EXISTENCE AND UNIQUENESS OF LINKING SYSTEMS: CHERMAK'S PROOF VIA OBSTRUCTION THEORY

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Abstract. We present a version of a proof by Andy Chermak of the existence and uniqueness of centric linking systems associated to arbitrary saturated fusion systems. This proof differs from the one in [Ch2] in that it is based on the computation of higher derived functors of certain inverse limits. This leads to a much shorter proof, but one which is aimed mostly at researchers familiar with homological algebra.

One of the central questions in the study of fusion systems is whether to each saturated fusion system one can associate a centric linking system, and if so, whether it is unique. This question was recently answered positively by Andy Chermak [Ch2], using direct constructions. His proof is quite lengthy, although some of the structures developed there seem likely to be of independent interest.

There is also a well established obstruction theory for studying this problem, involving higher derived functors of certain inverse limits. This is analogous to the use of group cohomology as an "obstruction theory" for the existence and uniqueness of group extensions. By using this theory, Chermak's proof can be greatly shortened, in part because it allows us to focus on the essential parts of Chermak's constructions, and in part by using results which are already established. The purpose of this paper is to present this shorter version of Chermak's proof, a form which we hope will be more easily accessible to researchers with a background in topology or homological algebra.

A *saturated fusion system* over a finite p-group S is a category whose objects are the subgroups of S, and whose morphisms are certain monomorphisms between the subgroups. This concept is originally due to Puig (see [P2]), and one version of his definition is given in Section 1 (Definition 1.1). One motivating example is the fusion system of finite group G with $S \in \mathrm{Syl}_p(G)$: the category $\mathcal{F}_S(G)$ whose objects are the subgroups of S and whose morphisms are those group homomorphisms which are conjugation by elements of G.

For $S \in \mathrm{Syl}_p(G)$ as above, there is a second, closely related category which can be defined, and which supplies the "link" between $\mathcal{F}_S(G)$ and the classifying space BG of G. A subgroup $P \le S$ is called *p-centric in G* if $Z(P) \in \mathrm{Syl}_p(C_G(P))$; equivalently, if $C_G(P) = Z(P) \times C'_G(P)$ for some (unique) subgroup $C'_G(P)$ of order

2000 *Mathematics Subject Classification.* Primary 55R35. Secondary 20J05, 20N99, 20D20, 20D05.

Key words and phrases. Finite groups, fusion, fusion systems, derived functors, classifying spaces, FF-offenders.

B. Oliver was partially supported by the DNRF through a visiting professorship at the Centre for Symmetry and Deformation in Copenhagen; and also by UMR 7539 of the CNRS and by project ANR BLAN08-2_338236, HGRT.

prime to p. Let $\mathcal{L}_S^c(G)$ (the *centric linking system of* G) be the category whose objects are the subgroups of S which are p-centric in G, and where for each pair of objects P, Q:

$$\mathrm{Mor}_{\mathcal{L}_S^c(G)}(P,Q) = \big\{ g \in G \,\big|\, {}^gP \le Q \big\} / C'_G(P) \,.$$

Such categories were originally defined by Puig in [P1].

To explain the significance of linking systems from a topologist's point of view, we must first define the *geometric realization* of an arbitrary small category $\mathcal{C}$. This is a space $|\mathcal{C}|$ built up of one vertex (point) for each object in $\mathcal{C}$, one edge for each nonidentity morphism (with endpoints attached to the vertices corresponding to its source and target), one 2-simplex (triangle) for each commutative triangle in $\mathcal{L}$, etc. (See, e.g., [AKO, § III.2.1–2] for more details.) By a theorem of Broto, Levi, and Oliver [BLO1, Proposition 1.1], for any G and S as above, the space $|\mathcal{L}_S^c(G)|$, after p-completion in the sense of Bousfield and Kan, is homotopy equivalent to the p-completed classifying space $BG^\wedge_p$ of G. Furthermore, many of the homotopy theoretic properties of the space $BG^\wedge_p$, such as its self homotopy equivalences, can be determined combinatorially by the properties (such as automorphisms) of the finite category $\mathcal{L}_S^c(G)$ [BLO1, Theorems B & C].

Abstract *centric linking systems* associated to a fusion system were defined in [BLO2] (see Definition 1.3). One of the motivations in [BLO2] for defining these categories was that it provides a way to associate a classifying space to a saturated fusion system. More precisely, if $\mathcal{L}$ is a centric linking system associated to a saturated fusion system $\mathcal{F}$, then we regard the p-completion $|\mathcal{L}|^\wedge_p$ of its geometric realization as a classifying space for $\mathcal{F}$. This is motivated by the equivalence $|\mathcal{L}_S^c(G)|^\wedge_p \simeq BG^\wedge_p$ noted above. To give one example of the role played by these classifying spaces, if $\mathcal{L}'$ is another centric linking system, associated to a fusion system $\mathcal{F}'$, and the classifying spaces $|\mathcal{L}|^\wedge_p$ and $|\mathcal{L}'|^\wedge_p$ are homotopy equivalent, then $\mathcal{L} \cong \mathcal{L}'$ and $\mathcal{F} \cong \mathcal{F}'$. We refer to [BLO2, Theorem A] for more details and discussion.

It is unclear from the definition whether there is a centric linking system associated to any given saturated fusion system, and if so, whether it is unique. Even when working with fusion systems of finite groups, which always have a canonical associated linking system, there is no simple reason why two groups with isomorphic fusion systems need have isomorphic linking systems, and hence equivalent p-completed classifying spaces. This question — whether $\mathcal{F}_S(G) \cong \mathcal{F}_T(H)$ implies $\mathcal{L}_S^c(G) \cong \mathcal{L}_T^c(H)$ and hence $BG^\wedge_p \simeq BH^\wedge_p$ — was originally posed by Martino and Priddy, and was what first got this author interested in the subject.

The main theorem of Chermak described in this paper is the following.

Theorem A (Chermak [Ch2])**.** *Each saturated fusion system has an associated centric linking system, which is unique up to isomorphism.*

Proof. This follows immediately from Theorem 3.4 in this paper, together with [BLO2, Proposition 3.1]. □

In particular, this provides a new proof of the Martino-Priddy conjecture, which was originally proven in [O1, O2] using the classification of finite simple groups. Chermak's theorem is much more general, but it also (indirectly) uses the classification in its proof.

Theorem A is proven by Chermak by directly and systematically constructing the linking system, and by directly constructing an isomorphism between two given linking systems. The proof given here follows the same basic outline, but uses as its main tool the obstruction theory which had been developed in [BLO2, Proposition 3.1] for dealing with this problem. So if this approach is shorter, it is only because we are able to profit by the results of [BLO2, § 3], and also by other techniques which have been developed more recently for computing these obstruction groups.

By [BLO3, Proposition 4.6], there is a bijective correspondence between centric linking systems associated to a given saturated fusion system $\mathcal{F}$ up to isomorphism, and homotopy classes of rigidifications of the homotopy functor $\mathcal{O}(\mathcal{F}^c) \longrightarrow \mathsf{hoTop}$ which sends P to BP. (See Definition 1.5 for the definition of $\mathcal{O}(\mathcal{F}^c)$.) Furthermore, if $\mathcal{L}$ corresponds to a rigidification $\widetilde{B}$, then $|\mathcal{L}|$ is homotopy equivalent to the homotopy direct limit of $\widetilde{B}$. Thus another consequence of Theorem A is:

Theorem B. *For each saturated fusion system $\mathcal{F}$, there is a functor*

$$\widetilde{B}\colon \mathcal{O}(\mathcal{F}^c) \longrightarrow \mathsf{Top},$$

together with a choice of homotopy equivalences $\widetilde{B}(P) \simeq BP$ for each object P, such that for each $[\varphi] \in \mathrm{Mor}_{\mathcal{O}(\mathcal{F}^c)}(P,Q)$, the composite

$$BP \simeq \widetilde{B}(P) \xrightarrow{\widetilde{B}([\varphi])} \widetilde{B}(Q) \simeq BQ$$

is homotopic to $B\varphi$. Furthermore, $\widetilde{B}$ is unique up to homotopy equivalence of functors, and $\bigl(\mathrm{hocolim}(\widetilde{B})\bigr)^\wedge_p$ is the (unique) classifying space for $\mathcal{F}$.

We also want to compare "outer automorphism groups" of fusion systems, linking systems, and their classifying spaces. When $\mathcal{F}$ is a saturated fusion system over a p-group S, set

$$\mathrm{Aut}(S,\mathcal{F}) = \{\alpha \in \mathrm{Aut}(S) \mid {}^\alpha\mathcal{F} = \mathcal{F}\} \quad \text{and} \quad \mathrm{Out}(S,\mathcal{F}) = \mathrm{Aut}(S,\mathcal{F})/\mathrm{Aut}_{\mathcal{F}}(S)\,.$$

Here, for $\alpha \in \mathrm{Aut}(S)$, ${}^\alpha\mathcal{F}$ is the fusion system over S for which $\mathrm{Hom}_{{}^\alpha\mathcal{F}}(P,Q) = \alpha\circ\mathrm{Hom}_{\mathcal{F}}(\alpha^{-1}(P),\alpha^{-1}(Q))\circ\alpha^{-1}$. Thus $\mathrm{Aut}(S,\mathcal{F})$ is the group of "fusion preserving" automorphisms of S.

When $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$, then for each object P of $\mathcal{L}$, there is a "distinguished monomorphism" $\delta_P\colon P \longrightarrow \mathrm{Aut}_{\mathcal{L}}(P)$ (Definition 1.3). An automorphism α of $\mathcal{L}$ (a bijective functor from $\mathcal{L}$ to itself) is called *isotypical* if it permutes the images of the distinguished monomorphisms; i.e., if $\alpha(\delta_P(P)) = \delta_{\alpha(P)}(\alpha(P))$ for each P. We denote by $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$ the group of isotypical automorphisms of $\mathcal{L}$ modulo natural transformations of functors. See also [AOV, § 2.2] or [AKO, Lemma III.4.9] for an alternative description of this group.

By [BLO2, Theorem D], $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}) \cong \mathrm{Out}(|\mathcal{L}|^\wedge_p)$, where $\mathrm{Out}(|\mathcal{L}|^\wedge_p)$ is the group of homotopy classes of self homotopy equivalences of the space $|\mathcal{L}|^\wedge_p$. This is one reason for the importance of this particular group of (outer) automorphisms of $\mathcal{L}$. Another reason is the role played by $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$ in the definition of a *tame fusion system* in [AOV, § 2.2].

The other main consequence of the results in this paper is the following.

Theorem C. *For each saturated fusion system $\mathcal{F}$ over a p-group S with associated centric linking system $\mathcal{L}$, the natural homomorphism*

$$\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}) \xrightarrow{\ \mu_{\mathcal{L}}\ } \mathrm{Out}(S,\mathcal{F})$$

induced by restriction to $\delta_S(S) \cong S$ is surjective, and is an isomorphism if p is odd.

Proof. By [AKO, III.5.12], $\mathrm{Ker}(\mu_{\mathcal{L}}) \cong \varprojlim{}^1(\mathcal{Z}_{\mathcal{F}})$, and $\mu_{\mathcal{L}}$ is onto whenever $\varprojlim{}^2(\mathcal{Z}_{\mathcal{F}}) = 0$. (This was shown in [BLO1, Theorem E] when $\mathcal{L}$ is the linking system of a finite group.) So the result follows from Theorem 3.4 in this paper. □

I would like to thank Assaf Libman for very carefully reading this manuscript, pointing out a couple gaps in the arguments, and making other suggestions for improving it. I also thank the three referees who read very carefully different parts of the paper and made detailed suggestions for simplifying or clarifying several of the arguments. And, of course, I very much want to thank Andy Chermak for solving this problem, which has taken up so much of my time for the past ten years.

1. Notation and background

We first briefly recall the definitions of saturated fusion systems and centric linking systems. For any group G and any pair of subgroups $H, K \leq G$, set

$$\mathrm{Hom}_G(H,K) = \big\{ c_g = (x \mapsto gxg^{-1}) \,\big|\, g \in G,\ {}^gH \leq K \big\} \subseteq \mathrm{Hom}(H,K).$$

A *fusion system* $\mathcal{F}$ over a finite p-group S is a category whose objects are the subgroups of S, and whose morphism sets $\mathrm{Hom}_{\mathcal{F}}(P,Q)$ satisfy the following two conditions:

- $\mathrm{Hom}_S(P,Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P,Q) \subseteq \mathrm{Inj}(P,Q)$ for all $P, Q \leq S$.
- For each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$, $\varphi^{-1} \in \mathrm{Hom}_{\mathcal{F}}(\varphi(P),P)$.

Two subgroups $P, P' \leq S$ are called *$\mathcal{F}$-conjugate* if they are isomorphic in the category $\mathcal{F}$. Let $P^{\mathcal{F}}$ denote the set of subgroups $\mathcal{F}$-conjugate to P.

The following is the definition of a saturated fusion system first formulated in [BLO2]. Other (equivalent) definitions, including the original one by Puig, are discussed and compared in [AKO, §§ I.2 & I.9].

Definition 1.1. Let $\mathcal{F}$ be a fusion system over a p-group S.

- A subgroup $P \leq S$ is *fully centralized in $\mathcal{F}$* if $|C_S(P)| \geq |C_S(Q)|$ for all $Q \in P^{\mathcal{F}}$.
- A subgroup $P \leq S$ is *fully normalized in $\mathcal{F}$* if $|N_S(P)| \geq |N_S(Q)|$ for all $Q \in P^{\mathcal{F}}$.
- A subgroup $P \leq S$ is *$\mathcal{F}$-centric* if $C_S(Q) \leq Q$ for all $Q \in P^{\mathcal{F}}$.
- The fusion system $\mathcal{F}$ is *saturated* if the following two conditions hold:
  - (I) For all $P \leq S$ which is fully normalized in $\mathcal{F}$, P is fully centralized in $\mathcal{F}$ and $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$.
  - (II) If $P \leq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ are such that $\varphi(P)$ is fully centralized, and if we set
  $$N_\varphi = \{ g \in N_S(P) \,|\, \varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(\varphi(P)) \},$$
  then there is $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ such that $\bar{\varphi}|_P = \varphi$.

The following technical result will be needed later.

Lemma 1.2 ([AKO, Lemma I.2.6(c)]). *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. Then for each $P \le S$, and each $Q \in P^{\mathcal{F}}$ which is fully normalized in $\mathcal{F}$, there is $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_S(P), S)$ such that $\varphi(P) = Q$.*

For any fusion system $\mathcal{F}$ over S, let $\mathcal{F}^c \subseteq \mathcal{F}$ be the full subcategory whose objects are the $\mathcal{F}$-centric subgroups of S, and also let $\mathcal{F}^c$ denote the set of $\mathcal{F}$-centric subgroups of S.

Definition 1.3 ([BLO2]). Let $\mathcal{F}$ be a fusion system over the p-group S. A *centric linking system* associated to $\mathcal{F}$ is a category $\mathcal{L}$ with $\mathrm{Ob}(\mathcal{L}) = \mathcal{F}^c$, together with a functor $\pi\colon \mathcal{L} \longrightarrow \mathcal{F}^c$ and *distinguished monomorphisms* $P \xrightarrow{\delta_P} \mathrm{Aut}_{\mathcal{L}}(P)$ for each $P \in \mathrm{Ob}(\mathcal{L})$, which satisfy the following conditions.

(A) π is the identity on objects and is surjective on morphisms. For each $P, Q \in \mathcal{F}^c$, $\delta_P(Z(P))$ acts freely on $\mathrm{Mor}_{\mathcal{L}}(P, Q)$ by composition, and π induces a bijection

$$\mathrm{Mor}_{\mathcal{L}}(P, Q)/\delta_P(Z(P)) \xrightarrow{\quad\cong\quad} \mathrm{Hom}_{\mathcal{F}}(P, Q).$$

(B) For each $g \in P \in \mathcal{F}^c$, π sends $\delta_P(g) \in \mathrm{Aut}_{\mathcal{L}}(P)$ to $c_g \in \mathrm{Aut}_{\mathcal{F}}(P)$.

(C) For each $P, Q \in \mathcal{F}^c$, $\psi \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$, and $g \in P$, $\psi \circ \delta_P(g) = \delta_Q(\pi(\psi)(g)) \circ \psi$ in $\mathrm{Mor}_{\mathcal{L}}(P, Q)$.

We next fix some notation for sets of subgroups of a given group. For any group G, let $\mathscr{S}(G)$ be the set of subgroups of G. If $H \le G$ is any subgroup, set

$$\mathscr{S}(G)_{\ge H} = \{K \in \mathscr{S}(G) \mid K \ge H\}.$$

Definition 1.4. Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. An *interval* of subgroups of S is a subset $\mathcal{R} \subseteq \mathscr{S}(S)$ such that $P < Q < R$ and $P, R \in \mathcal{R}$ imply $Q \in \mathcal{R}$. An interval is *$\mathcal{F}$-invariant* if it is invariant under $\mathcal{F}$-conjugacy.

Thus, for example, an $\mathcal{F}$-invariant interval $\mathcal{R} \subseteq \mathscr{S}(S)$ is closed under overgroups if and only if $S \in \mathcal{R}$. Each $\mathcal{F}$-invariant interval has the form $\mathcal{R} \smallsetminus \mathcal{R}_0$ for some pair of $\mathcal{F}$-invariant intervals $\mathcal{R}_0 \subseteq \mathcal{R}$ which are closed under overgroups.

