

REDUCTIONS TO SIMPLE FUSION SYSTEMS

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ABSTRACT. We prove that if $\mathcal{E} \trianglelefteq \mathcal{F}$ are saturated fusion systems over p -groups $T \trianglelefteq S$, such that $C_S(\mathcal{E}) \leq T$, and either $\text{Aut}_{\mathcal{F}}(T)/\text{Aut}_{\mathcal{E}}(T)$ or $\text{Out}(\mathcal{E})$ is p -solvable, then \mathcal{F} can be “reduced” to \mathcal{E} by alternately taking normal subsystems of p -power index or of index prime to p . In particular, this is the case whenever \mathcal{E} is simple and “tamely realized” by a known simple group K . This answers a question posed by Michael Aschbacher, and is useful when analyzing involution centralizers in simple fusion systems, in connection with his program for reproving parts of the classification of finite simple groups by classifying certain 2-fusion systems.

When p is a prime and S is a finite p -group, a *saturated fusion system* over S is a category whose objects are the subgroups of S , whose morphisms are injective group homomorphisms between the subgroups, and which satisfies a certain list of axioms motivated by the Sylow theorems for finite groups (Definition 1.1). For example, when G is a finite group and $S \in \text{Syl}_p(G)$, the *p -fusion system* of G is the category $\mathcal{F}_S(G)$ whose objects are the subgroups of S , and where for each $P, Q \leq S$, $\text{Hom}_{\mathcal{F}_S(G)}(P, Q)$ is the set of those homomorphisms induced by conjugation in G .

Normal fusion subsystems of a saturated fusion system are defined by analogy with normal subgroups of a group (Definition 1.4). Among the normal subsystems, we look at two particular classes: those of index prime to p (defined over the same Sylow subgroup), and those of p -power index (see the discussions before and after Lemma 1.10). A natural question arises: when $\mathcal{E} \trianglelefteq \mathcal{F}$, under what conditions can \mathcal{F} be “reduced” to \mathcal{E} via a sequence of steps, where one alternates taking normal subsystems of p -power index and normal subsystems of index prime to p ?

Our main theorem (Theorem 2.3) says that if $\mathcal{E} \trianglelefteq \mathcal{F}$ are saturated fusion systems over p -groups $T \trianglelefteq S$, such that $C_S(\mathcal{E}) \leq T$, and either $\text{Aut}_{\mathcal{F}}(T)/\text{Aut}_{\mathcal{E}}(T)$ or $\text{Out}(\mathcal{E})$ is p -solvable, then \mathcal{F} can be reduced to \mathcal{E} in the above sense. In particular, if \mathcal{E} is the fusion system of a known finite simple group K , and is “tamely realized” by K in the sense that $\text{Out}(K)$ surjects onto $\text{Out}(\mathcal{E})$ (see Section 2), then $\text{Out}(\mathcal{E})$ is solvable since $\text{Out}(K)$ is solvable by the Schreier conjecture, and hence \mathcal{F} reduces to \mathcal{E} .

This paper was motivated by a question posed by Michael Aschbacher. The above situation arises frequently when analyzing centralizers of involutions in simple fusion systems. If \mathcal{F} is the centralizer of an involution and $\mathcal{E} = F^*(\mathcal{F})$ denotes the generalized Fitting subsystem of \mathcal{F} (see [As, Chapter 9]), then the hypothesis $C_S(\mathcal{E}) \leq T$ always holds, and $\text{Out}(\mathcal{E})$ is solvable by Schreier’s conjecture if $\mathcal{E}/Z(\mathcal{E})$ is tamely realized by a known simple group K . Hence in this situation, Theorem 2.3 together with results in [AOV] imply that \mathcal{F} itself is realized by a certain extension of K . (See Corollary 2.5 for a slightly more general situation where this applies.)

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Throughout the paper, all p -groups are assumed to be finite. Composition is always taken from right to left.

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1. NORMAL SUBSYSTEMS WITH p -SOLVABLE QUOTIENT

Since the details of the definition of a saturated fusion system play an important role in the proofs here, we begin by recalling some of this terminology. When G is a group and $P, Q \leq G$ are subgroups, $\text{Inj}(P, Q)$ denotes the set of injective homomorphisms $P \rightarrow Q$, and $\text{Hom}_G(P, Q)$ is the set of homomorphisms of the form $(x \mapsto gxg^{-1})$ for $g \in G$. A *fusion system* over a p -group S is a category \mathcal{F} where $\text{Ob}(\mathcal{F})$ is the set of subgroups of S , and where for each $P, Q \leq S$,

- (i) $\text{Hom}_S(P, Q) \subseteq \text{Hom}_{\mathcal{F}}(P, Q) \subseteq \text{Inj}(P, Q)$, and
- (ii) $\varphi \in \text{Hom}_{\mathcal{F}}(P, Q)$ implies $\varphi^{-1} \in \text{Hom}_{\mathcal{F}}(\varphi(P), P)$.

By analogy with the terminology for groups, we say that $P, Q \leq S$ are \mathcal{F} -conjugate if they are isomorphic in \mathcal{F} , and let $P^{\mathcal{F}}$ denote the set of subgroups \mathcal{F} -conjugate to P .

Definition 1.1 ([RS]). Let \mathcal{F} be a fusion system over a p -group S .

