fix an object M in $\mathcal{A}$, and consider the unit morphism $\mathfrak{a}_M\colon M \longrightarrow \theta^*\theta_*(M)$. Since $\theta^*(\mathcal{B})$ is closed under subobjects, $\mathrm{Im}(\mathfrak{a}_M) \cong \theta^*(B)$ for some B in $\mathcal{B}$. Since $\theta^*(\mathcal{B})$ is a full subcategory of $\mathcal{A}$ by Lemma 1.2, each morphism in $\mathrm{Mor}_{\mathcal{A}}(\theta^*(B), \theta^*\theta_*(M))$ lies in $\theta^*(\mathcal{B})$. Thus $\mathfrak{a}_M$ factors as a composite

$$\mathfrak{a}_M\colon M \xrightarrow{\ g\ } \theta^*(B) \xrightarrow{\theta^*(\psi)} \theta^*\theta_*(M)$$

for some $\psi \in \mathrm{Mor}_{\mathcal{B}}(B, \theta_*(M))$, where g is surjective and $\theta^*(\psi)$ is injective.

Let $\gamma \in \mathrm{Mor}_{\mathcal{B}}(\theta_*(M), B)$ be the morphism adjoint to g. Then $\psi \circ \gamma = \mathrm{Id}_{\theta_*(M)}$ since $\mathfrak{a}_M$ is adjoint to the identity, and thus ψ is surjective. So $\theta^*(\psi)$ is also surjective by (OP3), and hence $\mathfrak{a}_M$ is surjective.

(a.ii $\Longrightarrow$ a.i) Now assume that $M \xrightarrow{\mathfrak{a}_M} \theta^*\theta_*(M)$ is an epimorphism for each M in $\mathcal{A}$. Let $M_1 \xrightarrow{f} M_2$ be a monomorphism in $\mathcal{A}$, where M_2 is in $\theta^*(\mathcal{B})$ and hence $\mathfrak{a}_{M_2}$ is an isomorphism. From the commutative square

$$\begin{tikzcd} M_1 \arrow[r, "f"] \arrow[d, "\mathfrak{a}_{M_1}"'] & M_2 \arrow[d, "\mathfrak{a}_{M_2}"', "\cong"] \\ \theta^*\theta_*(M_1) \arrow[r, "\theta^*\theta_*(f)"] & \theta^*\theta_*(M_2), \end{tikzcd}$$

we see that $\theta^*\theta_*(f) \circ \mathfrak{a}_{M_1} = \mathfrak{a}_{M_2} \circ f$ is a monomorphism, and hence that $\mathfrak{a}_{M_1}$ is a monomorphism. Since $\mathfrak{a}_{M_1}$ is also an epimorphism, we have $M_1 \cong \theta^*\theta_*(M_1) \in \mathrm{Ob}(\theta^*(\mathcal{B}))$.

(c) Let $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ be a short exact sequence in $\mathcal{A}$, where $M', M'' \in \mathrm{Ob}(\theta^*(\mathcal{B}))$ and M is not isomorphic to any object in $\theta^*(\mathcal{B})$. Consider the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "\alpha"] \arrow[d, "\mathfrak{a}_{M'}"', "\cong"] & M \arrow[r, "\beta"] \arrow[d, "\mathfrak{a}_M"'] & M'' \arrow[r] \arrow[d, "\mathfrak{a}_{M''}"', "\cong"] & 0 \\
& \theta^*\theta_*(M') \arrow[r, "\theta^*\theta_*(\alpha)"] & \theta^*\theta_*(M) \arrow[r, "\theta^*\theta_*(\beta)"] & \theta^*\theta_*(M'') \arrow[r] & 0\,.
\end{tikzcd}$$

Here, $\mathfrak{a}_{M'}$ and $\mathfrak{a}_{M''}$ are isomorphisms since M' and M'' are in $\theta^*(\mathcal{B})$, while $\mathfrak{a}_M$ is not an isomorphism since M is not isomorphic to any object in $\theta^*(\mathcal{B})$. Thus $\theta^*\theta_*(\alpha)$ is not injective in $\mathcal{A}$, so $\theta_*(\alpha)$ is not injective in $\mathcal{B}$ (Lemma 1.4(c)), and $(L_1\theta_*)(M'') \neq 0$.

(b) The implication (b.ii $\Longrightarrow$ b.i) follows immediately from (c), and it remains to prove the converse. So assume that (a.ii) holds, and that $\theta^*(\mathcal{B})$ is closed under extensions in $\mathcal{A}$. Fix M in $\theta^*(\mathcal{B})$, and let

$$0 \longrightarrow K \xrightarrow{\ \alpha\ } P \xrightarrow{\ \beta\ } M \longrightarrow 0$$

be a short exact sequence in $\mathcal{A}$ where P is projective. Set $K_0 = \mathrm{Ker}(\mathfrak{a}_K)$, and consider the following commutative diagram:

$$\begin{tikzcd}
& 0 \arrow[r] & K/K_0 \arrow[r, "\widehat{\alpha}"] \arrow[d, "\widehat{\mathfrak{a}}_K"', "\cong"] & P/\alpha(K_0) \arrow[r, "\widehat{\beta}"] \arrow[d, two heads, "\widehat{\mathfrak{a}}_P"'] & M \arrow[r] \arrow[d, "\mathfrak{a}_M"', "\cong"] & 0 \\
0 \arrow[r] & \theta^*\big((L_1\theta_*)(M)\big) \arrow[r] & \theta^*\theta_*(K) \arrow[r, "\theta^*\theta_*(\alpha)"] & \theta^*\theta_*(P) \arrow[r, "\theta^*\theta_*(\beta)"] & \theta^*\theta_*(M) \arrow[r] & 0\,.
\end{tikzcd} \tag{3}$$

Here, $\mathfrak{a}_M$ is an isomorphism since M is in $\theta^*(\mathcal{B})$, $\widehat{\mathfrak{a}}_K$ and $\widehat{\mathfrak{a}}_P$ are epimorphisms since $\mathfrak{a}_K$ and $\mathfrak{a}_P$ are surjective by (a.ii), and $\widehat{\mathfrak{a}}_K$ is injective by construction. The top row is exact by construction, and the bottom row since $(L_1\theta_*)(P) = 0$ (P is projective) and θ^* is exact.

Now, $K/K_0 \cong \theta^*\theta_*(K)$ and M are both isomorphic to objects of $\theta^*(\mathcal{B})$, and the same holds for $P/\alpha(K_0)$ since $\theta^*(\mathcal{B})$ is closed under extensions. Thus there is an object N in $\mathcal{B}$, a surjective morphism $f\colon P \longrightarrow \theta^*(N)$ with kernel $\alpha(K_0)$, and a morphism $\nu\colon \theta^*(N) \longrightarrow \theta^*\theta_*(P)$ such that $\nu \circ f = \mathfrak{a}_P$. Since $\theta^*(\mathcal{B})$ is a full subcategory of $\mathcal{A}$ (Lemma 1.2), $\nu = \theta^*(\chi)$ for some $\chi \in \mathrm{Mor}_{\mathcal{B}}(N, \theta_*(P))$.

Let $\varphi \in \mathrm{Mor}_{\mathcal{B}}(\theta_*(P), N)$ be adjoint to f. Then

$$\begin{tikzcd}
& P \arrow[dl, "\mathfrak{a}_P"'] \arrow[dr, "f"] & \\
\theta^*\theta_*(P) \arrow[rr, bend right, "\theta^*(\varphi)"'] & & \theta^*(N) \arrow[ll, bend right, "\nu=\theta^*(\chi)"']
\end{tikzcd}
\quad \text{is adjoint to} \quad
\begin{tikzcd}
& \theta_*(P) \arrow[dl, "\mathrm{Id}_{\theta_*(P)}"'] \arrow[dr, "\varphi"] & \\
\theta_*(P) \arrow[rr, bend right, "\varphi"'] & & N\,, \arrow[ll, bend right, "\chi"']
\end{tikzcd}$$

so $\theta^*(\varphi) \circ \mathfrak{a}_P = f$ and $\nu \circ f = \mathfrak{a}_P$. Since f and $\mathfrak{a}_P$ are both surjective, ν and $\theta^*(\varphi)$ are isomorphisms (and inverses to each other). So in diagram (3), $\widehat{\mathfrak{a}}_P$ is an isomorphism, $\theta^*\theta_*(\alpha)$ is injective, and thus $(L_1\theta_*)(M) = 0$. $\square$

The next proposition provides one tool for showing that Ω-resolutions do *not* exist in certain cases.

Proposition 2.6. *For each Ω-system $(\mathcal{A}, \mathcal{B}; \theta_*, \theta^*)$ for which $\mathcal{A}$ has enough projectives, and each $X \in \mathscr{P}(\mathcal{B})$, there is an Ω_1-resolution of X if and only if $(L_1\theta_*)(\theta^*(X)) = 0$.*

Proof. Assume that $(L_1\theta_*)(\theta^*(X)) = 0$. Let $\boldsymbol{R}_1 = \big(P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} \theta^*(X) \to 0\big)$ be an exact sequence in $\mathcal{A}$, where $P_0, P_1 \in \mathscr{P}(\mathcal{A})$. Then the sequence

$$0 \longrightarrow \mathrm{Im}(\partial_1) \longrightarrow P_0 \longrightarrow \theta^*(X) \longrightarrow 0$$

is short exact, and since $(L_1\theta_*)(\theta^*(X)) = 0$, the induced morphism $\theta_*(\mathrm{Im}(\partial_1)) \longrightarrow \theta_*(P_0)$ is a monomorphism. So $\boldsymbol{R}_1$ is an Ω_1-resolution of X, where (Ω_1-2) holds since θ_* is right exact.

Conversely, if $\boldsymbol{R}_1 = \big(P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} \theta^*(X) \to 0\big)$ is an Ω_1-resolution of X, then $\boldsymbol{R}_1$ is exact by (Ω_n-3). Since $P_0 \in \mathscr{P}(\mathcal{A})$, the sequence

$$0 \longrightarrow (L_1\theta_*)(\theta^*(X)) \longrightarrow \theta_*(\mathrm{Im}(\partial_1)) \xrightarrow{\theta_*(\mathrm{incl})} \theta_*(P_0) \xrightarrow{\theta_*(\varepsilon)} \theta_*(\theta^*(X)) \longrightarrow 0$$

is exact. Since $\theta_*(\mathrm{incl})$ is a monomorphism by condition (Ω_1-4), $(L_1\theta_*)(\theta^*(X)) = 0$. $\square$

The following lemma gives conditions for extending an Ω_n-resolution to an Ω_{n+1}-resolution.

Lemma 2.7. *Fix an Ω-system $(\mathcal{A}, \mathcal{B}; \theta_*, \theta^*)$, where $\mathcal{A}$ has enough projectives, and let X be a projective in $\mathcal{B}$. Let $\boldsymbol{R}_n = \big(P_n \xrightarrow{\partial_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \xrightarrow{\varepsilon} \theta^*(X) \to 0\big)$ be an Ω_n-resolution of X, for some $1 \le n < \infty$.*

(a) *The natural morphism $f_0\colon \theta_*(\mathrm{Ker}(\partial_n)) \longrightarrow \mathrm{Ker}(\theta_*(\partial_n))$ is a split epimorphism.*

(b) *If $P_{n+1} \in \mathscr{P}(\mathcal{A})$, and $\partial_{n+1} \in \mathrm{Mor}_{\mathcal{A}}(P_{n+1}, P_n)$ are such that $\partial_n \circ \partial_{n+1} = 0$, then the complex $\boldsymbol{R}_{n+1} = \big(P_{n+1} \xrightarrow{\partial_{n+1}} P_n \longrightarrow \cdots \xrightarrow{\varepsilon} \theta^*(X) \to 0\big)$ is an Ω_{n+1}-resolution of X if and only if*

  (b.i) $\mathrm{Ker}(\partial_n)/\mathrm{Im}(\partial_{n+1})$ *is in $\mathcal{B}$; and*

  (b.ii) *the composite $\theta_*(\mathrm{Im}(\partial_{n+1})) \xrightarrow{\theta_*(\mathrm{incl})} \theta_*(\mathrm{Ker}(\partial_n)) \xrightarrow{f_0} \mathrm{Ker}(\theta_*(\partial_n))$ is an isomorphism.*

(c) *The resolution $\boldsymbol{R}_n$ extends to an Ω_{n+1}-resolution if and only if for some splitting s of f_0, the composite*

$$\mathfrak{a}^{[s]}_{\mathrm{Ker}(\partial_n)}\colon \mathrm{Ker}(\partial_n) \xrightarrow{\mathfrak{a}_{\mathrm{Ker}(\partial_n)}} \theta^*\theta_*(\mathrm{Ker}(\partial_n)) \xrightarrow{\theta^*(\chi^{[s]})} \theta^*\big(\theta_*(\mathrm{Ker}(\partial_n))/\mathrm{Im}(s)\big)$$

(where $\chi^{[s]}$ is the natural projection) and the induced map

$$(L_1\theta_*)(\mathfrak{a}^{[s]}_{\mathrm{Ker}(\partial_n)})\colon (L_1\theta_*)\big(\mathrm{Ker}(\partial_n)\big) \longrightarrow (L_1\theta_*)\big(\theta^*(\theta_*(\mathrm{Ker}(\partial_n))/\mathrm{Im}(s))\big)$$

are both epimorphisms. If $\boldsymbol{R}_n$ does extend to $\boldsymbol{R}_{n+1}$ as in (b), then for some splitting s of f_0, $\mathrm{Im}(\partial_{n+1}) = \mathrm{Ker}(\mathfrak{a}^{[s]}_{\mathrm{Ker}(\partial_n)})$, and

$$\theta_*\big(H_n(P_*, \partial_*)\big) \cong \theta_*(\mathrm{Ker}(\partial_n))/\mathrm{Im}(s) \cong \mathrm{Ker}(f_0) \cong (L_1\theta_*)(\mathrm{Im}(\partial_n))\,. \tag{4}$$

Proof. **(a)** Since θ_* is right exact, we have the following commutative diagram in $\mathcal{B}$

$$\begin{tikzcd}
 & \theta_*(\mathrm{Ker}(\partial_n)) \arrow[r] \arrow[d, "f_0"'] & \theta_*(P_n) \arrow[r] \arrow[d, equal] & \theta_*(\mathrm{Im}(\partial_n)) \arrow[r] \arrow[d, "f_1"', "\cong"] & 0 \\
0 \arrow[r] & \mathrm{Ker}(\theta_*(\partial_n)) \arrow[r] & \theta_*(P_n) \arrow[r] & \mathrm{Im}(\theta_*(\partial_n)) \arrow[r] & 0
\end{tikzcd} \tag{5}$$

with exact rows. By condition (Ω_n-4), f_1 is a monomorphism (hence an isomorphism), and so f_0 is an epimorphism. Also, $\mathrm{Ker}(\theta_*(\partial_n)) \in \mathscr{P}(\mathcal{B})$ since the sequence $\theta_*(P_*) \longrightarrow \theta^*(X) \to 0$ is an exact sequence of projective objects, and hence f_0 splits.

(b) Assume that $\boldsymbol{R}_{n+1} = \big(P_{n+1} \xrightarrow{\partial_{n+1}} P_n \longrightarrow \cdots\big)$ is an Ω_{n+1}-resolution of X, and set $J = \mathrm{Im}(\partial_{n+1}) \le \mathrm{Ker}(\partial_n)$. Consider the following commutative diagram

$$\begin{tikzcd}
\theta_*(P_{n+1}) \arrow[r, "\theta_*(\partial^*_{n+1})"] \arrow[dr, two heads, "f_3"] & \theta_*(J) \arrow[r, "\theta_*(\mathrm{incl})"] \arrow[d, "f_4"'] & \theta_*(\mathrm{Ker}(\partial_n)) \arrow[d, "f_0"'] \\
& \mathrm{Im}(\theta_*(\partial_{n+1})) \arrow[r, equal] & \mathrm{Ker}(\theta_*(\partial_n))\,,
\end{tikzcd}$$

where $\partial^*_{n+1}\colon P_{n+1} \longrightarrow J$ is the restriction of ∂_{n+1} and is surjective, and f_3 is the restriction of $\theta_*(\partial_{n+1})$. By condition (Ω_{n+1}-4) on $\boldsymbol{R}_{n+1}$, the morphism $\theta_*(J) \longrightarrow \theta_*(P_n)$ is a monomorphism. Hence f_4 is a monomorphism, and is an isomorphism since f_3 is an epimorphism. This proves (b.ii), and (b.i) follows from condition (Ω_{n+1}-3).

Conversely, assume that (b.i) and (b.ii) hold. In particular, $\boldsymbol{R}_{n+1}$ satisfies (Ω_{n+1}-3), and it satisfies (Ω_{n+1}-1) (P_{n+1} is projective) by assumption. Condition (Ω_{n+1}-4) (that $\theta_*(\mathrm{Im}(\partial_{n+1}))$ injects into $\theta_*(P_n)$) follows from (b.ii).

It remains to prove (Ω_{n+1}-2); i.e., the exactness of $\theta_*(\boldsymbol{R}_{n+1})$. Since $\theta_*(\boldsymbol{R}_n)$ is exact, we need only show that $\mathrm{Im}(\theta_*(\partial_{n+1})) = \mathrm{Ker}(\theta_*(\partial_n))$. Consider the following diagram:

$$\begin{tikzcd}
P_{n+1} \arrow[r, two heads, "\partial^*_{n+1}"] \arrow[d, "\mathfrak{a}_{P_{n+1}}"] & \mathrm{Im}(\partial_{n+1}) \arrow[r, "i"] \arrow[d, "\mathfrak{a}_{\mathrm{Im}(\partial_{n+1})}"] & P_n \arrow[d, "\mathfrak{a}_{P_n}"] \\
\theta^*\theta_*(P_{n+1}) \arrow[r, two heads, "\theta^*\theta_*(\partial^*_{n+1})"] \arrow[rr, bend right, "\theta^*\theta_*(\partial_{n+1})"'] & \theta^*\theta_*(\mathrm{Im}(\partial_{n+1})) \arrow[r, "\theta^*\theta_*(i)"] & \theta^*\theta_*(P_n)
\end{tikzcd}$$

where ∂^*_{n+1} is surjective by definition and $\theta^*\theta_*(\partial^*_{n+1})$ is surjective since $\theta^*\theta_*$ is right exact. Hence $\mathrm{Im}(\theta^*\theta_*(\partial_{n+1})) = \mathrm{Im}(\theta^*\theta_*(i))$, and so $\mathrm{Im}(\theta_*(\partial_{n+1})) = \mathrm{Im}(\theta_*(i))$ by Lemma 1.4(c). Finally, $\mathrm{Im}(\theta_*(i)) = \mathrm{Ker}(\theta_*(\partial_n))$ by the following diagram

$$\begin{tikzcd}
\theta_*(\mathrm{Im}(\partial_{n+1})) \arrow[r, "\theta_*(i')"] \arrow[drr, "\theta_*(i)"'] & \theta_*(\mathrm{Ker}(\partial_n)) \arrow[r, "f_0"] \arrow[dr, "\theta_*(i'')"] & \mathrm{Ker}(\theta_*(\partial_n)) \arrow[d, "\mathrm{incl}"] \\
& & \theta_*(P_n)
\end{tikzcd}$$

and since $f_0 \circ \theta_*(i')$ is an isomorphism by (b.ii).

(c) Assume first that $\boldsymbol{R}_n$ does extend to an Ω_{n+1}-resolution

$$\boldsymbol{R}_{n+1} = \big(P_{n+1} \xrightarrow{\partial_{n+1}} P_n \longrightarrow \cdots\big)\,.$$

Set $J = \mathrm{Im}(\partial_{n+1})$. By (b.i) and (b.ii), $\mathrm{Ker}(\partial_n)/J$ is isomorphic to an object in $\theta^*(\mathcal{B})$, and the composite

$$\begin{tikzcd}
\theta_*(J) \arrow[r, "\theta_*(i_2)"'] \arrow[rr, bend left, "\cong"] & \theta_*(\mathrm{Ker}(\partial_n)) \arrow[r, "f_0"] & \mathrm{Ker}(\theta_*(\partial_n)) \arrow[l, dashed, bend left, "s"]
\end{tikzcd}$$

is an isomorphism. Set $s = \theta_*(i_2) \circ (f_0 \circ \theta_*(i_2))^{-1}$: a splitting for f_0.

Consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r, "i_2"] \arrow[d, "\mathfrak{a}_J"'] & \mathrm{Ker}(\partial_n) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathfrak{a}_{\mathrm{Ker}(\partial_n)}"'] \arrow[dr, "\omega"] & \mathrm{Ker}(\partial_n)/J \arrow[r] \arrow[d, "f_2"', "\cong"] & 0 \\
0 \arrow[r] & \theta^*\theta_*(J) \arrow[r, "\theta^*\theta_*(i_2)"] & \theta^*\theta_*(\mathrm{Ker}(\partial_n)) \arrow[r, "\theta^*\theta_*(\mathrm{pr}_2)"] & \theta^*\theta_*(\mathrm{Ker}(\partial_n)/J) \arrow[r] & 0\,.
\end{tikzcd} \tag{6}$$

Here, $f_2 = \mathfrak{a}_{\mathrm{Ker}(\partial_n)/J}$ is an isomorphism since $\mathrm{Ker}(\partial_n)/J$ is isomorphic to an object in $\theta^*(\mathcal{B})$ by (b.i). The bottom row of (6) is exact since $\theta^*\theta_*$ is right exact and $\theta_*(i_2)$ is a monomorphism by (b.ii) (and θ^* is left exact). Also, $\mathrm{Im}(\theta_*(i_2)) = \mathrm{Im}(s)$, and hence

$$\begin{aligned}
\Big(\mathrm{Ker}(\partial_n) \xrightarrow{\mathfrak{a}^{[s]}_{\mathrm{Ker}(\partial_n)}} \theta^*\big(\theta_*(\mathrm{Ker}(\partial_n))/\mathrm{Im}(s)\big)\Big) &\cong \Big(\mathrm{Ker}(\partial_n) \xrightarrow{\ \omega\ } \theta^*\theta_*(\mathrm{Ker}(\partial_n)/J)\Big) \\
&\cong \Big(\mathrm{Ker}(\partial_n) \xrightarrow{\mathrm{pr}_2} \mathrm{Ker}(\partial_n)/J\Big)\,.
\end{aligned}$$

Thus $\mathfrak{a}^{[s]}_{\mathrm{Ker}(\partial_n)}$ is an epimorphism, and $(L_1\theta_*)(\mathfrak{a}^{[s]}_{\mathrm{Ker}(\partial_n)}) \cong (L_1\theta_*)(\mathrm{pr}_2)$ is also an epimorphism since $\theta_*(i_2)$ is injective. This also proves that $\mathrm{Im}(\partial_{n+1}) = J = \mathrm{Ker}(\mathfrak{a}^{[s]}_{\mathrm{Ker}(\partial_n)})$, and proves (4) except for the isomorphism $\mathrm{Ker}(f_0) \cong (L_1\theta_*)(\mathrm{Im}(\partial_n))$ which follows from (5).