We next recall the obstruction theory to the existence and uniqueness of linking systems.

Definition 1.5. Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S.

(a) Let $\mathcal{O}(\mathcal{F}^c)$ be the *centric orbit category* of $\mathcal{F}$: $\mathrm{Ob}(\mathcal{O}(\mathcal{F}^c)) = \mathrm{Ob}(\mathcal{F}^c)$, and

$$\mathrm{Mor}_{\mathcal{O}(\mathcal{F}^c)}(P, Q) = \mathrm{Inn}(Q)\backslash \mathrm{Hom}_{\mathcal{F}}(P, Q).$$

(b) Let $\mathcal{Z}_{\mathcal{F}}\colon \mathcal{O}(\mathcal{F}^c)^{\mathrm{op}} \longrightarrow \mathsf{Ab}$ be the functor which sends P to $Z(P) = C_S(P)$. If $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$, and $[\varphi]$ denotes its class in $\mathrm{Mor}(\mathcal{O}(\mathcal{F}^c))$, then $\mathcal{Z}_{\mathcal{F}}([\varphi]) = \varphi^{-1}$ as a homomorphism from $Z(Q) = C_S(Q)$ to $Z(P) = C_S(P)$.

(c) For any $\mathcal{F}$-invariant interval $\mathcal{R} \subseteq \mathcal{F}^c$, let $\mathcal{Z}_{\mathcal{F}}^{\mathcal{R}}$ be the subquotient functor of $\mathcal{Z}_{\mathcal{F}}$ where $\mathcal{Z}_{\mathcal{F}}^{\mathcal{R}}(P) = Z(P)$ if $P \in \mathcal{R}$ and $\mathcal{Z}_{\mathcal{F}}^{\mathcal{R}}(P) = 0$ otherwise.

(d) For each $\mathcal{F}$-invariant interval $\mathcal{R} \subseteq \mathcal{F}^c$, we write for short

$$L^*(\mathcal{F}; \mathcal{R}) = \varprojlim_{\mathcal{O}(\mathcal{F}^c)}{}^{*}(\mathcal{Z}_{\mathcal{F}}^{\mathcal{R}})\,;$$

i.e., the higher derived functors of the inverse limit of $\mathcal{Z}_{\mathcal{F}}^{\mathcal{R}}$.

We refer to [AKO, § III.5.1] for more discussion of the functors $\varprojlim^*(-)$.

Thus $\mathcal{Z}_\mathcal{F} = \mathcal{Z}_\mathcal{F}^{\mathcal{F}^c}$, and $\varprojlim^*(\mathcal{Z}_\mathcal{F}) = L^*(\mathcal{F};\mathcal{F}^c)$. By [BLO2, Proposition 3.1], the obstruction to the existence of a centric linking system associated to $\mathcal{F}$ lies in $L^3(\mathcal{F};\mathcal{F}^c)$, and the obstruction to uniqueness lies in $L^2(\mathcal{F};\mathcal{F}^c)$.

For any $\mathcal{F}$ and any $\mathcal{F}$-invariant interval $\mathcal{R}$, $\mathcal{Z}_\mathcal{F}^\mathcal{R}$ is a quotient functor of $\mathcal{Z}_\mathcal{F}$ if $S \in \mathcal{R}$ (if $\mathcal{R}$ is closed under overgroups). If $\mathcal{R}_0 \subseteq \mathcal{R}$ are both $\mathcal{F}$-invariant intervals, and $P \in \mathcal{R}_0$ and $Q \in \mathcal{R}\smallsetminus\mathcal{R}_0$ implies $P \not\geq Q$, then $\mathcal{Z}_\mathcal{F}^{\mathcal{R}_0}$ is a subfunctor of $\mathcal{Z}_\mathcal{F}^\mathcal{R}$.

Lemma 1.6. *Fix a finite group Γ with Sylow subgroup $S \in \mathrm{Syl}_p(\Gamma)$, and set $\mathcal{F} = \mathcal{F}_S(\Gamma)$. Let $\mathcal{Q} \subseteq \mathcal{F}^c$ be an $\mathcal{F}$-invariant interval such that $S \in \mathcal{Q}$ (i.e., $\mathcal{Q}$ is closed under overgroups).*

(a) *Let $F\colon \mathcal{O}(\mathcal{F}^c)^{\mathrm{op}} \longrightarrow \mathsf{Ab}$ be a functor such that $F(P) = 0$ for each $P \in \mathcal{F}^c\smallsetminus\mathcal{Q}$. Let $\mathcal{O}(\mathcal{F}_\mathcal{Q}) \subseteq \mathcal{O}(\mathcal{F}^c)$ be the full subcategory with object set $\mathcal{Q}$. Then*

$$\varprojlim_{\mathcal{O}(\mathcal{F}^c)}{}^*(F) \cong \varprojlim_{\mathcal{O}(\mathcal{F}_\mathcal{Q})}{}^*(F|_{\mathcal{O}(\mathcal{F}_\mathcal{Q})})\,.$$

(b) *Assume $\mathcal{Q} = \mathscr{S}(S)_{\geq Y}$ for some p-subgroup $Y \trianglelefteq \Gamma$ such that $C_\Gamma(Y) \leq Y$. Then*

$$L^k(\mathcal{F};\mathcal{Q}) \overset{\mathrm{def}}{=} \varprojlim_{\mathcal{O}(\mathcal{F}^c)}{}^k(\mathcal{Z}_\mathcal{F}^\mathcal{Q}) \cong \begin{cases} Z(\Gamma) & \text{if } k=0 \\ 0 & \text{if } k>0. \end{cases}$$

Proof. **(a)** Set $\mathcal{C} = \mathcal{O}(\mathcal{F}^c)$ and $\mathcal{C}_0 = \mathcal{O}(\mathcal{F}_\mathcal{Q})$ for short. There is no morphism in $\mathcal{C}$ from any object of $\mathcal{C}_0$ to any object not in $\mathcal{C}_0$. Hence for any functor $F\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{Ab}$ such that $F(P) = 0$ for each $P \notin \mathrm{Ob}(\mathcal{C}_0)$, the two chain complexes $C^*(\mathcal{C};F)$ and $C^*(\mathcal{C}_0;F|_{\mathcal{C}_0})$ are isomorphic (see, e.g., [AKO, § III.5.1]). So $\varprojlim^*(F) \cong \varprojlim^*(F|_{\mathcal{C}_0})$ in this situation, and this proves (a). Alternatively, (a) follows upon showing that any $\mathcal{C}_0$-injective resolution of $F|_{\mathcal{C}_0}$ can be extended to an $\mathcal{C}$-injective resolution of F by assigning to all functors the value zero on objects not in $\mathcal{C}_0$.

(b) To simplify notation, set $\overline{H} = H/Y$ for each $H \in \mathscr{S}(\Gamma)_{\geq Y}$, and $\bar{g} = gY \in \bar{\Gamma}$ for each $g \in \Gamma$. Let $\mathcal{O}_{\bar{S}}(\bar{\Gamma})$ be the "orbit category" of $\bar{\Gamma}$: the category whose objects are the subgroups of $\bar{S}$, and where for $P,Q \in \mathcal{Q}$,

$$\mathrm{Mor}_{\mathcal{O}_{\bar{S}}(\bar{\Gamma})}(\bar{P},\bar{Q}) = \bar{Q}\backslash\{g \in \bar{\Gamma} \mid {}^g\bar{P} \leq \bar{Q}\}\,.$$

There is an isomorphism of categories $\Psi\colon \mathcal{O}(\mathcal{F}_\mathcal{Q}) \xrightarrow{\ \cong\ } \mathcal{O}_{\bar{S}}(\bar{\Gamma})$ which sends $P \in \mathcal{Q}$ to $\bar{P} = P/Y$ and sends $[c_g] \in \mathrm{Mor}_{\mathcal{O}(\mathcal{F}_\mathcal{Q})}(P,Q)$ to $\bar{Q}\bar{g}$. Then $\mathcal{Z}_\mathcal{F}^\mathcal{Q} \circ \Psi^{-1}$ sends $\bar{P}$ to $Z(P) = C_{Z(Y)}(\bar{P})$. Hence for $k \geq 0$,

$$\varprojlim_{\mathcal{O}(\mathcal{F}^c)}{}^k(\mathcal{Z}_\mathcal{F}^\mathcal{Q}) \cong \varprojlim_{\mathcal{O}(\mathcal{F}_\mathcal{Q})}{}^k(\mathcal{Z}_\mathcal{F}^\mathcal{Q}|_{\mathcal{O}(\mathcal{F}_\mathcal{Q})}) \cong \varprojlim_{\mathcal{O}_{\bar{S}}(\bar{\Gamma})}{}^k(\mathcal{Z}_\mathcal{F}^\mathcal{Q} \circ \Psi^{-1}) \cong \begin{cases} C_{Z(Y)}(\bar{\Gamma}) = Z(\Gamma) & \text{if } k=0 \\ 0 & \text{if } k>0 \end{cases}$$

where the first isomorphism holds by (a), and the last by a theorem of Jackowski and McClure [JM, Proposition 5.14]. We refer to [JMO, Proposition 5.2] for more details on the last isomorphism. □

More tools for working with these groups come from the long exact sequence of derived functors induced by a short exact sequence of functors.

Lemma 1.7. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. Let $\mathcal{Q}$ and $\mathcal{R}$ be $\mathcal{F}$-invariant intervals such that*

(i) $\mathcal{Q}\cap\mathcal{R}=\varnothing$,

(ii) $\mathcal{Q}\cup\mathcal{R}$ *is an interval, and*

(iii) $Q\in\mathcal{Q}$, $R\in\mathcal{R}$ *implies* $Q\not\leq R$.

Then $\mathcal{Z}_{\mathcal{F}}^{\mathcal{R}}$ is a subfunctor of $\mathcal{Z}_{\mathcal{F}}^{\mathcal{Q}\cup\mathcal{R}}$, $\mathcal{Z}_{\mathcal{F}}^{\mathcal{Q}\cup\mathcal{R}}/\mathcal{Z}_{\mathcal{F}}^{\mathcal{R}}\cong\mathcal{Z}_{\mathcal{F}}^{\mathcal{Q}}$, and there is a long exact sequence

$$0\longrightarrow L^0(\mathcal{F};\mathcal{R})\longrightarrow L^0(\mathcal{F};\mathcal{Q}\cup\mathcal{R})\longrightarrow L^0(\mathcal{F};\mathcal{Q})\longrightarrow\cdots$$
$$\longrightarrow L^{k-1}(\mathcal{F};\mathcal{Q})\longrightarrow L^k(\mathcal{F};\mathcal{R})\longrightarrow L^k(\mathcal{F};\mathcal{Q}\cup\mathcal{R})\longrightarrow L^k(\mathcal{F};\mathcal{Q})\longrightarrow\cdots.$$

In particular, the following hold.

(a) *If $L^k(\mathcal{F};\mathcal{R})\cong L^k(\mathcal{F};\mathcal{Q})=0$ for some $k\geq 0$, then $L^k(\mathcal{F};\mathcal{Q}\cup\mathcal{R})=0$.*

(b) *Assume $\mathcal{F}=\mathcal{F}_S(\Gamma)$, where $S\in\mathrm{Syl}_p(\Gamma)$, and there is a normal p-subgroup $Y\trianglelefteq\Gamma$ such that $C_\Gamma(Y)\leq Y$ and $\mathcal{Q}\cup\mathcal{R}=\mathscr{S}(S)_{\geq Y}$. Then for each $k\geq 2$, $L^{k-1}(\mathcal{F};\mathcal{Q})\cong L^k(\mathcal{F};\mathcal{R})$. Also, there is a short exact sequence*

$$1\longrightarrow C_{Z(Y)}(\Gamma)\longrightarrow C_{Z(Y)}(\Gamma^*)\longrightarrow L^1(\mathcal{F};\mathcal{R})\longrightarrow 1,$$

where $\Gamma^*=\langle g\in\Gamma\,|\,{}^gP\in\mathcal{Q}\text{ for some }P\in\mathcal{Q}\rangle$.

Proof. Condition (iii) implies that $\mathcal{Z}_{\mathcal{F}}^{\mathcal{R}}$ is a subfunctor of $\mathcal{Z}_{\mathcal{F}}^{\mathcal{Q}\cup\mathcal{R}}$, and it is then immediate from the definitions (and (i) and (ii)) that $\mathcal{Z}_{\mathcal{F}}^{\mathcal{Q}\cup\mathcal{R}}/\mathcal{Z}_{\mathcal{F}}^{\mathcal{R}}\cong\mathcal{Z}_{\mathcal{F}}^{\mathcal{Q}}$. The long exact sequence is induced by this short exact sequence of functors and the snake lemma. Point (a) now follows immediately.

Under the hypotheses in (b), by Lemma 1.6(b), $L^k(\mathcal{F};\mathcal{Q}\cup\mathcal{R})=0$ for $k>0$ and $L^0(\mathcal{F};\mathcal{Q}\cup\mathcal{R})\cong Z(\Gamma)=C_{Z(Y)}(\Gamma)$. The first statement in (b) thus follows immediately from the long exact sequence, and the second since $L^0(\mathcal{F};\mathcal{Q})\cong C_{Z(Y)}(\Gamma^*)$ (by definition of inverse limits). □

We next consider some tools for making computations in the groups $\varprojlim^*(-)$ for functors on orbit categories.

Definition 1.8. Fix a finite group G and a $\mathbb{Z}[G]$-module M. Let $\mathcal{O}_p(G)$ be the category whose objects are the p-subgroups of G, and where $\mathrm{Mor}_{\mathcal{O}_p(G)}(P,Q)=Q\backslash\{g\in G\,|\,{}^gP\leq Q\}$. Define a functor $F_M\colon\mathcal{O}_p(G)^{\mathrm{op}}\longrightarrow\mathsf{Ab}$ by setting

$$F_M(P)=\begin{cases}M & \text{if } P=1\\ 0 & \text{if } P\neq 1.\end{cases}$$

Here, $F_M(1)=M$ has the given action of $\mathrm{Aut}_{\mathcal{O}_p(G)}(1)=G$. Set

$$\Lambda^*(G;M)=\varprojlim_{\mathcal{O}_p(G)}{}^*(F_M).$$

These groups $\Lambda^*(G;M)$ provide a means of computing higher limits of functors on orbit categories which vanish except on one conjugacy class.

Proposition 1.9 ([BLO2, Proposition 3.2])**.** *Let $\mathcal{F}$ be a saturated fusion system over a p-group S. Let*

$$F\colon\mathcal{O}(\mathcal{F}^c)^{\mathrm{op}}\longrightarrow\mathbb{Z}_{(p)}\text{-}\mathsf{mod}$$

be any functor which vanishes except on the isomorphism class of some subgroup $Q \in \mathcal{F}^c$. Then

$$\varprojlim_{\mathcal{O}(\mathcal{F}^c)}{}^{*}(F) \cong \Lambda^*(\mathrm{Out}_{\mathcal{F}}(Q); F(Q)).$$

Upon combining Proposition 1.9 with the exact sequences of Lemma 1.7, we get the following corollary.

Corollary 1.10. *Let $\mathcal{F}$ be a saturated fusion system over a p-group S, and let $\mathcal{R} \subseteq \mathcal{F}^c$ be an $\mathcal{F}$-invariant interval. Assume, for some $k \geq 0$, that $\Lambda^k(\mathrm{Out}_{\mathcal{F}}(P); Z(P)) = 0$ for each $P \in \mathcal{R}$. Then $L^k(\mathcal{F}; \mathcal{R}) = 0$.*

What makes these groups $\Lambda^*(-;-)$ so useful is that they vanish in many cases, as described by the following proposition.

Proposition 1.11 ([JMO, Proposition 6.1(i,ii,iii,iv)])**.** *The following hold for each finite group G and each $\mathbb{Z}_{(p)}[G]$-module M.*

(a) *If $p \nmid |G|$, then* $\Lambda^i(G; M) = \begin{cases} M^G & \text{if } i = 0 \\ 0 & \text{if } i > 0. \end{cases}$

(b) *Let $H = C_G(M)$ be the kernel of the G-action on M. Then $\Lambda^*(G; M) \cong \Lambda^*(G/H; M)$ if $p \nmid |H|$, and $\Lambda^*(G; M) = 0$ if $p \big| |H|$.*

(c) *If $O_p(G) \neq 1$, then $\Lambda^*(G; M) = 0$.*

(d) *If $M_0 \leq M$ is a $\mathbb{Z}_{(p)}[G]$-submodule, then there is an exact sequence*

$$0 \longrightarrow \Lambda^0(G; M_0) \longrightarrow \Lambda^0(G; M) \longrightarrow \Lambda^0(G; M/M_0) \longrightarrow \cdots$$
$$\cdots \longrightarrow \Lambda^{n-1}(G; M/M_0) \longrightarrow \Lambda^n(G; M_0) \longrightarrow \Lambda^n(G; M) \longrightarrow \cdots.$$

The next lemma allows us in certain cases to replace the orbit category for one fusion system by that for a smaller one. For any saturated fusion system $\mathcal{F}$ over S and any $Q \leq S$, the *normalizer fusion system* $N_{\mathcal{F}}(Q)$ is defined as a fusion system over $N_S(Q)$ (cf. [AKO, Definition I.5.3]). If Q is fully normalized, then $N_{\mathcal{F}}(Q)$ is always saturated (cf. [AKO, Theorem I.5.5]).