- (a) A subgroup $P \leq S$ is *fully automized in \mathcal{F}* if $\text{Aut}_S(P) \in \text{Syl}_p(\text{Aut}_{\mathcal{F}}(P))$.
- (b) A subgroup $P \leq S$ is *receptive in \mathcal{F}* if for each $Q \in P^{\mathcal{F}}$ and each $\varphi \in \text{Iso}_{\mathcal{F}}(Q, P)$, φ extends to a homomorphism $\bar{\varphi} \in \text{Hom}_{\mathcal{F}}(N_{\varphi}, S)$, where

$$N_{\varphi} = \{g \in N_S(Q) \mid \varphi c_g \varphi^{-1} \in \text{Aut}_S(P)\}.$$

- (c) The fusion system \mathcal{F} is *saturated* if each \mathcal{F} -conjugacy class of subgroups of S contains at least one member which is fully automized and receptive in \mathcal{F} .
- (d) When \mathcal{H} is a set of subgroups of \mathcal{F} closed under \mathcal{F} -conjugacy, we say that \mathcal{F} is *\mathcal{H} -saturated* if each member of \mathcal{H} is \mathcal{F} -conjugate to a subgroup which is fully automized and receptive in \mathcal{F} . We say that \mathcal{F} is *\mathcal{H} -generated* if each morphism in \mathcal{F} is a composite of restrictions of morphisms between members of \mathcal{H} .
- (e) When X is a set of injective homomorphisms between subgroups of S , $\langle X \rangle$ denotes the fusion system over S generated by X : the smallest fusion system which contains X . When \mathcal{F}_0 is a category whose objects are subgroups of S and whose morphisms are injective homomorphisms, we write $\langle \mathcal{F}_0 \rangle = \langle \text{Mor}(\mathcal{F}_0) \rangle$.

We next recall some of the terminology for subgroups in a fusion system. When \mathcal{F} is a fusion system over S and $P \leq S$, we write $\text{Out}_{\mathcal{F}}(P) = \text{Aut}_{\mathcal{F}}(P)/\text{Inn}(P)$.

Definition 1.2. Let \mathcal{F} be a fusion system over a p -group S , and let $P \leq S$ be a subgroup. Then

- (a) P is *fully centralized (fully normalized) in \mathcal{F}* if $|C_S(P)| \geq |C_S(Q)|$ ($|N_S(P)| \geq |N_S(Q)|$) for each $Q \in P^{\mathcal{F}}$;
- (b) P is *\mathcal{F} -centric* if $C_S(Q) \leq Q$ for each $Q \leq P$;
- (c) P is *\mathcal{F} -radical* if $O_p(\text{Out}_{\mathcal{F}}(P)) = 1$; and
- (d) P is *strongly closed in \mathcal{F}* if $\varphi(P_0) \leq P$ for each $P_0 \leq P$ and each $\varphi \in \text{Hom}_{\mathcal{F}}(P_0, S)$.

We also let $\mathcal{F}^{cr} \subseteq \mathcal{F}^c$ denote the sets of subgroups of S which are \mathcal{F} -centric and \mathcal{F} -radical, or \mathcal{F} -centric, respectively.

The following lemma describes the relations between some of these conditions on subgroups. Point (a) and (b) are due to Roberts and Shpectorov [RS] and are also shown in [AKO, Lemma I.2.6(c)], while (c) is immediate from the definitions.

Lemma 1.3. *In a saturated fusion system \mathcal{F} over a p -group S , for each subgroup $P \leq S$,*

- (a) *P is fully centralized if and only if P is receptive;*
- (b) *P is fully normalized if and only if P is fully automized and receptive; and*
- (c) *if P is \mathcal{F} -centric, then P is fully centralized and hence receptive.*

As described in the introduction, normal fusion subsystems play a central role here.

Definition 1.4. Let \mathcal{F} be a saturated fusion system over a p -group S , and let $\mathcal{E} \leq \mathcal{F}$ be a saturated fusion subsystem over $T \leq S$. The subsystem \mathcal{E} is *normal* in \mathcal{F} ($\mathcal{E} \trianglelefteq \mathcal{F}$) if

- T is strongly closed in \mathcal{F} (in particular, $T \trianglelefteq S$);
- (invariance condition) each $\alpha \in \text{Aut}_{\mathcal{F}}(T)$ is fusion preserving with respect to \mathcal{E} (i.e., extends to $(\alpha, \hat{\alpha}) \in \text{Aut}(\mathcal{E})$);
- (Frattni condition) for each $P \leq T$ and each $\varphi \in \text{Hom}_{\mathcal{F}}(P, T)$, there are $\alpha \in \text{Aut}_{\mathcal{F}}(T)$ and $\varphi_0 \in \text{Hom}_{\mathcal{E}}(P, T)$ such that $\varphi = \alpha \circ \varphi_0$; and
- (extension condition) each $\alpha \in \text{Aut}_{\mathcal{E}}(T)$ extends to some $\bar{\alpha} \in \text{Aut}_{\mathcal{F}}(TC_S(T))$ such that $[\bar{\alpha}, C_S(T)] \leq Z(T)$.

Finally, we will frequently need to use the following version of Alperin's fusion theorem for fusion systems. This is the version shown in [BLO2, Theorem A.10]. For a stronger version due to Puig, see, e.g., [AKO, Theorems I.3.5–6].

Theorem 1.5. *If \mathcal{F} is a saturated fusion system over a p -group S , then*

$$\mathcal{F} = \langle \text{Aut}_{\mathcal{F}}(P) \mid P \leq S \text{ is } \mathcal{F}\text{-centric, } \mathcal{F}\text{-radical, and fully normalized in } \mathcal{F} \rangle.$$

Thus each morphism in \mathcal{F} is a composite of restrictions of \mathcal{F} -automorphisms of such subgroups.

If $\mathcal{E} \trianglelefteq \mathcal{F}$ are saturated fusion systems, then \mathcal{E} has *index prime to p* in \mathcal{F} if they are both over the same p -group S , and for each $P \leq S$, $\text{Aut}_{\mathcal{E}}(P) \geq O^{p'}(\text{Aut}_{\mathcal{F}}(P))$. Let $O^{p'}(\mathcal{F}) \trianglelefteq \mathcal{F}$ be the smallest normal subsystem of index prime to p in \mathcal{F} [BCGLO2, Theorem 5.4]. We first fix some tools for constructing normal subsystems of this type. The following lemma is basically the same as Theorem I.7.7 in [AKO], but stated in a slightly more general setting.