Conversely, assume, for some splitting s of f_0, that $\mathfrak{a}^{[s]}_{\mathrm{Ker}(\partial_n)}$ and $(L_1\theta_*)(\mathfrak{a}^{[s]}_{\mathrm{Ker}(\partial_n)})$ are both epimorphisms. Set $J = \mathfrak{a}^{-1}_{\mathrm{Ker}(\partial_n)}(\theta^*(\mathrm{Im}(s))) \le \mathrm{Ker}(\partial_n)$, and consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r, "i_2"] \arrow[d] & \mathrm{Ker}(\partial_n) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathfrak{a}_{\mathrm{Ker}(\partial_n)}"'] & \mathrm{Ker}(\partial_n)/J \arrow[r] \arrow[d, "f_3"', "\cong"] & 0 \\
0 \arrow[r] & \theta^*(\mathrm{Im}(s)) \arrow[r] & \theta^*\theta_*(\mathrm{Ker}(\partial_n)) \arrow[r, "\chi^{[s]}"] & \theta^*\big(\theta_*(\mathrm{Ker}(\partial_n))/\mathrm{Im}(s)\big) \arrow[r] & 0\,.
\end{tikzcd} \tag{7}$$

The left square in (7) is a pullback square by definition of J, so f_3 is a monomorphism, and f_3 is an epimorphism since $\chi^{[s]} \circ \mathfrak{a}_{\mathrm{Ker}(\partial_n)} = \mathfrak{a}^{[s]}_{\mathrm{Ker}(\partial_n)}$ is an epimorphism.

In particular, $\mathrm{Ker}(\partial_n)/J$ is isomorphic to an object in $\theta^*(\mathcal{B})$, and (b.i) holds. Also, f_2 is an isomorphism in (6), and the bottom row in (6) is exact since $(L_1\theta_*)(\mathfrak{a}^{[s]}_{\mathrm{Ker}(\partial_n)})$ is an epimorphism. Upon comparing (6) and (7), we see that $\mathrm{Im}(s) = \theta_*(i_2)(\theta_*(J))$, and (b.ii) now follows since $\mathrm{Im}(s)$ is the image of a splitting of f_0. So $\boldsymbol{R}_n$ extends to an Ω_{n+1}-resolution by (b). $\square$

The next proposition is our most general result on the existence of Ω-resolutions.

Proposition 2.8. *Fix an Ω-system $(\mathcal{A}, \mathcal{B}; \theta_*, \theta^*)$, where $\mathcal{A}$ has enough projectives. Assume that $\theta^*(\mathcal{B})$ is closed under subobjects and extensions in $\mathcal{A}$. Then each $X \in \mathscr{P}(\mathcal{B})$ admits an Ω-resolution. Furthermore, for $n \ge 0$, each Ω_n-resolution of X extends to an Ω-resolution of X.*

Proof. This follows immediately from Proposition 2.6, Lemma 2.7(c), and Lemma 2.5. $\square$

Under the assumptions of Example 1.3, we can now describe exactly under what conditions there are Ω-resolutions. Recall, for a commutative ring R, that a group G is R-perfect if $H_1(G; R) = 0$.

Example 2.9. Fix a commutative ring R and a surjective homomorphism $\theta\colon G \longrightarrow \pi$ of groups. Let $(RG\text{-}\mathtt{mod}, R\pi\text{-}\mathtt{mod}; \theta_*, \theta^*)$ be the Ω-system of Example 1.3. Then $\theta^*(R\pi\text{-}\mathtt{mod})$ is closed under subobjects.

(a) If $\mathrm{Ker}(\theta)$ is R-perfect, then $\theta^*(R\pi\text{-}\mathsf{mod})$ is closed under extensions in $RG\text{-}\mathsf{mod}$. So by Proposition 2.8, Ω-resolutions exist of all projective objects in $R\pi\text{-}\mathsf{mod}$.

(b) If $\mathrm{Ker}(\theta)$ is not R-perfect, then $\theta^*(R\pi\text{-}\mathsf{mod})$ is not closed under extensions, and for each nonzero object X in $R\pi\text{-}\mathsf{mod}$ that is free as an R-module, $(L_1\theta_*)(\theta^*(X)) \neq 0$. So by Proposition 2.6, no nonzero projective object in $R\pi\text{-}\mathsf{mod}$ that is free as an R-module has an Ω-resolution.

Proof. Set $K = \mathrm{Ker}(\theta)$, and note that an RG-module M is isomorphic to an object in $\theta^*(R\pi\text{-}\mathsf{mod})$ if and only if K acts trivially on M. Thus $\theta^*(R\pi\text{-}\mathsf{mod})$ is closed under subobjects. If $H_1(K;R) = 0$, then $\theta^*(R\pi\text{-}\mathsf{mod})$ is closed under extensions by Lemma A.1, and the existence of Ω-resolutions follows from Proposition 2.8.

If $H_1(K;R) \neq 0$, then for each nonzero object X in $R\pi\text{-}\mathsf{mod}$ that is free as an R-module, $(L_1\theta_*)(\theta^*(X)) \cong H_1(K;\theta^*(X)) \neq 0$ since $H_1(K;R) \neq 0$ and K acts trivially on $\theta^*(X)$. Thus $\theta^*(R\pi\text{-}\mathsf{mod})$ is not closed under extensions in $RG\text{-}\mathsf{mod}$ by Lemma 2.5(b). If in addition, X is projective in $R\pi\text{-}\mathsf{mod}$, Proposition 2.6 implies that it has no Ω-resolution. □

Note that the "squeezed resolutions" defined and studied by Benson [Be2] are Ω-resolutions in the context of Example 2.9(a), when G is a finite group and $K = O^p(G)$.

Remark 2.10. Proposition 2.8 gives some general conditions for the existence of Ω-resolutions: conditions which are satisfied by the Ω-systems of Example 1.3 (as just seen), and also by the much larger family of examples to be described in Proposition 3.5(b). However, they do not hold for the family of examples constructed in Proposition 3.12(a), even though Ω-resolutions are shown to exist in those cases (at least for certain projective objects) in Proposition 4.3. This suggests that there should be a more general existence result that covers all of these cases.

In fact, there are two questions of this type that one can ask. First, of course, we want to find conditions as general as possible on an Ω-system $(\mathcal{A},\mathcal{B};\theta_*,\theta^*)$ that imply the existence of Ω-resolutions of all projectives in $\mathcal{B}$. But we will see in Example 3.6 that there are Ω-systems for which some nonzero projectives have Ω-resolutions and others do not, and so we would also like to find more general conditions on a pair $\big((\mathcal{A},\mathcal{B};\theta_*,\theta^*),X\big)$, for $X \in \mathscr{P}(\mathcal{B})$, that imply the existence of an Ω-resolution of X.

3. Ω-systems of functor categories

We next look at a large family of examples of Ω-systems and Ω-resolutions involving functor categories; especially categories of $R\mathcal{C}$-modules for a small category $\mathcal{C}$ and a commutative ring R. At the end of the section, in Propositions 3.5 and 3.12, we give two large families of examples of Ω-systems where we can say fairly precisely in which cases Ω-resolutions exist.

We refer to [Mac1, § II.6 and § X.3] for the definitions and properties of overcategories and left Kan extensions. As usual, when $\mathcal{A}$ and $\mathcal{C}$ are categories and $\mathcal{C}$ is small, $\mathcal{A}^{\mathcal{C}}$ denotes the functor category whose objects are the functors $\mathcal{C} \longrightarrow \mathcal{A}$, and whose morphisms are the natural transformations of functors.

In the following proposition, we define a functor $\theta\colon \mathcal{C} \longrightarrow \mathcal{D}$ between small categories to be *quasisurjective* if it is surjective on objects, and each morphism in $\mathcal{D}$ is a composite of morphisms $\theta(\varphi)$ and $\theta(\psi)^{-1}$ for $\varphi,\psi \in \mathrm{Mor}(\mathcal{C})$ such that $\theta(\psi)$ is an isomorphism in $\mathcal{D}$. In other words, $\mathcal{D}$ is generated by the image of θ together with inverses of isomorphisms in the image of θ.

Proposition 3.1. *Let $\mathcal{A}$ be an abelian category with colimits, and let $\theta\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a quasisurjective functor between small categories. For each d in $\mathcal{D}$, let $\mathcal{I}(\theta{\downarrow}d)$ be the full subcategory of $\theta{\downarrow}d$ with objects (c,φ) for $\varphi \in \mathrm{Iso}_{\mathcal{D}}(\theta(c),d)$, and assume that all objects (c,Id_d) for $c \in \theta^{-1}(d)$ lie in the same connected component of $\mathcal{I}(\theta{\downarrow}d)$. Let $\theta^*\colon \mathcal{A}^{\mathcal{D}} \longrightarrow \mathcal{A}^{\mathcal{C}}$ be composition with θ, and let $\theta_*\colon \mathcal{A}^{\mathcal{C}} \longrightarrow \mathcal{A}^{\mathcal{D}}$ be left Kan extension along θ. Then $(\mathcal{A}^{\mathcal{C}},\mathcal{A}^{\mathcal{D}};\theta_*,\theta^*)$ is an Ω-system.*

Proof. Conditions (OP1) and (OP3) are clear. So the only difficulty is to show that condition (OP2) holds: that the counit $\mathfrak{b}\colon \theta_*\theta^* \longrightarrow \mathrm{Id}_{\mathcal{A}^{\mathcal{D}}}$ associated to the adjunction is an isomorphism.

Fix a functor $\alpha\colon \mathcal{D} \longrightarrow \mathcal{A}$ and an object d in $\mathcal{D}$, and let $\underline{\alpha(\underline{d})}\colon \theta{\downarrow}d \longrightarrow \mathcal{A}$ be the constant functor sending all objects to $\alpha(d)$. Let $\alpha_d\colon \theta{\downarrow}d \longrightarrow \mathcal{A}$ be the functor that sends an object (c,φ), where $\varphi \in \mathrm{Mor}_{\mathcal{D}}(\theta(c),d)$, to $(\theta^*\alpha)(c) = \alpha(\theta(c))$. Let $\alpha_*\colon \alpha_d \longrightarrow \underline{\alpha(\underline{d})}$ be the natural transformation of functors that sends (c,φ) to $\alpha(\varphi) \in \mathrm{Mor}_{\mathcal{A}}(\alpha_d(c,\varphi),\alpha(d))$. Then

$$\mathfrak{b}(\alpha)(d) = \operatorname*{colim}_{\theta\downarrow d}(\alpha_*)\colon (\theta_*\theta^*(\alpha))(d) = \operatorname*{colim}_{\theta\downarrow d}(\alpha_d) \xrightarrow{\qquad} \alpha(d)\,,$$

and we must show that this is an isomorphism (for all α and d).

To see this, choose $\widehat{d} \in \mathrm{Ob}(\mathcal{C})$ such that $\theta(\widehat{d}) = d$. For each (c,φ) in $\theta{\downarrow}d$, let $\iota_{(c,\varphi)}$ be the natural morphism from $\alpha_d(c,\varphi) = \alpha(\theta(c))$ to the colimit. Set

$$\beta = \iota_{(\widehat{d},\mathrm{Id}_d)}\colon \alpha(d) = \alpha_d(\widehat{d},\mathrm{Id}_d) \xrightarrow{\qquad} \mathrm{colim}(\alpha_d) = (\theta_*\theta^*(\alpha))(d).$$

Then $\mathfrak{b}(\alpha)(d)\circ\beta = \mathrm{Id}_{\alpha(d)}$, and it remains to show that $\beta\circ\mathfrak{b}(\alpha)(d)$ is also the identity. This means showing, for each object (c,φ) in $\theta{\downarrow}d$, that $\beta\circ\mathfrak{b}(\alpha)(d)\circ\iota_{(c,\varphi)} = \iota_{(c,\varphi)}$. Since

$$\mathfrak{b}(\alpha)(d)\circ\iota_{(c,\varphi)} = \alpha(\varphi)\colon \alpha_d(c,\varphi) = \alpha(\theta(c)) \xrightarrow{\qquad} \alpha(d) = \alpha_d(\widehat{d},\mathrm{Id}_d),$$

we are reduced to showing, for each (c,φ), that

$$\iota_{(\widehat{d},\mathrm{Id}_d)}\circ\alpha(\varphi) = \iota_{(c,\varphi)}\,. \tag{1}$$

We now claim the following:

(i) Equation (1) holds for $(\widehat{d},\mathrm{Id}_d)$.

(ii) If there is $\chi \in \mathrm{Mor}_{\theta\downarrow d}((c,\varphi),(c',\varphi'))$, then
  - (ii.1) if (1) holds for (c',φ'), then it also holds for (c,φ); and
  - (ii.2) if $\theta(\chi)$ is an isomorphism, then (1) holds for (c,φ) if and only if it holds for (c',φ').

(iii) If $\theta(c) = \theta(c')$ and $\varphi \in \mathrm{Mor}_{\mathcal{D}}(\theta(c),d)$, then (1) holds for (c,φ) if and only if it holds for (c',φ).

Point (i) is clear. If $\chi \in \mathrm{Mor}_{\theta\downarrow d}((c,\varphi),(c',\varphi'))$, then $\iota_{(c',\varphi')}\circ\alpha(\theta(\chi)) = \iota_{(c,\varphi)}$ by definition of colimits, and (ii) follows immediately from this. Point (iii) follows from (ii.2) and the assumption that $(c,\mathrm{Id}_{\theta(c)})$ and $(c',\mathrm{Id}_{\theta(c)})$ are in the same connected component of $\mathcal{I}(\theta{\downarrow}\theta(c))$.

Now let (c,φ) be arbitrary. Since θ is quasisurjective, for some $m \geq 1$, there are objects $\theta(c) = d_0, d_1, \dots, d_m = d$ in $\mathcal{D}$ and morphisms $\varphi_i \in \mathrm{Mor}_{\mathcal{D}}(d_{i-1},d_i)$ for $1 \leq i \leq m$ such that $\varphi = \varphi_m\circ\cdots\circ\varphi_2\circ\varphi_1$, and for each i, either $\varphi_i \in \theta(\mathrm{Mor}(\mathcal{C}))$ or $\varphi_i \in \mathrm{Iso}(\mathcal{D})$ and $\varphi_i^{-1} \in \theta(\mathrm{Mor}(\mathcal{C}))$. Since (1) holds for $(\widehat{d},\mathrm{Id}_d)$ by (i), it also holds for (c,Id_d) for all $c \in \theta^{-1}(d) = \theta^{-1}(d_m)$ by (iii). If $\varphi_m \in \theta(\mathrm{Mor}(\mathcal{C}))$, then (1) holds for (c,φ_m) for some $c \in \theta^{-1}(d_{m-1})$ by (ii.1), while if $\varphi_m \in \mathrm{Iso}(\mathcal{D})$ and $\varphi_m^{-1} \in \theta(\mathrm{Mor}(\mathcal{C}))$, then (1) holds for (c,φ_m) for some $c \in \theta^{-1}(d_{m-1})$ by (ii.2). In either case, (1) holds for (c,φ_m) for all $c \in \theta^{-1}(d_{m-1})$ by (iii). Upon continuing this argument, we see by downward induction that for each $1 \le i \le m$, (1) holds for $(c, \varphi_m \circ \cdots \circ \varphi_i)$ for all $c \in \theta^{-1}(d_{i-1})$. In particular, (1) holds for (c, φ) (the case $i = 1$). □

We now specialize to the case where $\mathcal{A} = R\text{-}\mathtt{mod}$: the category of modules over a commutative ring R.

Definition 3.2. *Let $\mathcal{C}$ be a small category, and let R be a commutative ring.*

(a) *An $R\mathcal{C}$-module is a* covariant *functor $M\colon \mathcal{C} \longrightarrow R\text{-}\mathtt{mod}$, and a morphism of $R\mathcal{C}$-modules is a natural transformation of functors. Let $R\mathcal{C}\text{-}\mathtt{mod}$ denote the category of $R\mathcal{C}$-modules.*

(b) *Let $\theta\colon \mathcal{C} \to \mathcal{D}$ be a functor between small categories. When M is an $R\mathcal{C}$-module, let $\theta_*(M)$ denote the left Kan extension of M along θ. When N is an $R\mathcal{D}$-module, let $\theta^*(N) = N \circ \theta$ be the $R\mathcal{C}$-module induced by composition with θ.*

(c) *An $R\mathcal{C}$-module M is* locally constant *on $\mathcal{C}$ if it sends all morphisms in $\mathcal{C}$ to isomorphisms of R-modules.*

(d) *An $R\mathcal{C}$-module M is* essentially constant *if M is isomorphic to a constant $R\mathcal{C}$-module; i.e., isomorphic to a functor $\mathcal{C} \longrightarrow R\text{-}\mathtt{mod}$ that sends each object to the same R-module V and each morphism to Id_V.*

The next lemma characterizes essentially constant modules in terms of an action of $\pi_1(|\mathcal{C}|)$.

Lemma 3.3. *Assume $\mathcal{C}$ is a small category, and let R be a commutative ring.*

(a) *If M is a locally constant $R\mathcal{C}$-module, then for each object c_0 in $\mathcal{C}$, there is a unique homomorphism*
$$M_\#\colon \pi_1(|\mathcal{C}|, c_0) \xrightarrow{\qquad} \mathrm{Aut}_R(M(c_0))$$
with the following property: for each sequence
$$\sigma = \Bigl(c_0 \xrightarrow{\ f_1\ } c_1 \xleftarrow{\ f_2\ } c_2 \xrightarrow{\ f_3\ } \cdots \xleftarrow{\ f_{2m}\ } c_{2m} = c_0\Bigr)$$
of morphisms in $\mathcal{C}$ ($m \ge 1$), beginning and ending at c_0, regarded as a loop in $|\mathcal{C}|$,
$$M_\#([\sigma]) = M(f_{2m})^{-1} \circ M(f_{2m-1}) \circ \cdots \circ M(f_2)^{-1} \circ M(f_1) \in \mathrm{Aut}_R(M(c_0))\,.$$

(b) *If $\mathcal{C}$ is connected, then a locally constant $R\mathcal{C}$-module M is essentially constant if and only if $M_\#$ (as defined in (a)) is the trivial homomorphism for some object c_0 in $\mathcal{C}$.*

Proof. **(a)** Let $\mathtt{Is}(R\text{-}\mathtt{mod})$ be the category of R-modules with only isomorphisms as morphisms, and regard M as a functor $M\colon \mathcal{C} \longrightarrow \mathtt{Is}(R\text{-}\mathtt{mod})$. This induces a map between the geometric realizations, and hence a homomorphism of fundamental groups
$$M_\#\colon \pi_1(|\mathcal{C}|, c_0) \xrightarrow{\qquad} \pi_1(|\mathtt{Is}(R\text{-}\mathtt{mod})|, M(c_0)) \cong \mathrm{Aut}_R(M(c_0))\,.$$
For each sequence σ as described above, $M_\#$ sends the class $[\sigma] \in \pi_1(|\mathcal{C}|, c_0)$ to
$$M(c_0) \xrightarrow{M(f_1)} M(c_1) \xleftarrow{M(f_2)} M(c_2) \xrightarrow{M(f_3)} \cdots \xleftarrow{M(f_{2m})} M(c_{2m}) = M(c_0)\,,$$
regarded as a loop in $|\mathtt{Is}(R\text{-}\mathtt{mod})|$, and this is homotopic to the composite
$$M(f_{2m})^{-1} \circ M(f_{2m-1}) \circ \cdots \circ M(f_1)^{-1} \circ M(f_0) \in \mathrm{Aut}_R(M(c_0))$$
when also regarded as a loop in $|\mathtt{Is}(R\text{-}\mathtt{mod})|$.

(b) If M is isomorphic to a constant functor, then it clearly sends all morphisms to isomorphisms, and sends a loop σ as above to a sequence whose composite is the identity. Thus for each c_0 in $\mathcal{C}$, the homomorphism $M_\#$ defined in (a) is trivial. It remains to prove the converse.

Assume that M is locally constant, and that for some object c_0 in $\mathcal{C}$, the homomorphism $M_\#$ defined in (a) is trivial. Set $M_\mathcal{C} = \operatorname{colim}_\mathcal{C}(M)$. We claim that the natural morphism $\iota_c \colon M(c) \longrightarrow M_\mathcal{C}$ is an isomorphism for each object c. Once this has been shown, the ι_c define an isomorphism of functors from M to the constant functor with value $M_\mathcal{C}$.

For each pair of objects c, d and each $\varphi \in \operatorname{Mor}_\mathcal{C}(c,d)$, we have $\iota_c = \iota_d \circ M(\varphi)$, where $M(\varphi)$ is an isomorphism since M is locally constant. Thus $\operatorname{Im}(\iota_c) = \operatorname{Im}(\iota_d)$ whenever $\operatorname{Mor}_\mathcal{C}(c,d) \neq \varnothing$, and so $\operatorname{Im}(\iota_c) = \operatorname{Im}(\iota_d)$ for each pair of objects c, d since $\mathcal{C}$ is connected. So ι_c is surjective for each c in $\mathcal{C}$.

For each object d in $\mathcal{C}$, since $\mathcal{C}$ is connected, there is a sequence

$$c_0 \xrightarrow{f_1} c_1 \xleftarrow{f_2} c_2 \xrightarrow{f_3} \cdots \xleftarrow{f_{2m}} c_{2m} = d$$

($m \geq 1$) of morphisms in $\mathcal{C}$ connecting c_0 to d. Set

$$\eta_d = M(f_1)^{-1} \circ M(f_2) \circ \cdots \circ M(f_{2m-1})^{-1} \circ M(f_{2m}) \colon M(d) \xrightarrow{\quad\cong\quad} M(c_0)\,.$$

Then η_d is independent of the choice of the f_i since $M_\# = 1$. This independence of the choice of sequence of morphisms also implies that for each pair of objects d and d' and each morphism $\varphi \in \operatorname{Mor}_\mathcal{C}(d,d')$, we have $\eta_{d'} \circ M(\varphi) = \eta_d$. We thus get a natural morphism $\eta \colon M_\mathcal{C} \longrightarrow M(c_0)$ such that $\eta \circ \iota_d = \eta_d$ for each d, and ι_d is injective for each d in $\mathcal{C}$ since η_d is. We already showed that ι_d is surjective for each d, so ι is a natural isomorphism of functors from M to the constant functor $\underline{M_\mathcal{C}}$. $\square$

The following description of certain projective $R\mathcal{C}$-modules will be needed later.