Lemma 1.12. *Let $\mathcal{F}$ be a saturated fusion system over a p-group S, fix a subgroup $Q \in \mathcal{F}^c$ which is fully normalized in $\mathcal{F}$, and set $\mathcal{E} = N_{\mathcal{F}}(Q)$. Set $\mathcal{E}^\bullet = \mathcal{F}^c \cap \mathcal{E}^c$, a full subcategory of $\mathcal{E}^c$, and let $\mathcal{O}(\mathcal{E}^\bullet) \subseteq \mathcal{O}(\mathcal{E}^c)$ be its orbit category. Define*

$$\mathcal{T} = \bigl\{ P \leq S \bigm| Q \trianglelefteq P, \text{ and } R \in Q^{\mathcal{F}},\, R \trianglelefteq P \text{ implies } R = Q \bigr\}.$$

Let $F \colon \mathcal{O}(\mathcal{F}^c)^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{-}\mathsf{mod}$ be any functor which vanishes except on subgroups $\mathcal{F}$-conjugate to subgroups in $\mathcal{T}$, set $F_0 = F|_{\mathcal{O}(\mathcal{E}^\bullet)}$, and let $F_1 \colon \mathcal{O}(\mathcal{E}^c)^{\mathrm{op}} \longrightarrow \mathsf{Ab}$ be such that $F_1|_{\mathcal{O}(\mathcal{E}^\bullet)} = F_0$ and $F_1(P) = 0$ for all $P \in \mathcal{E}^c \smallsetminus \mathcal{E}^\bullet$. Then restriction to $\mathcal{E}^\bullet$ induces isomorphisms

$$\varprojlim_{\mathcal{O}(\mathcal{F}^c)}{}^{*}(F) \xrightarrow[\cong]{R} \varprojlim_{\mathcal{O}(\mathcal{E}^\bullet)}{}^{*}(F_0) \xleftarrow[\cong]{R_1} \varprojlim_{\mathcal{O}(\mathcal{E}^c)}{}^{*}(F_1). \tag{1}$$

Proof. Since R_1 is an isomorphism by Lemma 1.6(a), we only need to show that R is an isomorphism. If $F' \subseteq F$ is a pair of functors from $\mathcal{O}(\mathcal{F}^c)^{\mathrm{op}}$ to $\mathbb{Z}_{(p)}$-mod, and the lemma holds for F' and for F/F', then it also holds for F by the 5-lemma (and since R is natural with respect to functors on $\mathcal{O}(\mathcal{F}^c)^{\mathrm{op}}$). It thus suffices to prove that R is an isomorphism when F vanishes except on the $\mathcal{F}$-conjugacy class of one subgroup in $\mathcal{T}$.

Fix $P \in \mathcal{T}$, and assume $F(R) = 0$ for all $R \notin P^{\mathcal{F}}$. Then $Q \trianglelefteq P$ by definition of $\mathcal{T}$. If $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ is such that $Q \trianglelefteq \varphi(P)$, then $\varphi^{-1}(Q) \trianglelefteq P$, and $\varphi(Q) = Q$ since $P \in \mathcal{T}$. Thus $\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Out}_{\mathcal{E}}(P)$, and

$$P^{\mathcal{E}} = \{R \in P^{\mathcal{F}} \,|\, Q \trianglelefteq R\}\,. \tag{2}$$

Since $P^{\mathcal{E}} \subseteq \mathcal{T}$, we can assume P was chosen to be fully normalized in $\mathcal{E}$.

Let $F_2 \subseteq F_1$ be the subfunctor on $\mathcal{O}(\mathcal{E}^c)$ defined by setting $F_2(R) = F_1(R)$ if $R \not\geq Q$, and $F_2(R) = 0$ if $R \geq Q$. Then $P^{\mathcal{E}}$ contains all subgroups in $P^{\mathcal{F}}$ (hence all objects in $\mathcal{E}^c$) on which F_1/F_2 is nonvanishing. If $R \in \mathcal{E}^c$ and $R \not\geq Q$, then $O_p(\mathrm{Out}_{\mathcal{E}}(R)) \neq 1$ (R is not $\mathcal{E}$-radical) since $Q \trianglelefteq \mathcal{E}$ and $R \in \mathcal{E}^c$ (cf. [AKO, Proposition I.4.5(b)]), so $\Lambda^*(\mathrm{Out}_{\mathcal{E}}(R); F_1(R)) = 0$ by Proposition 1.11(c). Thus $\varprojlim^*(F_2) = 0$ by Corollary 1.10. Set $F_2^\bullet = F_2|_{\mathcal{O}(\mathcal{E}^\bullet)} \subseteq F_0$; then $\varprojlim^*(F_2^\bullet) \cong \varprojlim^*(F_2) = 0$ by Lemma 1.6(a), so $\varprojlim^*(F_0) \cong \varprojlim^*(F_0/F_2^\bullet)$. By (2), $(F_0/F_2^\bullet)(R) = 0$ for all $R \in \mathrm{Ob}(\mathcal{E}^\bullet) \smallsetminus P^{\mathcal{E}}$.

This yields the following diagram:

$$\begin{array}{ccccccc}
\varprojlim\limits_{\mathcal{O}(\mathcal{F}^c)}{}^*(F) & \xrightarrow{\;R\;} & \varprojlim\limits_{\mathcal{O}(\mathcal{E}^\bullet)}{}^*(F_0) & \xrightarrow[\cong]{\;R_2\;} & \varprojlim\limits_{\mathcal{O}(\mathcal{E}^\bullet)}{}^*(F_0/F_2^\bullet) & \xleftarrow[\cong]{\;R_3\;} & \varprojlim\limits_{\mathcal{O}(\mathcal{E}^c)}{}^*(F_1/F_2) \\
& \searrow^{\Phi^*}_{\cong} & & & & \swarrow^{\Phi_1^*}_{\cong} & \\
& & & \Lambda^*(\mathrm{Out}_{\mathcal{F}}(P); F(P)) & & &
\end{array}$$

where Φ^* and Φ_1^* are the isomorphisms of Proposition 1.9, where R_2 is induced by $(F_0 \twoheadrightarrow F_0/F_2^\bullet)$, and where R_3 is induced by restriction (and is an isomorphism by Lemma 1.6(a)).

Let $\mathcal{O}_{\mathrm{Out}_S(P)}(\mathrm{Out}_{\mathcal{F}}(P)) \subseteq \mathcal{O}_p(\mathrm{Out}_{\mathcal{F}}(P))$ be the full subcategory whose objects are the subgroups of $\mathrm{Out}_S(P) \in \mathrm{Syl}_p(\mathrm{Out}_{\mathcal{F}}(P))$. (Recall that P is fully normalized in $\mathcal{E}$ and $\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Out}_{\mathcal{E}}(P)$.) By the proof of [BLO2, Proposition 3.2], Φ^* and Φ_1^* are both induced by restriction via an embedding of $\mathcal{O}_{\mathrm{Out}_S(P)}(\mathrm{Out}_{\mathcal{F}}(P))$ into $\mathcal{O}(\mathcal{E}^\bullet)$: the embedding which sends $\mathrm{Out}_R(P)$ to R (for $P \leq R \leq N_S(P)$), and sends a morphism (the coset of some $\gamma \in \mathrm{Out}_{\mathcal{F}}(P)$) to the class of the appropriate extension of γ. Hence the above diagram commutes, and R is an isomorphism. $\square$

The following lemma can also be stated and proven as a result about extending automorphisms from a linking system to a group [Ch2, Lemma 4.15].

Lemma 1.13 ([Ch2, 4.15])**.** *Fix a pair of finite groups $H \trianglelefteq G$, together with $S \in \mathrm{Syl}_p(G)$ and $T = S \cap H \in \mathrm{Syl}_p(H)$. Set $\mathcal{F} = \mathcal{F}_S(G)$ and $\mathcal{E} = \mathcal{F}_T(H)$. Assume $Y \leq T$ is such that $Y \trianglelefteq G$ and $C_G(Y) \leq Y$. Let $\mathcal{Q}$ be an $\mathcal{F}$-invariant interval in $\mathscr{S}(S)_{\geq Y}$ such that $S \in \mathcal{Q}$, and such that $Q \in \mathcal{Q}$ implies $H \cap Q \in \mathcal{Q}$. Set $\mathcal{Q}_0 = \{Q \in \mathcal{Q} \,|\, Q \leq H\}$. Then restriction induces an injective homomorphism*

$$L^1(\mathcal{F}; \mathcal{Q}) \xrightarrow{\;\;R\;\;} L^1(\mathcal{E}; \mathcal{Q}_0).$$

Proof. Since $\mathcal{E}^c$ need not be contained in $\mathcal{F}^c$, we must first check that there is a well defined "restriction" homomorphism. Set $\mathcal{E}^\bullet = \mathcal{E}^c \cap \mathcal{F}^c$: a full subcategory of $\mathcal{E}^c$. Since the functor $\mathcal{Z}_{\mathcal{E}}^{\mathcal{Q}_0}$ vanishes on all subgroups in $\mathcal{E}^c$ not in $\mathcal{Q}_0 \subseteq \mathcal{E}^\bullet$, the higher limits are the same whether taken over $\mathcal{O}(\mathcal{E}^\bullet)$ or $\mathcal{O}(\mathcal{E}^c)$ (Lemma 1.6(a)). Thus R is defined as the restriction map to $\varprojlim^1(\mathcal{Z}_{\mathcal{E}}^{\mathcal{Q}_0}|_{\mathcal{E}^\bullet}) \cong L^1(\mathcal{E}; \mathcal{Q}_0)$.

We work with the bar resolutions for $\mathcal{O}(\mathcal{F}^c)$ and $\mathcal{O}(\mathcal{E}^\bullet)$, using the notation of [AKO, § III.5.1]. Fix a cocycle $\eta \in Z^1(\mathcal{O}(\mathcal{F}^c); \mathcal{Z}_\mathcal{F}^\mathcal{Q})$ such that $[\eta] \in \mathrm{Ker}(R)$. Thus η is a function from $\mathrm{Mor}(\mathcal{O}(\mathcal{F}^c))$ to $Z(Y)$ which sends the class $[\varphi]$ of $\varphi \in \mathrm{Hom}_G(P,Q)$ to an element of $Z(P)$ if $P \in \mathcal{Q}$, and to 1 if $P \notin \mathcal{Q}$. We can assume, after adding an appropriate coboundary, that $\eta\bigl(\mathrm{Mor}(\mathcal{O}(\mathcal{E}^\bullet))\bigr) = 1$.

Define $\widehat{\eta} \in Z^1(N_G(T)/T; Z(T))$ to be the restriction of η to $\mathrm{Aut}_{\mathcal{O}(\mathcal{F}^c)}(T) = N_G(T)/T$. For $g \in N_G(T)$, let $\bar{g}$ be its class in $N_G(T)/T$. Set $\gamma = \eta([\mathrm{incl}_T^S]) \in Z(T)$, so $d\gamma \in Z^1(N_G(T)/T; Z(T))$ is the cocycle $d\gamma(\bar{g}) = \gamma^g{\cdot}\gamma^{-1}$. For each $g \in S$, $[\mathrm{incl}_T^S]{\circ}[c_g] = [\mathrm{incl}_T^S]$ in $\mathcal{O}(\mathcal{F}^c)$, so $\gamma^g{\cdot}\eta([c_g]) = \gamma$, and thus $\widehat{\eta}(\bar{g}) = \eta([c_g]) = (d\gamma(\bar{g}))^{-1}$. In other words, $\widehat{\eta}|_{S/T}$ is a coboundary, and since $S/T \in \mathrm{Syl}_p(N_G(T)/T)$, $[\widehat{\eta}] = 1 \in H^1(N_G(T)/T; Z(T))$ (cf. [CE, Theorem XII.10.1]). Hence there is $\beta \in Z(T)$ such that $\widehat{\eta} = d\beta$. Since $\eta([c_h]) = 1$ for all $h \in N_H(T)$, $[\beta, h] = 1$ for all $h \in N_H(T)$, and thus $\beta \in Z(N_H(T))$.

Let $G^* \le G$ be the subgroup generated by all $g \in G$ such that for some $Q \in \mathcal{Q}$, ${}^gQ \in \mathcal{Q}$. Define $H^* \le H$ similarly. Since $S \le N_G(T) \le G^*$ and $N_H(T) \le H^*$, $S \in \mathrm{Syl}_p(G^*)$, $T \in \mathrm{Syl}_p(H^*)$, and $HG^* \ge HN_G(T) = G$ by the Frattini argument (Lemma 1.14(b)). If $g = ha$ where $h \in H$, $a \in N_G(T)$, and ${}^gQ \in \mathcal{Q}$ for some $Q \in \mathcal{Q}$, then ${}^a(Q \cap H)$ and ${}^g(Q \cap H) = {}^gQ \cap H$ are both in $\mathcal{Q}_0$, and thus $h \in H^*$. Since $N_G(T)$ normalizes H^*, this shows that $G^* = H^*N_G(T)$. So $G^* \cap H = (H^*N_G(T)) \cap H = H^*N_H(T) = H^*$. In particular, $H^* \trianglelefteq G^*$ and $G^*/H^* \cong G/H$.

For each $\varphi \in \mathrm{Hom}_H(P,Q)$ (where $Y \le P, Q \le T$), and each $g \in N_G(T)$, set ${}^g\varphi = c_g\varphi c_g^{-1} \in \mathrm{Hom}_H({}^gP, {}^gQ)$. Since $\eta([\varphi]) = \eta([{}^g\varphi]) = 1$, we have $\varphi^{-1}(\widehat{\eta}(\bar{g})) = \widehat{\eta}(\bar{g})$. Thus for each $g \in N_G(T)$, $\widehat{\eta}(\bar{g}) = \beta^g\beta^{-1}$ is invariant under the action of H^*; i.e., $\beta^g\beta^{-1} \in Z(H^*)$. So the class $[\beta] \in Z(N_H(T))/Z(H^*)$ is fixed under the action of $N_G(T)$ on this quotient.

Since $p\nmid[H^*{:}N_H(T)]$, and since $N_G(T)$ normalizes H^* and $N_H(T)$, the inclusion of $Z(H^*) = C_{Z(Y)}(H^*)$ into $Z(N_H(T)) = C_{Z(Y)}(N_H(T))$ is $N_G(T)$-equivariantly split by the trace homomorphism for the actions of $H^* \ge N_H(T)$ on $Z(Y)$. So the fixed subgroup for the $N_G(T)$-action on the quotient group $Z(N_H(T))/Z(H^*)$ is $Z(N_G(T))/Z(G^*)$. Thus $\beta \in Z(N_G(T))Z(H^*)$, and we can assume $\beta \in Z(H^*)$ without changing $d\beta = \widehat{\eta}$.

Define a 0-cochain $\widehat{\beta} \in C^0(\mathcal{O}(\mathcal{F}^c); \mathcal{Z}_\mathcal{F}^\mathcal{Q})$ by setting $\widehat{\beta}(P) = \beta$ if $P \in \mathcal{Q}_0$ and $\widehat{\beta}(P) = 1$ otherwise. Then $\eta([\varphi]) = d\widehat{\beta}([\varphi])$ for all $\varphi \in \mathrm{Mor}(\mathcal{E}^\bullet)$ (since both vanish) and also for all $\varphi \in \mathrm{Aut}_G(T)$. Since $G = HN_G(T)$, each morphism in $\mathcal{F}$ between subgroups of T is the composite of a morphism in $\mathcal{E}$ and the restriction of a morphism in $\mathrm{Aut}_\mathcal{F}(T)$. Hence $\eta([\varphi]) = d\widehat{\beta}([\varphi])$ for all such morphisms φ (since η and $d\widehat{\beta}$ are both cocycles). Upon replacing η by $\eta(d\widehat{\beta})^{-1}$, we can assume η vanishes on all morphisms in $\mathcal{F}$ between subgroups of T.

For each $P \in \mathcal{Q}$, set $P_0 = P \cap T$ and let $i_P \in \mathrm{Hom}_G(P_0, P)$ be the inclusion. Then $\eta([i_P]) \in Z(P_0)$ (and $\eta([i_P]) = 1$ if $P \notin \mathcal{Q}$). For each $g \in P$, the relation $[i_P] = [i_P] \circ [c_g]$ in $\mathcal{O}(\mathcal{F}^c)$ (where $[c_g] \in \mathrm{Aut}_{\mathcal{O}(\mathcal{F}^c)}(P_0)$) implies that $\eta([i_P])$ is c_g-invariant. Thus $\eta([i_P]) \in Z(P)$. Let $\rho \in C^0(\mathcal{O}(\mathcal{F}^c); \mathcal{Z}_\mathcal{F}^\mathcal{Q})$ be the 0-cochain $\rho(P) = \eta([i_P])$ when $P \in \mathcal{Q}$ and $\rho(P) = 1$ if $P \in \mathcal{F}^c{\smallsetminus}\mathcal{Q}$. Thus $\rho(P) = 1$ if $P \le T$ by the initial assumptions on η. Then $d\rho([i_P]) = \eta([i_P])$ for each P, and $d\rho(\varphi) = 1 = \eta(\varphi)$ for each φ between subgroups of T. For each $\varphi \in \mathrm{Hom}_G(P,Q)$ in $\mathcal{F}^c$, let $\varphi_0 \in \mathrm{Hom}_G(P_0, Q_0)$ be its restriction; the relation $[\varphi] \circ [i_Q] = [i_P] \circ [\varphi_0]$ in $\mathcal{O}(\mathcal{F}^c)$ implies that $\eta([\varphi]) = d\rho([\varphi])$ since this holds for $[\varphi_0]$ and the inclusions. Thus $\eta = d\rho$, and so $[\eta] = 1$ in $L^1(\mathcal{F}; \mathcal{Q})$. □

We end the section by recalling a few elementary results about finite groups.