Lemma 1.6. *Let \mathcal{F} be a saturated fusion system over a p -group S . Let $\mathcal{H} \subseteq \mathcal{F}^c$ be a nonempty set of \mathcal{F} -centric subgroups of S such that*

- (i) *\mathcal{H} is closed under \mathcal{F} -conjugacy and overgroups; and*
- (ii) *for each $P \in \mathcal{F}^c \setminus \mathcal{H}$, there is $P^* \in P^{\mathcal{F}}$ such that $\text{Out}_S(P^*) \cap O_p(\text{Out}_{\mathcal{F}}(P^*)) \neq 1$.*

Let $\mathcal{F}^ \subseteq \mathcal{F}$ be the full subcategory with object set \mathcal{H} , let Δ be a finite group of order prime to p , and let $\chi: \text{Mor}(\mathcal{F}^*) \longrightarrow \Delta$ be such that*

- (iii) *$\chi(\text{incl}_P^S) = 1$ for all $P \in \mathcal{H}$,*

- (iv) $\chi(\psi\varphi) = \chi(\psi)\chi(\varphi)$ for each composable pair of morphisms ψ and φ in \mathcal{F}^* , and
- (v) $\chi(\text{Aut}_{\mathcal{F}}(S)) = \Delta$.

Let $\mathcal{F}_0^* \subseteq \mathcal{F}^*$ be the subcategory with the same objects, and with $\text{Mor}(\mathcal{F}_0^*) = \chi^{-1}(1)$. Set $\mathcal{F}_0 = \langle \mathcal{F}_0^* \rangle$: the fusion system over S generated by \mathcal{F}_0^* . Then $\mathcal{F}_0 \trianglelefteq \mathcal{F}$ is a normal saturated fusion subsystem, $O^{p'}(\mathcal{F}) \leq \mathcal{F}_0 \leq \mathcal{F}$, and $\text{Aut}_{\mathcal{F}_0}(S) = \text{Ker}(\chi|_{\text{Aut}_{\mathcal{F}}(S)})$.

Proof. By (iv), \mathcal{F}_0^* is a subcategory of \mathcal{F}^* . Since $|\Delta|$ is prime to p , $\chi(\text{Inn}(S)) = 1$, and hence $\text{Inn}(S) \leq \text{Aut}_{\mathcal{F}_0^*}(S)$. Also, by (iii) and (iv),

$$\chi(\varphi') = \chi(\varphi) \text{ when } \varphi' \in \text{Mor}(\mathcal{F}^*) \text{ is any restriction of } \varphi \in \text{Mor}(\mathcal{F}^*). \quad (1)$$

Since \mathcal{H} is closed under overgroups by (i), we thus have $\text{Hom}_{\mathcal{F}_0}(P, Q) = \text{Hom}_{\mathcal{F}_0^*}(P, Q)$ for all $P, Q \in \mathcal{H}$.

We next show the following:

- (2) For each $P, Q \in \mathcal{H}$ and $\varphi \in \text{Hom}_{\mathcal{F}^*}(P, Q)$, there are morphisms $\alpha \in \text{Aut}_{\mathcal{F}}(S)$ and $\varphi_0 \in \text{Hom}_{\mathcal{F}_0^*}(P, Q^*)$, where $Q^* = \alpha^{-1}(Q)$, such that $\varphi = \alpha|_{Q^*} \circ \varphi_0$.
- (3) If $P, Q \in \mathcal{H}$ and $Q \in P^{\mathcal{F}}$, then there is $\alpha \in \text{Aut}_{\mathcal{F}}(S)$ such that $\alpha(Q) \in P^{\mathcal{F}_0}$.
- (4) If $P \in \mathcal{H}$ is fully automized in \mathcal{F} , then it is fully automized in \mathcal{F}_0 .
- (5) If $P \in \mathcal{H}$ is receptive in \mathcal{F} , then it is receptive in \mathcal{F}_0 .

Point (2) follows from (v) (and (1)): choose α such that $\chi(\alpha) = \chi(\varphi)$, and set $\varphi_0 = \alpha^{-1}\varphi \in \text{Mor}(\mathcal{F}_0^*)$. Point (3) follows immediately from (2). Point (4) holds since if $\text{Aut}_S(P) \in \text{Syl}_p(\text{Aut}_{\mathcal{F}}(P))$, then $\text{Aut}_S(P)$ is also a Sylow p -subgroup in $\text{Aut}_{\mathcal{F}_0}(P)$.

If $P \in \mathcal{H}$ is receptive in \mathcal{F} , then each $\varphi \in \text{Hom}_{\mathcal{F}_0}(Q, S)$ with $\varphi(Q) = P$ extends to some $\bar{\varphi} \in \text{Hom}_{\mathcal{F}}(N_{\varphi}, S)$, where $N_{\varphi} = \{x \in N_S(Q) \mid \varphi c_x \varphi^{-1} \in \text{Aut}_S(P)\}$. Then $N_{\varphi} \in \mathcal{H}$ by (i) and $\chi(\bar{\varphi}) = \chi(\varphi) = 1$ by (1), so $\bar{\varphi} \in \text{Hom}_{\mathcal{F}_0}(N_{\varphi}, S)$. Thus P is receptive in \mathcal{F}_0 in this case, and this proves (5).

Since \mathcal{F} is saturated, for each $P \in \mathcal{H}$, there is $Q \in P^{\mathcal{F}}$ which is fully automized and receptive in \mathcal{F} . By (3), there is $\alpha \in \text{Aut}_{\mathcal{F}}(S)$ such that $\alpha(Q) \in P^{\mathcal{F}_0}$. Then $\alpha(Q)$ is also fully automized and receptive in \mathcal{F} , hence in \mathcal{F}_0 by (4) and (5). Thus \mathcal{F}_0 is \mathcal{H} -saturated, and it is \mathcal{F}_0 -generated by definition. So by (ii) and [BCGLO1, Theorem 2.2], \mathcal{F}_0 is saturated.