Lemma 3.4. *Let R be a commutative ring, and let $\mathcal{C}$ be a small category. For each object c in $\mathcal{C}$, let $F_c^{R\mathcal{C}}$ be the $R\mathcal{C}$-module that sends an object d to $R(\operatorname{Mor}_\mathcal{C}(c,d))$ (the free R-module with basis $\operatorname{Mor}_\mathcal{C}(c,d)$); and sends a morphism $\varphi \in \operatorname{Mor}_\mathcal{C}(d,d')$ to composition with φ. Then $F_c^{R\mathcal{C}}$ is projective, and for each $R\mathcal{C}$-module M, evaluation at $\operatorname{Id}_c \in F_c^{R\mathcal{C}}(c)$ defines a bijection $\operatorname{Mor}_{R\mathcal{C}}(F_c^{R\mathcal{C}}, M) \cong M(c)$.*

Proof. The bijection $\operatorname{Mor}_{R\mathcal{C}}(F_c^{R\mathcal{C}}, M) \cong M(c)$ is clear. In particular, $\operatorname{Mor}_{R\mathcal{C}}(F_c^{R\mathcal{C}}, -)$ is an exact functor, and so $F_c^{R\mathcal{C}}$ is projective. $\square$

We now restrict further to two different cases: one where $\theta \colon \mathcal{C} \longrightarrow \mathcal{D}$ is bijective on objects, and the second where $\mathcal{D}$ is the category of a group. In each of these cases, we are able to get much more precise results about the existence of Ω-resolutions. In both cases, for a commutative ring R, we call a group G *R-perfect* if $H_1(G;R) = 0$; i.e., if $(G/[G,G]) \otimes_\mathbb{Z} R = 0$.

3.1. **Functors bijective on objects.**

We begin with the case where θ is bijective on objects. When R is a commutative ring and one additional technical assumption holds, we can say quite precisely in which cases there always exist Ω-resolutions.

Proposition 3.5. *Fix a commutative ring R. Let $\theta \colon \mathcal{C} \longrightarrow \mathcal{D}$ be a functor between small categories that is bijective on objects and surjective on morphism sets. Then*

(a) *$(R\mathcal{C}\text{-}\mathtt{mod}, R\mathcal{D}\text{-}\mathtt{mod}; \theta_*, \theta^*)$ is an Ω-system, and the subcategory $\theta^*(R\mathcal{D}\text{-}\mathtt{mod})$ is closed under subobjects in $R\mathcal{C}\text{-}\mathtt{mod}$.*

For each object c in $\mathcal{C}$, set

$$K_c = \operatorname{Ker}\bigl[\theta_c\colon \operatorname{Aut}_{\mathcal{C}}(c) \longrightarrow \operatorname{Aut}_{\mathcal{D}}(\theta(c))\bigr],$$

and assume that θ has the following property:

$$\text{for each pair of objects } c,c' \text{ in } \mathcal{C}, \text{ and each pair of morphisms } \varphi,\varphi' \in \operatorname{Mor}_{\mathcal{C}}(c,c') \\ \text{such that } \theta_{c,c'}(\varphi)=\theta_{c,c'}(\varphi'), \text{ there is some } \alpha\in K_{c'} \text{ such that } \varphi=\alpha\varphi'. \tag{2}$$

Then the following hold.

(b) *If K_c is R-perfect for each $c \in \operatorname{Ob}(\mathcal{C})$, then $\theta^*(R\mathcal{D}\text{-}\mathfrak{mod})$ is closed under extensions, and hence all projectives in $R\mathcal{D}\text{-}\mathfrak{mod}$ have Ω-resolutions.*

(c) *If K_c is not R-perfect for some $c \in \operatorname{Ob}(\mathcal{C})$, then $\theta^*(R\mathcal{D}\text{-}\mathfrak{mod})$ is not closed under extensions, and there is a projective object X in $R\mathcal{D}\text{-}\mathfrak{mod}$ that does not have an Ω-resolution.*

Proof. **(a)** Since θ is surjective on objects and morphisms, it is quasisurjective. Since it is bijective on objects, the condition on $\mathcal{I}(\theta{\downarrow}d)$ in Proposition 3.1 holds for all objects d in $\mathcal{D}$, and so $(R\mathcal{C}\text{-}\mathfrak{mod}, R\mathcal{D}\text{-}\mathfrak{mod}; \theta_*, \theta^*)$ is an Ω-system by that proposition.

Since θ is bijective on objects and surjective on morphisms, an $R\mathcal{C}$-module M is isomorphic to an object in $\theta^*(R\mathcal{D}\text{-}\mathfrak{mod})$ if and only if it has the following property: if $\varphi,\psi \in \operatorname{Mor}_{\mathcal{C}}(c,c')$ are such that $\theta(\varphi)=\theta(\psi)$ (some $c,c' \in \operatorname{Ob}(\mathcal{C})$), then $M_{c,c'}(\varphi) = M_{c,c'}(\psi)$. In particular, $\theta^*(R\mathcal{D}\text{-}\mathfrak{mod})$ is closed under subobjects.

(b,c) Now assume that (2) holds. For each $R\mathcal{C}$-module M, let M_K be the $R\mathcal{D}$-module defined by setting, for each $d \in \operatorname{Ob}(\mathcal{D})$ and $c \in \theta^{-1}(d)$,

$$(M_K)(d) \cong M(c)_{K_c} \overset{\text{def}}{=} M(c)/\bigl\langle \alpha(x)-x \bigm| x\in M(c),\ \alpha\in K_c\bigr\rangle.$$

For each morphism $\varphi \in \operatorname{Mor}_{\mathcal{C}}(c,c')$ and each $\alpha\in K_c$, $\theta_{c,c'}(\varphi\circ\alpha)=\theta_{c,c'}(\varphi)$, so by (2), there is $\beta \in K_{c'}$ such that $\varphi\circ\alpha=\beta\circ\varphi$. Hence for each $x\in M(c)$, $\varphi_*(x)$ and $\varphi_*(\alpha(x))$ are in the same orbit of $K_{c'}$. It follows that $\varphi_* \in \operatorname{Mor}_R(M(c),M(c'))$ induces a homomorphism between the quotient modules $M_K(c)$ and $M_K(c')$. So by (2) and since θ is surjective on morphisms, there is a unique functor M_K on $\mathcal{D}$ such that the natural surjections $M(c) \longrightarrow M_K(\theta(c))$ define a morphism of $R\mathcal{C}$-modules $M \longrightarrow \theta^*(M_K)$, and hence a morphism of $R\mathcal{D}$-modules $\theta_*(M) \longrightarrow M_K$.

By (2) and the surjectivity of θ again, we have a natural bijection $\operatorname{Mor}_{R\mathcal{C}}(M,\theta^*N) \cong \operatorname{Mor}_{R\mathcal{D}}(M_K,N)$ for each $R\mathcal{C}$-module M and each $R\mathcal{D}$-module N, and thus $M_K \cong \theta_*(M)$. We have now shown that

for each $R\mathcal{C}$-module M and each $c\in\operatorname{Ob}(\mathcal{C})$, the natural morphism

$$\mathfrak{a}_M\colon M(c) \xrightarrow{\quad\quad} (\theta^*\theta_*M)(c) = (\theta_*M)(\theta(c)) \tag{3}$$

induces an isomorphism $M(c)_{K_c} \cong (\theta_*M)(\theta(c))$.

(b) Assume K_c is R-perfect for each $c\in\operatorname{Ob}(\mathcal{C})$. Let $0\longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be an extension of $R\mathcal{C}$-modules such that M' and M'' are in $\theta^*(R\mathcal{D}\text{-}\mathfrak{mod})$. For each $c\in\operatorname{Ob}(\mathcal{C})$, K_c acts trivially on $M'(c)$ and on $M''(c)$, and hence also acts trivially on $M(c)$ by Lemma A.1. So $M\cong\theta^*(\theta_*(M))$ by (3).

This proves that $\theta^*(R\mathcal{D}\text{-}\mathfrak{mod})$ is closed under extensions in $R\mathcal{C}\text{-}\mathfrak{mod}$, and hence by Proposition 2.8 that Ω-resolutions exist of all projectives in $R\mathcal{D}\text{-}\mathfrak{mod}$.

(c) Assume, for some object c_0 in $\mathcal{C}$, that K_{c_0} is not R-perfect, and set $d_0=\theta(c_0)$. Let $F^{R\mathcal{C}}_{c_0}$ and $F^{R\mathcal{D}}_{d_0}$ be the projective $R\mathcal{C}$- and $R\mathcal{D}$-modules defined in Lemma 3.4; thus

$$F^{R\mathcal{C}}_{c_0}(c)=R(\mathrm{Mor}_{\mathcal{C}}(c_0,c)) \qquad \text{and} \qquad F^{R\mathcal{D}}_{d_0}(d)=R(\mathrm{Mor}_{\mathcal{D}}(d_0,d))$$

for each $c\in\mathrm{Ob}(\mathcal{C})$ and $d\in\mathrm{Ob}(\mathcal{D})$. Since θ is surjective on morphisms, there is a natural surjection of $R\mathcal{C}$-modules $\chi\colon F^{R\mathcal{C}}_{c_0}\longrightarrow\theta^*(F^{R\mathcal{D}}_{d_0})$ that sends $\varphi\in F^{R\mathcal{C}}_{c_0}(c)$ to $\theta(\varphi)\in\theta^*(F^{R\mathcal{D}}_{d_0})(c)=F^{R\mathcal{D}}_{d_0}(\theta(c))$.

Set $Q_0=\mathrm{Ker}(\chi)$, and consider the exact sequence

$$0\longrightarrow(L_1\theta_*)(\theta^*(F^{R\mathcal{D}}_{d_0}))\longrightarrow\theta_*(Q_0)\longrightarrow\theta_*(F^{R\mathcal{C}}_{c_0})\longrightarrow F^{R\mathcal{D}}_{d_0}\longrightarrow 0.$$

Here, $\theta_*(F^{R\mathcal{C}}_{c_0})\cong F^{R\mathcal{D}}_{d_0}$ by (3) and since $\mathrm{Mor}_{\mathcal{C}}(c_0,c)/K_c\cong\mathrm{Mor}_{\mathcal{D}}(d_0,\theta(c))$ for each c in $\mathcal{C}$ by (2). Thus $(L_1\theta_*)(\theta^*F^{R\mathcal{D}}_{d_0})\cong\theta_*(Q_0)$. We will show that $\theta_*(Q_0)(d_0)\neq0$; then $(L_1\theta_*)(\theta^*F^{R\mathcal{D}}_{d_0})\neq0$, so $F^{R\mathcal{D}}_{d_0}$ has no Ω-resolution by Proposition 2.8, and $\theta^*(R\mathcal{D}\text{-}\mathsf{mod})$ is not closed under extensions by Lemma 2.5(b).

Set $\mathrm{End}^{(1)}_{\mathcal{C}}(c_0)=\mathrm{Aut}_{\mathcal{C}}(c_0)$, and let $\mathrm{End}^{(2)}_{\mathcal{C}}(c_0)$ be its complement (as a set) in $\mathrm{End}_{\mathcal{C}}(c_0)$. Set $\mathrm{End}^{(i)}_{\mathcal{D}}(d_0)=\theta_{c_0}(\mathrm{End}^{(i)}_{\mathcal{C}}(c_0))$ for $i=1,2$. Thus

$$\mathrm{End}_{\mathcal{C}}(c_0)=\mathrm{End}^{(1)}_{\mathcal{C}}(c_0)\amalg\mathrm{End}^{(2)}_{\mathcal{C}}(c_0) \qquad\text{and}\qquad \mathrm{End}_{\mathcal{D}}(d_0)=\mathrm{End}^{(1)}_{\mathcal{D}}(d_0)\amalg\mathrm{End}^{(2)}_{\mathcal{D}}(d_0):$$

the first by definition, and the second by (2) and since θ_{c_0} is surjective. Set $U^{(i)}_{\mathcal{C}}=R(\mathrm{End}^{(i)}_{\mathcal{C}}(c_0))$ and $U^{(i)}_{\mathcal{D}}=R(\mathrm{End}^{(i)}_{\mathcal{D}}(d_0))$ $(i=1,2)$, so that $F^{R\mathcal{C}}_{c_0}(c_0)=U^{(1)}_{\mathcal{C}}\oplus U^{(2)}_{\mathcal{C}}$ and $F^{R\mathcal{D}}_{d_0}(d_0)=U^{(1)}_{\mathcal{D}}\oplus U^{(2)}_{\mathcal{D}}$. Thus $Q_0(c_0)=Q^{(1)}_0\oplus Q^{(2)}_0$ where $Q^{(i)}_0$ is the kernel of the surjection $U^{(i)}_{\mathcal{C}}\longrightarrow U^{(i)}_{\mathcal{D}}$.

By (3), we must show that $Q_0(c_0)_{Kc_0}\cong\theta_*(Q_0)(d_0)\neq0$, and to do this, it suffices to show that $(Q^{(1)}_0)_{Kc_0}\neq0$. Set $A=R[\mathrm{Aut}_{\mathcal{C}}(c_0)]\cong U^{(1)}_{\mathcal{C}}$, and identify it with the group ring. We can also identify $Q^{(1)}_0=I$: the 2-sided ideal in A generated as an R-module by the elements $g-h$ for $g,h\in\mathrm{Aut}_{\mathcal{C}}(c_0)$ such that $gh^{-1}\in K_{c_0}$. Then $X_{K_{c_0}}=X/IX$ for each A-module X. In particular, $(Q^{(1)}_0)_{Kc_0}\cong I/I^2$.

Consider the short exact sequence $0\to I\to A\to A/I\to0$ of $R[K_{c_0}]$-modules. Since A is projective, this induces an isomorphism $I/I^2\cong H_1(K_{c_0};A/I)$. Since K_{c_0} is not R-perfect and acts trivially on the free R-module $A/I\cong R[\mathrm{Aut}_{\mathcal{D}}(d_0)]$, we now conclude that $I/I^2\neq0$. $\square$

Example 3.6. In the situation of Proposition 3.5(c), there can also be nonzero $R\mathcal{C}$-modules that do have Ω-resolutions. For example, fix a prime p, set $R=\mathbb{F}_p$, and assume that $\mathrm{Ob}(\mathcal{C})=\mathrm{Ob}(\mathcal{D})=\{x,y\}$, where $\mathrm{End}_{\mathcal{D}}(x)=\mathrm{End}_{\mathcal{C}}(y)=\mathrm{End}_{\mathcal{D}}(y)=\{\mathrm{Id}\}$ and $\mathrm{End}_{\mathcal{C}}(x)\cong C_p$, and each category has a unique morphism from x to y and none from y to x. Then the unique functor $\theta\colon\mathcal{C}\longrightarrow\mathcal{D}$ satisfies the hypotheses of Proposition 3.5, and $K_x\cong C_p$ is not R-perfect while $K_y=1$ is. Set $X=F^{R\mathcal{D}}_y$; then $\theta^*(X)\cong F^{R\mathcal{C}}_y$ is projective as an $R\mathcal{C}$-module, so X has as Ω-resolution the sequence $0\longrightarrow F^{R\mathcal{C}}_y\xrightarrow{\ \varepsilon\ }\theta^*(F^{R\mathcal{D}}_y)\longrightarrow0$.

3.2. **Categories over a group.**

The other large family of examples we consider are those where $\mathcal{D}=\mathcal{B}(\pi)$ for a group π.

Definition 3.7. *A* category over a group π *consists of a pair $(\mathcal{C},\theta)$, where $\mathcal{C}$ is a nonempty small connected category, and $\theta\colon\mathcal{C}\longrightarrow\mathcal{B}(\pi)$ is a functor such that the homomorphism $\pi_1(|\mathcal{C}|)\longrightarrow\pi$ induced by θ is surjective.*

For example, if G and π are groups, and $\theta\colon\mathcal{B}(G)\longrightarrow\mathcal{B}(\pi)$ is the functor induced by a surjective homomorphism $G\longrightarrow\pi$, then $(\mathcal{B}(G),\theta)$ is a category over π.

As another example, one that helped motivate this work, let $(S,\mathcal{F},\mathcal{L})$ be a p-local compact group as defined in [BLO]. Set $\pi=\pi_1(|\mathcal{L}|_p^\wedge)$. Then π is a finite p-group, and there is a natural functor $\theta\colon \mathcal{L}\longrightarrow\mathcal{B}(\pi)$ whose restriction to $\mathcal{B}(S)$ is surjective. It follows from properties of linking systems that $(\mathcal{L},\theta)$ is a category over π. We refer to the introduction to Section 5 for more details.

Lemma 3.8. *Let R be a commutative ring, and let $(\mathcal{C},\theta)$ be a category over π. Then*

(a) *the overcategory $\theta{\downarrow}\circ_\pi$ is connected, $(R\mathcal{C}\text{-}\mathsf{mod},R\pi\text{-}\mathsf{mod};\theta_*,\theta^*)$ is an Ω-system, and the projection $|\theta{\downarrow}\circ_\pi|\longrightarrow|\mathcal{C}|$ is a covering space with covering group π.*

For an $R\mathcal{C}$-module M,

(b) *$\theta_*(M)\cong\operatorname*{colim}_{\theta\downarrow\circ_\pi}(M)$; and*

(c) *$M\cong\theta^*(N)$ for some $R\pi$-module N if and only if M is locally constant on $\mathcal{C}$ and essentially constant on $\theta{\downarrow}\circ_\pi$.*

Proof. **(a)** Since π acts freely on $|\theta{\downarrow}\circ_\pi|$ with orbit space $|\mathcal{C}|$, the projection to $|\mathcal{C}|$ is a covering space with covering group π. In particular, $|\theta{\downarrow}\circ_\pi|$ is connected since $\pi_1(|\mathcal{C}|)$ surjects onto π. Also, θ is quasisurjective since $\pi_1(|\mathcal{C}|)$ surjects onto π, and so $(R\mathcal{C}\text{-}\mathsf{mod},R\pi\text{-}\mathsf{mod};\theta_*,\theta^*)$ is an Ω-system by Proposition 3.1.

(b) By definition of left Kan extension, $\theta_*(M)=\operatorname*{colim}_{\theta\downarrow\circ_\pi}(M)$.

(c) Assume M is locally constant on $\mathcal{C}$ and essentially constant on $\theta{\downarrow}\circ_\pi$. We claim that the natural morphism $\mathfrak{a}_M\colon M\longrightarrow\theta^*\theta_*(M)$ is an isomorphism. This means showing, for each c in $\mathcal{C}$, that the natural morphism from $M(c)$ to $\theta_*(M)$ is an isomorphism. By (b), this is equivalent to showing that the natural morphism $\xi_c\colon M(c)\longrightarrow M_{\theta\downarrow\circ_\pi}$ is an isomorphism for each c. But this holds since by assumption, the restriction of M to $\theta{\downarrow}\circ_\pi$ is isomorphic to a constant functor.

Conversely, if $M\cong\theta^*(N)$, then M is locally constant on $\mathcal{C}$, and isomorphic to a constant functor on $\theta{\downarrow}\circ_\pi$. $\square$

Lemma 3.9. *Let R be a commutative ring, let $(\mathcal{C},\theta)$ be a category over a group π, and set $H=\operatorname{Ker}\big[\pi_1(|\theta|)\colon\pi_1(|\mathcal{C}|)\longrightarrow\pi\big]$.*

(a) *If H is R-perfect, then $(L_1\theta_*)(\theta^*X)=0$ for each $R\pi$-module X; and*

(b) *if H is not R-perfect, then $(L_1\theta_*)(\theta^*X)\neq0$ for each $R\pi$-module $X\neq0$ that (as an R-module) contains R as a direct summand.*

Proof. For each $R\pi$-module X, $(L_1\theta_*)(\theta^*X)\cong(L_1(\operatorname*{colim}_{\theta\downarrow\circ_\pi}))(\theta^*X)$ as R-modules by Lemma 3.8(b), and $(L_1(\operatorname*{colim}_{\theta\downarrow\circ_\pi}))(\theta^*X)\cong H_1(|\theta{\downarrow}\circ_\pi|;X)$ since the two sides are homology groups of the same chain complex by [GZ, Appendix II, Proposition 3.3]. Here, the homology is with untwisted coefficients since $H\cong\pi_1(|\theta{\downarrow}\circ_\pi|)$ (Lemma 3.8(a)) acts trivially on the $R\pi$-module X. Thus $(L_1\theta_*)(\theta^*X)=0$ if and only if $H_1(|\theta{\downarrow}\circ_\pi|;X)=0$. Points (a) and (b) now follow since $H\cong\pi_1(|\theta{\downarrow}\circ_\pi|)$ is R-perfect if and only if $H_1(|\theta{\downarrow}\circ_\pi|;R)\cong H_1(H;R)=0$. $\square$

Since a category over a group π gives rise to an Ω-system, we can now work with Ω-resolutions in this situation.

Lemma 3.10. *Let $(\mathcal{C},\theta)$ be a category over a group π, and let R be a commutative ring. A complex of $R\mathcal{C}$-modules*

$$\dots \longrightarrow P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} \theta^*(R\pi) \longrightarrow 0$$

is an Ω-resolution of $\theta^*(R\pi)$ with respect to the Ω-system $(R\mathcal{C}\text{-}\mathtt{mod}, R\pi\text{-}\mathtt{mod}; \theta_*, \theta^*)$ of Proposition 3.1 if and only if

(1) *P_n is a projective $R\mathcal{C}$-module for each $n \geq 0$;*

(2) *the complex $\theta_*(P_*)$ is acyclic, and ε induces an isomorphism $H_0(P_*)(c) \cong R\pi$ for each $c \in \mathrm{Ob}(\mathcal{C})$; and*

(3) *for each $n \geq 0$, $H_n(P_*)$ is locally constant on $\mathcal{C}$ and essentially constant on $\theta{\downarrow}\circ_\pi$.*

Proof. By Lemma 3.8(c), (3) is equivalent to the first statement in (Ω-3) (that $H_n(P_*,\partial_*)$ is isomorphic to an object in $\theta^*(R\pi\text{-}\mathtt{mod})$). The equivalence of (1) with (Ω-1), and of (2) with (Ω-2) and the second part of (Ω-3) (that $H_0(P_*,\partial_*) \cong X$), is clear. □

By Proposition 1.6, if R is a commutative ring and $(\mathcal{C},\theta)$ is a category over a group π, and there is at least one Ω-resolution of $R\pi$, then all Ω-resolutions are chain homotopy equivalent to each other. This allows us to define "Ω-homology" in this situation.