Lemma 1.14. (a) *If $Q > P$ are p-groups for some prime p, then $N_Q(P) > P$.*

(b) (Frattini argument) *If $H \trianglelefteq G$ are finite groups and $T \in \mathrm{Syl}_p(H)$, then $G = HN_G(T)$.*

Proof. See, for example, [Sz1, Theorems 2.1.6 & 2.2.7]. □

As usual, for any finite p-group P, $\Omega_1(P) = \langle g \in P \mid g^p = 1 \rangle$.

Lemma 1.15. *Let G be a finite group such that $O_p(G) = 1$, and assume G acts faithfully on an abelian p-group D. Then G acts faithfully on $\Omega_1(D)$.*

Proof. The subgroup $C_G(\Omega_1(D))$ is a normal p-subgroup of G (cf. [G, Theorem 5.2.4]), and hence is contained in $O_p(G) = 1$. □

2. The Thompson subgroup and offenders

The proof of the main theorem is centered around the Thompson subgroup of a p-group, and the FF-offenders for an action of a group on an abelian p-group. We first fix the terminology and notation which will be used.

Definition 2.1. (a) For any p-group S, set $d(S) = \sup\bigl\{|A| \bigm| A \leq S \text{ abelian}\bigr\}$, let $\mathcal{A}(S)$ be the set of all abelian subgroups of S of order $d(S)$, and set $J(S) = \langle \mathcal{A}(S) \rangle$.

(b) Let G be a finite group which acts faithfully on the abelian p-group D. A *best offender* in G on D is an abelian subgroup $A \leq G$ such that $|A||C_D(A)| \geq |B||C_D(B)|$ for each $B \leq A$. (In particular, $|A||C_D(A)| \geq |D|$.) Let $\mathcal{A}_D(G)$ be the set of best offenders in G on D, and set $J_D(G) = \langle \mathcal{A}_D(G) \rangle$.

(c) Let Γ be a finite group, and let $D \trianglelefteq \Gamma$ be a normal abelian p-subgroup. Let $J(\Gamma, D) \leq \Gamma$ be the subgroup such that $J(\Gamma, D)/C_\Gamma(D) = J_D(\Gamma/C_\Gamma(D))$.

Note, in the situation of point (c) above, that

$$D \leq C_\Gamma(D) \leq J(\Gamma, D) \leq \Gamma \qquad \text{and} \qquad J(J(\Gamma, D), D) = J(\Gamma, D).$$

The relation between the Thompson subgroup $J(-)$ and best offenders is described by the next lemma and corollary.

Lemma 2.2. (a) *Assume G acts faithfully on a finite abelian p-group D. If A is a best offender in G on D, and U is an A-invariant subgroup of D, then $A/C_A(U)$ is a best offender in $N_G(U)/C_G(U)$ on U.*

(b) *Let S be a finite p-group, let $D \trianglelefteq S$ be a normal abelian subgroup, and set $G = S/C_S(D)$. Assume $A \in \mathcal{A}(S)$. Then the image of A in G is a best offender on D.*

Proof. We give here the standard proofs.

(a) Set $\bar{A} = A/C_A(U)$ for short. For each $\bar{B} = B/C_A(U) \le \bar{A}$,

$$|C_U(B)||C_D(A)| = |C_U(B)C_D(A)||C_U(B) \cap C_D(A)| \le |C_D(B)||C_U(A)|.$$

Also, $|B||C_D(B)| \le |A||C_D(A)|$ since A is a best offender on D, and hence

$$|\bar{B}||C_U(\bar{B})| = \frac{|B||C_U(B)|}{|C_A(U)|} \le \frac{|B||C_D(B)|}{|C_D(A)|} \cdot \frac{|C_U(A)|}{|C_A(U)|} \le |A| \cdot \frac{|C_U(A)|}{|C_A(U)|} = |\bar{A}||C_U(\bar{A})|.$$

Thus $\bar{A}$ is a best offender on U.

(b) Set $\bar{A} = A/C_A(D)$, identified with the image of A in G. Fix some $\bar{B} = B/C_A(D) \le \bar{A}$, and set $B^* = C_D(B)B$. This is an abelian group since D and B are abelian, and hence $|B^*| \le |A|$ since $A \in \mathcal{A}(S)$. Since $B \cap C_D(B) \le C_D(A)$,

$$|\bar{B}||C_D(\bar{B})| = \frac{|B||C_D(B)|}{|C_A(D)|} = \frac{|B^*||B \cap C_D(B)|}{|C_A(D)|} \le \frac{|A||C_D(A)|}{|C_A(D)|} = |\bar{A}||C_D(\bar{A})|.$$

Since this holds for all $\bar{B} \le \bar{A}$, $\bar{A}$ is a best offender on D. □

The following corollary reinterprets Lemma 2.2 in terms of the groups $J(\Gamma, D)$ defined above.

Corollary 2.3. *Let Γ be a finite group, and let $D \trianglelefteq \Gamma$ be a normal abelian p-subgroup.*

(a) *If $U \le D$ is also normal in Γ, then $J(\Gamma, U) \ge J(\Gamma, D)$.*

(b) *If Γ is a p-group, then $J(\Gamma) \le J(\Gamma, D)$.*

An action of a group G on a group D is *quadratic* if $[G, [G, D]] = 1$. If D is abelian and G acts faithfully, then a *quadratic best offender* in G on D is an abelian subgroup $A \le G$ which is a best offender and whose action is quadratic.

Lemma 2.4. *Let G be a finite group which acts faithfully on an elementary abelian p-group V. If the action of G on V is quadratic, then G is also an elementary abelian p-group.*

Proof. We write V additively for convenience; thus $[g, v] = gv - v$ for $g \in G$ and $v \in V$. By an easy calculation, and since the action is quadratic, $[gh, v] = [g, v] + [h, v]$ for each $g, h \in G$ and $v \in V$. Thus $g \mapsto (v \mapsto [g, v])$ is a homomorphism from G to the additive group $\mathrm{End}(V)$, and is injective since the action is faithful. Since $\mathrm{End}(V)$ is an elementary abelian p-group, so is G. □

We will also need the following form of Timmesfeld's replacement theorem.

Theorem 2.5. *Let $A \ne 1$ and $V \ne 1$ be abelian p-groups. Assume A acts faithfully on V and is a best offender on V. Then there is $1 \ne B \le A$ such that B is a quadratic best offender on V. More precisely, we can take $B = C_A([A, V]) \ne 1$, in which case $|A||C_V(A)| = |B||C_V(B)|$ and $C_V(B) = [A, V] + C_V(A) < V$.*

Proof. We follow the proof given by Chermak in [Ch1, § 1]. Set $m = |A||C_V(A)|$. Since A is a best offender,

$$m = \sup\{|B||C_U(B)| \mid B \le A,\ U \le V\}. \tag{1}$$

For each $U \leq V$, consider the set

$$\mathcal{M}_U = \bigl\{ B \leq A \bigm| |B||C_U(B)| = m \bigr\} .$$

By the maximality of m in (1),

$$B \in \mathcal{M}_U \quad \Longrightarrow \quad |C_U(B)| = |C_V(B)| \quad \Longrightarrow \quad C_V(B) \leq U . \tag{2}$$

Step 1: (Thompson's replacement theorem) For each $x \in V$, set

$$V_x = [A, x] \stackrel{\text{def}}{=} \bigl\langle [a, x] = ax - x \bigm| a \in A \bigr\rangle \qquad \text{and} \qquad A_x = C_A(V_x).$$

Note that V_x is A-invariant. We will show that

$$|A_x||C_V(A_x)| = |A||C_V(A)| = m \qquad \text{and} \qquad C_V(A_x) = V_x + C_V(A). \tag{3}$$

Define $\Phi\colon A \longrightarrow V_x$ (a map of sets) by setting $\Phi(a) = [a, x] = ax - x$ for each $a \in A$. We first claim that Φ induces an injective map of sets

$$\phi\colon A/A_x \longrightarrow V_x/C_{V_x}(A)$$

between these quotient groups. Since A is abelian, $[a, [b, x]] = abx - bx - ax + x = [b, [a, x]]$ for all $a, b \in A$. Hence for all $g, h \in A$,

$$\begin{aligned} \Phi(g) - \Phi(h) = gx - hx \in C_{V_x}(A) &\iff h([h^{-1}g, x]) \in C_{V_x}(A) \\ &\iff 1 = [A, [h^{-1}g, x]] = [h^{-1}g, [A, x]] = [h^{-1}g, V_x] \\ &\iff h^{-1}g \in C_A(V_x) = A_x . \end{aligned}$$

Thus ϕ is well defined and injective.

Now,

$$|V_x||C_V(A)| = |C_{V_x}(A)||V_x + C_V(A)| \leq |C_{V_x}(A)||C_V(A_x)|, \tag{4}$$

since $V_x \leq C_V(A_x)$ by definition of A_x. Together with the injectivity of ϕ, this implies that

$$\frac{|A|}{|A_x|} \leq \frac{|V_x|}{|C_{V_x}(A)|} \leq \frac{|C_V(A_x)|}{|C_V(A)|},$$

and so $m = |A||C_V(A)| \leq |A_x||C_V(A_x)|$. The opposite inequality holds by (1), so $A_x \in \mathcal{M}_V$ and the inequality in (4) is an equality. Thus $|V_x + C_V(A)| = |C_V(A_x)|$, finishing the proof of (3).

Step 2: Assume, for some $U \leq V$, that $B_0, B_1 \in \mathcal{M}_U$. Then $m = |B_0||C_U(B_0)| \geq |B_0B_1||C_U(B_0B_1)|$ by (1), and hence

$$\frac{|B_1|}{|B_0 \cap B_1|} = \frac{|B_0B_1|}{|B_0|} \leq \frac{|C_U(B_0)|}{|C_U(B_0B_1)|} = \frac{|C_U(B_0) + C_U(B_1)|}{|C_U(B_1)|} \leq \frac{|C_U(B_0 \cap B_1)|}{|C_U(B_1)|}.$$

So $m = |B_1||C_U(B_1)| \leq |B_0 \cap B_1||C_U(B_0 \cap B_1)|$ with equality by (1) again, and we conclude that $B_0 \cap B_1 \in \mathcal{M}_U$.

Step 3: Set $B = C_A([A, V])$ and $U = [A, V] + C_V(A)$. For each $x \in V$, (3) implies that $C_V(A_x) = [A, x] + C_V(A) \leq U$ and $A_x \in \mathcal{M}_U$. Hence $B = \bigcap_{x \in V} A_x \in \mathcal{M}_U$ by Step 2, so B is a best offender on V, and is quadratic since $[B, [A, V]] = 1$ by definition. Also, $C_V(B) \leq U$ by (2). Since $U = [A, V] + C_V(A) \leq C_V(B)$ by definition, we conclude that $U = C_V(B)$.

If $U = V$, then $V = [A, V] \oplus W$ is an A-invariant splitting for some $W \leq C_V(A)$. But this would imply $[A, V] = [A, [A, V]] + [A, W] = [A, [A, V]]$, which is impossible since $[A, X] < X$ for each finite nontrivial p-group X on which A acts. We conclude that $U = C_V(B) < V$, and hence that $B \neq 1$. □

We finish the section with two lemmas which were suggested to us by one of the referees, and which will be very useful in Section 4.

Lemma 2.6. *Let $A \neq 1$ be a finite abelian p-group, and let $V \neq 0$ be an $\mathbb{F}_p[A]$-module such that A acts freely on a basis of V and quadradically on V. Then $p = 2$ and $|A| = 2$. If A is a best offender on V, then $\mathrm{rk}(V) = 2$.*

Proof. Assume $|A| > 2$, and choose $g, h \in A$ such that $g \neq h$ and $g \neq 1 \neq h$. Let $\mathcal{B}$ be a basis permuted freely by A. For $b \in \mathcal{B}$, $(1-g)(1-h)b = b - gb - hb + ghb \neq 0$ since the elements b, gb, and hb are independent in V, contradicting the assumption that A acts quadratically. Thus $|A| = 2$ (and hence $p = 2$).

If A is a best offender on V, then $|A||C_V(A)| \geq |V|$, so $\mathrm{rk}(C_V(A)) = \mathrm{rk}(V) - 1$. But $\mathrm{rk}(C_V(A)) = \frac{1}{2}|\mathcal{B}| = \frac{1}{2}\mathrm{rk}(V)$ since $\mathcal{B}$ is permuted freely by A, so $\mathrm{rk}(V) = 2$. □

Lemma 2.7. *Let G be a nontrivial finite group, and let V be a faithful $\mathbb{F}_p[G]$-module. Fix p-subgroups $Q \trianglelefteq P \leq G$, where $Q < P$, and $|P/Q| \geq 4$ if $p = 2$. Assume $C_V(Q)$, with its induced action of P/Q, contains a copy of the free module $\mathbb{F}_p[P/Q]$. Then for each quadratic best offender $A \leq P$ on V, $A \leq Q$.*

Proof. Set $A_0 = A \cap Q$, and assume $A > A_0$. By assumption, there is an $\mathbb{F}_p[P/Q]$-submodule $1 \neq W \leq C_V(Q)$ with a basis on which P/Q acts freely. Thus $\mathrm{rk}(W) \geq |P/Q|$, $|P/Q| \geq 3$ by assumption, and A/A_0 permutes freely the basis. By Lemma 2.2(a), A/A_0 is a quadratic best offender on W. Hence by Lemma 2.6, $|A/A_0| = 2$, $p = 2$, and $\mathrm{rk}(W) = 2$, which is a contrdiction. Thus $A = A_0 \leq Q$. □

3. Proof of the main theorem

The following terminology will be very useful when carrying out the reduction procedures used in this section.

Definition 3.1 ([Ch2, 6.3])**.** A *general setup* is a triple (Γ, S, Y), where Γ is a finite group, $S \in \mathrm{Syl}_p(\Gamma)$, $Y \trianglelefteq \Gamma$ is a normal p-subgroup, and $C_\Gamma(Y) \leq Y$ (Y is centric in Γ). A *reduced setup* is a general setup (Γ, S, Y) such that $Y = O_p(\Gamma)$, $C_S(Z(Y)) = Y$, and $O_p(\Gamma/C_\Gamma(Z(Y))) = 1$.

The next proposition, which will be shown in Section 4, is the key technical result needed to prove the main theorem. Its proof uses the classification by Meierfrankenfeld and Stellmacher [MS] of FF-offenders, and through that depends on the classification of finite simple groups.

Proposition 3.2 (Compare [Ch2, 6.10])**.** *Let (Γ, S, Y) be a reduced setup, set $D = Z(Y)$, and assume $\Gamma/C_\Gamma(D)$ is generated by quadratic best offenders on D. Set $\mathcal{F} = \mathcal{F}_S(\Gamma)$, and let $\mathcal{R} \subseteq \mathcal{F}^c$ be the set of all $R \geq Y$ such that $J(R, D) = Y$. Then $L^2(\mathcal{F}; \mathcal{R}) = 0$ if $p = 2$, and $L^1(\mathcal{F}; \mathcal{R}) = 0$ if p is odd.*

Since this distinction between the cases where $p = 2$ or where p is odd occurs throughout this section and the next, it will be convenient to define

$$k(p) = \begin{cases} 2 & \text{if } p = 2 \\ 1 & \text{if } p \text{ is an odd prime.} \end{cases}$$

Thus under the hypotheses of Proposition 3.2, we claim that $L^{k(p)}(\mathcal{F};\mathcal{R}) = 0$.

Proposition 3.2 seems very restricted in scope, but it can be generalized to the following situation.

Proposition 3.3 (Compare [Ch2, 6.11]). *Let (Γ, S, Y) be a general setup. Set $\mathcal{F} = \mathcal{F}_S(\Gamma)$ and $D = Z(Y)$. Let $\mathcal{R} \subseteq \mathscr{S}(S)_{\geq Y}$ be an $\mathcal{F}$-invariant interval such that for each $Q \in \mathscr{S}(S)_{\geq Y}$, $Q \in \mathcal{R}$ if and only if $J(Q, D) \in \mathcal{R}$. Then $L^k(\mathcal{F};\mathcal{R}) = 0$ for all $k \geq k(p)$.*

Proof. Assume the proposition is false. Let $(\Gamma, S, Y, \mathcal{R}, k)$ be a counterexample for which the 4-tuple $(k, |\Gamma|, |\Gamma/Y|, |\mathcal{R}|)$ is the smallest possible under the lexicographical ordering.

We will show in Step 1 that $\mathcal{R} = \{P \leq S \,|\, J(P, D) = Y\}$, in Step 2 that $k = k(p)$, in Step 3 that (Γ, S, Y) is a reduced setup, and in Step 4 that $\Gamma/C_\Gamma(D)$ is generated by quadratic best offenders on D. The result then follows from Proposition 3.2.