We claim that

$$\alpha \in \text{Aut}_{\mathcal{F}}(S), \varphi \in \text{Mor}(\mathcal{F}_0) \implies \alpha\varphi\alpha^{-1} \in \text{Mor}(\mathcal{F}_0). \quad (6)$$

Here, $\alpha\varphi\alpha^{-1}$ means composition on each side with the appropriate restriction of α or α^{-1} . This holds for $\varphi \in \text{Mor}(\mathcal{F}_0^*)$ by definition (and (1)), and hence holds for all composites of restrictions of such morphisms. Since $\mathcal{F}_0 = \langle \mathcal{F}_0^* \rangle$, (6) holds for all $\varphi \in \text{Mor}(\mathcal{F}_0)$.

We next check that

$$\forall \varphi \in \text{Mor}(\mathcal{F}), \exists \varphi_0 \in \text{Mor}(\mathcal{F}_0), \alpha \in \text{Aut}_{\mathcal{F}}(S) \text{ such that } \varphi = \alpha\varphi_0. \quad (7)$$

Since $\mathcal{H} \supseteq \mathcal{F}^{cr}$ by (ii), $\mathcal{F} = \langle \mathcal{F}^* \rangle$ by Alperin's fusion theorem. Hence $\mathcal{F} = \langle \mathcal{F}_0, \text{Aut}_{\mathcal{F}}(S) \rangle$ by (2). By [BCGLO2, Lemma 3.4.c] and (6), this suffices to prove (7).

Since the extension condition for normality holds trivially in this case, this proves that $\mathcal{F}_0 \trianglelefteq \mathcal{F}$ (see [AKO, Definition I.6.1]). Hence $\mathcal{F}_0 \geq O^{p'}(\mathcal{F})$, since \mathcal{F} and \mathcal{F}_0 are both saturated fusion systems over S (see [AOV, Lemma 1.26]). \square

The next lemma is needed to check that point (ii) holds when we apply Lemma 1.6.

Lemma 1.7. *Let $\mathcal{E} \trianglelefteq \mathcal{F}$ be saturated fusion systems over $T \trianglelefteq S$. Then for each $P \in \mathcal{F}^c$ such that $P \cap T \notin \mathcal{E}^c$, there is $P^* \in P^{\mathcal{F}}$ such that $\text{Out}_S(P^*) \cap O_p(\text{Out}_{\mathcal{F}}(P^*)) \neq 1$. In particular, $P \notin \mathcal{F}^{cr}$.*

Proof. By [AKO, Lemma I.2.6.c], there is $\varphi \in \text{Hom}_{\mathcal{F}}(N_S(P \cap T), S)$ such that $\varphi(P \cap T)$ is fully normalized in \mathcal{F} . Set $P^* = \varphi(P)$; then $\varphi(P \cap T) = P^* \cap T$ since T is strongly closed in \mathcal{F} , and hence $P^* \cap T$ is fully normalized in \mathcal{F} . Since $P \cap T \notin \mathcal{E}^c$, there is $Q \in (P \cap T)^{\mathcal{E}} \subseteq (P^* \cap T)^{\mathcal{F}}$ such that $C_T(Q) \not\leq Q$, each $\psi \in \text{Iso}_{\mathcal{F}}(Q, P^* \cap T)$ extends to $QC_S(Q)$, and hence $C_T(P^* \cap T) \not\leq P^*$.

Thus $P^*C_T(P^* \cap T) > P^*$, so $N_{P^*C_T(P^* \cap T)}(P^*) > P^*$, and there is $x \in N_T(P^*) \setminus P^*$ such that $[x, P^* \cap T] = 1$. Conjugation by x induces the identity on $P^* \cap T$ and on $P^*/(P^* \cap T)$, so $c_x \in O_p(\text{Aut}_{\mathcal{F}}(P^*))$. Also, $c_x \notin \text{Inn}(P^*)$ since $P^* \in \mathcal{F}^c$, so $1 \neq [c_x] \in \text{Out}_S(P^*) \cap O_p(\text{Out}_{\mathcal{F}}(P^*))$. \square

The next proposition provides a more explicit way to construct proper normal subsystems of index prime to p . Note that the existence of a normal subsystem over the strongly closed subgroup T is crucial, as is clearly seen by considering the case where $T = S$. In fact, the proposition is rather trivial when $T = S$ or $T = 1$, and is useful only when $1 \neq T < S$.

Proposition 1.8. *Let $\mathcal{E} \trianglelefteq \mathcal{F}$ be saturated fusion systems over p -groups $T \trianglelefteq S$. Let $\chi_0: \text{Aut}_{\mathcal{F}}(T) \rightarrow \Delta$ be a surjective homomorphism, for some $\Delta \neq 1$ of order prime to p , such that $\text{Aut}_{\mathcal{E}}(T) \leq \text{Ker}(\chi_0)$. Then there is a unique proper normal subsystem $\mathcal{F}_0 \trianglelefteq \mathcal{F}$ over S such that*

$$\text{Aut}_{\mathcal{F}_0}(S) = \{\alpha \in \text{Aut}_{\mathcal{F}}(S) \mid \alpha|_T \in \text{Ker}(\chi_0)\} \quad (1)$$

and $\mathcal{F}_0 \geq \mathcal{E}$. In particular, $O^{p'}(\mathcal{F}) \leq \mathcal{F}_0 < \mathcal{F}$.

Proof. The uniqueness of \mathcal{F}_0 follows from (1) and [BCGLO2, Theorem 5.4]: a saturated fusion subsystem of index prime to p in \mathcal{F} is uniquely determined by the automizer of S .