Definition 3.11. *Let $(\mathcal{C},\theta)$ be a category over a group π. For a commutative ring R, if there is an Ω-resolution (P_*,∂_*) of $\theta^*(R\pi)$ with respect to $(\mathcal{C},\theta)$, then we define*

$$H_*^{\Omega}(\mathcal{C},\theta;R) = \theta_*\big(H_*(P_*,\partial_*)\big).$$

The following proposition is a first step towards determining for which categories over π the free module $R\pi$ has an Ω-resolution. In the next section, we will show that Ω-resolutions of $R\pi$ do exist in all cases not excluded here. Recall that $\mathcal{C}$ is an *EI-category* if all endomorphisms of objects in $\mathcal{C}$ are automorphisms.

Proposition 3.12. *Fix a commutative ring R. Let $(\mathcal{C},\theta)$ be a category over a group π, and set $H = \mathrm{Ker}\big[\pi_1(|\theta|)\colon \pi_1(|\mathcal{C}|) \longrightarrow \pi\big]$. Thus $(R\mathcal{C}\text{-}\mathtt{mod}, R\pi\text{-}\mathtt{mod}; \theta_*, \theta^*)$ is an Ω-system by Lemma 3.8(a).*

(a) *If H is R-perfect, then $(L_1\theta_*)(\theta^*(X)) = 0$ for each X in $R\pi\text{-}\mathtt{mod}$, and $\theta^*(R\pi\text{-}\mathtt{mod})$ is closed under extensions in $R\mathcal{C}\text{-}\mathtt{mod}$. If $\mathcal{C} = \mathcal{B}(G)$ for a group G, then $\theta^*(R\pi\text{-}\mathtt{mod})$ is closed under subobjects in $R\mathcal{C}\text{-}\mathtt{mod}$, and each projective $R\pi$-module has an Ω-resolution. If $\mathcal{C}$ is an EI-category with more than one isomorphism class, then $\theta^*(R\pi\text{-}\mathtt{mod})$ is not closed under subobjects in $R\mathcal{C}\text{-}\mathtt{mod}$.*

(b) *If H is not R-perfect, then $\theta^*(R\pi\text{-}\mathtt{mod})$ is not closed under extensions in $R\mathcal{C}\text{-}\mathtt{mod}$, and the projective $R\pi$-module $R\pi$ does not have an Ω-resolution. More generally, if X is a nonzero projective $R\pi$-module that is free as an R-module, then X has no Ω-resolution.*

Proof. **(a)** Assume that H is R-perfect. Then $(L_1\theta_*)(\theta^*(X)) = 0$ for each $R\pi$-module X by Lemma 3.9(a). Hence by Lemma 2.5(c), $\theta^*(R\pi\text{-}\mathtt{mod})$ is closed under extensions in $R\mathcal{C}\text{-}\mathtt{mod}$. If $\mathcal{C} \cong \mathcal{B}(G)$ for some G, then θ is surjective on morphisms, $H = \mathrm{Ker}[G \longrightarrow \pi]$, and $\theta^*(R\pi\text{-}\mathtt{mod})$ is closed under subobjects in $R\mathcal{C}\text{-}\mathtt{mod}$ by Proposition 3.5(a).

Assume $\mathcal{C}$ is an EI-category with more than one isomorphism class, and let $x, y \in \mathrm{Ob}(\mathcal{C})$ be a pair of nonisomorphic objects. At least one of the sets $\mathrm{Mor}_{\mathcal{C}}(x,y)$ and $\mathrm{Mor}_{\mathcal{C}}(y,x)$ must be empty; we can assume that $\mathrm{Mor}_{\mathcal{C}}(x,y) = \varnothing$. Let $\underline{R}$ be the constant $R\mathcal{C}$-module with value R, and let $M \leq \underline{R}$ be the submodule where $M(c) = 0$ if $\mathrm{Mor}_{\mathcal{C}}(x,c) = \varnothing$ and $M(c) = R$ otherwise. Then $M(c) = R$ and $M(c') = 0$ imply that $\mathrm{Mor}_{\mathcal{C}}(c,c') = \varnothing$; thus M is well defined as a submodule of $\underline{R}$. Also, $M(x) = R$ so $M \neq 0$, and $M(y) = 0$ so M is properly contained in $\underline{R}$. Since $\mathcal{C}$ is connected, M is not locally constant, and hence not isomorphic to an object in $\theta^*(R\pi\text{-}\mathsf{mod})$. So $\theta^*(R\pi\text{-}\mathsf{mod})$ is not closed under subobjects in $R\mathcal{C}\text{-}\mathsf{mod}$.

(b) Fix an object c_0 in $\mathcal{C}$, and set $G = \pi_1(|\mathcal{C}|, c_0)$ for short. Let $\eta\colon G \longrightarrow \pi$ be the homomorphism induced by $|\theta|\colon |\mathcal{C}| \longrightarrow |\mathcal{B}(\pi)| = B\pi$. Thus η is surjective and $H = \mathrm{Ker}(\eta)$.

Assume H is not R-perfect. By Lemma 3.9(b), for each nonzero $R\pi$-module X that is free as an R-module, $(L_1\theta_*)(\theta^*X) \neq 0$. So X has no Ω-resolution by Proposition 2.6, and it remains to show that $\theta^*(R\pi\text{-}\mathsf{mod})$ is not closed under extensions in $R\mathcal{C}\text{-}\mathsf{mod}$.

Set $N_0 = H^{\mathrm{ab}} \otimes_{\mathbb{Z}} R \cong H_1(H; R)$, regarded as an R-module, and let $\chi\colon H \longrightarrow N_0$ be the homomorphism $\chi(h) = [h] \otimes 1$. Since H is not R-perfect, $N_0 \neq 0$, and χ is not the trivial homomorphism. Let M_0 be the RH-module with underlying R-module $N_0 \times N_0$, where $h \in H$ acts via the matrix $\left(\begin{smallmatrix} 1 & \chi(h) \\ 0 & 1 \end{smallmatrix}\right)$. Thus there is a submodule $M_0' = \{(x, 0) \,|\, x \in R\} \leq M_0$ such that H acts trivially on M_0' and on M_0/M_0'.

Now set $M = RG \otimes_{RH} M_0$. Thus M is an RG-module, and contains a submodule M' such that M' and M/M' are both isomorphic to $\eta^*(R\pi)$.

We now use this to construct a counterexample to $\theta^*(R\pi\text{-}\mathsf{mod})$ being closed under extensions. For each $c \in \mathrm{Ob}(\mathcal{C})$, choose a path $\widetilde{\phi}_c$ in $|\theta{\downarrow}\circ_\pi|$ from (c_0, Id) to (c, Id) ($|\theta{\downarrow}\circ_\pi|$ is connected by Lemma 3.8(a)), and let ϕ_c be its image in $|\mathcal{C}|$. In particular, let $\widetilde{\phi}_{c_0}$ and ϕ_{c_0} be the constant paths at (c_0, Id) and c_0, respectively. Define a functor $\widetilde{\theta}\colon \mathcal{C} \longrightarrow \mathcal{B}(G)$ by sending each object in $\mathcal{C}$ to the unique object $\circ_G$, and by sending each morphism $\omega \in \mathrm{Mor}_{\mathcal{C}}(c, c')$ to the class of the loop $\phi_c{\cdot}\omega{\cdot}\phi_{c'}^{-1}$ (where we compose paths from left to right). We claim that

(i) $\pi_1(|\widetilde{\theta}|)\colon \pi_1(|\mathcal{C}|, c_0) \longrightarrow \pi_1(\mathcal{B}(G), \circ_G) = G$ is the identity on G; and

(ii) $\theta = \mathcal{B}(\eta) \circ \widetilde{\theta}$.

Point (i) is immediate from the definition of $\widetilde{\theta}$ (and since ϕ_{c_0} is the constant path). Point (ii) holds since the paths ϕ_c all lift to $|\theta{\downarrow}\circ_\pi|$ and hence are sent to trivial loops in $\mathcal{B}(\pi)$, and since $\eta\colon G \longrightarrow \pi$ is induced by θ.

Now, $\widetilde{\theta}^*(M)$ is an $R\mathcal{C}$-module with submodule $\widetilde{\theta}^*(M')$, such that by (ii),

$$\widetilde{\theta}^*(M') \cong \widetilde{\theta}^*(\eta^*(R\pi)) \cong \theta^*(R\pi) \qquad \text{and} \qquad \widetilde{\theta}^*(M)\big/\widetilde{\theta}^*(M') \cong \widetilde{\theta}^*(M/M') \cong \theta^*(R\pi).$$

Thus $\widetilde{\theta}^*(M')$ and $\widetilde{\theta}^*(M)\big/\widetilde{\theta}^*(M')$ are both isomorphic to objects in $\theta^*(R\pi\text{-}\mathsf{mod})$. As for $\widetilde{\theta}^*(M)$, by (i), the homomorphism

$$(\widetilde{\theta}^*(M))_\#\colon G = \pi_1(|\mathcal{C}|, c_0) \xrightarrow{\qquad} \mathrm{Aut}_R(\widetilde{\theta}^*(M)(c_0)) = \mathrm{Aut}_R(M)$$

of Lemma 3.3(a) is just the given action of G on the RG-module M. So its restriction to $H = \pi_1(|\theta{\downarrow}\circ_\pi|)$ is nontrivial, and by Lemma 3.3(b), $\widetilde{\theta}^*(M)$ is not essentially constant on $\theta{\downarrow}\circ_\pi$. By Lemma 3.8(c), it is not isomorphic to an object in $\theta^*(R\pi\text{-}\mathsf{mod})$, and thus $\theta^*(R\pi\text{-}\mathsf{mod})$ is not closed under extensions in $R\mathcal{C}\text{-}\mathsf{mod}$. □

Note that Proposition 2.8 does not apply under the hypotheses of Proposition 3.12(a), although at least Ω_1-resolutions exist by Proposition 2.6. If $\mathcal{C}$ is the category of a group, then Ω-resolutions always exist by Proposition 3.5(b). We will show in Theorem 4.4 that in fact, Ω-resolutions of $R\pi$ always exist in under the hypotheses of Proposition 3.12(a).

Example 3.13. In the situation of Proposition 3.12(a), if $\mathcal{C}$ is not an EI-category, then $\theta^*(R\pi\text{-}\mathsf{mod})$ can fail to be closed under subobjects even when $\mathcal{C}$ has only one object, and can be closed under subobjects even when $\mathcal{C}$ has more than one isomorphism class of object:

(a) Set $R = \mathbb{Z}$, $\pi = \mathbb{Z}$, and $\mathcal{C} = \mathcal{B}(\mathbb{N})$, and let $\theta\colon \mathcal{B}(\mathbb{N}) \longrightarrow \mathcal{B}(\pi)$ be the inclusion. Then $(\mathcal{C}, \theta)$ is a category over π. Let N be the $R\pi$-module with underlying group $\mathbb{Q}$, where $\pi = \mathbb{Z}$ acts via $n(x) = 2^n x$. Let M be the $R\mathbb{N}$-module with underlying group $\mathbb{Z}$, where $n \in \mathbb{N}$ acts in the same way. Thus M is a submodule of $\theta^*(N)$, but is not isomorphic to an object in $\theta^*(R\pi\text{-}\mathsf{mod})$.

(b) Let $\mathcal{C}$ be a category with two objects x and y, where $\mathrm{End}_{\mathcal{C}}(x) = \{0_x, 1_x\}$, $\mathrm{End}_{\mathcal{C}}(y) = \{0_y, 1_y\}$, and there are unique morphisms $0_{xy} \in \mathrm{Mor}_{\mathcal{C}}(x, y)$ and $0_{yx} \in \mathrm{Mor}_{\mathcal{C}}(y, x)$. Composition is defined by multiplication of the labels 0 or 1. Set $\pi = \mathbb{Z}$, and let $\theta\colon \mathcal{C} \longrightarrow \mathcal{B}(\mathbb{Z})$ be the functor that sends all endomorphisms to 0 and the other two morphisms to 1 and -1, respectively. Via generators and relations, one checks that θ induces an isomorphism $\pi_1(|\mathcal{C}|) \cong \mathbb{Z}$. We are thus in the situation of Proposition 3.12(a) with $H = 1$. An $R\mathcal{C}$-module M is isomorphic to an object in $\theta^*(R\pi\text{-}\mathsf{mod})$ if and only if all endomorphisms induce the identity, in which case the other two morphisms induce inverse isomorphisms between $M(x)$ and $M(y)$. So $\theta^*(R\pi\text{-}\mathsf{mod})$ is closed under subobjects in this case.

4. Homology of loop spaces of categories over groups

We next show, in the situation of Proposition 3.12(a), that Ω-resolutions of $R\pi$ with respect to $(\mathcal{C}, \theta)$ do exist, and that their homology is the R-homology of a certain loop space (Theorem 4.4). For example, when k is a field of characteristic p for some prime p and π is a finite p-group, the homology of the Ω-resolution is isomorphic to $H_*(\Omega(|\mathcal{C}|^\wedge_p); k)$ (Theorem 4.7).

Throughout this section, we work mostly with simplicial sets and their realizations, referring to [GJ, Chapter I] and [Cu] for the definitions and basic properties that we use. In particular, Kan fibrations of simplicial sets (called "fibre maps" by Curtis) play an important role here, and we refer to [GJ, § I.3] and [Cu, Definition 2.5] for their definitions. We let $|K|$ denote the geometric realization of a simplicial set K, let $C_*(K)$ denote its simplicial chain complex, and write $H_*(K) = H_*(C_*(K))$ $(\cong H_*(|K|))$. Thus $|\mathcal{C}| = |\mathcal{N}(\mathcal{C})|$ when $\mathcal{C}$ is a small category and $\mathcal{N}(\mathcal{C})$ is its nerve. Note that if $f\colon E \longrightarrow K$ is a Kan fibration and $\mu\colon L \longrightarrow K$ is a simplicial map, then the pullback of f along μ is also a Kan fibration.

For a small category $\mathcal{C}$, a *$\mathcal{C}$-diagram of simplicial sets* is a functor from $\mathcal{C}$ to simplicial sets, and a morphism of $\mathcal{C}$-diagrams is a natural transformation of such functors. Let $\underline{K}$ denote the constant $\mathcal{C}$-diagram that sends each object to the simplicial set K, and let $\underline{f}\colon \underline{K} \longrightarrow \underline{L}$ denote the morphism induced by a map $f\colon K \longrightarrow L$ of simplicial sets.

Let $E\mathcal{C}$ denote the $\mathcal{C}$-diagram of simplicial sets where $E\mathcal{C}(c) = \mathcal{N}(\mathrm{Id}_{\mathcal{C}}{\downarrow}c)$, and a morphism φ in $\mathcal{C}$ induces a map between spaces $E\mathcal{C}(-)$ by composition with φ. Then $|E\mathcal{C}(c)|$ is contractible for each c in $\mathcal{C}$, so $|E\mathcal{C}|$ is the "$\mathcal{C}$-CW-approximation" of the trivial (point) $\mathcal{C}$-space in the sense of [DL, Definitions 3.6 and 3.8]. The forgetful functors $\mathrm{Id}_{\mathcal{C}}{\downarrow}c \to \mathcal{C}$ induce a natural transformation $\eta\colon E\mathcal{C} \to \underline{\mathcal{N}(\mathcal{C})}$.

For each Kan fibration $f\colon K \longrightarrow \mathcal{N}(\mathcal{C})$, let $\mu\colon E_f \longrightarrow E\mathcal{C}$ denote the pullback of $\underline{K}$ along η. Thus E_f is the $\mathcal{C}$-diagram of simplicial sets that sends an object c in $\mathcal{C}$ to the pullback $E_f(c)$ of the system

$$K \xrightarrow{\;f\;} \mathcal{N}(\mathcal{C}) \xleftarrow{\;\eta_c\;} E\mathcal{C}(c).$$

Lemma 4.1. *Fix a commutative ring R and a small category $\mathcal{C}$, and let $f\colon K \to \mathcal{N}(\mathcal{C})$ be a Kan fibration. Then for each $n \geq 0$, the $R\mathcal{C}$-module $C_n(E_f; R)$ is projective, and the morphism $\omega\colon E_f \longrightarrow \underline{K}$ induces an isomorphism $\operatorname*{colim}_{\mathcal{C}}\bigl(C_*(E_f; R)\bigr) \cong C_*(K; R)$.*

Proof. For each $n \geq 0$ and each object $c \in \mathcal{C}$, $C_n(E\mathcal{C};R)(c)$ has as basis the set of all chains $(c_0 \to c_1 \to \cdots \to c_n \to c)$. So in the notation of Lemma 3.4, the $R\mathcal{C}$-module $C_n(E\mathcal{C};R)$ is the direct sum of one copy of $F^{R\mathcal{C}}_{c_n}$ for each n-simplex $(c_0 \to c_1 \to \cdots \to c_n)$ in $\mathcal{N}(\mathcal{C})$. In particular, it is projective, and since $\operatorname{colim}_{\mathcal{C}}(F^{R\mathcal{C}}_{c_n}) \cong R$, the natural transformation $\eta\colon E\mathcal{C} \longrightarrow \underline{\mathcal{N}(\mathcal{C})}$ induces an isomorphism $\operatorname{colim}_{\mathcal{C}}(C_n(E\mathcal{C};R)) \cong C_n(\mathcal{N}(\mathcal{C});R)$.

This proves the lemma when f is the identity fibration, and the general case is similar. An n-simplex in the pullback $E_f(c)$ is a pair $(\sigma, c_0 \to \cdots \to c_n \to c)$ where $\sigma \in K_n$ is such that $f(\sigma) = (c_0 \to \cdots \to c_n)$. Hence the $R\mathcal{C}$-module $C_n(E_f;R)$ is the direct sum of copies of $F^{R\mathcal{C}}_{c_n}$, one for each pair $(\sigma, c_0 \to \cdots \to c_n)$ as above, hence is projective, and $\operatorname{colim}_{\mathcal{C}}(C_n(E_f;R)) \cong C_n(K;R)$. □

We next define plus constructions, which play an important role in this section.

Definition 4.2. *Fix a commutative ring R, a connected CW complex X, and a normal subgroup $H \trianglelefteq \pi_1(X)$. An* R-plus construction *for (X,H) consists of a CW complex X^+_R together with a map $\kappa\colon X \longrightarrow X^+_R$, such that $\pi_1(\kappa)$ is surjective with kernel H, and $H_*(\kappa;N)$ is an isomorphism for each $R[\pi_1(X)/H]$-module N.*

A few results about R-plus constructions are collected in the appendix. For example, we show there that (X,H) has an R-plus construction if and only if H is R-perfect (i.e., $H_1(H;R) = 0$), and that the R-completion of a space in the sense of Bousfield and Kan is an R-plus construction under certain hypotheses.

For $n \geq 0$, let Δ^n denote the n-simplex as a simplicial set, and let $v_0, \ldots, v_n$ be its vertices. For $0 \leq k \leq n$, let $\Lambda^n_k \subseteq \Delta^n$ be the simplicial subset whose realization is the union of all proper (closed) faces in Δ^n containing v_k. Thus a Kan fibration is a simplicial map $f\colon K \longrightarrow L$ with the following lifting property: for each $0 \leq k \leq n$, each $\sigma\colon \Delta^n \longrightarrow L$, and each $\tau\colon \Lambda^n_k \longrightarrow K$ such that $f \circ \tau = \sigma|_{\Lambda^n_k}$, there is a simplicial map $\widetilde{\sigma}\colon \Delta^n \longrightarrow K$ such that $\widetilde{\sigma}|_{\Lambda^n_k} = \tau$ and $f \circ \widetilde{\sigma} = \sigma$. A *Kan complex* is a simplicial set K for which the (unique) map to Δ^0 is a Kan fibration; equivalently, a simplicial set for which each simplicial map $\Lambda^n_k \longrightarrow K$ extends to Δ^n (see [GJ, § I.3], [Cu, Definition 1.12], or [GZ, § IV.3]). For example, for each space X, the singular simplicial set $S.(X)$ is a Kan complex [GJ, Lemma I.3.3].

For any connected simplicial set K with basepoint $x_0 \in K_0$, let $\mathcal{P}(K) = \mathcal{P}(K, x_0)$ be the simplicial set of paths in K based at x_0. Thus an n-simplex in $\mathcal{P}(K)$ is a map of simplicial sets $\Delta^1 \times \Delta^n \longrightarrow K$ that sends $\{v_0\} \times \Delta^n$ to x_0 (more precisely, to the image of x_0 under the degeneracy map $K_0 \longrightarrow K_n$). Let $e = e_K\colon \mathcal{P}(K) \longrightarrow K$ denote the path-loop fibration over K: the simplicial map that sends an n-simplex $\Delta^1 \times \Delta^n \longrightarrow K$ to the image of $\{v_1\} \times \Delta^n$. If K is a Kan complex, then $e_K\colon \mathcal{P}(K) \longrightarrow K$ is a Kan fibration and $|\mathcal{P}(K)|$ is weakly contractible (see [GJ, Lemma I.7.5]). Thus the fibre of e_K over x_0 is the loop simplicial set $\Omega(K, x_0)$ based at x_0 [GJ, p. 31]. Using the fact that the realization of a Kan fibration is a Serre fibration (see [GJ, Theorem I.10.10]), one can show that $|\Omega(K, x_0)|$ is weakly equivalent to $\Omega(|K|, x_0)$.

If $f\colon K \longrightarrow L$ is a Kan fibration, and $\chi\colon \widehat{L} \longrightarrow L$ is an arbitrary simplicial map, then the pullback $\widehat{f}\colon \widehat{K} \longrightarrow \widehat{L}$ is defined levelwise: $\widehat{K}_n$ is the pullback (as a set) of $f_n\colon K_n \longrightarrow L_n$ along $\chi_n\colon \widehat{L}_n \longrightarrow L_n$. It is immediate from the definitions that $\widehat{f}$ is also a Kan fibration. By [GZ, Theorem III.3.1], pullbacks commute with geometric realization; i.e., $|\widehat{K}|$ is the pullback of $|K| \longrightarrow |L|$ along $|\widehat{L}|$. Note, however, that this requires that the pullbacks of realizations be taken in the category of compactly generated Hausdorff spaces (called "Kelley spaces" in [GZ]).