Step 1: Let $R_0 \in \mathcal{R}$ be a minimal element of $\mathcal{R}$ which is fully normalized in $\mathcal{F}$. Since $J(R_0, D) \in \mathcal{R}$ by assumption (and $J(R_0, D) \leq R_0$), $J(R_0, D) = R_0$. Let $\mathcal{R}_0$ be the set of all $R \in \mathcal{R}$ such that $J(R, D)$ is $\mathcal{F}$-conjugate to R_0, and set $\mathcal{Q}_0 = \mathcal{R} \smallsetminus \mathcal{R}_0$. Then $\mathcal{R}_0$ and $\mathcal{Q}_0$ are both $\mathcal{F}$-invariant intervals, and satisfy the conditions $Q \in \mathcal{R}_0$ ($Q \in \mathcal{Q}_0$) if and only if $J(Q, D) \in \mathcal{R}_0$ ($J(Q, D) \in \mathcal{Q}_0$). Since $L^k(\mathcal{F};\mathcal{R}) \neq 0$, Lemma 1.7(a) implies $L^k(\mathcal{F};\mathcal{R}_0) \neq 0$ or $L^k(\mathcal{F};\mathcal{Q}_0) \neq 0$. Hence $\mathcal{Q}_0 = \varnothing$ and $\mathcal{R} = \mathcal{R}_0$ by the minimality assumption on $|\mathcal{R}|$ (and since $\mathcal{R}_0 \neq \varnothing$).

Set $\Gamma_1 = N_\Gamma(R_0)$, $S_1 = N_S(R_0)$, $\mathcal{F}_1 = N_\mathcal{F}(R_0) = \mathcal{F}_{S_1}(\Gamma_1)$ (see [AKO, Proposition I.5.4]), and $\mathcal{R}_1 = \{R \in \mathcal{R} \,|\, J(R, D) = R_0\}$. Since R_0 is fully normalized, each subgroup in $\mathcal{R}$ is $\mathcal{F}$-conjugate to a subgroup in $\mathcal{R}_1$ (Lemma 1.2). Also, for $P \in \mathcal{R}_1$, if $R \in R_0{}^\mathcal{F}$ and $R \trianglelefteq P$, then $J(P, D) \geq J(R, D) = R$ implies $R = R_0$. The hypotheses of Lemma 1.12 are thus satisfied, and so $L^k(\mathcal{F}_1;\mathcal{R}_1) \cong L^k(\mathcal{F};\mathcal{R}) \neq 0$. For $R \in \mathscr{S}(S_1)_{\geq Y}$, $R \in \mathcal{R}_1$ if and only if $J(R, D) = R_0$. Thus $(\Gamma_1, S_1, Y, \mathcal{R}_1, k)$ is another a counterexample to the proposition. By the minimality assumption, $\Gamma_1 = \Gamma$, and thus $R_0 \trianglelefteq \Gamma$.

We have now shown that there is a p-subgroup $R_0 \trianglelefteq \Gamma$ such that $\mathcal{R} = \{R \leq S \,|\, J(R, D) = R_0\}$. Set $Y_1 = R_0 \geq Y$ and $D_1 = Z(Y_1) \leq D$. For each $R \leq S$ such that $R \geq R_0$ and $R \notin \mathcal{R}$, $J(R, D_1) \geq J(R, D)$ by Corollary 2.3(a), $J(R, D) \notin \mathcal{R}$ by the remarks just after Definition 2.1, and hence $J(R, D_1) \notin \mathcal{R}$. Thus $(\Gamma, S, Y_1, \mathcal{R}, k)$ is a counterexample to the proposition, and so $Y = Y_1 = R_0$ by the minimality assumption on $|\Gamma/Y|$. We conclude that $\mathcal{R} = \{R \leq S \,|\, J(R, D) = Y\}$.

Step 2: Let $\mathcal{Q}$ be the set of all overgroups of Y in S which are not in $\mathcal{R}$. Equivalently, $\mathcal{Q} = \{Q \leq S \,|\, J(Q, D) > Y\}$. If $k \geq 2$, then $L^{k-1}(\mathcal{F};\mathcal{Q}) \cong L^k(\mathcal{F};\mathcal{R}) \neq 0$ by Lemma 1.7(b). Since k was assumed to be the smallest degree $\geq k(p)$ for which the proposition is not true, we conclude that $k = k(p)$.

Step 3: Assume (Γ, S, Y) is not a reduced setup. Let $K \trianglelefteq \Gamma$ be such that $K \geq C_\Gamma(D)$ and $K/C_\Gamma(D) = O_p(\Gamma/C_\Gamma(D))$, and set $Y_2 = S \cap K \trianglelefteq S$. Then $Y_2 > Y$, since either $Y_2 \geq O_p(\Gamma) > Y$, or $Y_2 \geq C_S(D) > Y$, or $p\big||K/C_\Gamma(D)|$ and hence $Y_2 > C_S(D) \geq Y$. Set $\Gamma_2 = N_\Gamma(Y_2)$, and set $\mathcal{R}_2 = \{P \in \mathcal{R} \,|\, P \geq Y_2\}$. Note that $S \in \mathrm{Syl}_p(\Gamma_2)$, and also that $\mathcal{R}_2$ is an $\mathcal{F}$-invariant interval since Y_2 is strongly closed in S with respect to Γ. Set $\mathcal{F}_2 = \mathcal{F}_S(\Gamma_2) = N_\mathcal{F}(Y_2)$ [AKO, Proposition I.5.4].

Assume $P \in \mathcal{R}\smallsetminus\mathcal{R}_2$. Then $P \not\geq Y_2$, so $PY_2 > P$, and hence $N_{PY_2}(P) > P$ (Lemma 1.14(a)). Set $G = \mathrm{Out}_\Gamma(P)$ and $G_0 = \mathrm{Out}_K(P)$. Then $G_0 \trianglelefteq G$ since $K \trianglelefteq \Gamma$, and $C_{G_0}(Z(P)) = \mathrm{Out}_{C_K(Z(P))}(P) \geq \mathrm{Out}_{C_\Gamma(D)}(P)$ since $K \geq C_\Gamma(D)$ and $Z(P) \leq Z(Y) = D$. Hence $G_0/C_{G_0}(Z(P))$ is a p-group since $K/C_\Gamma(D)$ is a p-group. For any $g \in N_{PY_2}(P)\smallsetminus P$, $\mathrm{Id} \neq [c_g] \in \mathrm{Out}_K(P) = G_0$ since $Y_2 \leq K$ (and since $C_\Gamma(P) \leq C_\Gamma(Y) \leq Y \leq P$). Thus $\mathrm{Out}_K(P) = G_0 \trianglelefteq G$ contains a nontrivial element of p-power order, and its action on $Z(P)$ factors through the p-group $G_0/C_{G_0}(Z(P))$. Proposition 1.11(b,c) now implies that $\Lambda^*(\mathrm{Out}_\Gamma(P); Z(P)) = 0$.

Since this holds for all $P \in \mathcal{R}\smallsetminus\mathcal{R}_2$, $L^*(\mathcal{F}; \mathcal{R}\smallsetminus\mathcal{R}_2) = 0$ by Corollary 1.10. Hence $L^*(\mathcal{F}; \mathcal{R}_2) \cong L^*(\mathcal{F}; \mathcal{R})$ by the exact sequence in Lemma 1.7. Also, the hypotheses of Lemma 1.12 hold for the functor $\mathcal{Z}_\mathcal{F}^{\mathcal{R}_2}$ on $\mathcal{O}(\mathcal{F}^c)$ (with $Q = Y_2$) since Y_2 is strongly closed. So $L^*(\mathcal{F}; \mathcal{R}_2) \cong L^*(\mathcal{F}_2; \mathcal{R}_2)$. Since $L^k(\mathcal{F}; \mathcal{R}) \neq 0$ by assumption, $L^k(\mathcal{F}_2; \mathcal{R}_2) \neq 0$.

Set $D_2 = Z(Y_2) \leq D$. For each $P \in \mathscr{S}(S)_{\geq Y_2}$,

$$P \geq J(P, D_2) \geq J(P, D) \geq C_P(D) \geq Y \tag{1}$$

by Corollary 2.3(a) and by definition of $J(P, -)$. We must show that for all $P \geq Y_2$, $P \in \mathcal{R}_2$ if and only if $J(P, D_2) \in \mathcal{R}_2$. If $P \in \mathcal{R}_2 \subseteq \mathcal{R}$, then $J(P, D) \in \mathcal{R}$ by assumption, so $J(P, D_2) \in \mathcal{R}$ by (1) since $\mathcal{R}$ is an interval, and $J(P, D_2) \in \mathcal{R}_2$ since $J(P, D_2) \geq C_P(D_2) \geq Y_2$. If $P \notin \mathcal{R}_2$, then $P \notin \mathcal{R}$, so $J(P, D) \notin \mathcal{R}$, and $J(P, D_2) \notin \mathcal{R}$ (hence $J(P, D_2) \notin \mathcal{R}_2$) by (1) again and since $\mathcal{R}$ is an interval containing Y.

Thus $(\Gamma_2, S, Y_2, \mathcal{R}_2, k)$ is a counterexample to the proposition. So $\Gamma_2 = \Gamma$ and $Y_2 = Y$ by the minimality assumption, which contradicts the above claim that $Y_2 > Y$. We conclude that (Γ, S, Y) is a reduced setup.

Step 4: It remains to prove that $\Gamma/C_\Gamma(D)$ is generated by quadratic best offenders on D; the result then follows from Proposition 3.2.

Let $\Gamma_3 \trianglelefteq \Gamma$ be such that $\Gamma_3 \geq C_\Gamma(D)$ and $\Gamma_3/C_\Gamma(D)$ is generated by all quadratic best offenders on D. If $\Gamma_3 = \Gamma$ we are done, so assume $\Gamma_3 < \Gamma$. Set $S_3 = \Gamma_3 \cap S$ and $\mathcal{F}_3 = \mathcal{F}_{S_3}(\Gamma_3)$. Set

$$\mathcal{Q} = \mathscr{S}(S)_{\geq Y}\smallsetminus\mathcal{R}, \quad \mathcal{Q}_3 = \mathcal{Q} \cap \mathscr{S}(S_3)_{\geq Y}, \quad \text{and} \quad \mathcal{R}_3 = \mathcal{R} \cap \mathscr{S}(S_3)_{\geq Y}.$$

Since $L^k(\mathcal{F}; \mathcal{R}) \neq 0$, $\mathcal{R} \subsetneqq \mathscr{S}(S)_{\geq Y}$ by Lemma 1.6(b), and $\mathcal{Q} \neq \varnothing$. The proposition holds for $(\Gamma_3, S_3, Y, \mathcal{R}_3, k)$ by the minimality assumption, and thus $L^k(\mathcal{F}_3; \mathcal{R}_3) = 0$.

For $Q \in \mathcal{Q}$, $J(Q, D) > Y$, so Q/Y has nontrivial best offenders on D, hence has nontrivial quadratic best offenders on D by Theorem 2.5, and thus $J(Q\cap\Gamma_3, D) > Y$. So $Q \in \mathcal{Q}$ implies $Q \cap \Gamma_3 \in \mathcal{Q}_3$ by Step 1. In particular, $S_3 \in \mathcal{Q}_3$.

If $k = 2$ (i.e., if $p = 2$), then $L^1(\mathcal{F}; \mathcal{Q}) \cong L^2(\mathcal{F}; \mathcal{R}) \neq 0$ and $L^1(\mathcal{F}_3; \mathcal{Q}_3) \cong L^2(\mathcal{F}_3; \mathcal{R}_3) = 0$ by Lemma 1.7(b), which is impossible by Lemma 1.13.

If $k = 1$ (if p is odd), set

$$\Gamma^* = \big\langle g \in \Gamma \,\big|\, {}^gP \in \mathcal{Q} \text{ for some } P \in \mathcal{Q} \big\rangle \le \Gamma$$
$$\Gamma_3^* = \big\langle g \in \Gamma_3 \,\big|\, {}^gP \in \mathcal{Q}_3 \text{ for some } P \in \mathcal{Q}_3 \big\rangle \le \Gamma_3\,.$$

Then $\Gamma_3^* \le \Gamma^*$ since $\Gamma_3 \le \Gamma$ and $\mathcal{Q}_3 \subseteq \mathcal{Q}$. By Lemma 1.7(b), there are exact sequences

$$\begin{aligned} 1 \longrightarrow C_{Z(Y)}(\Gamma) \longrightarrow C_{Z(Y)}(\Gamma^*) \longrightarrow L^1(\mathcal{F};\mathcal{R}) \ne 1 \\ 1 \longrightarrow C_{Z(Y)}(\Gamma_3) \longrightarrow C_{Z(Y)}(\Gamma_3^*) \longrightarrow L^1(\mathcal{F}_3;\mathcal{R}_3) = 1. \end{aligned} \tag{2}$$

Also, $\Gamma^*\Gamma_3 \ge N_\Gamma(S_3)\Gamma_3 = \Gamma$ since $S_3 \in \mathcal{Q}_3$, where the equality follows from the Frattini argument (Lemma 1.14(b)), so

$$C_{Z(Y)}(\Gamma) = C_{Z(Y)}(\Gamma^*\Gamma_3) = C_{Z(Y)}(\Gamma^*) \cap C_{Z(Y)}(\Gamma_3).$$

But this is impossible, since $C_{Z(Y)}(\Gamma) < C_{Z(Y)}(\Gamma^*) \le C_{Z(Y)}(\Gamma_3^*) = C_{Z(Y)}(\Gamma_3)$ by the exactness in (2). □

We now have the tools needed to prove the main vanishing result.

Theorem 3.4. *For each saturated fusion system $\mathcal{F}$ over a p-group S, $\varprojlim^k_{\mathcal{Q}(\mathcal{F}^c)}(\mathcal{Z}_\mathcal{F}) = 0$ for all $k \ge 2$, and for all $k \ge 1$ if p is odd.*

Proof. As in [Ch2, § 6], we choose inductively subgroups $X_0, X_1, \ldots, X_N \in \mathcal{F}^c$ and $\mathcal{F}$-invariant intervals $\varnothing = \mathcal{Q}_{-1} \subseteq \mathcal{Q}_0 \subseteq \cdots \subseteq \mathcal{Q}_N = \mathcal{F}^c$ as follows. Assume $\mathcal{Q}_{n-1}$ has been defined ($n \ge 0$), and $\mathcal{Q}_{n-1} \subsetneqq \mathcal{F}^c$. Consider the sets of subgroups

$$\begin{aligned} \mathcal{U}_1 &= \mathcal{U}_1^{(n)} = \big\{ P \in \mathcal{F}^c \smallsetminus \mathcal{Q}_{n-1} \,\big|\, d(P) \text{ maximal} \big\} \\ \mathcal{U}_2 &= \mathcal{U}_2^{(n)} = \big\{ P \in \mathcal{U}_1 \,\big|\, |J(P)| \text{ maximal} \big\} \\ \mathcal{U}_3 &= \mathcal{U}_3^{(n)} = \big\{ P \in \mathcal{U}_2 \,\big|\, J(P) \in \mathcal{F}^c \big\} \\ \mathcal{U}_4 &= \mathcal{U}_4^{(n)} = \begin{cases} \big\{ P \in \mathcal{U}_3 \,\big|\, |P| \text{ minimal} \big\} & \text{if } \mathcal{U}_3 \ne \varnothing \\ \big\{ P \in \mathcal{U}_2 \,\big|\, |P| \text{ maximal} \big\} & \text{otherwise.} \end{cases} \end{aligned}$$

(See Definition 2.1(a) for the definition of $d(P)$.) Let X_n be any subgroup in $\mathcal{U}_4$ such that X_n and $J(X_n)$ are both fully normalized in $\mathcal{F}$.

We first check that there is such an X_n. For each $X \in \mathcal{U}_4$ and each $Y \in J(X)^\mathcal{F}$ which is fully normalized in $\mathcal{F}$, there is $\varphi \in \mathrm{Hom}_\mathcal{F}(N_S(J(X)), N_S(Y))$ such that $\varphi(J(X)) = Y$ (Lemma 1.2), and $\varphi(X)$ is also fully normalized since $N_S(X) \le N_S(J(X))$. Since $\mathcal{U}_4$ is invariant under $\mathcal{F}$-conjugacy, this shows that $X_n \in \mathcal{U}_4$ can be chosen as required.

Let $\mathcal{Q}_n$ be the union of $\mathcal{Q}_{n-1}$ with the set of all overgroups of subgroups $\mathcal{F}$-conjugate to X_n. Set $\mathcal{R}_n = \mathcal{Q}_n \smallsetminus \mathcal{Q}_{n-1}$ for each $0 \le n \le N$. Thus the sets $\mathcal{Q}_n$ are all closed under overgroups, and the $\mathcal{R}_n$ are intervals. By definition of $\mathcal{U}_4$, $X_n = J(X_n)$ if $J(X_n) \in \mathcal{F}^c$, while $\mathcal{R}_n = {X_n}^\mathcal{F}$ if $J(X_n) \notin \mathcal{F}^c$. Note also that $X_0 = J(S)$ and $\mathcal{R}_0 = \mathcal{Q}_0 = \mathscr{S}(S)_{\ge J(S)}$.