Let $\mathcal{F}|_{\mathcal{E}^c} \subseteq \mathcal{F}$ be the full subcategory with objects in \mathcal{E}^c . We first show that there is a map

$$\chi: \text{Mor}(\mathcal{F}|_{\mathcal{E}^c}) \longrightarrow \Delta$$

which extends χ_0 , which sends composites to products, and which sends $\text{Mor}(\mathcal{E}^c)$ to the identity. By the Frattini condition for a normal fusion subsystem [AKO, Definition I.6.1], for each $P, Q \in \mathcal{E}^c$ and each $\varphi \in \text{Hom}_{\mathcal{F}}(P, Q)$, there are $\alpha \in \text{Aut}_{\mathcal{F}}(T)$ and $\varphi_0 \in \text{Hom}_{\mathcal{E}}(P, Q_1)$, where $Q_1 = \alpha^{-1}(Q)$, such that $\varphi = \alpha|_{Q_1} \circ \varphi_0$. In this situation, the conditions imposed on χ imply that $\chi(\varphi) = \chi(\alpha) = \chi_0(\alpha)$. It remains to prove that this is independent of the choice of decomposition of φ , and that it sends composites to products.

To see that χ sends composites to products when it is uniquely defined, fix a composable pair of morphisms $\psi, \varphi \in \text{Mor}(\mathcal{F})$: a pair such that $\psi \circ \varphi$ is defined. Assume $\varphi = \alpha\varphi_0$ and $\psi = \beta\psi_0$ (after appropriate restrictions of α and β), where $\varphi_0, \psi_0 \in \text{Mor}(\mathcal{E})$ and $\alpha, \beta \in \text{Aut}_{\mathcal{F}}(T)$. Thus $\chi(\varphi) = \chi_0(\alpha)$ and $\chi(\psi) = \chi_0(\beta)$. Also,

$$\psi \circ \varphi = \beta\psi_0\alpha\varphi_0 = (\beta\alpha)(\alpha^{-1}\psi_0\alpha)\varphi_0$$

where $\alpha^{-1}\psi_0\alpha \in \text{Mor}(\mathcal{E})$ since $\mathcal{E} \trianglelefteq \mathcal{F}$. So $\chi(\psi \circ \varphi) = \chi_0(\beta\alpha)$.

Again fix $P, Q \in \mathcal{E}^c$ and $\varphi \in \text{Hom}_{\mathcal{F}}(P, Q)$. Let $\varphi = \alpha|_{Q_1} \circ \varphi_0 = \beta|_{Q_2} \circ \psi_0$ be two decompositions, where $\alpha, \beta \in \text{Aut}_{\mathcal{F}}(T)$, $Q_1 = \alpha^{-1}(Q)$, $Q_2 = \beta^{-1}(Q)$, and $\varphi_0, \psi_0 \in \text{Mor}(\mathcal{E})$. If $P = Q = T$, then all of these morphisms lie in $\text{Aut}_{\mathcal{F}}(T)$, and $\chi_0(\alpha) = \chi_0(\beta)$ since $\chi_0(\text{Aut}_{\mathcal{E}}(T)) = 1$. So assume $P < T$, and also assume inductively that

χ is uniquely defined on all morphisms between subgroups in \mathcal{E}^c strictly larger than P . Then $(\beta^{-1}\alpha|_{Q_1})\varphi_0 = \psi_0 \in \text{Hom}_{\mathcal{E}}(P, Q_2)$. Since $\beta^{-1}\alpha|_{Q_1} \in \text{Hom}_{\mathcal{E}}(Q_1, Q_2)$ extends in \mathcal{F} to T , it also extends in \mathcal{E} to $N_T(Q_1) > Q_1$ (recall that all \mathcal{E} -centric subgroups are receptive by Lemma 1.3(c)). Let $\gamma \in \text{Hom}_{\mathcal{E}}(N_T(Q_1), T)$ be such that $\beta^{-1}\alpha|_{Q_1} = \gamma|_{Q_1}$. Since $Q_1 \in \mathcal{E}^c$, $\beta^{-1}\alpha(N_T(Q_1)) = \gamma(N_T(Q_1))$, so $\alpha^{-1}\beta\gamma \in \text{Aut}_{\mathcal{F}}(N_T(Q_1))$. This automorphism has p -power order since it is the identity on Q_1 and on $N_T(Q_1)/Q_1$, and hence $\chi(\alpha^{-1}\beta\gamma) = 1$. Since all of these homomorphisms involve subgroups strictly larger than P , $\chi(\alpha)^{-1}\chi(\beta)\chi(\gamma) = 1$, where $\chi(\gamma) = 1$ since $\gamma \in \text{Mor}(\mathcal{E}^c)$. So $\chi_0(\alpha) = \chi_0(\beta)$, and the two decompositions of φ give the same value for $\chi(\varphi)$.

Thus χ is uniquely defined. Set

$$\mathcal{H}^* = \{P \in \mathcal{F}^c \mid P \cap T \in \mathcal{E}^c\},$$

and let $\widehat{\chi}$ be the composite

$$\widehat{\chi}: \text{Mor}(\mathcal{F}|_{\mathcal{H}^*}) \xrightarrow{R} \text{Mor}(\mathcal{F}|_{\mathcal{E}^c}) \xrightarrow{x} \Delta,$$

where R sends $\varphi \in \text{Hom}_{\mathcal{F}}(P, Q)$ to $\varphi|_{P \cap T} \in \text{Hom}_{\mathcal{F}}(P \cap T, Q \cap T)$. Since $T \trianglelefteq S$, T is fully normalized, and hence is fully automized and receptive (Lemma 1.3(b)). Hence $\text{Aut}_S(T)$ lies in $\text{Syl}_p(\text{Aut}_{\mathcal{F}}(T)) = \text{Syl}_p(O^{p'}(\text{Aut}_{\mathcal{F}}(T)))$, and so

$$\text{Aut}_{\mathcal{F}}(T) = N_{\text{Aut}_{\mathcal{F}}(T)}(\text{Aut}_S(T)) \cdot O^{p'}(\text{Aut}_{\mathcal{F}}(T))$$

by the Frattini argument. Since T is receptive, each $\alpha \in N_{\text{Aut}_{\mathcal{F}}(T)}(\text{Aut}_S(T))$ extends to some $\bar{\alpha} \in \text{Aut}_{\mathcal{F}}(S)$, and $\widehat{\chi}(\alpha) = \widehat{\chi}(\bar{\alpha})\widehat{\chi}(\text{incl}_T^S) = \widehat{\chi}(\bar{\alpha})$. Since $\widehat{\chi}(O^{p'}(\text{Aut}_{\mathcal{F}}(T))) = 1$, $\widehat{\chi}(\text{Aut}_{\mathcal{F}}(S)) = \widehat{\chi}(\text{Aut}_{\mathcal{F}}(T)) = \Delta$.