Proposition 4.3. *Fix a commutative ring R and a group π, let $(\mathcal{C},\theta)$ be a category over π, and set $H = \mathrm{Ker}[\pi_1(|\mathcal{C}|) \xrightarrow{\pi_1(|\theta|)} \pi]$. Assume that H is R-perfect, let $\kappa\colon |\mathcal{C}| \longrightarrow |\mathcal{C}|_R^+$ be an R-plus construction for $(|\mathcal{C}|, H)$, and let $\widehat{\kappa}\colon \mathcal{N}(\mathcal{C}) \longrightarrow S.(|\mathcal{C}|_R^+)$ be the simplicial map adjoint to κ. Fix an object c_0 in $\mathcal{C}$, regarded as a vertex in $\mathcal{N}(\mathcal{C})$, set $x_0 = \widehat{\kappa}(c_0)$, and let $e = e_{S.(|\mathcal{C}|_R^+)}$ be the path-loop fibration over $S.(|\mathcal{C}|_R^+)$ based at x_0. Let $\nu\colon A\mathcal{C} \longrightarrow \mathcal{N}(\mathcal{C})$ be the pullback of e along $\widehat{\kappa}$, and let $\mu\colon E_\nu \longrightarrow E\mathcal{C}$ denote the fibration of $\mathcal{C}$-diagrams of simplicial sets obtained as the pullback of ν along η. We thus have, for each object c in $\mathcal{C}$, the following diagram of simplicial sets with pullback squares*

$$\begin{tikzcd}
E_\nu(c) \arrow[r] \arrow[d, "\mu_c"] & A\mathcal{C} \arrow[r] \arrow[d, "\nu"] & \mathcal{P}(S.(|\mathcal{C}|_R^+), x_0) \arrow[d, "e"] \\
E\mathcal{C}(c) \arrow[r, "\eta_c"] & \mathcal{N}(\mathcal{C}) \arrow[r, "\widehat{\kappa}"] & S.(|\mathcal{C}|_R^+)\,.
\end{tikzcd} \tag{1}$$

Then the following hold.

(a) *For each $n \geq 0$, $C_n(E_\nu; R)$ is a projective $R\mathcal{C}$-module.*

(b) *The complex $\theta_*(C_*(E_\nu;R)) \cong \operatorname*{colim}_{\theta\downarrow\circ_\pi}(C_*(E_\nu;R))$ is acyclic, and ε induces an isomorphism $H_0\big(\theta_*(C_*(E_\nu;R))\big) \cong R\pi$.*

(c) *For each $n \geq 0$, $H_n(E_\nu;R)$ is locally constant on $\mathcal{C}$ and essentially constant on $\theta\downarrow\circ_\pi$.*

(d) *For each object c in $\mathcal{C}$, $|E_\nu(c)|$ is weakly equivalent to $\Omega(|\mathcal{C}|_R^+)$.*

In particular, by (a)–(c), *$C_*(E_\nu;R)$ is an Ω-resolution of $R\pi$ with respect to $(\mathcal{C},\theta)$.*

Proof. We write $C_*(-) = C_*(-;R)$ and $H_*(-) = H_*(-;R)$ for short, and refer to diagram (1), where by construction, μ_c, ν, and e are all Kan fibrations with fibre $\Omega(S.(|\mathcal{C}|_R^+)) \cong S.(\Omega(|\mathcal{C}|_R^+))$. Then $A\mathcal{C}$ is R-acyclic since $\mathcal{P}(S.(|\mathcal{C}|_R^+), x_0)$ is contractible and $H_*(\widehat{\kappa}; N)$ is an isomorphism for each $R\pi$-module N. Point (a) follows from Lemma 4.1, applied with $A\mathcal{C}$ and ν in the roles of K and f, and point (d) holds since each $E\mathcal{C}(c)$ is the nerve of a category with final object and hence contractible.

Let $\sigma\colon \theta\downarrow\circ_\pi \longrightarrow \mathcal{C}$ be the forgetful functor, and define $\widetilde{A\mathcal{C}}$, $\widetilde{\nu}$, and $E_{\widetilde{\nu}}$ as pullbacks in the following diagram (for each object (c,g) in $\theta\downarrow\circ_\pi$):

$$\begin{tikzcd}
E_{\widetilde{\nu}}(c,g) \arrow[rr, "\cong"'] \arrow[dr] \arrow[dd] & & E_\nu(c) \arrow[dr] \arrow[dd, "\mu_c" near start] & \\
& \widetilde{A\mathcal{C}} \arrow[rr] \arrow[dd, "\widetilde{\nu}" near start] & & A\mathcal{C} \arrow[dd, "\nu"] \\
E(\theta\downarrow\circ_\pi)(c,g) \arrow[rr, "E\sigma_{(c,g)}", "\cong"' near end] \arrow[dr] & & E\mathcal{C}(c) \arrow[dr, "\eta_c"] & \\
& \mathcal{N}(\theta\downarrow\circ_\pi) \arrow[rr, "\mathcal{N}(\sigma)"] & & \mathcal{N}(\mathcal{C})
\end{tikzcd}$$

Note that $E\sigma_{(c,g)}$ is an isomorphism of simplicial sets since for each morphism $\varphi \in \mathrm{Mor}_{\mathcal{C}}(c',c)$ and each $g \in \pi$, there is a unique $g' \in \pi$ such that $\varphi \in \mathrm{Mor}_{\theta\downarrow\circ_\pi}((c',g'),(c,g))$. Hence $E_{\widetilde{\nu}}(c,g) \cong E_\nu(c)$. So by Lemma 3.8(b), and Lemma 4.1 applied with $\theta\downarrow\circ_\pi$ in the role of $\mathcal{C}$,

$$H_*\big(\theta_*(C_*(E_\nu))\big) \cong H_*\big(\operatorname*{colim}_{\theta\downarrow\circ_\pi} \sigma^*(C_*(E_\nu))\big) \cong H_*\big(\operatorname*{colim}_{\theta\downarrow\circ_\pi} C_*(E_{\widetilde{\nu}})\big) \cong H_*\big(C_*(\widetilde{A\mathcal{C}})\big) \cong H_*(\widetilde{A\mathcal{C}})\,.$$

But $|\theta\downarrow\circ_\pi|$ is the covering space of $|\mathcal{C}|$ with fundamental group H and covering group π (Lemma 3.8(a)), the image of $\pi_1(|A\mathcal{C}|)$ in $\pi_1(|\mathcal{C}|)$ is contained in $H = \mathrm{Ker}(\pi_1(|\kappa|))$ since it vanishes in $\pi_1(|\mathcal{C}|_R^+)$, and hence $|\widetilde{A\mathcal{C}}| \cong \pi \times |A\mathcal{C}|$. Since $|A\mathcal{C}|$ is R-acyclic, this proves (b): $\theta_*(C_*(E_\nu))$ is acyclic and $H_0(\theta_*(C_*(E_\nu))) \cong R\pi$.

For each object c in $\mathcal{C}$, let $F(c) = \nu^{-1}(c)$ be the fibre of ν over the vertex c in $\mathcal{N}(\mathcal{C})$. Via homotopy lifting, this is extended to a homotopy functor F from $\mathcal{C}$ to simplicial sets, and this in turn defines a locally constant graded $R\mathcal{C}$-module $M_* = H_*(F)$. For each c in $\mathcal{C}$, the action of $\pi_1(|\mathcal{C}|, c)$ on $M_*(c) = H_*(F(c))$ described in Lemma 3.3(a) is the usual action of the fundamental group of the base on the homology of a fibre, and since ν is a pullback of e, this action factors through $\pi_1(|\mathcal{C}|_R^+) \cong \pi$. So M_* is essentially constant on $\theta{\downarrow}\circ_\pi$ by Lemma 3.3(b). Also, since each $E\mathcal{C}(c)$ contracts to the vertex (c, Id_c) in a natural way, where $\eta_c(c, \mathrm{Id}_c) = c$, we have homotopy equivalences $E_\nu(c) \simeq F(c)$ natural in $\mathcal{C}$ up to homotopy. So $H_*(E_\nu) \cong M_*$ as $R\mathcal{C}$-modules, and this proves (c).

Thus by Lemma 3.8(c), $H_*(E_\nu) \cong \theta^*(N_*)$ for some graded $R\pi$-module N_*. Since θ_* is right exact,

$$N_0 \cong \theta_*\theta^*(N_0) \cong \theta_*(H_0(E_\nu)) \cong H_0\big(\theta_*(C_*(E_\nu))\big) \cong R\pi,$$

and so $H_0(E_\nu) \cong \theta^*(R\pi)$. This defines a surjective homomorphism $\varepsilon\colon C_0(E_\nu) \longrightarrow \theta^*(R\pi)$, and finishes the proof that $(C_*(E_\nu), \partial_*) \longrightarrow \theta^*(R\pi) \longrightarrow 0$ is an Ω-resolution of $R\pi$. □

Upon combining Proposition 4.3 with Lemma A.3, we get the following theorem.

Theorem 4.4. *Fix a commutative ring R and a group π. Let $(\mathcal{C}, \theta)$ be a category over π, set $H = \mathrm{Ker}[\pi_1(|\mathcal{C}|) \xrightarrow{\pi_1(|\theta|)} \pi]$, and assume that H is R-perfect. Then*

(a) *$(|\mathcal{C}|, H)$ admits an R-plus construction;*

(b) *the free $R\pi$-module $R\pi$ has Ω-resolutions with respect to $(\mathcal{C}, \theta)$; and*

(c) *for each R-plus construction $|\mathcal{C}|_R^+$ for $(|\mathcal{C}|, H)$, $H_*^\Omega(\mathcal{C}, \theta; R) \cong H_*(\Omega(|\mathcal{C}|_R^+); R)$.*

In the special case where $\pi \cong \pi_1(|\mathcal{C}|)$, this takes the form:

Corollary 4.5. *Let $\mathcal{C}$ be a small, connected category, and set $\pi = \pi_1(|\mathcal{C}|)$. Then there is a functor $\theta\colon \mathcal{C} \longrightarrow \mathcal{B}(\pi)$ such that $\pi_1(|\theta|)$ is an isomorphism. For such θ, the 4-tuple $(R\mathcal{C}\text{-}\mathsf{mod}, R\pi\text{-}\mathsf{mod}; \theta_*, \theta^*)$ is an Ω-system, the free module $\mathbb{Z}\pi$ has an Ω-resolution with respect to $(R\mathcal{C}\text{-}\mathsf{mod}, R\pi\text{-}\mathsf{mod}; \theta_*, \theta^*)$, and*

$$H_*^\Omega(\mathcal{C}, \theta; \mathbb{Z}) \cong H_*(\Omega(|\mathcal{C}|); \mathbb{Z}).$$

It is not hard to find examples of small categories that are not groupoids, but whose geometric realizations are aspherical. This leads to the following curious situation.

Remark 4.6. Let $\theta\colon \mathcal{C} \longrightarrow \mathcal{B}(\pi)$ be as in Corollary 4.5, and assume in addition that θ induces a homotopy equivalence $|\mathcal{C}| \simeq B\pi$ but is not an equivalence of categories. Then $H_i^\Omega(\mathcal{C}, \theta; \mathbb{Z}) = 0$ for $i > 0$ by the corollary, and hence each Ω-resolution of $\theta^*(\mathbb{Z}\pi)$ is a projective resolution in the usual sense in the category $\mathbb{Z}\mathcal{C}$. Equivalently, by condition (Ω-2) in Definition 1.5, $(L_i\theta_*)(\mathbb{Z}\pi) = 0$ for each $i > 0$.

The R-plus construction of $(\mathcal{N}(\mathcal{C}), H)$ as defined in Definition 4.2 is not in general unique, not even up to homotopy. However, in certain cases, we can choose it to be a completion or a fibrewise completion of $|\mathcal{C}|$ in the sense of Bousfield and Kan. Recall [BK, III.5.1] that for $R \subseteq \mathbb{Q}$, a group π is R-nilpotent if it has a central series for which each quotient is an R-module.

Theorem 4.7. *Let $(\mathcal{C}, \theta)$ be a category over a group π, and set $H = \mathrm{Ker}[\pi_1(|\mathcal{C}|) \xrightarrow{\pi_1(|\theta|)} \pi]$.*

(a) *Assume that R is a subring of $\mathbb{Q}$ or $R = \mathbb{F}_p$ for some prime p, and that H is R-perfect. Let $|\mathcal{C}|^\wedge$ be the fibrewise R-completion of $|\mathcal{C}|$ over $B\pi$. Then*

$$H_*^\Omega(\mathcal{C}, \theta; R) \cong H_*(\Omega(|\mathcal{C}|^\wedge); R).$$

(b) *If $R \subseteq \mathbb{Q}$ is such that H is R-perfect, and π is R-nilpotent with nilpotent action on $H_i(|\theta{\downarrow}\circ_\pi|; R)$ for each i, then*

$$H_*^\Omega(\mathcal{C}, \theta; R) \cong H_*(\Omega(|\mathcal{C}|_R^\wedge); R),$$

where $|\mathcal{C}|_R^\wedge$ is the R-completion of $|\mathcal{C}|$.

(c) *If for some prime p, k is a field of characteristic p, π is a finite p-group, and H is p-perfect, then*

$$H_*^\Omega(\mathcal{C}, \theta; k) \cong H_*(\Omega(|\mathcal{C}|_p^\wedge); k)$$

where $|\mathcal{C}|_p^\wedge$ is the p-completion of $|\mathcal{C}|$.

Proof. By Lemma A.4, the natural map from $|\mathcal{C}|$ to $|\mathcal{C}|^\wedge$, $|\mathcal{C}|_R^\wedge$, or $|\mathcal{C}|_p^\wedge$ is an R- or k-plus construction for $(|\mathcal{C}|, H)$ under the hypotheses of (a), (b), or (c), respectively. So this theorem follows as a special case of Theorem 4.4(c). □

The following corollary includes the case proven by Benson in [Be2]. Note that when G is a finite group, its quotient by the maximal normal p-perfect subgroup is always a p-group.

Corollary 4.8. *Fix a prime p. Let G be a (possibly infinite) discrete group, and let $O^p(G)$ be the maximal normal p-perfect subgroup of G. Set $\pi = G/O^p(G)$, let $\chi\colon G \longrightarrow \pi$ be the natural surjection, and assume that π is a finite p-group. Then for each field k of characteristic p, $H_*(\Omega(BG_p^\wedge); k) \cong H_*^\Omega(\mathcal{B}(G), \mathcal{B}(\chi); k)$: the homology of an Ω-resolution of $k\pi$ with respect to $(\mathcal{B}(G), \mathcal{B}(\chi))$ as a category over π.*

Proof. This is just Theorem 4.7(c) when $\mathcal{C} = \mathcal{B}(G)$. □

The results in this section lead in a natural way to the following question.

Question 4.9. Are there more general conditions on an Ω-system $(\mathcal{A}, \mathcal{B}; \theta_*, \theta^*)$ and $X \in \mathscr{P}(\mathcal{B})$ under which $H_*^\Omega(\mathcal{A}, \mathcal{B}; X)$, or a functorial image, describes the homology of a space (e.g., of a loop space)? In particular, can the homology of the Ω-resolutions of Proposition 3.5(b) be realized as the homology of some space determined by the Ω-systems?

5. Examples: Ω-resolutions for some p-local compact groups

One problem that motivated this work was that of finding a way to characterize the p-compact groups among the more general p-local compact groups. As already noted in the introduction, we did not succeed in doing so. The aim of this section is to give some very simple examples that demonstrate how complicated this problem can be, for example, by analyzing some p-local compact groups that are not p-compact. We also give some results, and one explicit computation, that follow from knowing that Ω-resolutions determine homology of loop spaces without having to explicitly construct the resolutions themselves.

Throughout this section, we fix a prime p and a field k of characteristic p. We first recall some definitions. A *p-compact group* consists of a loop space X and its classifying space BX, such that $X \simeq \Omega(BX)$, $H_*(X; \mathbb{F}_p)$ is finite (in particular, $H_n(X; \mathbb{F}_p) = 0$ for n large enough), and BX is p-complete. This concept was first introduced by Dwyer and Wilkerson [DW], and developed by them and others in several papers. If G is a compact Lie group whose group of components $\pi_0(G)$ is a p-group, then $\Omega(BG_p^\wedge)$ is a p-compact group, but this need not be the case if $\pi_0(G)$ is not a p-group. Every p-compact group contains a maximal torus with properties very similar to those of maximal tori in compact Lie groups.

A *p-local compact group* consists of a discrete p-toral group S (i.e., an extension of a discrete p-torus $(\mathbb{Z}/p^\infty)^r$ for some $r \geq 0$ by a finite p-group), together with a fusion system $\mathcal{F}$ over S and a linking system $\mathcal{L}$ associated to $\mathcal{F}$. We refer to [BLO, Definitions 2.2 and 4.1] for the precise definitions of fusion and linking systems in this context; here, we just note that $\mathcal{F}$ and $\mathcal{L}$ are categories, $\mathrm{Ob}(\mathcal{F})$ is the set of subgroups of S, each morphism in $\mathcal{F}$ is a homomorphism between subgroups, and there is a functor $\mathcal{L} \longrightarrow \mathcal{F}$ that is an inclusion on objects and surjective on each morphism set. The classifying space of such a triple $(S, \mathcal{F}, \mathcal{L})$ is the p-completed space $|\mathcal{L}|^\wedge_p$. By [BLO, §§ 9–10], each compact Lie group G or p-compact group X has a maximal discrete p-toral subgroup S (unique up to conjugacy), together with a fusion system $\mathcal{F}$ and a linking system $\mathcal{L}$ such that $|\mathcal{L}|^\wedge_p$ is homotopy equivalent to $BG^\wedge_p$ or BX, respectively.

By [BLO, Proposition 4.4], for each p-local compact group $(S, \mathcal{F}, \mathcal{L})$, the fundamental group of the classifying space $|\mathcal{L}|^\wedge_p$ is a finite p-group. So as a special case of Theorem 4.7(c), we get:

Theorem 5.1. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local compact group, and set $\pi = \pi_1(|\mathcal{L}|^\wedge_p)$. Then there is $\theta \colon \mathcal{L} \longrightarrow \mathcal{B}(\pi)$ such that $(\mathcal{L}, \theta)$ is a category over π and*

$$H_*(\Omega(|\mathcal{L}|^\wedge_p); \mathbb{F}_p) \cong H^\Omega_*(\mathcal{L}, \theta; \mathbb{F}_p).$$

If Γ is an extension of a discrete p-torus by a finite p-group, then $B\Gamma^\wedge_p$ is the classifying space of a p-compact group. In contrast, if Γ is an extension of a discrete p-torus by an arbitrary finite group, then $B\Gamma^\wedge_p$ need not be the classifying space of a p-compact group (nor the p-completion of BG for a compact Lie group G), but it is always the classifying space of a p-local compact group. For example, if p is an odd prime and $r \geq 2$, and $\Gamma = (\mathbb{Z}/p^\infty)^r \rtimes C_2$ where C_2 inverts $(\mathbb{Z}/p^\infty)^r$, then $\Omega(B\Gamma^\wedge_p)$ is not a p-compact group since its mod p homology is nonvanishing in arbitrarily large degrees (see Example 5.10 for the case $r = 2$).

What we want to do now is to give some explicit examples of such Ω-resolutions. We focus on p-local compact groups associated to extensions of discrete p-tori by finite groups, especially by those of order prime to p.

Proposition 5.2. *Let $T \trianglelefteq \Gamma$ be a pair of groups such that $T \cong (\mathbb{Z}/p^\infty)^r$ for some $r \geq 1$ and Γ/T is finite. Then there is a p-local compact group $(S, \mathcal{F}, \mathcal{L})$ associated to Γ, where $T \leq S \leq \Gamma$ and $S/T \in \mathrm{Syl}_p(\Gamma/T)$, and where $|\mathcal{L}|^\wedge_p \simeq B\Gamma^\wedge_p$. If, in addition, $\Omega(B\Gamma^\wedge_p)$ is a p-compact group, then $O^p(\Gamma/T)$ has order prime to p.*

Proof. Embed T in $GL_r(\mathbb{C})$ as the subgroup of diagonal matrices of p-power order. Then via induction, Γ embeds as a subgroup of $GL_{r\cdot|\Gamma/T|}(\mathbb{C})$, and hence is a linear torsion group in the sense of [BLO, § 8]. So by [BLO, Theorem 8.10], it has an associated p-local compact group $(S, \mathcal{F}, \mathcal{L})$, where $T \leq S \leq \Gamma$, $S/T \in \mathrm{Syl}_p(\Gamma/T)$, and $B\Gamma^\wedge_p \simeq |\mathcal{L}|^\wedge_p$. Also, $\mathcal{F} = \mathcal{F}_S(\Gamma)$: the fusion system over S whose morphisms are those homomorphisms between subgroups of S induced by conjugation in Γ.

Now assume that $\Omega(B\Gamma^\wedge_p)$ is a p-compact group; i.e., that $H_*(\Omega(B\Gamma^\wedge_p); \mathbb{F}_p)$ is finite. Let $O^p(\Gamma) \trianglelefteq \Gamma$ be such that $O^p(\Gamma) \geq T$ and $O^p(\Gamma)/T = O^p(\Gamma/T)$, and set $\pi = \Gamma/O^p(\Gamma)$ (a finite p-group). By the mod-R fibre lemma [BK, II.5.2.iv] and since $B\pi$ is p-complete [BK, VI.3.4 and VI.5.4], $BO^p(\Gamma)^\wedge_p$ is the homotopy fibre of the natural map $B\Gamma^\wedge_p \longrightarrow B(\Gamma/O^p(\Gamma))$, and hence is equivalent to the covering space of $B\Gamma^\wedge_p$ with covering group π. So $\Omega(BO^p(\Gamma)^\wedge_p)$ also has finite mod p homology. We can thus assume that $\Gamma = O^p(\Gamma)$ is p-perfect, and hence that $\Omega(B\Gamma^\wedge_p)$ is connected.