We will show, for each n, that

$$L^k(\mathcal{F};\mathcal{R}_n) = 0 \text{ for all } k \ge k(p). \tag{3}$$

Then by Lemma 1.7(a), for all $k \ge k(p)$, $L^k(\mathcal{F};\mathcal{Q}_{n-1}) = 0$ implies $L^k(\mathcal{F};\mathcal{Q}_n) = 0$. The theorem now follows by induction on n.

Case 1: Assume n is such that $J(X_n) \notin \mathcal{F}^c$, and hence that $\mathcal{R}_n = X_n{}^{\mathcal{F}}$. Since $J(X_n)$ is fully normalized and not $\mathcal{F}$-centric, $C_S(J(X_n)) \nleq J(X_n)$. Then $X_nC_S(J(X_n)) > X_n$, since $J(X_n)$ *is* centric in X_n. Hence $N_{X_nC_S(J(X_n))}(X_n) > X_n$ by Lemma 1.14(a), so there is $g \in N_S(X_n) \smallsetminus X_n$ such that $[g, J(X_n)] = 1$. Then g acts trivially on $Z(X_n) \le J(X_n)$, so the kernel of the $\mathrm{Out}_{\mathcal{F}}(X_n)$-action on $Z(X_n)$ has order a multiple of p, and $\Lambda^*(\mathrm{Out}_{\mathcal{F}}(X_n); Z(X_n)) = 0$ by Proposition 1.11(b). Hence (3) holds by Proposition 1.9.

Case 2: Assume n is such that $J(X_n) \in \mathcal{F}^c$, and hence $X_n = J(X_n)$ by definition of $\mathcal{U}_4$. By definition of $\mathcal{U}_1$ and $\mathcal{U}_2$, for each $P \ge X_n$ in $\mathcal{R}_n$, $d(P) = d(X_n)$ and $J(P) = X_n$. Hence

$$P \in \mathcal{R}_n \implies J(P) \text{ is the unique subgroup of } P\ \mathcal{F}\text{-conjugate to } X_n. \tag{4}$$

Set $T = N_S(X_n)$ and $\mathcal{E} = N_{\mathcal{F}}(X_n)$. Then $\mathcal{E}$ is a saturated fusion system over T (cf. [AKO, Theorem I.5.5]), and contains X_n as normal centric subgroup. Hence there is a *model* for $\mathcal{E}$ (cf. [AKO, Theorem III.5.10]): a finite group Γ such that $T \in \mathrm{Syl}_p(\Gamma)$, $X_n \trianglelefteq \Gamma$, $C_\Gamma(X_n) \le X_n$, and $\mathcal{F}_T(\Gamma) = \mathcal{E}$.

Let $\mathcal{R}$ be the set of all $P \in \mathcal{R}_n$ such that $P \ge X_n$. Then (Γ, T, X_n) is a general setup, and $\mathcal{R}$ is an $\mathcal{E}$-invariant interval containing X_n. If $P \in \mathcal{R}$ and $Y \le P$ is $\mathcal{F}$-conjugate to X_n, then $Y = X_n$ by (4). Also, each subgroup in $\mathcal{R}_n$ is $\mathcal{F}$-conjugate to a subgroup in $\mathcal{R}$ by (4) and Lemma 1.2 (recall X_n is fully normalized). The hypotheses of Lemma 1.12 thus hold, and hence

$$L^*(\mathcal{F}; \mathcal{R}_n) \cong L^*(\mathcal{E}; \mathcal{R}). \tag{5}$$

Set $D = Z(X_n)$. We claim that for each $P \in \mathscr{S}(T)_{\ge X_n}$,

$$P \in \mathcal{R} \iff J(P, D) \in \mathcal{R}. \tag{6}$$

Fix such a P. By Corollary 2.3(b), $J(P, D) \ge J(P)$, and $X_n \ge J(X_n, D) \ge J(X_n) = X_n$. If $P \in \mathcal{R}$, then $J(P, D) \in \mathcal{R}$ since $X_n = J(X_n, D) \le J(P, D) \le P$ and $\mathcal{R}$ is an interval. If $P \notin \mathcal{R}$, then $P \in \mathcal{R}_i$ for some $0 \le i \le n-1$. By definition of $\mathcal{U}_1^{(i)}$ and $\mathcal{U}_2^{(i)}$, either $d(P) = d(X_i) > d(X_n)$, or $d(P) = d(X_i) = d(X_n)$ and $J(P) > J(X_n) = X_n$, or $J(P) = X_n \in J(X_i)^{\mathcal{F}}$. The latter is not possible since by definition of $\mathcal{U}_4^{(i)}$, either $J(X_i) = X_i$ or $J(X_i) \notin \mathcal{F}^c$. If $d(P) > d(X_n)$, then $d(J(P, D)) = d(P) > d(X_n)$ since $J(P) \le J(P, D) \le P$, and $J(P, D) \notin \mathcal{R}$ since $d(R) = d(X_n)$ for all $R \in \mathcal{R}$. If $J(P) > X_n$, then $J(P) \notin \mathcal{R}$ since $J(R) = X_n$ for all $R \in \mathcal{R}$, and hence $J(P, D) \notin \mathcal{R}$ since $J(P, D) \ge J(P)$ and $\mathcal{R}$ is an interval. This proves (6).

Thus by Proposition 3.3, $L^k(\mathcal{E}; \mathcal{R}) = 0$ for all $k \ge k(p)$. Together with (5), this finishes the proof of (3), and hence of the theorem. □

4. Proof of Proposition 3.2

It remains to prove Proposition 3.2, which we restate here as:

Proposition 4.1. *Let (Γ, S, Y) be a reduced setup, set $D = Z(Y)$, and assume $\Gamma/C_\Gamma(D)$ is generated by quadratic best offenders on D. Set $\mathcal{F} = \mathcal{F}_S(\Gamma)$, and let $\mathcal{R} \subseteq \mathcal{F}^c$ be the set of all $R \ge Y$ such that $J(R, D) = Y$. Then $L^{k(p)}(\mathcal{F}; \mathcal{R}) = 0$.*

It is in this section that we use the classification of offenders by Meierfrankenfeld and Stellmacher [MS], and through that the classification of finite simple groups. The following theorem is a summary of those parts of [MS, Theorems 1 & 2] which we need here. The complete results in [MS] give a much more precise description of all representations of groups containing elementary abelian best offenders.

We adopt the notation in [GLS3], and let $\mathfrak{Lie}(p)$ denote the class of groups $G = O^{p'}(C_{\overline{G}}(\sigma))$, where $\overline{G}$ is a connected, (quasi-)simple algebraic group over $\bar{\mathbb{F}}_p$, and $\sigma \in \mathrm{End}(\overline{G})$ is an algebraic endomorphism with finite fixed subgroup. Most of these groups are quasisimple, with a few exceptions such as $SL_2(2)$, $SL_2(3)$, and $G_2(2)$. Note that $SO^{\pm}_{2m}(2^k) \notin \mathfrak{Lie}(2)$ $(m \geq 3)$, since $SO_{2m}(\bar{\mathbb{F}}_2)$ is not connected.

When $G \cong A_n$ or Σ_n, the "natural module for G" in characteristic two is the simple $\mathbb{F}_2[G]$-module of rank $n-1$ (n odd) or $n-2$ (n even) which is a subquotient of the permutation module of rank n.

A *$\mathcal{CK}$-group* is a finite group all of whose composition factors are known simple groups.

Theorem 4.2. *Fix a finite $\mathcal{CK}$-group G such that $O_p(G) = 1$, and a faithful finite dimensional $\mathbb{F}_p[G]$-module V. Assume G is generated by elementary abelian p-groups which are best offenders on V. Let $\mathcal{J}$ be the set of all subgroups $1 \neq K \trianglelefteq G$ which are minimal with the property that $[K, G] = K$. Set $W = [\mathcal{J}, V]C_V(\mathcal{J})/C_V(\mathcal{J})$. Then*

(a) $O^p(G) = \langle \mathcal{J} \rangle = \times \mathcal{J}$*;*

(b) *W is a faithful, semisimple $\mathbb{F}_p[G]$-module; and*

(c) *each elementary abelian best offender on V is a best offender on W.*

(d) *If W is a simple $\mathbb{F}_p[G]$-module, then either*
   (1) *$G \in \mathfrak{Lie}(p)$ (possibly one of the non-quasisimple groups $G \cong SL_2(2)$, $SL_2(3)$, $Sp_4(2)$, or $G_2(2)$); or*
   (2) *$G \cong SO^{\pm}_{2m}(2^k)$, where $p = 2$, $m \geq 3$, and V is the natural module for G; or*
   (3) *$G \cong 3A_6$ or A_7 and $p = 2$; or*
   (4) *$G \cong A_n$ ($n \geq 6$, n even) or Σ_n ($n \geq 3$, $n \neq 4$), $p = 2$, and V is the natural module for G.*

Proof. Points (a)–(c) are points (c)–(e) in [MS, Theorem 1], while (d) follows from [MS, Theorem 2] (which gives a much more explicit list). Note that we have dropped the group $SO^-_4(2) \cong \Sigma_5$ from point (d.2), since its natural module is isomorphic to that of Σ_5 (case (d.4)). □

When $H_1 \leq H_2 \leq \cdots \leq H_k$ are subgroups of a group G, we let $N_G(H_1, \ldots, H_k)$ denote the intersection of their normalizers.

Definition 4.3. Let G be a finite group.

(a) A *radical p-subgroup* of G is a p-subgroup $P \leq G$ such that $O_p(N_G(P)) = P$; i.e., $O_p(N_G(P)/P) = 1$.

(b) A *radical p-chain of length k* in G is a sequence of p-subgroups $P_0 < P_1 < \cdots < P_k \le G$ such that P_0 is radical in G, P_i is radical in $N_G(P_0,\ldots,P_{i-1})$ for each $i \ge 1$, and $P_k \in \mathrm{Syl}_p(N_G(P_0,\ldots,P_{k-1}))$.

The reason for defining this here is the following vanishing result, which involves only radical p-chains with $P_0 = 1$.

Proposition 4.4 ([AKO, Lemma III.5.27] and [O2, Proposition 3.5])**.** *Fix a finite group G, a finite $\mathbb{F}_p[G]$-module M, and $k \ge 1$ such that $\Lambda^k(G;M) \ne 0$. Then there is a radical p-chain $1 = P_0 < P_1 < \cdots < P_k$ of length k such that M contains a copy of the free module $\mathbb{F}_p[P_k]$.*

Since the trivial subgroup is a radical p-subgroup of G only if $O_p(G) = 1$, Proposition 4.4 includes the statement that $\Lambda^k(G;M) = 0$ if $O_p(G) \ne 1$ (Proposition 1.11(c)). The reason for defining radical p-chains more generally here — to also allow chains where $P_0 \ne 1$ — will be seen in the proof of the next proposition and in that of Proposition 4.1.

Proposition 4.5. *Let G be a nontrivial finite group with $O_p(G) = 1$, and let V be a faithful $\mathbb{F}_p[G]$-module. Let $\mathcal{U}$ be the set of quadratic best offenders in G on V, and assume $G = \langle \mathcal{U} \rangle$. Set $G_0 = O^p(G)$ and $W = C_V(G_0)[G_0,V]/C_V(G_0)$. Assume, for some p-subgroup $P_0 \le G$, some $\mathbb{F}_p[N_G(P_0)/P_0]$-submodule $X \le C_W(P_0)$, and some $k \ge k(p)$, that $\Lambda^k(N_G(P_0)/P_0;X) \ne 0$. Then each $U \in \mathcal{U}$ is G-conjugate to a subgroup of P_0.*

Proof. Quadratic offenders with faithful action on V are elementary abelian by Lemma 2.4. So Theorem 4.2 ([MS, Theorems 1 & 2]) applies. Then $O^p(G) = \langle \mathcal{J} \rangle$ by Theorem 4.2(a), and hence W as defined here is the same as W defined in that theorem.

Case 1: Assume V is a simple $\mathbb{F}_p[G]$-module. Thus $V = W$. Set $H_0 = C_{N_G(P_0)}(C_W(P_0))$. Thus $P_0 \trianglelefteq H_0$, and $p \nmid |H_0/P_0|$ by Proposition 1.11(b). By Proposition 1.11(c), P_0 is radical in G. By Proposition 4.4, there is a radical p-chain $1 < R_1/H_0 < \cdots < R_k/H_0$ of length k in $N_G(P_0)/H_0$ such that X, and hence $C_W(P_0)$, contains a copy of $\mathbb{F}_p[R_k/H_0]$. Hence by Lemma A.4 (applied with $H_0/P_0 \trianglelefteq N_G(P_0)/P_0$ in the role of $H \trianglelefteq G$), there is a radical p-chain $1 < P_1/P_0 < \cdots < P_k/P_0$ in $N_G(P_0)/P_0$ such that $P_iH_0 = R_i$ for each i. Then $P_0 < P_1 < \cdots < P_k$ is a radical p-chain in G. Also, $P_k/P_0 \cong R_k/H_0$, so $C_{W_i}(P_0)$ contains a copy of $\mathbb{F}_p[P_k/P_0]$.

By Lemma 4.6, $P_k \in \mathrm{Syl}_p(G)$. Hence each quadratic best offender in G on W is G-conjugate to some $U \le P_k$. By Lemma 2.7 (and since $C_W(P_0)$ contains a copy of $\mathbb{F}_p[P_k/P_0]$), $U \le P_0$.

Case 2: Now assume V is arbitrary. By Theorem 4.2(b,c), W is a semisimple $\mathbb{F}_p[G]$-module, and each $U \in \mathcal{U}$ is a quadratic best offender on W. Set $W = W_1 \oplus \cdots \oplus W_m$, where each W_i is a simple $\mathbb{F}_p[G]$-module. For $0 \le i \le m$, set $X_i = X \cap (W_1 \oplus \cdots \oplus W_i)$. Thus $0 = X_0 \le X_1 \le \cdots \le X_m = X$ are $\mathbb{F}_p[N_G(P_0)/P_0]$-submodules, and X_i/X_{i-1} is isomorphic to a submodule of $C_{W_i}(P_0)$ for each $i \ge 1$.

Since $\Lambda^k(N_G(P_0)/P_0;X) \ne 0$, the exact sequences for the pairs $X_{i-1} \le X_i$ (Proposition 1.11(d)) imply that $\Lambda^k(N_G(P_0)/P_0;X_i/X_{i-1}) \ne 0$ for some $1 \le i \le m$. Set

$K = C_G(W_i)$, $\bar{G} = G/K$, and $\bar{H} = HK/K \le \bar{G}$ for each $H \le G$. The action of $N_G(P_0)/P_0$ on $C_{W_i}(P_0)$ factors through

$$N_{\bar{G}}(\bar{P}_0)/\bar{P}_0 = N_{G/K}(P_0K/K)/(P_0K/K) \cong N_G(P_0K)/P_0K\,,$$

and $N_G(P_0)/P_0$ surjects onto $N_{\bar{G}}(\bar{P}_0)/\bar{P}_0$ with kernel $N_{P_0K}(P_0)/P_0$. By Proposition 1.11(b), $p \nmid |N_{P_0K}(P_0)/P_0|$ and $\Lambda^k(N_{\bar{G}}(\bar{P}_0)/\bar{P}_0; X_i/X_{i-1}) \ne 0$. By Lemma 1.14(a), $P_0 \in \mathrm{Syl}_p(P_0K)$.

Since $G = \langle \mathcal{U} \rangle$ where $\mathcal{U}$ is the set of quadratic best offenders on W, $\bar{G} = \langle \bar{\mathcal{U}} \rangle$ where $\bar{\mathcal{U}} = \{\bar{U} \,|\, U \in \mathcal{U}\}$ is a set of quadratic best offenders on W_i by Lemma 2.2(a). By assumption, W_i is a faithful, simple $\mathbb{F}_p[\bar{G}]$-module. So by Case 1, each $\bar{U} \in \bar{\mathcal{U}}$ is $\bar{G}$-conjugate to a subgroup of $\bar{P}_0 = P_0K/K$. Hence each $U \in \mathcal{U}$ is G-conjugate to a subgroup of $P_0 \in \mathrm{Syl}_p(P_0K)$. □

The following lemma was needed to prove Proposition 4.5. This is where the explicit list in Theorem 4.2 was needed.

Lemma 4.6. *Let G be a nontrivial finite group, let W be a faithful, simple $\mathbb{F}_p[G]$-module, and assume G is generated by its quadratic best offenders on W. Let $P_0 < P_1 < \cdots < P_k$ be a radical p-chain in G with $k \ge k(p)$. Set*

$$H_0 = C_{N_G(P_0)}(C_W(P_0))\,,$$

and assume also that $p \nmid |H_0/P_0|$ and that $1 < P_1H_0/H_0 < \cdots < P_kH_0/H_0$ is a radical p-chain in $N_G(P_0)/H_0$. Then either

(a) *$C_W(P_0)$, with its induced action of $N_G(P_0)/P_0$, does not contain a copy of the free module $\mathbb{F}_p[P_k/P_0]$; or*

(b) *$P_k \in \mathrm{Syl}_p(G)$.*

Proof. Quadratic offenders with faithful action are elementary abelian by Lemma 2.4. Also, $O_p(G) = 1$, since $C_W(O_p(G))$ is a nontrivial $\mathbb{F}_p[G]$-submodule of W and G acts faithfully. So we are in the situation of Theorem 4.2(d). The cases listed there will be considered individually.