Condition (ii) in Lemma 1.6 holds by Lemma 1.7. We just checked condition (v), condition (iv) holds for $\widehat{\chi}$ since it holds for χ , and the other two are clear. So by that lemma, there is $\mathcal{F}_0 < \mathcal{F}$ which is normal in \mathcal{F} and contains $O^{p'}(\mathcal{F})$, and such that $\text{Aut}_{\mathcal{F}_0}(S)$ is as required.

It remains to show that $\mathcal{F}_0 \geq \mathcal{E}$. If $P \in \mathcal{E}^c$ and P is fully centralized in \mathcal{F} , then each $\varphi \in \text{Aut}_{\mathcal{E}}(P)$ extends to some $\bar{\varphi} \in \text{Aut}_{\mathcal{F}}(PC_S(P))$, where $PC_S(P) \in \mathcal{F}^c$ and $\widehat{\chi}(\bar{\varphi}) = \chi(\varphi) = 1$, so $\bar{\varphi}$ and hence φ are in $\text{Mor}(\mathcal{F}_0)$. If $P \in \mathcal{E}^c$ is arbitrary, then $\psi(P)$ is fully centralized in \mathcal{F} for some $\psi \in \text{Hom}_{\mathcal{F}}(P, T)$, and $\text{Aut}_{\mathcal{E}}(P) = (\text{Aut}_{\mathcal{E}}(\varphi(P)))^\varphi$ and $\text{Aut}_{\mathcal{F}_0}(P) = (\text{Aut}_{\mathcal{F}_0}(\varphi(P)))^\varphi$ since \mathcal{E} and \mathcal{F}_0 are normal in \mathcal{F} (see [AKO, Proposition I.6.4(d)], applied with $Q = P$). Hence $\text{Aut}_{\mathcal{E}}(P) \leq \text{Aut}_{\mathcal{F}_0}(P)$ for all $P \in \mathcal{E}^c$, and $\mathcal{E} \leq \mathcal{F}_0$ by Alperin's fusion theorem. \square

Corollary 1.9. *If $\mathcal{E} \trianglelefteq \mathcal{F}$ are saturated fusion systems over $T \trianglelefteq S$, and*

$$\text{Aut}_{\mathcal{E}}(T)O^{p'}(\text{Aut}_{\mathcal{F}}(T)) < \text{Aut}_{\mathcal{F}}(T),$$

then $O^{p'}(\mathcal{F}) < \mathcal{F}$.

We now turn to constructions of normal subsystems of p -power index. Recall first the definition of the hyperfocal subgroup $\text{h}\eta\text{p}(\mathcal{F})$ for a saturated fusion system \mathcal{F} over a p -group S :

$$\text{h}\eta\text{p}(\mathcal{F}) = \langle [O^p(\text{Aut}_{\mathcal{F}}(P)), P] \mid P \leq S \rangle \trianglelefteq S.$$

Lemma 1.10. *Let \mathcal{F} be a saturated fusion system over a p -group S , and assume $T \trianglelefteq S$ is strongly closed in \mathcal{F} . Then*

- (a) $\text{h}\eta\text{p}(\mathcal{F}/T) = T \cdot \text{h}\eta\text{p}(\mathcal{F})/T$;
- (b) $TC_S(T)$ is strongly closed in \mathcal{F} ; and

- (c) *the natural isomorphism $S/TC_S(T) \cong \text{Out}_S(T)$ extends to an isomorphism of fusion systems $\mathcal{F}/TC_S(T) \cong \mathcal{F}_{\text{Out}_S(T)}(\text{Out}_{\mathcal{F}}(T))$.*

Proof. By [AKO, Theorem II.5.12], and since T is strongly closed in \mathcal{F} , there is a morphism of fusion systems $(\Psi, \widehat{\Psi}): \mathcal{F} \rightarrow \mathcal{F}/T$ which sends P to PT/T and sends $\varphi \in \text{Hom}_{\mathcal{F}}(P, Q)$ to the induced homomorphism between quotient groups.

Set $\widehat{T} = TC_S(T)$ for short.

(a) If $P \leq S$, and $\varphi \in \text{Aut}_{\mathcal{F}}(P)$ has order prime to p , then $\widehat{\Psi}(\varphi) \in \text{Aut}_{\mathcal{F}/T}(PT/T)$ also has order prime to p . Hence $T[\varphi, P]/T \leq \mathfrak{hnp}(\mathcal{F}/T)$. Since $\mathfrak{hnp}(\mathcal{F})$ is generated by such commutators $[\varphi, P]$ by definition, this proves that $T \cdot \mathfrak{hnp}(\mathcal{F})/T \leq \mathfrak{hnp}(\mathcal{F}/T)$.

Conversely, for each $P/T \leq S/T$, and each $\psi \in \text{Aut}_{\mathcal{F}/T}(P/T)$ of order prime to p , ψ lifts to some $\widehat{\psi} \in \text{Aut}_{\mathcal{F}}(P)$ by definition of \mathcal{F}/T , and $[\psi, P/T] = T[\widehat{\psi}, P]/T$ where $[\widehat{\psi}, P] \leq \mathfrak{hnp}(\mathcal{F})$. Since $\mathfrak{hnp}(\mathcal{F}/T)$ is generated by such commutators $[\psi, P/T]$, this proves that $\mathfrak{hnp}(\mathcal{F}/T) \leq T \cdot \mathfrak{hnp}(\mathcal{F})/T$.