For a finite p-group Q, set $\mathrm{Rep}(Q,\mathcal{L}) = \mathrm{Hom}(Q,S)/{\sim}$, where $\rho_1 \sim \rho_2$ if $\rho_1 = \alpha\rho_2$ for some $\alpha \in \mathrm{Iso}_{\mathcal{F}}(\rho_2(Q),\rho_1(Q))$. In other words, it is the set Γ-conjugacy classes in $\mathrm{Hom}(Q,S)$. Let $[BQ, B\Gamma^\wedge_p]$ be the set of homotopy classes of maps $BQ \longrightarrow B\Gamma^\wedge_p$. By [BLO, Theorem 6.3(a)], there is a bijection $\mathrm{Rep}(Q,\mathcal{L}) \longrightarrow [BQ, B\Gamma^\wedge_p]$ that sends the conjugacy class of a homomorphism ρ to the homotopy class of $B\rho$.

By [DW, Proposition 5.6] and since $\Omega(B\Gamma^\wedge_p)$ is connected, for each $n \geq 1$ and each $f\colon BC_{p^n} \longrightarrow B\Gamma^\wedge_p$, f extends (up to homotopy) to a map from $BC_{p^{n+1}}$ to $B\Gamma^\wedge_p$. Hence each $\rho \in \mathrm{Hom}(C_{p^n},S)$ extends, up to Γ-conjugacy, to some $\bar\rho \in \mathrm{Hom}(C_{p^{n+1}},S)$. Since $T \trianglelefteq \Gamma$ and S/T is finite, we conclude that $S=T$, and thus that Γ/T has order prime to p. $\square$

In fact, whenever $T \trianglelefteq \Gamma$ are such that T is a discrete p-torus and Γ/T is finite, $\Omega(B\Gamma^\wedge_p)$ is a p-compact group if and only if $O^p(\Gamma/T)$ has order prime to p and $\mathrm{Aut}_{O^p(\Gamma/T)}(T)$ is generated by pseudoreflections on T. The necessity of this last condition was shown by Dwyer and Wilkerson [DW, Theorem 9.7.ii]. Conversely, Clark and Ewing [CE, Corollary, p. 426] showed that if $O^p(\Gamma/T)$ has order prime to p and its action is generated by pseudoreflections, then $H^*(B\Gamma^\wedge_p;\mathbb{F}_p)$ is a polynomial algebra over $\mathbb{F}_p$, and hence the (co)homology of its loop space is finite.

Remark 5.3. Assume $T \trianglelefteq \Gamma$ are as in Proposition 5.2, where in addition, $p \nmid |\Gamma/T|$ and the conjugation action of Γ/T on T is faithful (i.e., $C_\Gamma(T) = T$). Then $S=T$, $\mathrm{Ob}(\mathcal{L}) = \{T\}$, and $\mathrm{Aut}_{\mathcal{L}}(T) = \Gamma$, so that $\mathcal{L} \cong \mathcal{B}(\Gamma)$, $\pi_1(|\mathcal{L}|) \cong \Gamma$, and $\pi_1(|\mathcal{L}|^\wedge_p) = 1$ since Γ is p-perfect. Thus by Proposition 3.12 and Theorem 4.7(b), the Ω-system associated to $(S,\mathcal{F},\mathcal{L})$ is $(k\Gamma\text{-}\mathsf{mod}, k\text{-}\mathsf{mod}; \theta_*,\theta^*)$, where $\theta_*(M) = \operatorname{colim}_\Gamma(M)$ for a $k\Gamma$-module M and θ^* sends a k-module N to the corresponding $k\Gamma$-module with trivial action; and $H_*(\Omega(|\mathcal{L}|^\wedge_p);k)$ is the homology of an Ω-resolution of k.

Note also, in the situation of Remark 5.3, that since $|\Gamma/T|$ has order prime to p, the group T is uniquely $|\Gamma/T|$-divisible. Hence $H^i(\Gamma/T;T) = 0$ for all $i>0$, and Γ must be a semidirect product: $\Gamma \cong T \rtimes H$ where $H \cong \Gamma/T$.

Along the lines of Remark 4.6, we also note the following:

Remark 5.4. Let Γ be a linear torsion group: a subgroup of $GL_n(K)$, for some field K of characteristic different from p, all of whose elements have finite order. By [BLO, Theorem 8.10], there is a p-local compact group $(S,\mathcal{F},\mathcal{L})$, where $S \leq \Gamma$ is a maximal discrete p-toral subgroup and $|\mathcal{L}|^\wedge_p \simeq B\Gamma^\wedge_p$. Set $\pi = \pi_1(B\Gamma^\wedge_p) \cong \pi_1(|\mathcal{L}|^\wedge_p)$ (a finite p-group by [BLO, Proposition 4.4]), and let $\theta\colon \mathcal{B}(\Gamma) \longrightarrow \mathcal{B}(\pi)$ and $\eta\colon \mathcal{L} \longrightarrow \mathcal{B}(\pi)$ be functors that induce these isomorphisms. By Theorem 4.7(c), $H^\Omega_*(\mathcal{B}(\Gamma),\theta;k) \cong H^\Omega_*(\mathcal{L},\eta;k)$; i.e., Ω-resolutions with respect to these two different Ω-systems have the same homology.

Throughout the rest of the section, whenever Γ is a group whose maximal normal p-perfect subgroup $O^p(\Gamma)$ is such that $\pi \overset{\text{def}}{=} \Gamma/O^p(\Gamma)$ is a finite p-group, we write "Ω-resolution of $k\pi$ with respect to Γ" to mean an Ω-resolution of $k\pi$ with respect to the category $(\mathcal{B}(\Gamma),\theta)$ over π or the Ω-system $(k\Gamma\text{-}\mathsf{mod}, k\pi\text{-}\mathsf{mod}; \theta_*,\theta^*)$, where $\theta\colon \mathcal{B}(\Gamma) \longrightarrow \mathcal{B}(\pi)$ is the natural projection.

5.1. Ω-resolutions with respect to discrete p-tori.

Let $T \trianglelefteq \Gamma$ be a pair of groups, where $T \cong (\mathbb{Z}/p^\infty)^r$ is a discrete p-torus of rank $r \geq 1$ and Γ/T is finite of order prime to p. Thus $\Gamma = T \rtimes H$ for some finite subgroup $H \leq \Gamma$ of order prime to p. We regard the group ring kT as a left $k\Gamma$-module, where for $t \in T$, $h \in H$, and $x \in kT$, $t(x) = tx$ and $h(x) = hxh^{-1}$. We will construct complexes of projective

$k\Gamma$-modules which, as complexes of kT-modules, are Ω-resolutions of k with respect to T. The $k\Gamma$-module structure on these complexes will be used in the next two subsections.

Set $V = \Omega_1(T) \cong (C_p)^r$. For each $n \geq 1$, set $T_n = \Omega_n(T) \cong (C_{p^n})^r$ (thus $V = T_1$), and regard kT_n as a subring of kT. Let $I(kT) \leq kT$ and $I(kT_n) \leq kT_n$ be the augmentation ideals.

For each $n \geq 1$, and each kT_n-module M and proper submodule $M_0 < M$, M/M_0 has a nontrivial quotient module with trivial T_n-action. Hence $I(kT_n)\cdot(M/M_0) < M/M_0$, and so $M_0 + I(kT_n)\cdot M < M$.

For each $n \geq 0$, let $\bar{\varphi}_n \colon V \longrightarrow I(kT_n)/I(kT_n)^2$ be the map $\bar{\varphi}_n(t) = [t-1]$. This is a homomorphism of groups, and extends to a kH-linear isomorphism $k \otimes_{\mathbb{F}_p} V \cong I(kT_n)/I(kT_n)^2$. Lift $\bar{\varphi}_n$ to an $\mathbb{F}_pH$-linear homomorphism $\widetilde{\varphi}_n \colon V \longrightarrow I(kT_n)$ (the ring $\mathbb{F}_pH$ is semisimple since $p \nmid |H|$), and extend that to a $k\Gamma$-linear homomorphism

$$\varphi_n \colon kT\cdot V \overset{\text{def}}{=} kT \otimes_{\mathbb{F}_p} V \xrightarrow{\qquad} kT$$

by setting $\varphi_n(\xi \otimes v) = \xi\cdot\widetilde{\varphi}_n(v)$ for $\xi \in kT$ and $v \in V$. Since φ_n induces an isomorphism $k \otimes V \cong I(kT_n)/I(kT_n)^2$ by assumption, $\varphi_n(kT_n\cdot V) + I(kT_n)^2 = I(kT_n)$, and so $\varphi_n(kT_n\cdot V) = I(kT_n)$ by the last paragraph (applied with $\varphi_n(kT_n\cdot V)$ and $I(kT_n)$ in the roles of M_0 and M). Hence $\mathrm{Im}(\varphi_n) = kT\cdot I(kT_n)$.

In particular, for each $n \geq 1$, $\mathrm{Im}(\varphi_n) \leq I(kT)\cdot\mathrm{Im}(\varphi_{n+1})$. Since $kT\cdot V$ is projective as a $k\Gamma$-module, there is a $k\Gamma$-linear homomorphism $\psi_n \colon kT\cdot V \longrightarrow kT\cdot V$ such that $\varphi_{n+1} \circ \psi_n = \varphi_n$. Note that

$$\psi_n(kT\cdot V) \leq I(kT)\cdot V. \tag{1}$$

Let $\Lambda^m_R(M)$ denote the m-th exterior power over a commutative ring R of an R-module M. Define, for each $0 \leq m \leq r$ and each $n \geq 1$,

$$D_m = \Lambda^m_{kT}\big(kT\cdot V\big) \cong kT \otimes_{\mathbb{F}_p} \Lambda^m_{\mathbb{F}_p}(V)\,,$$

regarded as a $k\Gamma$-module. In particular, $D_0 = kT$. For each $n \geq 1$ and each $1 \leq m \leq r$, define a boundary map $\partial^{(n)}_m \colon D_m \longrightarrow D_{m-1}$ by setting

$$\partial^{(n)}_m\big(v_1 \wedge v_2 \wedge \cdots \wedge v_m\big) = \sum_{i=1}^{m} (-1)^{i-1} \varphi_n(v_i)\cdot v_1 \wedge \cdots \widehat{v_i} \cdots \wedge v_m$$

for $v_1, \ldots, v_m \in kT\cdot V$. Then

$$\boldsymbol{D}^{(n)} \overset{\text{def}}{=} (D_*, \partial^{(n)}_*) = \Big(0 \longrightarrow D_r \xrightarrow{\partial^{(n)}_r} D_{r-1} \longrightarrow \cdots \longrightarrow D_1 \xrightarrow{\partial^{(n)}_1} kT \longrightarrow 0\Big)$$

is a chain complex of projective $k\Gamma$-modules, and $\big\{\Lambda^m(\psi_n)\big\}_{m=0}^{r}$ defines a morphism of chain complexes $\Psi^{(n)} \colon \boldsymbol{D}^{(n)} \longrightarrow \boldsymbol{D}^{(n+1)}$ where $\Lambda^0(\psi_n) = \mathrm{Id}_{kT}$.

We claim that

$$\text{for each } n \geq 1,\ T_n \text{ acts trivially on the homology of } \boldsymbol{D}^{(n)}. \tag{2}$$

To see this, fix $n \geq 1$, $0 \leq m \leq r$, and $x \in D_m = \Lambda^m_{kT}(kT\cdot V)$ such that $\partial^{(n)}_m(x) = 0$. For each $v \in V$,

$$\partial^{(n)}_{m+1}(x \wedge v) = \partial^{(n)}_m(x) \wedge v + (-1)^m \varphi_n(v)\cdot x = (-1)^m \varphi_n(v)\cdot x.$$

Thus $\varphi_n(v)\cdot[x] = 0$ in $H_m(\boldsymbol{D}^{(n)})$ for each $v \in V$, so $\mathrm{Im}(\varphi_n)$ annihilates $H_*(\boldsymbol{D}^{(n)})$. Since $\mathrm{Im}(\varphi_n) \geq I(kT_n)$, we now conclude that T_n acts trivially.

As usual, whenever $\psi_*\colon (C_*,\partial_*) \longrightarrow (C'_*,\partial'_*)$ is a morphism of chain complexes, the *mapping cone* of ψ is the chain complex

$$C_\psi = \Bigl(\cdots \xrightarrow{\left(\begin{smallmatrix}\partial'_4 & -\psi_3\\ 0 & \partial_3\end{smallmatrix}\right)} C'_3\oplus C_2 \xrightarrow{\left(\begin{smallmatrix}\partial'_3 & \psi_2\\ 0 & \partial_2\end{smallmatrix}\right)} C'_2\oplus C_1 \xrightarrow{\left(\begin{smallmatrix}\partial'_2 & -\psi_1\\ 0 & \partial_1\end{smallmatrix}\right)} C'_1\oplus C_0 \xrightarrow{(\partial'_1,\psi_0)} C'_0\Bigr).$$

The signs are chosen so that $C'_* \le C_\psi$ as chain complexes, and $C_\psi/C'_* \cong \Sigma C_*$. See [W, § 1.5] for more details.

Let $\boldsymbol{D}$ be the mapping cone of the chain map

$$\Psi = \bigl(\mathrm{Id} - \oplus\Psi^{(n)}\bigr)\colon \bigoplus_{n=1}^{\infty} \boldsymbol{D}^{(n)} \xrightarrow{\qquad\quad} \bigoplus_{n=1}^{\infty} \boldsymbol{D}^{(n)}$$

(see [W, § 1.5]). More explicitly,

$$\Psi(x_1,x_2,x_3,\dots) = (x_1,\, x_2-\Psi^{(1)}(x_1),\, x_3-\Psi^{(2)}(x_2),\dots).$$

Since Ψ is injective, $H_*(\boldsymbol{D}) \cong H_*(\mathrm{Coker}(\Psi))$, where $\mathrm{Coker}(\Psi)\cong \mathrm{colim}(\boldsymbol{D}^{(n)},\Psi^{(n)})$. So

$$H_*(\boldsymbol{D}) \cong \mathrm{colim}\Bigl(H_*(\boldsymbol{D}^{(1)}) \xrightarrow{H_*(\Psi^{(1)})} H_*(\boldsymbol{D}^{(2)}) \xrightarrow{H_*(\Psi^{(2)})} H_*(\boldsymbol{D}^{(3)}) \xrightarrow{\qquad} \cdots\Bigr),$$

since colimits are exact. Also, T acts trivially on $H_*(\boldsymbol{D})$ by (2).

Now, $H_*(k\otimes_{kT}\boldsymbol{D})$ is isomorphic to the homology of the cokernel of the chain map

$$\overline{\Psi} = \bigl(\mathrm{Id} - \oplus\overline{\Psi}^{(n)}\bigr)\colon \bigoplus_{n=1}^{\infty} k\otimes_{kT}\boldsymbol{D}^{(n)} \xrightarrow{\qquad\quad} \bigoplus_{n=1}^{\infty} k\otimes_{kT}\boldsymbol{D}^{(n)}.$$

Recall that $\boldsymbol{D}^{(n)} = \bigl(\Lambda^m_{kT}(kT{\cdot}V),\partial^{(n)}_m\bigr)_{m=0}^{r}$ and $\Psi^{(n)} = \bigl\{\Lambda^m(\psi_n)\bigr\}_{m=0}^{r}$. Here, $\Lambda^0(\psi_n)=\mathrm{Id}$, while by (1), $(\Lambda^m(\psi_n))(\Lambda^m_{kT}(kT{\cdot}V)) \le I(kT){\cdot}\Lambda^m_{kT}(kT{\cdot}V)$ for $m>0$. So each $\overline{\Psi}^{(n)}$ is zero in positive degrees and the identity in degree 0, and hence the quotient complex $\mathrm{Coker}(\overline{\Psi})$ is zero in positive degrees and isomorphic to k in degree 0. Thus $k\otimes_{kT}\boldsymbol{D}$ is acyclic with $H_0(k\otimes_{kT}\boldsymbol{D})\cong k$. Also,

$$H_0(\boldsymbol{D}) \cong k\otimes_{kT} H_0(\boldsymbol{D}) \cong H_0(k\otimes_{kT}\boldsymbol{D}) \cong k\colon \tag{3}$$

the first isomorphism since T acts trivially on $H_0(\boldsymbol{D})$ and the second since $(k\otimes_{kT}-)$ is right exact.

We have now proven:

Proposition 5.5. *For k, Γ, and $\boldsymbol{D}$ as above, $\boldsymbol{D}$ is a chain complex of length $r+1$ of projective $k\Gamma$-modules, and $\boldsymbol{D}\xrightarrow{\varepsilon} k \longrightarrow 0$ is an Ω-resolution of k with respect to T.*

5.2. Ω-resolutions with respect to the Sullivan spheres.

We now restrict to the special case of Remark 5.3 where p is odd and $r=1$. Thus $T\trianglelefteq\Gamma$ where $T\cong\mathbb{Z}/p^\infty$, $p\nmid|\Gamma/T|$, and $C_\Gamma(T)=T$. Then $\mathrm{Aut}(T)\cong(\mathbb{Z}_p)^\times\cong C_{p-1}\times\mathbb{Z}_p$, and since Γ/T is finite and acts faithfully on T, it must be cyclic of order dividing $p-1$. Again, we will construct explicit Ω-resolutions of k with respect to Γ.

Spaces $B\Gamma^\wedge_p$ for Γ of this form are the simplest and oldest examples constructed of p-compact groups (other than compact Lie groups). They were originally constructed from the space $K(\mathbb{Z}_p,2)$ $(\simeq B(\mathbb{Z}/p^\infty)^\wedge_p)$, by taking the Borel construction $B(p,m)$ of the faithful action of a cyclic group C_m (for $m|(p-1)$) on $K(\mathbb{Z}_p,2)$. The p-completion of $B(p,m)$ is a classifying space for the p-completed sphere $(S^{2m-1})^\wedge_p$, and hence $B(p,m)^\wedge_p$ is the classifying space of a p-compact group that is often referred to as a "Sullivan sphere". What we will show is that not only do these spaces have finite dimensional homology, but also that the associated Ω-systems have Ω-resolutions of finite length.

Write $\Gamma = T \rtimes H \cong \mathbb{Z}/p^\infty \rtimes H$, where H is cyclic of order $m|(p-1)$. Let $\chi\colon H \longrightarrow \mathbb{F}_p^\times$ be the injective homomorphism such that $hth^{-1} = t^{\chi(h)}$ for all $h \in H$ and all $t \in V = \Omega_1(T)$. Set $kT_{(1)} = kT{\cdot}V = kT \otimes_k V$ as a $k\Gamma$-module, which we identify with kT but with H-action $h(x) = \chi(h){\cdot}hxh^{-1}$ for $h \in H$ and $x \in kT$. More generally, for arbitrary $j \geq 0$, we write

$$k_{(j)} = V^{\otimes j} \qquad \text{and} \qquad kT_{(j)} = kT \otimes_k k_{(j)}$$

as $k\Gamma$-modules. Thus $k_{(j)} \cong k$ and $kT_{(j)} \cong kT$ as kT-modules, but $h \in H$ acts on the first via multiplication by $\chi(h)^j$ and on the second via that and conjugation. We also write $kT = kT_{(0)}$ and $k = k_{(0)}$ for short.

Let $\varphi_n \in \mathrm{Hom}_{k\Gamma}(kT_{(1)}, kT)$ and $\psi_n \in \mathrm{Hom}_{k\Gamma}(kT_{(1)}, kT_{(1)})$ be as above, and set $\mu_n = \varphi_n(1)$ and $\nu_n = \psi_n(1)$. Then for all $n \geq 1$,

$$\mu_{n+1}\nu_n = \mu_n \qquad \text{and} \qquad kT_n{\cdot}\mu_n = I(kT_n), \tag{4}$$

the first since $\varphi_{n+1} \circ \psi_n = \varphi_n$ and the second since $\varphi_n(kT_n{\cdot}V) = I(kT_n)$. Also,

$$h(\mu_n) = \chi(h){\cdot}\mu_n \tag{5}$$

for all $n \geq 1$ and $h \in H$ since φ_n is $k\Gamma$-linear.

The complex $\boldsymbol{D}$ of Proposition 5.5 has the form

$$\boldsymbol{D} = \Big(0 \longrightarrow \bigoplus_{n=1}^{\infty} kT_{(1)}{\cdot}\boldsymbol{a}_n \xrightarrow{\ \partial_2\ } \bigoplus_{n=1}^{\infty} kT_{(1)}{\cdot}\boldsymbol{a}_n \oplus \bigoplus_{n=1}^{\infty} kT{\cdot}\boldsymbol{b}_n \xrightarrow{\ \partial_1\ } \bigoplus_{n=1}^{\infty} kT{\cdot}\boldsymbol{b}_n \longrightarrow 0\Big),$$

where

$$\partial_2(\boldsymbol{a}_n) = -(\boldsymbol{a}_n - \nu_n\boldsymbol{a}_{n+1}) + \mu_n\boldsymbol{b}_n\,, \qquad \partial_1(\boldsymbol{a}_n) = \mu_n\boldsymbol{b}_n\,, \qquad \text{and} \qquad \partial_1(\boldsymbol{b}_n) = \boldsymbol{b}_n - \boldsymbol{b}_{n+1}.$$

Since $\big(\bigoplus_{n=1}^\infty kT{\cdot}\boldsymbol{b}_n \xrightarrow{(\boldsymbol{b}_n \mapsto \boldsymbol{b}_n - \boldsymbol{b}_{n+1})} \bigoplus_{n=1}^\infty kT{\cdot}\boldsymbol{b}_n\big)$ is injective with cokernel kT, the complex $\boldsymbol{D}$ is equivalent to

$$\overline{\boldsymbol{D}} = \Big(0 \longrightarrow \bigoplus_{n=1}^{\infty} kT_{(1)}{\cdot}\boldsymbol{a}_n \xrightarrow{\ \partial_2\ } \bigoplus_{n=1}^{\infty} kT_{(1)}{\cdot}\boldsymbol{a}_n \xrightarrow{\ \partial_1\ } kT_{(0)}{\cdot}\boldsymbol{a}_0 \longrightarrow 0\Big),$$

where this time

$$\partial_2(\boldsymbol{a}_n) = \boldsymbol{a}_n - \nu_n\boldsymbol{a}_{n+1} \qquad \text{and} \qquad \partial_1(\boldsymbol{a}_n) = \mu_n\boldsymbol{a}_0.$$

By (4) and (5), ∂_1 and ∂_2 are $k\Gamma$-linear and $\partial_1 \circ \partial_2 = 0$.