Assume $C_W(P_0)$ does contain a copy of $\mathbb{F}_2[P_k/P_0]$. We must show that $P_k \in \mathrm{Syl}_p(G)$. Note that if $p = 2$, then $\mathrm{rk}(C_W(P_0)) \ge |P_k/P_0| \ge 4$ since $k \ge k(2) = 2$.

Case 1: Assume $G \in \mathfrak{Lie}(p)$. The nontrivial radical p-subgroups of G are well known: by a theorem of Borel and Tits (see [GLS3, Corollary 3.1.5]), they are all conjugate to maximal normal unipotent subgroups in parabolic subgroups. Hence the normalizers $N_G(P_0, \ldots, P_i)$ all contain Sylow p-subgroups of G, and the quotients $N_G(P_0, \ldots, P_i)/P_i$ (the Levi complements) are central products of groups in $\mathfrak{Lie}(p)$ (see [GLS3, Theorem 2.6.5(f)]). Since $P_k \in \mathrm{Syl}_p(N_G(P_0, \ldots, P_{k-1}))$, $P_k \in \mathrm{Syl}_p(G)$ in this case.

Case 2: Now assume $p = 2$ and $G \cong SO^{\pm}_{2m}(q)$, where $2m \ge 6$, $q = 2^a$ ($a \ge 1$), and W is the natural $\mathbb{F}_2[G]$-module of rank $2am$. Set $G_0 = \Omega^{\pm}_{2m}(q)$, so $[G{:}G_0] = 2$.

For any radical 2-subgroup $P \le G$, $P \cap G_0$ is a radical 2-subgroup of G_0 by Lemma A.2, and hence is either trivial, or is a maximal normal unipotent subgroup in a parabolic subgroup. If $P \cap G_0 = 1$ and $P \ne 1$, then $P = \langle t \rangle$ for some involution $t \in SO^{\pm}_{2m}(q) \smallsetminus \Omega^{\pm}_{2m}(q)$. Set $W_1 = C_W(t)$ and $W_2 = [t, W] \le W_1$. Then $W_1 \perp W_2$, so the quadratic form $\mathfrak{q}$ on W is linear on W_2 with $W_3 \overset{\text{def}}{=} \text{Ker}(\mathfrak{q}|_{W_2}) \le W_2$ of index at most 2. If $W_3 \ne 0$, then by Witt's theorem (cf. [Ta, Theorem 7.4]), each $\alpha \in \text{Aut}_{\mathbb{F}_q}(W_1)$ which induces the identity on W_2 and on W_1/W_3 extends to some $\bar{\alpha} \in G$, then $\bar{\alpha} \in O_2(N_G(P))$, so P is not radical. Thus $W_3 = 0$, $\text{rk}(W_2) = 1$, and t is a transvection. By Witt's theorem again, restriction to W_1 induces an isomorphism $N_G(P)/P \cong SO_{2m-1}(q) \cong Sp_{2m-2}(q)$.

Assume first that $P_0 = 1$. If $P_1 \cap G_0 = 1$, then P_1 is generated by a transvection, so P_1 is a quadratic best offender, which contradicts Lemma 2.7. Thus $P_1 \cap G_0$ is a maximal normal unipotent subgroup of a parabolic subgroup. So $|P_1| \ge q^{2m-3}$ by Lemma A.5, $|P_2| \ge q^{2m-2}$, and $q^{2m-2} > \text{rk}_{\mathbb{F}_2}(W) = 2am$ since $m \ge 3$. Hence this case is impossible.

Next assume $P_0 \ne 1$ and $P_0 \cap G_0 = 1$. As noted above, P_0 is generated by a transvection (hence $\text{rk}_{\mathbb{F}_q}(C_W(P_0)) = 2m-1$), and $N_G(P_0)/P_0 \cong Sp_{2m-2}(q)$. By Case 1 (applied with $P_0 = 1$), $P_k/P_0 \in \text{Syl}_2(N_G(P_0)/P_0)$. Hence $|P_k/P_0| = q^{(m-1)^2} \le 2m-1$ (cf. [Ta, p. 70]), which is impossible since $m \ge 3$. (Alternatively, $N_G(P_0) \smallsetminus P_0$ and hence $P_k \smallsetminus P_0$ contains a transvection, which contradicts Lemma 2.7.)

Finally, assume $P_0 \cap G_0 \ne 1$, and hence is a maximal normal unipotent subgroup of a parabolic subgroup. Set $W_0 = C_W(P_0) < W$. Then $W_0 \le C_W(P_0 \cap G_0)$ is a totally isotropic subspace of W of rank at most m, and $N_G(P_0)/P_0 \cong GL(W_0) \times N_{SO(W_0^\perp/W_0)}(P_0)/P_0$ acts on it via projection to the first factor. Thus $N_G(P_0)/H_0 \cong GL(W_0)$, and $P_k/P_0 \cong P_kH_0/H_0 \in \text{Syl}_2(N_G(P_0)/H_0)$ by Case 1 (applied with $G = SL(W_0)$ and $P_0 = 1$). Set $r = \text{rk}(W_0)$; then $|P_k/P_0| = 2^{r(r-1)/2} \le r$ implies $r \le 2$, which contradicts the above observation that $r \ge 4$.

Case 3: Assume $p = 2$, and $G \cong 3A_6$ or A_7. Then the Sylow 2-subgroups of G have order 8, the nontrivial radical 2-subgroups have order 4 or 8, hence are normal in Sylow 2-subgroups, and thus $P_k \in \text{Syl}_2(G)$ $(k \ge 2)$.

Case 4: Assume $p = 2$, $G \cong \Sigma_m$ or A_m, and W is a natural module for G. Set $\boldsymbol{m} = \{1, 2, \dots, m\}$, with the canonical action of G. Set $V = \mathbb{F}_2(\boldsymbol{m})$: the $\mathbb{F}_2$-vector space with basis $\boldsymbol{m}$, and G-action induced by that on $\boldsymbol{m}$. Set $\Delta = C_V(G)$: the subgroup generated by the sum of all elements in $\boldsymbol{m}$. Identify $W = V/\Delta$ if m is odd (so $\text{rk}(W) = m-1$), and $W \le V/\Delta$ with index two if m is even (so $\text{rk}(W) = m-2$). Since $\text{rk}(W) \ge 4$, $m \ge 5$.

For any $H \le G$, let $\boldsymbol{m}/H$ be the set of orbits of H acting on $\boldsymbol{m}$ (with induced action of $N_G(H)/H$), and let $\mathbb{F}_2(\boldsymbol{m}/H)$ be the permutation module with basis $\boldsymbol{m}/H$. Since $C_V(H)$ is the group of elements of $V = \mathbb{F}_2(\boldsymbol{m})$ whose coefficients are constant on each H-orbit, we can identify $C_V(H)$ with $\mathbb{F}_2(\boldsymbol{m}/H)$ as $\mathbb{F}_2[N_G(H)/H]$-modules.

If P_0 acts on $\boldsymbol{m}$ with more than one orbit, then $C_{V/\Delta}(P_0) = C_V(P_0)/\Delta$ by Lemma A.8(a). Thus $C_W(P_0) \le C_V(P_0)/\Delta$, so $C_V(P_0)$ also contains a copy of the free module $\mathbb{F}_2[P_k/P_0]$, and its basis $\boldsymbol{m}/P_0$ contains a free (P_k/P_0)-orbit by Lemma A.1. Since $k \ge 2$, $\boldsymbol{m}/P_0$ contains $|P_k/P_1| \ge 2$ free (P_1/P_0)-orbits, which contradicts Lemma A.7(i).

Now assume P_0 acts transitively on $\boldsymbol{m}$. Let $U \le V$ be such that $U/\Delta = C_{V/\Delta}(P_0) \ge C_W(P_0)$. Each $g \in N_G(P_0)$ normalizes U/Δ, so $N_G(P_0) \le N_G(U)$, and $P_0 < \cdots < P_k$ is also a radical p-chain in $N_G(U)$. Also, U/Δ contains a copy of $\mathbb{F}_2[P_k/P_0]$, and in particular, $P_k \cap C_G(U) = P_0 \cap C_G(U)$. By Lemma A.3, applied with $C_G(U) \trianglelefteq N_G(U)$ in the role of $H \trianglelefteq G$, $C_{P_0}(U) \in \mathrm{Syl}_p(C_G(U))$.

By Lemma A.8(b), $N_G(P_0)/H_0 \cong GL(U/\Delta)$ with the canonical action on $U/\Delta = C_{V/\Delta}(P_0)$. Hence $C_W(P_0) = U/\Delta$. By Case 1 (applied with $G = SL(C_W(P_0))$ and $P_0 = 1$), $P_k/P_0 \cong P_kH_0/H_0 \in \mathrm{Syl}_2(N_G(P_0)/H_0)$. Set $r = \mathrm{rk}(C_W(P_0))$. Then $|P_k/P_0| = 2^{r(r-1)/2} \le r$ since $C_W(P_0)$ contains a copy of $\mathbb{F}_2[P_k/P_0]$. Thus $r = 2$ and $|P_k/P_0| = 2$, which is impossible since $k \ge 2$. □

Proof of Proposition 4.1. Fix a reduced setup (Γ, S, Y), set $D = Z(Y)$, $V = \Omega_1(D)$, and $G = \Gamma/C_\Gamma(D)$, and assume $G = \langle \mathcal{U} \rangle$ where

$$\mathcal{U} = \{1 \ne P \le G \mid P \text{ a quadratic best offender on } D\}.$$

Since $O_p(G) = 1$ by definition of a reduced setup, G acts faithfully on V by Lemma 1.15. Hence $\mathcal{U}$ is a set of quadratic best offenders on V by Lemma 2.2(a).

Recall that $\mathcal{R} = \{P \in \mathcal{F}^c \mid J(P, D) = Y\}$. By Timmesfeld's replacement theorem (Theorem 2.5), $\mathcal{R}$ is the set of all $P \in \mathscr{S}(S)_{\ge Y}$ such that $P/Y = P/C_S(D)$ contains no nontrivial quadratic best offender on D; i.e., no subgroups in $\mathcal{U}$.

Set $D_0 = 1$. For each $i \ge 1$, set $D_i = \Omega_i(D) = \{g \in D \mid g^{p^i} = 1\}$ and $V_i = D_i/D_{i-1}$. Thus each V_i is an $\mathbb{F}_p[G]$-module, and $(x \mapsto x^p)$ sends V_i injectively to V_{i-1} for each $i > 0$.

Set $k = k(p)$. We will show that $\Lambda^k(\mathrm{Out}_\Gamma(R); Z(R)) = 0$ for each $R \in \mathcal{R}$; the proposition then follows from Corollary 1.10. Here, $Z(R) = C_D(R)$ and $\mathrm{Out}_\Gamma(R) \cong N_\Gamma(R)/R$ since $R \ge Y$ and $C_S(Y) = Z(Y) = D$. So by Proposition 1.11(d), it suffices to show, for each R and i, that $\Lambda^k\bigl(N_\Gamma(R)/R; C_{D_i}(R)/C_{D_{i-1}}(R)\bigr) = 0$. Also, for each i, $C_{D_i}(R)/C_{D_{i-1}}(R)$ can be identified with an $N_\Gamma(R)$-invariant subgroup of $C_{V_i}(R) \le C_V(R)$. It thus suffices to show that

$$\Lambda^k\bigl(N_\Gamma(R)/R; X\bigr) = 0 \quad \forall\, R \in \mathcal{R},\ \forall\, N_\Gamma(R)\text{-invariant } X \le C_V(R). \tag{1}$$

Set $W_1 = C_V(O^p(G))$, $W_2 = W_1[O^p(G), V]$, and $W = W_2/W_1$. Thus the G-actions on W_1 and on V/W_2 factor through the quotient p-group $G/O^p(G)$. So by Proposition 1.11(a,b,c), for each $R \in \mathcal{R}$ and each $X \le C_V(R)$ as in (1),

$$\Lambda^k(N_\Gamma(R)/R; X \cap W_1) = 0 \qquad \text{and} \qquad \Lambda^k(N_\Gamma(R)/R; X/(X \cap W_2)) = 0.$$

By the exact sequences of Proposition 1.11(d), we are now reduced to showing that $\Lambda^k(N_\Gamma(R)/R; (X \cap W_2)/(X \cap W_1)) = 0$ for all such X; or more generally that

$$\Lambda^k\bigl(N_\Gamma(R)/R; X\bigr) = 0 \quad \forall\, R \in \mathcal{R},\ \forall\, N_\Gamma(R)\text{-invariant } X \le C_W(R). \tag{2}$$

Since $R \in \mathcal{R}$, no $U \in \mathcal{U}$ is contained in R, and hence (2) follows from Proposition 4.5. □

Appendix A. Radical p-chains and free submodules

We collect here some lemmas needed in the proofs in Section 4.

Lemma A.1. *Let P be a p-group, and let V be an $\mathbb{F}_p[P]$-module. Assume V has an $\mathbb{F}_p$-basis $\mathcal{B}$ which is permuted by P, and also contains a copy of the free module $\mathbb{F}_p[P]$. Then $\mathcal{B}$ contains a free P-orbit.*

Proof. Write $V = V_1 \oplus \cdots \oplus V_n$, where each V_i has as basis one P-orbit $\mathcal{B}_i \subseteq \mathcal{B}$. For each i, let $\mathrm{pr}_i \colon V \to V_i$ be the projection.

Let $F \le V$ be a submodule isomorphic to $\mathbb{F}_p[P]$. Then $C_F(P) \cong \mathbb{F}_p$. Choose i such that $C_F(P) \nleq \mathrm{Ker}(\mathrm{pr}_i)$. Then $\mathrm{Ker}(\mathrm{pr}_i|_F) = 0$ (otherwise it contains nontrivial elements fixed by P), so pr_i sends F injectively into V_i. Thus $|P| \le \mathrm{rk}(V_i) = |\mathcal{B}_i|$, so $\mathcal{B}_i$ is a free orbit. □

Lemma A.2. *Assume $H \trianglelefteq G$ are finite groups, and let $P \le G$ be a radical p-subgroup. Then $P \cap H$ is a radical p-subgroup of H.*

Proof. Set $Q = O_p(N_H(P \cap H))$. Then $N_G(P)$ normalizes Q, so $N_{QP}(P) \trianglelefteq N_G(P)$. Hence $N_{QP}(P) \le O_p(N_G(P)) = P$, so $Q \le P$ by Lemma 1.14(a), and $P \cap H$ is radical in H. □

The next two lemmas are useful when manipulating radical p-chains. The first was suggested by one of the referees.

Lemma A.3. *Fix a finite group G and a normal subgroup $H \trianglelefteq G$. Let $P_0 < P_1 < \cdots < P_k$ be a radical p-chain in G such that $P_0 \cap H = P_k \cap H$. Then $P_0 \cap H \in \mathrm{Syl}_p(H)$.*

Proof. Choose $S \in \mathrm{Syl}_p(G)$ such that $S \ge P_k$, and set $Q = S \cap H \in \mathrm{Syl}_p(H)$. Then $Q \ge P_k \cap H = P_0 \cap H$. We must show that $Q = P_0 \cap H$.

Assume otherwise: assume $Q > P_0 \cap H$. By construction, Q is normalized by S and hence by each of the P_i. Set $Q = Q_{-1}$, and define recursively $Q_i = Q_{i-1} \cap N_{P_iQ_{i-1}}(P_i)$ for each $0 \le i \le k$. Since P_k normalizes Q and each of the P_i, it normalizes each Q_i (so the P_iQ_{i-1} are subgroups of G). By Lemma 1.14(a), if $Q_{i-1} > P_0 \cap H = P_i \cap H$, then $N_{P_iQ_{i-1}}(P_i) > P_i$ so that $Q_i > P_i \cap H = P_0 \cap H$. Thus $P_kQ_k > P_k$, $Q_k \le N_G(P_0, P_1, \ldots, P_{k-1})$, and this is impossible since $P_k \in \mathrm{Syl}_p(N_G(P_0, P_1, \ldots, P_{k-1}))$ by definition of a radical p-chain. □

Lemma A.4. *Let G be a finite group, and let $H \trianglelefteq G$ be a normal subgroup of order prime to p. Set $\overline{G} = G/H$, and set $\overline{X} = XH/H$ for each $X \le G$.*

(a) *If $P \le G$ is a p-subgroup such that $\overline{P}$ is radical in $\overline{G}$, then P is radical in G.*

(b) *If $1 < R_1/H < \cdots < R_k/H$ is a radical p-chain in $\overline{G}$, then there is a radical p-chain $1 < P_1 < \cdots < P_k$ in G such that $\overline{P}_i = R_i/H$ for each $i \le k$.*

Proof. For each p-subgroup $P \le G$, the Frattini argument (Lemma 1.14(b)), applied with $P \le PH \trianglelefteq N_G(PH)$ in the role of $T \le H \trianglelefteq G$, implies that $N_G(PH) = PH \cdot (N_{N_G(PH)}(P)) = HN_G(P)$.

(a) By assumption, $PH/H = O_p(N_G(PH)/H) = O_p(N_G(P)H/H)$. Set $Q = O_p(N_G(P))$; then $QH/H \le PH/H$, and $Q = P$ since $p \nmid |H|$. Thus P is radical in G.