(b) Fix $P \leq \widehat{T}$ and $\varphi \in \text{Hom}_{\mathcal{F}}(P, S)$. Choose $\widehat{\varphi} \in \text{Hom}_{\mathcal{F}}(PT, S)$ such that $\widehat{\Psi}(\widehat{\varphi}) = \widehat{\Psi}(\varphi) \in \text{Hom}_{\mathcal{F}/T}(PT/T, S/T)$. Then $\widehat{\varphi}(T) = T$, $PT \leq \widehat{T} = TC_S(T)$, so $PT = TC_{PT}(T)$, and $\varphi(P) \leq \widehat{\varphi}(PT) = TC_{\widehat{\varphi}(PT)}(T) \leq \widehat{T}$. Thus \widehat{T} is strongly closed.

(c) Fix $P, Q \leq S$ which contain \widehat{T} , and $\varphi \in \text{Hom}_{\mathcal{F}}(P, Q)$. Let $\widehat{\varphi} \in \text{Hom}_{\mathcal{F}/\widehat{T}}(P/\widehat{T}, Q/\widehat{T})$ be the induced homomorphism, and let $[\varphi|_T] \in \text{Out}_{\mathcal{F}}(T)$ be the class of $\varphi|_T \in \text{Aut}_{\mathcal{F}}(T)$. Then for $x \in P$ and $c_x \in \text{Aut}_P(T)$, $(\varphi|_T)c_x(\varphi|_T)^{-1} = c_{\varphi(x)} \in \text{Aut}_Q(T)$. So if we identify $\text{Out}_P(T) \cong P/\widehat{T}$ and $\text{Out}_Q(T) \cong Q/\widehat{T}$, then $\widehat{\varphi}$ is conjugation by $[\varphi|_T]$, and hence a morphism in $\mathcal{F}_{\text{Out}_S(T)}(\text{Out}_{\mathcal{F}}(T))$.

Conversely, if conjugation by the class of $\psi \in \text{Aut}_{\mathcal{F}}(T)$ sends $\text{Out}_P(T)$ into $\text{Out}_Q(T)$ for $P, Q \geq \widehat{T}$, then ψ extends to some $\psi^* \in \text{Hom}_{\mathcal{F}}(P, Q)$ since $T \trianglelefteq S$ is receptive (Lemma 1.3(b)), and hence $c_{[\psi]} \in \text{Hom}_{\text{Out}_{\mathcal{F}}(T)}(\text{Out}_P(T), \text{Out}_Q(T))$ is identified with $\widehat{\psi}^* \in \text{Hom}_{\mathcal{F}/\widehat{T}}(P/\widehat{T}, Q/\widehat{T})$. \square

In [BCGLO2, §3], a fusion subsystem $\mathcal{F}_0 \leq \mathcal{F}$ over $U \leq S$ is defined to have *p-power index* if $U \geq \mathfrak{hnp}(\mathcal{F})$ and $\text{Aut}_{\mathcal{F}_0}(P) \geq O^p(\text{Aut}_{\mathcal{F}}(P))$ for each $P \leq U$. By [AKO, Theorem I.7.4], if \mathcal{F} is saturated, then for each $U \leq S$ containing $\mathfrak{hnp}(\mathcal{F})$, there is a unique saturated fusion subsystem $\mathcal{F}_U \leq \mathcal{F}$ over U of *p-power index* in \mathcal{F} , and $\mathcal{F}_U \trianglelefteq \mathcal{F}$ if $U \trianglelefteq S$.

Proposition 1.11. *If \mathcal{F} is a saturated fusion system over a p -group S , and $T \trianglelefteq S$ is strongly closed in \mathcal{F} , then*

$$\mathfrak{hnp}(\mathcal{F}) \leq \{x \in S \mid c_x \in O^p(\text{Aut}_{\mathcal{F}}(T))\text{Inn}(T)\}.$$

*If, in addition, $\mathcal{E} \trianglelefteq \mathcal{F}$ and $\mathcal{F}_0 \trianglelefteq \mathcal{F}$ are normal subsystems over T and U , respectively, where $U \geq T \cdot \mathfrak{hnp}(\mathcal{F})$ and \mathcal{F}_0 has *p-power index* in \mathcal{F} , then $\mathcal{E} \leq \mathcal{F}_0$.*

Proof. Set $\widehat{T} = TC_S(T)$ and $Q = \{x \in S \mid c_x \in O^p(\text{Aut}_{\mathcal{F}}(T))\text{Inn}(T)\} \geq \widehat{T}$, for short, and let $\omega: S/\widehat{T} \xrightarrow{\cong} \text{Out}_S(T)$ be the natural isomorphism. By Lemma 1.10(b), \widehat{T} is strongly closed in \mathcal{F} , and by Lemma 1.10(c), ω induces an isomorphism of fusion systems

$\mathcal{F}/\widehat{T} \cong \mathcal{F}_{\text{Out}_S(T)}(\text{Out}_{\mathcal{F}}(T))$. Puig's hyperfocal theorem for groups now implies that

$$\begin{aligned} \omega(\mathfrak{hnp}(\mathcal{F}/\widehat{T})) &= \mathfrak{hnp}(\mathcal{F}_{\text{Out}_S(T)}(\text{Out}_{\mathcal{F}}(T))) = O^p(\text{Out}_{\mathcal{F}}(T)) \cap \text{Out}_S(T) \\ &= (O^p(\text{Aut}_{\mathcal{F}}(T))\text{Inn}(T) \cap \text{Aut}_S(T))/\text{Inn}(T) \\ &= \text{Aut}_Q(T)/\text{Inn}(T) = \omega(Q/\widehat{T}). \end{aligned}$$

Hence $\mathfrak{hnp}(\mathcal{F}/\widehat{T}) = Q/\widehat{T}$, so $Q = \widehat{T} \cdot \mathfrak{hnp}(\mathcal{F})$ by Lemma 1.10(a).