We now want to identify $H_1(\boldsymbol{D})$ more precisely, and use this complex to construct an Ω-resolution with respect to Γ. To do this, define elements σ_n (all $n \geq 1$) and ν_0 in kT_n by setting

$$\sigma_n = \sum_{t \in T_n} t \quad (\text{all } n \geq 1) \qquad \text{and} \qquad \nu_0 = \sigma_1. \tag{6}$$

To better understand the relation between the μ_n, ν_n, and σ_n, fix $n \geq 2$ and a generator $t_n \in T_n$, and set $X = t_n - 1 \in I(kT_n)$. Then $X^{p^n} = t_n^{p^n} - 1 = 0$ and $\{1, X, X^2, \ldots, X^{p^n-1}\}$ is a basis for kT_n, so $kT_n \cong k[X]/(X^{p^n})$ as rings, and each ideal in kT_n is a power of $I(kT_n) = (X)$. Thus $(\mu_n) = (X)$, $(\mu_{n-1}) = (t_n^p - 1) = (X)^p$, and hence $(\nu_{n-1}) = (X)^{p-1}$ by (4). Also, $(\sigma_n) = (X)^{p^n-1}$ and $(\sigma_{n-1}) = (X)^{p(p^{n-1}-1)}$ (see (6)), so $(\nu_{n-1}\sigma_{n-1}) = (X)^{p^n-1} = k{\cdot}\sigma_n$. Thus for each $n \geq 2$, $\nu_{n-1}\sigma_{n-1} = a_n{\cdot}\sigma_n$ for some $0 \neq a_n \in k$. To simplify notation, we can replace the μ_n (all $n \geq 2$) and ν_n (all $n \geq 1$) by appropriate scalar multiples, and arrange that

$$\text{for each } n \geq 1, \quad \nu_{n-1}\sigma_{n-1} = \sigma_n\,. \tag{7}$$

Each element in $\mathrm{Coker}(\partial_2)$ is the class of $\xi{\cdot}\boldsymbol{a}_n$ for some n and some $\xi \in kT_m$, and we can always arrange (modulo $\mathrm{Im}(\partial_2)$) that $m = n$. If in addition, $\partial_1(\xi{\cdot}\boldsymbol{a}_n) = 0$, then $\mu_n\xi = 0$, so $I(kT_n){\cdot}\xi = 0$ by (4), and hence $\xi = a{\cdot}\sigma_n$ for some $a \in k$. Thus $H_1(\boldsymbol{D})$ is generated by the classes $[\sigma_n\boldsymbol{a}_n]$ for $n \geq 1$, where for each n, $[\sigma_n\boldsymbol{a}_n] = [\sigma_n\nu_n\boldsymbol{a}_{n+1}] = [\sigma_{n+1}\boldsymbol{a}_{n+1}]$ by the definition of ∂_2 and (7). Also, $h(\sigma_n) = \chi(h)\sigma_n$ in $kT_{(1)}$, so $H_1(\boldsymbol{D}) \cong k_{(1)}$ as $k\Gamma$-modules. To summarize,

$$H_0(\boldsymbol{D}) = \big\langle[\boldsymbol{a}_0]\big\rangle \cong k \qquad \text{and} \qquad H_1(\boldsymbol{D}) = \big\langle[\sigma_1\boldsymbol{a}_1]\big\rangle \cong k_{(1)}\,. \tag{8}$$

Thus $\boldsymbol{D}$ is an Ω-resolution of k with respect to Γ if $H = 1$, but is not an Ω-resolution if $H \neq 1$ since Γ acts nontrivially on $H_1(\boldsymbol{D})$ (i.e., condition (Ω-3) fails). In this case, we construct an Ω-resolution by "pasting together" several copies of the above sequence.

Define a complex $\boldsymbol{C}_\infty$ of projective $k\Gamma$-modules of infinite length

$$\boldsymbol{C}_\infty = \Big(\cdots \longrightarrow \bigoplus_{n=0}^{\infty} kT_{(3)}{\cdot}\boldsymbol{a}_n \xrightarrow{\partial_6} \bigoplus_{n=1}^{\infty} kT_{(3)}{\cdot}\boldsymbol{a}_n \xrightarrow{\partial_5} \bigoplus_{n=0}^{\infty} kT_{(2)}{\cdot}\boldsymbol{a}_n \xrightarrow{\partial_4}$$
$$\bigoplus_{n=1}^{\infty} kT_{(2)}{\cdot}\boldsymbol{a}_n \xrightarrow{\partial_3} \bigoplus_{n=0}^{\infty} kT_{(1)}{\cdot}\boldsymbol{a}_n \xrightarrow{\partial_2} \bigoplus_{n=1}^{\infty} kT_{(1)}{\cdot}\boldsymbol{a}_n \xrightarrow{\partial_1} kT_{(0)}{\cdot}\boldsymbol{a}_0 \longrightarrow 0\Big),$$

where $\partial_1(\boldsymbol{a}_n) = \mu_n\boldsymbol{a}_0$ (as in $\boldsymbol{D}$), and for $i \geq 2$,

$$\partial_i(\boldsymbol{a}_n) = \begin{cases} \boldsymbol{a}_n - \nu_n\boldsymbol{a}_{n+1} & \text{if } i \text{ is even} \\ \mu_n(\boldsymbol{a}_0 + \sigma_1\boldsymbol{a}_1 + \sigma_2\boldsymbol{a}_2 + \cdots + \sigma_{n-1}\boldsymbol{a}_{n-1}) & \text{if } i \text{ is odd.} \end{cases}$$

Here, it is understood that $\boldsymbol{a}_0 = 0$ in the terms of odd degree. By (4), (5), (6), and (7), all boundary maps are $k\Gamma$-linear and $\partial_{i-1}\circ\partial_i = 0$ for all $i \geq 2$.

For each $j \geq 1$, let $\boldsymbol{C}_j \subseteq \boldsymbol{C}_\infty$ be the subcomplex consisting of all terms in $\boldsymbol{C}_\infty$ of degree at most $2j-1$ together with the summands $\bigoplus_{n=1}^{\infty} kT_{(j)}{\cdot}\boldsymbol{a}_n$ in degree $2j$ (thus omitting only the summand $kT_{(j)}{\cdot}\boldsymbol{a}_0$). Thus $\boldsymbol{C}_1 \cong \overline{\boldsymbol{D}}$. More generally, if we set $\boldsymbol{C}_0 = 0$, then for each $j \geq 0$, $\boldsymbol{C}_{j+1}/\boldsymbol{C}_j$ is isomorphic to the $2j$-fold suspension of $\overline{\boldsymbol{D}}$ tensored by $k_{(j)}$, and hence by (8) has homology isomorphic to $k_{(j+1)}$ in degree $2j+1$ and $k_{(j)}$ in degree $2j$. If $j \geq 1$, then the homology of $\boldsymbol{C}_{j+1}/\boldsymbol{C}_j$ in degree $2j$ is represented by the class of $\boldsymbol{a}_0$ in that degree, $\partial_{2j}(\boldsymbol{a}_0) = -\nu_0\boldsymbol{a}_1 = -\sigma_1\boldsymbol{a}_1$, and by (8) again, this represents the homology class in $\boldsymbol{C}_j/\boldsymbol{C}_{j-1}$ of degree $2j-1$. Together, these observations imply that $\boldsymbol{C}_\infty$ is acyclic, and that for each $j \geq 1$,

$$H_0(\boldsymbol{C}_j) \cong k, \qquad H_{2j-1}(\boldsymbol{C}_j) \cong k_{(j)}, \qquad \text{and} \qquad H_i(\boldsymbol{C}_j) = 0 \text{ for } i \neq 0, 2j-1.$$

Set $\boldsymbol{R} = \boldsymbol{C}_m$ (recall $m = |H|$). We claim that $\boldsymbol{R} \longrightarrow k \longrightarrow 0$ is an Ω-resolution with respect to Γ. Condition (Ω-1) clearly holds (each of the terms is projective), and Γ acts trivially on $H_*(\boldsymbol{R})$ since $\chi^m = 1$. It remains to show (Ω-2): that $k \otimes_{k\Gamma} \boldsymbol{R}$ is acyclic. Since $k \otimes_{k\Gamma} kT_{(i)} \cong k$ whenever $m|i$ and is zero otherwise,

$$k \otimes_{k\Gamma} \boldsymbol{R} \cong \Big(0 \longrightarrow \bigoplus_{n=1}^{\infty} k{\cdot}\boldsymbol{a}_n \xrightarrow{\bar\partial_{2m}=\mathrm{Id}} \bigoplus_{n=1}^{\infty} k{\cdot}\boldsymbol{a}_n \longrightarrow 0 \longrightarrow \ldots \longrightarrow 0 \longrightarrow k \longrightarrow 0\Big)$$

and is acyclic.

We have now shown:

Proposition 5.6. *Let $\Gamma = T \rtimes H$, where $T \cong \mathbb{Z}/p^\infty$ and H acts on $\Omega_1(T)$ via an injective character $\chi\colon H \longrightarrow \mathbb{F}_p^\times$. Set $m = |H|$. Then the complex $\boldsymbol{R} \longrightarrow k \longrightarrow 0$ defined above is an Ω-resolution of k with respect to the Ω-system $(k\Gamma\text{-}\mathsf{mod}, k\text{-}\mathsf{mod}; \theta_*, \theta^*)$, and $H_*(\Omega(B\Gamma^\wedge_p); k) \cong H_*(S^{2m-1}; k)$.*

Note that $\boldsymbol{C}_j$ is not an Ω-resolution when $j > m$ since $k \otimes_{k\Gamma} \boldsymbol{C}_j$ has nonzero homology in degree $2m$ (the class of $\boldsymbol{a}_0$), and is not an Ω-resolution when $1 \leq j < m$ since Γ acts nontrivially on $H_{2j-1}(\boldsymbol{C}_j)$. Note also that since $\Omega(B\Gamma^\wedge_p)$ has nonzero homology in degree $2m-1$, $\boldsymbol{R}$ has the shortest possible length of any Ω-resolution of k with respect to $(k\Gamma\text{-}\mathfrak{mod}, k\text{-}\mathfrak{mod}; \theta_*, \theta^*)$. The equivalence $\Omega(B\Gamma^\wedge_p) \simeq (S^{2m-1})^\wedge_p$ follows from [Su, pp. 103–105] (from the proof of the proposition), and since $B(\mathbb{Z}/p^\infty)^\wedge_p \simeq K(\mathbb{Z}_p, 2)$ (see [BK, VI.2.1–2.2]).

We will see later (Proposition 5.11 and Remark 5.12) that there are similar constructions of Ω-resolutions when $T \trianglelefteq \Gamma$ are such that $T \cong (\mathbb{Z}/p^\infty)^r$ for $r > 1$ and $p \nmid |\Gamma/T|$.

Remark 5.7. When $H \neq 1$, the parameters μ_n and ν_n can be defined more explicitly as follows. Fix generators $t_n \in T_n$ for each $n \geq 1$, chosen so that $(t_n)^p = t_{n-1}$ when $n \geq 2$, and set $\mu_n = \sum_{h\in H} \chi(h)^{-1} t_n^{\chi(h)} \in kT_n$ for all $n \geq 1$. It is straightforward to check that $h(\mu_n) = \chi(h)\mu_n$ for $h \in H$, that $kT_n{\cdot}\mu_n = I(kT_n)$, and that $(\mu_1)^p = 0$ while $(\mu_n)^p = \mu_{n-1}$ for $n \geq 2$. Also, $(\mu_n)^{p-1}\sigma_{n-1} = \sigma_n$, so we can set $\nu_{n-1} = (\mu_n)^{p-1}$ for each n, and use these parameters to define the Ω-resolution $\boldsymbol{R}$.

5.3. Groups with discrete p-tori of index prime to p.

We now make some more computations of Ω-homology in the situation of Proposition 5.2: this time by using the existence of Ω-resolutions without constructing them explicitly. The key to doing this is the following spectral sequence.

Lemma 5.8. *Let $T \trianglelefteq \Gamma$ be a pair of groups such that T is p-perfect. Let (C_*, ∂_*) be a positively graded chain complex of $k\Gamma$-modules that are projective as kT-modules, and assume that T acts trivially on $H_*(C_*, \partial_*)$. Then there is a first quadrant spectral sequence of $k[\Gamma/T]$-modules of the form*

$$E^2_{ij} = H_i(k \otimes_{kT} C_*) \otimes_k H_j(\Omega(BT^\wedge_p); k) \qquad \Longrightarrow \qquad H_{i+j}(C_*)$$

where the action of Γ/T on $H_*(\Omega(BT^\wedge_p); k)$ is that induced by conjugation on T.

Proof. Since T is p-perfect, BT is p-good and $BT^\wedge_p$ is simply connected [BK, Proposition VII.3.2], and so $\Omega(BT^\wedge_p)$ is connected. Consider the following diagram of spaces

$$\begin{tikzcd} L^pT \arrow[r] \arrow[d] & A^pT \arrow[r] \arrow[d, "\nu"] & \mathcal{P}(BT^\wedge_p) \arrow[d] \\ ET \arrow[r] & BT \arrow[r] & BT^\wedge_p \end{tikzcd}$$

where $BT = |\mathcal{B}(T)|$ and ET is its universal covering space, $\mathcal{P}(BT^\wedge_p)$ is the space of paths in $BT^\wedge_p$ originating at the image of the (unique) vertex in BT, both squares are pullbacks, and thus the vertical maps are all fibrations with fibre $\Omega(BT^\wedge_p)$. In particular, A^pT is the homotopy fibre of the completion map and is mod p acyclic, $L^pT \simeq \Omega(BT^\wedge_p)$ since $ET \simeq *$, and T acts freely on L^pT with orbit space A^pT.

Now, Γ acts on the right on all of these spaces via the conjugation action. More precisely, we identify the vertices in ET with T, and let Γ act on ET by setting $x * g = g^{-1}xg$ for $x \in T$ and $g \in \Gamma$, in contrast to the free right action of T defined by $x{\cdot}t = xt$. This induces actions of Γ on BT, $BT^\wedge_p$, and $\mathcal{P}(BT^\wedge_p)$, and hence on L^pT; and the actions of T and Γ on ET and on L^pT satisfy the relation $((x * g){\cdot}t) * g^{-1} = x{\cdot}(gtg^{-1})$ for $g \in \Gamma$ and $t \in T$.

In particular, $C_*(L^pT; k)$ is a complex of $k[T \rtimes \Gamma]$-modules that are free as kT-modules, and $C_*(A^pT; k) \cong C_*(L^pT; k) \otimes_{kT} k$ is an acyclic complex of $k\Gamma$-modules. The action of Γ on $H_*(L^pT; k)$ restricts to the conjugation action of $T = \pi_1(BT)$ on the fibre of ν over the basepoint, and this action is trivial since ν is pulled back from the simply connected space $BT^\wedge_p$. Thus the action of Γ on $H_*(L^pT;k) \cong H_*(\Omega(BT^\wedge_p);k)$ factors through Γ/T.

Consider the complex $C_*(L^pT;k) \otimes_{kT} C_*$. This is a double complex of $k\Gamma$-modules, where $g(x \otimes y) = x * g^{-1} \otimes gy$ for $g \in \Gamma$. This action of Γ is well defined on the tensor product over kT, since for $g \in \Gamma$ and $t \in T$,

$$\begin{aligned} g(xt \otimes t^{-1}y) &= (xt) * g^{-1} \otimes gt^{-1}y = (x * g^{-1}){\cdot}(gtg^{-1}) \otimes gt^{-1}y \\ &= x * g^{-1} \otimes (gtg^{-1})gt^{-1}y = x * g^{-1} \otimes gy = g(x \otimes y) \end{aligned}$$

by the relation shown above. As usual, we consider the two spectral sequences of $k\Gamma$-modules induced by this double complex.

If we first take homology in the left-hand factor, we obtain

$$E^1_{i,j} \cong H_j(\Omega(BT^\wedge_p);k) \otimes_{kT} C_i \qquad \text{and} \qquad E^2_{i,j} \cong H_j(\Omega(BT^\wedge_p);k) \otimes_k H_i(k \otimes_{kT} C_*),$$

where the first isomorphism holds since each C_i is projective as a kT-module, and the second since T acts trivially on $\Omega(BT^\wedge_p)$. On the other hand, if we first take homology of the right-hand factor, we obtain

$$\bar{E}^1_{i,j} \cong C_j(L^pT;k) \otimes_{kT} H_i(C_*) \cong \bigl(C_j(L^pT;k) \otimes_{kT} k\bigr) \otimes H_i(C_*),$$

the first isomorphism since each $C_j(L^pT;k)$ is free as a kT-module, and the second since the action of T on the homology of C_* is trivial. Then

$$\bar{E}^2_{i,j} \cong \begin{cases} H_i(C_*) & \text{if } j = 0 \\ 0 & \text{if } j > 0 \end{cases}$$

since $C_*(L^pT;k) \otimes_{kT} k$ is isomorphic to $C_*(A^pT;k)$ and hence is acyclic. □

In the following lemma, when we say that a graded vector space over k is "finite dimensional", we mean that it is finite dimensional in each degree and is nonzero in only finitely many degrees.

Lemma 5.9. *Fix a pair of groups $T \trianglelefteq \Gamma$, where $T \cong (\mathbb{Z}/p^\infty)^r$ for some $r \geq 1$ and Γ/T is finite. Let $O^p(\Gamma) \trianglelefteq \Gamma$ be the smallest normal subgroup such that $\Gamma/O^p(\Gamma)$ is a finite p-group (thus $O^p(\Gamma) \geq T$), set $\pi = \Gamma/O^p(\Gamma)$, and let $\theta\colon \mathcal{B}(\Gamma) \longrightarrow \mathcal{B}(\pi)$ be the natural functor. Let (C_*, ∂_*) be an Ω-resolution of $k\pi$ with respect to the Ω-system $(k\Gamma\text{-}\mathsf{mod}, k\pi\text{-}\mathsf{mod}; \theta_*, \theta^*)$. Then $H_*(C_*, \partial_*)$ is finite dimensional if and only if $H_*(C_* \otimes_{kT} k)$ is finite dimensional.*

Proof. Since $H_*(\Omega(BT^\wedge_p);k)$ is finite dimensional, Lemma 5.8 implies that if $H_*(C_* \otimes_{kT} k)$ is finite dimensional, then so is $H_*(C_*;k)$. Thus it remains to prove the converse. Since $H_*(-;k) \cong H_*(-;\mathbb{F}_p) \otimes_{\mathbb{F}_p} k$, it suffices to show this when $k = \mathbb{F}_p$.

Consider the following diagram:

$$\begin{tikzcd} L^p\Gamma \arrow[r] \arrow[d] & (L^p\Gamma)/T \arrow[r] \arrow[d] & A^p\Gamma \arrow[r] \arrow[d] & \mathcal{P}(B\Gamma^\wedge_p) \arrow[d] \\ E\Gamma \arrow[r] & (E\Gamma)/T \arrow[r] & B\Gamma \arrow[r] & B\Gamma^\wedge_p \end{tikzcd}$$

where the squares are pullback squares. By Proposition 4.3, $C_*(L^p\Gamma;\mathbb{F}_p)$ is an Ω-resolution of $\mathbb{F}_p\pi$ with respect to the given Ω-system. Hence by Proposition 1.6, we may assume $C_* = C_*(L^p\Gamma;\mathbb{F}_p)$, and so $C_* \otimes_{\mathbb{F}_pT} \mathbb{F}_p \cong C_*(L^p\Gamma/T;\mathbb{F}_p)$. From the above pullback diagram, we see that $L^p\Gamma/T$ is the homotopy fibre of the map $BT \simeq E\Gamma/T \longrightarrow B\Gamma^\wedge_p$. Since $\pi_1(B\Gamma^\wedge_p)$ is a finite p-group, it acts nilpotently on $H_*(L^p\Gamma/T;\mathbb{F}_p)$, and hence by the mod-$R$ fibre lemma of Bousfield and Kan [BK, Lemma II.5.1], $(L^p\Gamma/T)^\wedge_p$ is the homotopy fibre of the map $BT^\wedge_p \longrightarrow B\Gamma^\wedge_p$ induced by the inclusion. Since T is p-perfect and $|\Gamma/T|$ is finite, $BT^\wedge_p$ and $B\Gamma^\wedge_p$ are p-complete, and hence $(L^p\Gamma/T)^\wedge_p$ is also p-complete by the mod-R fibre lemma again.

Now assume that $H_*(C_*;\mathbb{F}_p)$ is finite dimensional, and hence that $\Omega(B\Gamma^\wedge_p)$ is a p-compact group. By Proposition 5.2, there is a p-local compact group $(S,\mathcal{F},\mathcal{L})$ with $T \le S \le \Gamma$ and $|\mathcal{L}|^\wedge_p \simeq B\Gamma^\wedge_p$. So for each $1 \neq t \in T$, [BLO, Theorem 6.3] implies that the map $B\langle t\rangle \longrightarrow B\Gamma^\wedge_p$ induced by the inclusion is not null homotopic. In the terminology of [DW, § 7], this means that the map $BT^\wedge_p \longrightarrow B\Gamma^\wedge_p$, regarded as a map of p-compact groups, has trivial kernel. So by [DW, Theorem 7.3], $BT^\wedge_p \longrightarrow B\Gamma^\wedge_p$ is a monomorphism of p-compact groups, and by definition [DW, 3.2], the $\mathbb{F}_p$-homology of its homotopy fibre $(L^p\Gamma/T)^\wedge_p$ is finite dimensional. □

We now give some consequences of Lemmas 5.8 and 5.9. The first example can also be carried out using the Serre spectral sequence for the path-loop fibration of $B\Gamma^\wedge_p$, but the argument given here is slightly easier, and it illustrates nicely how the action of Γ/T on the spectral sequence of Lemma 5.8 can be exploited.

Example 5.10. Set $\Gamma = T \rtimes C_2$, where p is odd, $T \cong (\mathbb{Z}/p^\infty)^2$, and $\Gamma/T \cong C_2$ inverts T. Then

$$H_i(\Omega(B\Gamma^\wedge_p);\mathbb{F}_p) \cong \begin{cases} \mathbb{F}_p & \text{if } i=0 \\ \mathbb{F}_p^3 & \text{if } i=3 \\ \mathbb{F}_p^4 & \text{if } i>3 \text{ and } i\in 3\mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. To simplify notation, we do this over an arbitary field k of characteristic p. Fix an Ω-resolution (C_*,∂_*) of k with respect to Γ, and let E be the spectral sequence of Lemma 5.8. Recall that this is a spectral sequence of $k[\Gamma/T]$-modules: the action of Γ/T plays a central role in the argument here.

Since $H_i(C_*\otimes_{k\Gamma} k) = 0$ for $i>0$ by (Ω-2), $\Gamma/T \cong C_2$ acts via $-\mathrm{Id}$ on $E^2_{i,0} \cong H_i(C_*\otimes_{kT} k)$ for each $i>0$. Also, since Γ acts trivially on $H_i(C_*)$ for all i by (Ω-3), Γ/T acts trivially on $E^\infty_{i,j}$ for all i,j.