(b) Choose $P_k \in \mathrm{Syl}_p(R_k)$, and set $P_i = R_i \cap P_k \in \mathrm{Syl}_p(R_i)$ for each $i < k$ (recall $R_i \trianglelefteq R_k$). Thus $P_iH = R_i$ for each $i \le k$ (hence $\overline{P}_i = R_i/H$), since R_i/H is a p-group and $p \nmid |H|$.

Since $N_G(P_iH) = N_G(P_i)H$ for each i, $N_G(R_1, \ldots, R_i) = N_G(P_1, \ldots, P_i)H$. Since $\overline{P}_i$ is radical in $N_{\overline{G}}(\overline{P}_1, \ldots, \overline{P}_{i-1})$ for each i, P_i is radical in $N_G(P_1, \ldots, P_{i-1})$ by (a). Also, $\bar{P}_k \in \mathrm{Syl}_p(N_G(R_1, \dots, R_{k-1})/H)$, so $P_k \in \mathrm{Syl}_p(N_G(P_1, \dots, P_{k-1}))$. Thus $1 < P_1 < \cdots < P_k$ is a radical p-chain in G. $\square$

We need the following lower bounds for orders of radical subgroups of $\Omega_{2m}^{\pm}(q)$.

Lemma A.5. *Let P be a radical 2-subgroup of $G = \Omega_{2m}^{\pm}(q)$, where $m \geq 2$ and $q = 2^a$. Then $|P| \geq q^{2m-3}$, and $|P| \geq q^{2m-2}$ if $m \geq 4$.*

Proof. By a theorem of Borel and Tits (see [GLS3, Corollary 3.1.5]), P is conjugate to the maximal normal unipotent subgroup in a parabolic subgroup $N < G$. We can assume that P is minimal, and hence that N is a maximal parabolic subgroup. Thus $P = O_2(N)$, N contains a Borel subgroup and hence a Sylow 2-subgroup of G, and N is the stabilizer of a totally isotropic subspace of dimension ℓ for some $1 \leq \ell \leq m$ (see [GL, pp. 100–101]). Let $L \cong N/O_2(N)$ be a Levi factor for N; then $O^{2'}(L) \cong SL_\ell(q) \times \Omega_{2m-2\ell}^{\pm}(q)$ ([GLS3, Example 3.2.3]).

Thus G has Sylow 2-subgroups of order $q^{m(m-1)}$, while $N/O_2(N)$ has Sylow 2-subgroups of order $q^{\ell(\ell-1)/2} \cdot q^{(m-\ell)(m-\ell-1)}$. So $|P| = q^{\ell(4m-3\ell-1)/2}$. Since $m \geq \ell \geq 1$,

$$\begin{aligned} \ell(4m-3\ell-1)/2 &= (2m-3) + \bigl(2m(\ell-1) - \tfrac{3}{2}\ell^2 - \tfrac{1}{2}\ell + 3\bigr) \\ &\geq (2m-3) + \bigl(\tfrac{1}{2}\ell(\ell-5) + 3\bigr) \geq 2m-3, \end{aligned}$$

with equality only when $\ell = m \in \{2, 3\}$. $\square$

The remaining lemmas involve symmetric and alternating groups. For any $m > 0$, we set $\boldsymbol{m} = \{1, \dots, m\}$, and regard $A_m < \Sigma_m$ as the alternating and symmetric groups on the set $\boldsymbol{m}$.

Lemma A.6. *Assume $4|m$, let $\{X_1, X_2\}$ be a partition of $\boldsymbol{m}$, and let $\sigma \in \Sigma_m$ be a permutation which exchanges X_1 and X_2. Set $\sigma^2 = \tau_i\tau_2$, where τ_i is a permutation of X_i for $i = 1, 2$. Then σ and the τ_i have the same partity.*

Proof. Assume σ is a product of disjoint cycles of length $2k_1, 2k_2, \dots, 2k_r$ (thus $m = \sum_{i=1}^r 2k_i$). Then τ_1 and τ_2 are each products of cycles of length $k_1, \dots, k_r$. Hence $\mathrm{sgn}(\sigma) = \prod_{i=1}^r (-1)^{2k_i-1} = (-1)^{m-r}$, while $\mathrm{sgn}(\tau_i) = \prod_{i=1}^r (-1)^{k_i-1} = (-1)^{(m/2)-r}$. Since m and $m/2$ are both even, $\mathrm{sgn}(\sigma) = (-1)^r = \mathrm{sgn}(\tau_i)$. $\square$

In the next two lemmas, when a group G acts on a set X, X/G denotes the set of G-orbits in X.

Lemma A.7. *Assume $G = \Sigma_m$ or A_m for some $m \geq 2$. Let $Q \trianglelefteq P \leq G$ be 2-subgroups such that $Q < P$, and either*

(i) *Q is radical in G and P is radical in $N_G(Q)$; or*

(ii) *P is radical in G.*

Then the action of P/Q on $\boldsymbol{m}/Q$ contains at most one free orbit.

Proof. Assume otherwise. Let $X_1, \dots, X_r \subseteq \boldsymbol{m}$ be the orbits under the action of P, arranged so that P/Q acts freely on X_1/Q and X_2/Q. For each $X \subseteq \boldsymbol{m}$, let $A_X \trianglelefteq \Sigma_X$ be the alternating and symmetric groups on X, regarded as subgroups of Σ_m (groups of permutations of $\boldsymbol{m}$ which leave $\boldsymbol{m} \setminus X$ pointwise fixed). Set $\overline{H} = \Sigma_{X_1} \times \cdots \times \Sigma_{X_r} \leq \Sigma_m$. Thus $P \leq \overline{H}$.

For each $i = 1, \ldots, r$, let $Q_i \le P_i \le \Sigma_{X_i}$ be the images of $Q < P$ under the i-th projection. Set $\bar{P} = P_1 \cdots P_r$, $\bar{Q} = Q_1 \cdots Q_r$, $P^* = \bar{P} \cap G$, and $Q^* = \bar{Q} \cap G$. Thus $Q \le Q^* \le \bar{Q}$, $P \le P^* \le \bar{P}$, and $\bar{Q} \trianglelefteq \bar{P} \le \bar{H}$.

In case (ii), $N_{P^*}(P) \trianglelefteq N_G(P)$, so $N_{P^*}(P) \le O_2(N_G(P)) = P$, and $P^* = P$ by Lemma 1.14(a). In case (i), $Q = Q^* \trianglelefteq P^*$ by a similar argument,

$$N_{P^*}(Q,P) = N_{P^*}(P) \trianglelefteq N_G(Q,P) \quad \Longrightarrow \quad N_{P^*}(P) \le O_2(N_G(Q,P)) = P\,,$$

and again $P^* = P$ by Lemma 1.14(a).

Let $I \subseteq \{1, \ldots, r\}$ be the set of all i such that $P_i \nleq Q$, and choose $\sigma_i \in P_i \smallsetminus Q$ for $i \in I$. For each $i \in I$ and $\sigma \in P_i \smallsetminus Q$, $\sigma \notin P$ since it acts trivially on at least one of the sets X_1 or X_2 (and P/Q acts freely on $(X_1 \cup X_2)/Q$), and hence σ is an odd permutation. Since P/Q acts nontrivially on X_i/Q for $i = 1, 2$, $1, 2 \in I$. If $i \in I$ for $i \ge 3$, then $\sigma_1\sigma_i \in P^* = P$ since it is an even permutation, but it acts trivially on X_2/Q, which is a contradiction. Thus $I = \{1, 2\}$. Also, $G = A_m$.

For $i \in I$, $P_i \cap Q = P_i \cap A_{X_i}$ has index two in P_i, while $P_i \le Q \le A_m$ and hence $P_i = Q_i \le A_{X_i}$ for $i \notin I$. Hence for $i \in I$, $Q_i = P_i$ or $Q_i = P_i \cap Q$. Since $Q \le G = A_m$, either $Q_i \le A_{X_i}$ for each i or $Q_i \nleq A_{X_i}$ for at least two indices i. In the latter case, $Q_i = P_i$ for $i = 1, 2$, so $Q^* = P^* = P > Q$, which would imply that the projections of Q into Σ_{X_i} for $i = 1, 2$ are $Q \cap P_i < Q_i$, a contradiction. Thus $Q_i \le Q$ for each i, so $Q = \bar{Q} = Q^*$. Then $[\bar{P}{:}Q] = \prod_{i=1}^r [P_i{:}Q_i] = 4$, so $[P{:}Q] = 2$, and $P = Q\langle\sigma\rangle$ where $\sigma = \sigma_1\sigma_2$.

For $i = 1, 2$, let Y_{i1}, Y_{i2} be the two orbits under the action of Q_i on X_i. Set $\sigma_i^2 = \tau_{i1}\tau_{i2}$, where $\tau_{ij} \in \Sigma_{Y_{ij}}$. By Lemma A.6 and since σ_i is an odd permutation, either $|X_i| = 2$, or τ_{i1}, τ_{i2} are also odd. Since $Q_i\langle\tau_{i1}\rangle \trianglelefteq N_{\Sigma_{X_i}}(Q_i)$, and $Q_i = 1$ if $|X_i| = 2$, we get $[O_2(N_{\Sigma_{X_i}}(Q_i)){:}Q_i] \ge 2$ in either case. Hence

$$[O_2(N_{\Sigma_m}(Q)){:}Q] \ge [O_2(N_{\Sigma_{X_1}}(Q_1)){:}Q_1][O_2(N_{\Sigma_{X_2}}(Q_2)){:}Q_2] \ge 4\,,$$

Q is not radical in A_m, and (i) does not hold. So P is radical in $G = A_m$.

Now, $O_2\bigl(N_{\Sigma_{X_1}}(P_1)N_{\Sigma_{X_2}}(P_2)\bigr) \le O_2(N_{\Sigma_m}(P))$. Since P is radical in A_m,

$$[O_2(N_{\Sigma_m}(P)){:}P] \le 2 \text{ and } [\bar{P}{:}P] = 2 \quad \Longrightarrow \quad O_2(N_{\Sigma_{X_i}}(P_i)) = P_i \;\text{ for } i = 1, 2.$$

So P_i is radical in Σ_{X_i} for $i = 1, 2$. Let $R_i \trianglelefteq P_i$ be the subgroup of elements of Q_i which act via even permutations on Y_{i1} and on Y_{i2}. If $|X_i| > 2$, then τ_{i1} and τ_{i2} are odd as just shown, so $\sigma_i^2 = \tau_{i1}\tau_{i2} \notin R_i$, and $P_i/R_i \cong C_4$. But by [AF, Proposition (2A)], each radical 2-subgroup of Σ_{X_i} is an iterated wreath product of elementary abelian 2-groups, and hence $P_i/[P_i, P_i]$ is elementary abelian. This is a contradiction, and we conclude that $|X_1| = |X_2| = 2$.

Thus $P = Q\langle\sigma\rangle$, where Q acts trivially on $X_1 \cup X_2$, and σ acts on it as a product of two transpositions. Also, for each $i \ge 3$, P_i contains only even permutations of X_i since $P_i \le Q$ ($i \notin I$). Hence each element of $N_G(P)$ sends $X_1 \cup X_2$ to itself, $O_2(N_G(P)) \ge O_2(A_{X_1 \cup X_2}) > \langle\sigma\rangle$, and P is not radical in G. □

We thank two of the referees for suggesting the following lemma and proof, both simpler than those in the original version. Whenever X is a set with G-action, $\mathbb{F}_2(X)$ denotes the permutation module over the group ring $\mathbb{F}_2[G]$ with $\mathbb{F}_2$-basis X.

Lemma A.8. *Let $G = \Sigma_m$ or A_m, and set $V = \mathbb{F}_2(\boldsymbol{m})$. Let $\Delta = C_V(\Sigma_m) \le V$ be the 1-dimensional submodule generated by the sum of the elements in $\boldsymbol{m}$. Let $P \le G$ be a radical 2-subgroup, and let $U \le V$ be such that $U/\Delta = C_{V/\Delta}(P)$.*

(a) *If P is not transitive on $\boldsymbol{m}$, then $U = C_V(P)$.*

(b) *If P is transitive on $\boldsymbol{m}$ and P_0 is the stabilizer of some point in $\boldsymbol{m}$, then $U/\Delta \cong (P/\mathrm{Fr}(P)P_0)^*$. If, in addition, $C_P(U) \in \mathrm{Syl}_p(C_G(U))$ and $m \ge 8$, then $N_G(P)$ acts on U/Δ via its full general linear group $GL(U/\Delta)$.*

Proof. We identify V with the power set of $\boldsymbol{m}$, with addition given by symmetric difference. Thus $\Delta = \langle \boldsymbol{m} \rangle$. If $X \in U \smallsetminus C_V(P)$, then $|X| = m/2$ and P acts transitively on the partition $\{X, X + \boldsymbol{m}\}$.

(b) If $m \ge 8$ and P is transitive on $\boldsymbol{m}$, then $C_V(P) = \Delta$, so all elements of $U \smallsetminus \Delta$ are partitions as just described. The map $R \mapsto \boldsymbol{m}/R$ defines a bijection from the set of subgroups of index two in P containing P_0 to the set of partitions of order two on which P acts transitively, and thus a natural bijection

$$\Psi\colon (P/\mathrm{Fr}(P)P_0)^* \xrightarrow{\ \cong\ } C_{V/\Delta}(P) = U/\Delta\,.$$

If $\varphi_1, \varphi_2, \varphi_3 \in (P/\mathrm{Fr}(P)P_0)^*$ are nonzero elements such that $\varphi_3 = \varphi_1 + \varphi_2$, then the $\mathrm{Ker}(\varphi_i)$ are the three subgroups of index two which contain some fixed subgroup R_0 of index four, so the $\Psi(\varphi_i)$ are the three partitions into sets of order $m/2$ refined by $\boldsymbol{m}/R_0$, and $\Psi(\varphi_3) = \Psi(\varphi_1) + \Psi(\varphi_2)$. Thus Ψ is an isomorphism.

We claim that

$$\mathrm{Aut}_P(U) = \big\{\alpha \in \mathrm{Aut}(U) \,\big|\, [\alpha, U] \le \Delta\big\} \tag{1}$$

$$\mathrm{Aut}_G(U/\Delta) = GL(U/\Delta)\,. \tag{2}$$

By definition ($U/\Delta = C_{V/\Delta}(P)$), $\mathrm{Aut}_P(U)$ is contained in the right hand side of (1). For each $g \in P \smallsetminus \mathrm{Fr}(P)P_0$, there is $R < P$ of index two which contains P_0 but not g, and g exchanges the two orbits of R on $\boldsymbol{m}$. Thus $C_P(U) \le \mathrm{Fr}(P)P_0$ so $|\mathrm{Aut}_P(U)| \ge |P/\mathrm{Fr}(P)P_0| = |U/\Delta|$. Since the right side of (1) has order $|U/\Delta|$, this proves (1).

To see (2), fix $\alpha \in GL(U/\Delta)$, and let $\beta \in \mathrm{Aut}(P/\mathrm{Fr}(P)P_0)$ be such that $\beta^* = \Psi^{-1}\alpha\Psi$. Choose an orbit $X \in \boldsymbol{m}/\mathrm{Fr}(P)P_0$, and choose any $\sigma \in \Sigma_m$ such that $\sigma(g(X)) = \beta(g)(X)$ for each $g \in P/\mathrm{Fr}(P)P_0$. For each $\varphi \in (P/\mathrm{Fr}(P)P_0)^*$, β sends $\mathrm{Ker}(\beta^*(\varphi))$ to $\mathrm{Ker}(\varphi)$, so σ sends $\Psi(\beta^*(\varphi)) = \alpha(\Psi(\varphi))$ to $\Psi(\varphi)$. Thus σ normalizes U and induces the automorphism α^{-1} on U/Δ. So $\mathrm{Aut}_{\Sigma_m}(U/\Delta) = GL(U/\Delta)$. If $|X| \ge 2$, then we can always arrange that $\sigma \in A_m \le G$. If $|X| = 1$, then $m = |U/\Delta| \ge 8$, so $\mathrm{rk}(U/\Delta) \ge 3$, and $GL(U/\Delta)$ has no subgroup of index two. Thus (2) holds in either case.

Now assume $C_P(U) \in \mathrm{Syl}_p(C_G(U))$. By definition of U, $N_G(P) \le N_G(U)$, and in particular, P normalizes $C_G(U)$. So $P \in \mathrm{Syl}_p(PC_G(U))$. By (1) (and since each element of $\mathrm{Aut}_G(U)$ fixes Δ)), $\mathrm{Aut}_P(U)$ is normal in $\mathrm{Aut}_G(U)$. Hence $PC_G(U) \trianglelefteq N_G(U)$. By the Frattini argument (Lemma 1.14(b)), $N_G(U) = PC_G(U){\cdot}N_{N_G(U)}(P) = C_G(U){\cdot}N_G(P)$. So by (2), $\mathrm{Aut}_{N_G(P)}(U/\Delta) = \mathrm{Aut}_G(U/\Delta) = GL(U/\Delta)$.

(a) Now assume $U > C_V(P)$; i.e., $C_{V/\Delta}(P) > C_V(P)/\Delta$. Then there is a partition $\{X, X'\}$ of $\boldsymbol{m}$ upon which P acts transitively. Let $Q < P$ be the subgroup of index two which stabilizes X and X'. Then P/Q acts freely on $\boldsymbol{m}/Q$, so it acts transitively on $\boldsymbol{m}/Q$ by Lemma A.7(ii), and P acts transitively on $\boldsymbol{m}$. □

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