We claim that $E^2_{i,j}$ takes the following form:

$$\begin{tikzcd}
k_+ & 0 & k_-^2 & 0 & 0 & k_-^2 & 0 & 0 & k_-^2 & \cdots \\
k_-^2 & 0 & k_+^4 \arrow[ull, dashed] & 0 & 0 & k_+^4 & 0 & 0 & k_+^4 & \cdots \\
k_+ & 0 & k_-^2 \arrow[ull, "\cong"'] & 0 & 0 & k_-^2 \arrow[uulll, "\cong"'] & 0 & 0 & k_-^2 \arrow[uulll, "\cong"'] & \cdots
\end{tikzcd}$$

where the subscripts $(\pm)$ describe the action of $\Gamma/T \cong C_2$, and where the pattern continues with $E^2_{3k+2,0} \cong k_-^2$ for $i \ge 0$ and $E^2_{i,0} = 0$ when $0 < i \not\equiv 2 \pmod 3$. To see this, note first that $E^2_{1,0} \cong E^\infty_{1,0} = 0$ since Γ/T inverts $E^2_{1,0}$ and fixes $E^\infty_{1,0}$, and that the differential sends $E^2_{2,0}$ isomorphically to $E^2_{0,1}$ since the action on E^∞ is trivial. Hence $E^2_{i,j}$ is as described for $i \le 2$. Also, $E^2_{3,0} = 0$ and $E^2_{4,0} = 0$ since there are no terms inverted by Γ/T which their differentials could hit, and $E^2_{5,0} \cong k_-^2$ must be sent isomorphically to $E^2_{2,2}$.

Upon continuing in this way, we see inductively that $E^2_{i,0} \cong k_-^2$ for $i = 3j+2$ (all $j \ge 0$) and is zero in other positive degrees, and the differentials are as shown with one possible exception. We claim that the differential from $E^2_{2,1}$ to $E^2_{0,2}$ is surjective: this holds since

$H_i(B\Gamma^\wedge_p;k) \cong (H_i(BT^\wedge_p;k))^{\Gamma/T} = 0$ for $i \le 3$ implies that $H_i(\Omega(B\Gamma^\wedge_p);k) = 0$ for $i \le 2$. Thus $E^\infty_{i,j}$ is zero in positive total degrees except $E^\infty_{21} \cong k^3$ and $E^\infty_{3i+2,1} \cong k^4$ (all $i \ge 1$). □

The next proposition shows that in these examples, at least, whenever the homology of the loop space is bounded, we get an Ω-resolution of finite length.

Proposition 5.11. *Let $T \trianglelefteq \Gamma$ be such that $T \cong (\mathbb{Z}/p^\infty)^r$ for some $r \ge 1$ and Γ/T is finite. If in addition, $\Omega(B\Gamma^\wedge_p)$ is a p-compact group, then there is an Ω-resolution of k with respect to Γ of finite length. More precisely, if $N \ge 1$ is maximal such that $H_N(\Omega(B\Gamma^\wedge_p);k) \ne 0$, then there is an Ω-resolution of k with respect to Γ of length $N+1$.*

Proof. Since $\Omega(B\Gamma^\wedge_p)$ is a p-compact group, $O^p(\Gamma/T)$ has order prime to p by Proposition 5.2. Let $O^p(\Gamma) \trianglelefteq \Gamma$ be such that $O^p(\Gamma)/T = O^p(\Gamma/T)$, and set $\pi = \Gamma/O^p(\Gamma)$. Since $\Omega(B(O^p(\Gamma))^\wedge_p)$ is homotopy equivalent to a connected component of $\Omega(B\Gamma^\wedge_p)$ (as shown in the proof of Proposition 5.2), $\Omega(BO^p(\Gamma)^\wedge_p)$ is also a p-compact group. If $\boldsymbol{C} \longrightarrow k \longrightarrow 0$ is an Ω-resolution of k with respect to $O^p(\Gamma)$, then $\mathrm{Ind}^{\Gamma}_{O^p(\Gamma)}(\boldsymbol{C}) \longrightarrow k\pi \longrightarrow 0$ is an Ω-resolution of $k\pi$ with respect to Γ. So it suffices to prove the proposition when $\Gamma = O^p(\Gamma)$; i.e., when Γ/T has order prime to p and $\pi = 1$.

Recall that r is the rank of T. By Proposition 5.5, there is an Ω-resolution $\boldsymbol{D} \longrightarrow k \longrightarrow 0$ of k with respect to T of length $r+1$ which is also a chain complex of projective $k\Gamma$-modules.

Step 1: By Proposition 2.8, there is an Ω-resolution $\boldsymbol{C} \longrightarrow k \longrightarrow 0$ of k with respect to Γ (possibly of infinite length). We will use this as a template for constructing an Ω-resolution of finite length $N+1$.

By Lemma 5.9, the homology of $k \otimes_{kT} \boldsymbol{C}$ is finitely generated. Let

$$0 = m_1 < m_2 < \cdots < m_\ell = m$$

be the degrees in which $H_*(k \otimes_{kT} \boldsymbol{C})$ is nonzero. Thus ℓ is the number of distinct degrees in which this homology is nonzero. By the spectral sequence $\{E^r_{*,*}\}$ of Lemma 5.8, $H_{m+r}(\boldsymbol{C}) \cong E^2_{m,r} \ne 0$, and so $N = m + r$ by Corollary 4.8.

Step 2: By Proposition 1.6, and since $\boldsymbol{C}$ satisfies condition (Ω-3) (Definition 1.5) as a complex of kT-modules, there is a kT-linear chain map

$$\psi^{(0)}_*\colon \boldsymbol{D} = H_0(k \otimes_{kT} \boldsymbol{C}) \otimes_k \Sigma^0 \boldsymbol{D} \longrightarrow \boldsymbol{C}$$

that induces a $k\Gamma$-linear isomorphism $H_0(\boldsymbol{D}) \xrightarrow{\cong} H_0(\boldsymbol{C})$. By averaging, i.e., by replacing $\psi^{(0)}_*$ by the map $x \mapsto \frac{1}{|\Gamma/T}\sum_{gT\in\Gamma/T} g(\psi^{(0)}_*(g^{-1}x))$ for $x \in \boldsymbol{D}$, we can arrange that $\psi^{(0)}_*$ is $k\Gamma$-linear without changing $H_0(\psi^{(0)}_*)$. Let $\boldsymbol{C}^{(1)}$ be the mapping cone of $\psi^{(0)}_*$ [W, § 1.5]; again a chain complex of projective $k\Gamma$-modules. Since the p-perfect group T acts trivially on the homology of $\boldsymbol{D}$ and of $\boldsymbol{C}$, it also acts trivially on $H_i(\boldsymbol{C}^{(1)})$ for each i (Lemma A.1). Also, the homology of $k \otimes_{kT} \boldsymbol{C}^{(1)}$ is isomorphic to that of $k \otimes_{kT} \boldsymbol{C}$ (as k-vector spaces), except in degree $0 = m_1$.

Step 3: Let t be the minimum of all i such that $H_i(\boldsymbol{C}^{(1)}) \ne 0$. If $t = \infty$ (i.e., if $\boldsymbol{C}^{(1)}$ is exact), then the sequence splits, so $k \otimes_{kT} \boldsymbol{C}^{(1)}$ is also exact, and $\ell = 1$.

Assume that $\boldsymbol{C}^{(1)}$ is not exact; i.e., that $t < \infty$. Then the exact sequence $C^{(1)}_t \longrightarrow C^{(1)}_{t-1} \longrightarrow \cdots \longrightarrow C^{(1)}_0 \longrightarrow 0$ of projective $k\Gamma$-modules splits. By this splitting, and since $(k \otimes_{kT} -)$ is right exact and T acts trivially on $H_t(\boldsymbol{C}^{(1)})$, we have $H_t(k \otimes_{kT} \boldsymbol{C}^{(1)}) \cong H_t(\boldsymbol{C}^{(1)}) \ne 0$, while $H_i(k \otimes_{kT} \boldsymbol{C}^{(1)}) = 0$ for all $i < t$. Thus $t = m_2$ and $\ell \ge 2$. By Proposition 1.6 again (and averaging), there is a $k\Gamma$-linear chain map

$$\psi_*^{(1)}\colon H_{m_2}(k\otimes_{kT}\boldsymbol{C})\otimes_k \Sigma^{m_2}\boldsymbol{D} \longrightarrow \boldsymbol{C}^{(1)}$$

that induces an isomorphism in $H_{m_2}(-)$. In other words, we shift $\boldsymbol{D}$ by degree m_2, tensor each term by the k-module

$$H_{m_2}(k\otimes_{kT}\boldsymbol{C}) \cong H_{m_2}(k\otimes_{kT}\boldsymbol{C}^{(1)}) \cong H_{m_2}(\boldsymbol{C}^{(1)}),$$

and then map the resulting complex into $\boldsymbol{C}^{(1)}$.

Let $\boldsymbol{C}^{(2)}$ be the mapping cone of $\psi_*^{(1)}$. By the arguments used in Step 2, T acts trivially on $H_*(\boldsymbol{C}^{(2)})$, and $H_i(k\otimes_{kT}\boldsymbol{C}^{(2)}) \cong H_i(k\otimes_{kT}\boldsymbol{C})$ for all $i>m_2$ while $H_i(k\otimes_{kT}\boldsymbol{C}^{(2)})=0$ for $i\le m_2$.

Step 4: We now repeat this procedure to obtain an increasing sequence

$$\boldsymbol{C}\le \boldsymbol{C}^{(1)}\le \boldsymbol{C}^{(2)}\le\cdots\le \boldsymbol{C}^{(\ell)}$$

of chain complexes of projective $k\Gamma$-modules, where for each $1\le r\le \ell$, T acts trivially on $H_*(\boldsymbol{C}^{(r)})$, and $H_i(k\otimes_{kT}\boldsymbol{C}^{(r)}) \cong H_i(k\otimes_{kT}\boldsymbol{C})$ for all $i>m_r$ while $H_i(k\otimes_{kT}\boldsymbol{C}^{(r)})=0$ for $i\le m_r$. Also, by the argument at the start of Step 3, $H_i(\boldsymbol{C}^{(r)})=0$ for $i<m_{r+1}$, while $H_{m_{r+1}}(\boldsymbol{C}^{(r)})\cong H_{m_{r+1}}(k\otimes_{kT}\boldsymbol{C}^{(r)})\cong H_{m_{r+1}}(k\otimes_{kT}\boldsymbol{C})\ne 0$.

In particular, $\boldsymbol{C}^{(\ell)}$ is an exact sequence of projective $k\Gamma$-modules. Set

$$\boldsymbol{R}=\Sigma^{-1}\big(\boldsymbol{C}^{(\ell)}/\boldsymbol{C}\big).$$

Then $\boldsymbol{C}^{(\ell)}$ is the mapping cone of a $k\Gamma$-linear chain map $\boldsymbol{R}\longrightarrow\boldsymbol{C}$, and Γ acts trivially on $H_*(\boldsymbol{R})\cong H_*(\boldsymbol{C})$. Also, $k\otimes_{k\Gamma}\boldsymbol{R}$ is acyclic since $k\otimes_{k\Gamma}\boldsymbol{C}$ is (and since the sequence $k\otimes_{k\Gamma}C_*^{(\ell)}$ is exact), and so $\boldsymbol{R}$ is an Ω-resolution of k with respect to Γ of length $m+r+1=N+1$. □

Note that the converse of Proposition 5.11 also holds: $\Omega(B\Gamma_p^\wedge)$ is a p-compact group if there is an Ω-resolution of finite length. As usual, this can be reduced to the case where Γ/T is p-perfect; i.e., where $B\Gamma_p^\wedge$ is simply connected. If there is an Ω-resolution of k with respect to Γ of finite length, then $H_i(\Omega(B\Gamma_p^\wedge);\mathbb{F}_p)=0$ for i large enough by Corollary 4.8. Also, $H_i(B\Gamma_p^\wedge;\mathbb{F}_p)\cong H_i(B\Gamma;\mathbb{F}_p)$ is finite for each i by the Serre spectral sequence for the fibration sequence $BT\longrightarrow B\Gamma\longrightarrow B(\Gamma/T)$ (and since $H_*(BT;\mathbb{F}_p)\cong H_*(B(S^1)^r;\mathbb{F}_p)$ and $H_*(B(\Gamma/T);\mathbb{F}_p)$ are finite in each degree), and hence $H_i(\Omega(B\Gamma_p^\wedge);\mathbb{F}_p)$ is finite for each i by [Se, Proposition 7] and since $B\Gamma_p^\wedge$ is simply connected. So $\Omega(B\Gamma_p^\wedge)$ is a p-compact group.

Remark 5.12. By a closer inspection, one can say more about the Ω-resolution constructed in the proof of Proposition 5.11. By construction, $\boldsymbol{R}$ has a filtration whose successive quotients are the suspended complexes $H_{m_r}(k\otimes_{kT}\boldsymbol{R})\otimes_k\Sigma^{m_r}\boldsymbol{D}$ for $1\le r\le\ell$, where $\boldsymbol{D}$ is the complex constructed in Proposition 5.5 and $0=m_1<m_2<\cdots<m_\ell=m$ are the degrees in which $H_*(k\otimes_{kT}\boldsymbol{R})$ is nonzero.

Appendix A. R-perfect groups and R-plus constructions

Recall that for a group G, we write $G^{\mathrm{ab}}=G/[G,G]\cong H_1(G;\mathbb{Z})$ for short. For a commutative ring R, we say that G is R-perfect if $H_1(G;R)=0$; equivalently, if $G^{\mathrm{ab}}\otimes_{\mathbb{Z}}R=0$. When p is a prime, G is p-perfect if it is $\mathbb{F}_p$-perfect.

Lemma A.1. *Fix a commutative ring R and an R-perfect group G. Let $M_0\subseteq M$ be RG-modules such that G acts trivially on M_0 and on M/M_0. Then G also acts trivially on M.*

Proof. Let $\mathrm{Aut}_R^0(M)$ be the group of all R-linear automorphisms of M that induce the identity on M_0 and on M/M_0. Then $\mathrm{Aut}_R^0(M) \cong \mathrm{Hom}_R(M/M_0, M_0)$ is abelian and has the structure of an R-module, and G acts on M via a homomorphism $G \longrightarrow \mathrm{Aut}_R^0(M)$. Each homomorphism from G to an R-module factors through $G^{\mathrm{ab}} \otimes_{\mathbb{Z}} R = 0$ and hence is trivial, so G acts trivially on M. $\square$

We now turn our attention to plus constructions (see Definition 4.2). The following lemma will be needed when checking the condition in the definition about homology with twisted coefficients.

Lemma A.2. *Fix a commutative ring R and a group π. Let $f\colon X \longrightarrow Y$ be a map between connected spaces, and let $\eta\colon \pi_1(Y) \longrightarrow \pi$ be a homomorphism such that η and $\eta\circ\pi_1(f)$ are both surjective. Let $\widetilde{X}$ and $\widetilde{Y}$ be the covering spaces of X and Y with fundamental groups* $\mathrm{Ker}(\eta\circ\pi_1(f))$ *and* $\mathrm{Ker}(\eta)$, *respectively, and assume that a covering map $\widetilde{f}\colon \widetilde{X} \longrightarrow \widetilde{Y}$ is an R-homology equivalence. Then $H_*(f;M)$ is an isomorphism for each $R\pi$-module M.*

Proof. Let $\widehat{C}_*$ be the mapping cone of the chain map $C_*(\widetilde{f})\colon C_*(\widetilde{X};R) \longrightarrow C_*(\widetilde{Y};R)$ (see the remark just before Proposition 5.11). Since $\widehat{C}_*$ is an exact sequence of free $R\pi$-modules and is bounded below, $\widehat{C}_* \otimes_{R\pi} M$ is also exact, and hence $H_*(f;M)$ is an isomorphism. $\square$

The conditions for the existence of an R-plus construction are essentially the same as those for the existence of the usual plus construction.

Lemma A.3. *Let R be a commutative ring, let X be a connected CW complex, and let $H \trianglelefteq \pi_1(X)$ be a normal subgroup. Let $\widetilde{X}$ be the covering space of X with fundamental group H. Then the following are equivalent:*

(a) *The pair (X,H) admits an R-plus construction.*

(b) *The subgroup H is R-perfect.*

(c) $H_1(\widetilde{X};R) = 0$.

Proof. Set $\pi = \pi_1(X)/H$.

(a $\Longrightarrow$ c) Let $\kappa\colon X \longrightarrow X_R^+$ be an R-plus construction for (X,H), and let $\widetilde{X}_R^+$ be the universal cover of X_R^+. Then by definition,

$$H_1(\widetilde{X};R) \cong H_1(X;R\pi) \cong H_1(X_R^+;R\pi) \cong H_1(\widetilde{X}_R^+;R) = 0\,.$$

(b $\Longleftrightarrow$ c) This holds since $H_1(\widetilde{X};R) \cong \pi_1(\widetilde{X})^{\mathrm{ab}} \otimes_{\mathbb{Z}} R \cong H_1(H;R)$.

(c $\Longrightarrow$ a) This is essentially Quillen's construction. Assume (c) holds, and attach 2-cells to $\widetilde{X}$ in free π-orbits to obtain a free, simply connected π-space $\widetilde{X}_0^+ \supseteq \widetilde{X}$. Then $H_2(\widetilde{X}_0^+;R)$ surjects onto the free $R\pi$-module $H_2(\widetilde{X}_0^+,\widetilde{X};R)$ since $H_1(\widetilde{X};R)=0$ by assumption. By the Hurewicz theorem, $H_2(\widetilde{X}_0^+;R) \cong \pi_2(\widetilde{X}_0^+)\otimes_{\mathbb{Z}} R$, and hence free π-orbits of 3-cells can be attached to $\widetilde{X}_0^+$ to obtain a free π-space $\widetilde{X}_R^+$ with $H_*(\widetilde{X}_R^+,\widetilde{X};R)=0$. Set $X_R^+ = \widetilde{X}_R^+/\pi$ and let $\kappa\colon X \longrightarrow X_R^+$ be the inclusion; then $\pi_1(\kappa)$ is surjective with kernel H.

By Lemma A.2, and since the inclusion of $\widetilde{X}$ in $\widetilde{X}_R^+$ is an R-homology equivalence, the inclusion of X in X_R^+ induces an isomorphism in homology with coefficients in any $R\pi$-module. So X_R^+ is an R-plus construction for (X,H). $\square$

Under certain conditions, completion or fibrewise completion as defined by Bousfield and Kan gives another, more functorial way to construct plus constructions.

Lemma A.4. *Let π be a group, and let $\theta\colon X \longrightarrow B\pi$ be a map of spaces where X is connected and $\pi_1(\theta)$ is onto. Set $H = \mathrm{Ker}(\pi_1(\theta))$.*

(a) *Assume that R is a subring of $\mathbb{Q}$ or $R = \mathbb{F}_p$ for some prime p, and also that H is R-perfect. Let $\widehat{\theta}\colon X^\wedge \longrightarrow B\pi$ be the fibrewise R-completion of X over $B\pi$. Then $\kappa\colon X \longrightarrow X^\wedge$ is an R-plus construction for (X,H).*

(b) *Let $\widetilde{X}$ be the covering space of X with covering group π and fundamental group H. Assume, for some $R \subseteq \mathbb{Q}$, that H is R-perfect, and that π is R-nilpotent and has nilpotent action on $H_i(\widetilde{X};R)$ for each i. Then the R-completion map $\kappa\colon X \longrightarrow X^\wedge_R$ is an R-plus construction for (X,H).*

(c) *Assume, for some prime p, that π is a finite p-group and H is p-perfect. Then the p-completion map $\kappa\colon X \longrightarrow X^\wedge_p$ is a k-plus construction for (X,H) for each field k of characteristic p.*

Proof. We can assume that $\theta\colon X \longrightarrow B\pi$ is a fibration, with fibre F. Thus the inclusion $F \subseteq X$ induces an isomorphism $\pi_1(F) \cong H$. Let $\widetilde{X}$ be the covering space of X with covering group π and fundamental group H; thus $\widetilde{X} \simeq F$.

(a) Since H is R-perfect, $H_1(F;R) \cong H_1(H;R) = 0$ (Lemma A.3), and hence $F^\wedge_R$ is simply connected by [BK, Lemma I.6.1] (applied with $k = 1$). Since $F^\wedge_R$ is the fibre of the fibration $\widehat{\theta}$ by [BK, Corollary I.8.3], $\widehat{\theta}$ induces an isomorphism $\pi_1(X^\wedge) \cong \pi$ and $\widetilde{X}^\wedge_R$ is the universal cover of $X^\wedge$.

Since $F^\wedge_R$ is simply connected, it is R-good by Proposition V.3.4 or VI.5.3 in [BK], and hence F is R-good by [BK, Proposition I.5.2]. So $\kappa_0\colon F \longrightarrow F^\wedge_R$ is an R-homology equivalence. Hence $\widetilde{X} \longrightarrow \widetilde{X}^\wedge_R$ is also an R-homology equivalence, and by Lemma A.2, $\kappa\colon X \longrightarrow X^\wedge$ induces an isomorphism in homology with coefficients in arbitrary $R\pi$-modules. So κ is an R-plus construction for (X,H).

(b) If $R \subseteq \mathbb{Q}$ and π is R-nilpotent, then $B\pi$ is R-complete by [BK, Proposition V.2.2]. If, in addition, H is R-perfect and the action of π on $H_i(\widetilde{X};R)$ (equivalently, on $H_i(F;R)$) is nilpotent for each i, then fibrewise completion over $B\pi$ is the same as R-completion by the mod-R fibration lemma [BK, II.5.1], and the result follows from (a).

(c) If π is a finite p-group, then $B\pi$ is p-complete by [BK, VI.3.4 and VI.5.4]. Hence fibrewise completion over $B\pi$ is the same as p-completion by the mod-R fibration lemma [BK, II.5.1 and II.5.2.iv], and $\kappa\colon X \longrightarrow X^\wedge_p$ is an $\mathbb{F}_p$-plus construction by (a) (applied with $R = \mathbb{F}_p$) when H is p-perfect. If k is an arbitrary field of characteristic p, then κ is also a k-plus construction since $H_*(-;k) \cong H_*(-;\mathbb{F}_p) \otimes_{\mathbb{F}_p} k$. □

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