Now let $U \trianglelefteq S$ be any normal subgroup containing $T \cdot \mathfrak{hnp}(\mathcal{F})$, and assume that \mathcal{E} and \mathcal{F}_0 are normal subsystems in \mathcal{F} over T and U , respectively, where \mathcal{F}_0 has p -power index in \mathcal{F} . If $P \leq T$ is fully normalized in \mathcal{E} , then $\text{Aut}_{\mathcal{E}}(P) = \text{Aut}_T(P)O^p(\text{Aut}_{\mathcal{E}}(P))$ since $\text{Aut}_T(P) \in \text{Syl}_p(\text{Aut}_{\mathcal{E}}(P))$, $\text{Aut}_T(P) \leq \text{Aut}_{\mathcal{F}_0}(P)$ since \mathcal{F}_0 is a fusion system over $U \geq T$, and $O^p(\text{Aut}_{\mathcal{E}}(P)) \leq O^p(\text{Aut}_{\mathcal{F}}(P)) \leq \text{Aut}_{\mathcal{F}_0}(P)$ where the second inclusion holds since $\mathcal{F}_0 \trianglelefteq \mathcal{F}$ has p -power index. Hence $\text{Aut}_{\mathcal{E}}(P) \leq \text{Aut}_{\mathcal{F}_0}(P)$ for all such P , and $\mathcal{E} \leq \mathcal{F}_0$ since \mathcal{E} is generated by such automorphisms by Alperin's fusion theorem. \square

The next lemma will be needed to show that certain subnormal systems are normal, when iterating inductively Propositions 1.8 and 1.11.

Lemma 1.12 ([As, 7.4]). *Let $\mathcal{F}_2 \trianglelefteq \mathcal{F}_1 \trianglelefteq \mathcal{F}$ be saturated fusion systems over p -groups $S_2 \trianglelefteq S_1 \trianglelefteq S$. Assume, for each $\alpha \in \text{Aut}_{\mathcal{F}}(S_1)$, that ${}^{\alpha}\mathcal{F}_2 = \mathcal{F}_2$. Assume also that $C_S(S_2) \leq S_2$. Then $\mathcal{F}_2 \trianglelefteq \mathcal{F}$.*

For an arbitrary saturated fusion system \mathcal{F} , let $\mathcal{F}^{\infty} \trianglelefteq \mathcal{F}$ be the limit after applying $O^p(-)$ and $O^{p'}(-)$ until the sequence stabilizes. More precisely, set $\mathcal{F}^{\infty} = \bigcap_{i=0}^{\infty} \mathcal{F}^{(i)}$, where the sequence $\{\mathcal{F}^{(i)}\}$ is defined by setting $\mathcal{F}^{(0)} = \mathcal{F}$ and $\mathcal{F}^{(i+1)} = O^{p'}(O^p(\mathcal{F}^{(i)}))$.

Lemma 1.13. *Let $\mathcal{F} = \mathcal{F}_0 > \mathcal{F}_1 > \cdots > \mathcal{F}_m = \mathcal{E}$ be any sequence of saturated fusion systems, each normal in \mathcal{F} and each of p -power index or of index prime to p in the preceding one. Then $\mathcal{F}^{\infty} = \mathcal{E}^{\infty}$.*

Proof. We prove that $\mathcal{E}^{\infty} = \mathcal{F}^{\infty}$ by induction on $|\text{Mor}(\mathcal{F})|$. In particular, it suffices to do this when $m = 1$. If $\mathcal{E} \trianglelefteq \mathcal{F}$ has p -power index, then $O^p(\mathcal{E}) = O^p(\mathcal{F})$ [Cr, Theorem 7.53(ii)], and $\mathcal{E}^{\infty} = \mathcal{F}^{\infty}$ by definition. So assume \mathcal{E} has index prime to p in \mathcal{F} . Set $T = \mathfrak{hnp}(\mathcal{F})$. Since $\mathfrak{hnp}(\mathcal{E}) \leq T$ by definition, there is a unique normal subsystem $\mathcal{E}_T \trianglelefteq \mathcal{E}$ of p -power index over T , and by [AKO, Theorem I.7.4], $\mathcal{E}_T = \langle \text{Inn}(T), O^p(\text{Aut}_{\mathcal{E}}(P)) \mid P \leq T \rangle \leq O^p(\mathcal{F})$. Also, $\mathcal{E}_T \trianglelefteq \mathcal{F}$ by Lemma 1.12 and the uniqueness of \mathcal{E}_T (and since $T \trianglelefteq S$). Hence $\mathcal{E}_T \trianglelefteq O^p(\mathcal{F})$, and \mathcal{E}_T has index prime to p in $O^p(\mathcal{F})$ since they are both fusion systems over T [AOV, Lemma 1.26]. Hence

$$\mathcal{E}^{\infty} = (\mathcal{E}_T)^{\infty} = (O^{p'}(O^p(\mathcal{F})))^{\infty} = \mathcal{F}^{\infty},$$

where the second equality holds by the induction hypothesis. \square

We are now ready to combine Propositions 1.8 and 1.11 to get the following:

Theorem 1.14. *Let $\mathcal{E} \trianglelefteq \mathcal{F}$ be saturated fusion systems over p -groups $T \trianglelefteq S$. Assume that $\text{Aut}_{\mathcal{F}}(T)/\text{Aut}_{\mathcal{E}}(T)$ is p -solvable (equivalently, that $\text{Out}_{\mathcal{F}}(T)$ is p -solvable). Then there is a normal saturated subsystem $\mathcal{F}_0 \trianglelefteq \mathcal{F}$ over $TC_S(T)$, such that $\mathcal{F}_0 \geq \mathcal{E}$, $\mathcal{F}_0 \geq \mathcal{F}^{\infty}$, $(\mathcal{F}_0)^{\infty} = \mathcal{F}^{\infty}$, and $\text{Aut}_{\mathcal{F}_0}(T) = \text{Aut}_{\mathcal{E}}(T)$.*

Proof. Choose a sequence of subgroups $\text{Aut}_{\mathcal{F}}(T) = G_m > G_{m-1} > \cdots > G_0 = \text{Aut}_{\mathcal{E}}(T)$, all normal in G_m , and such that for each i , G_i/G_{i-1} is a p -group or a p' -group. For each i , set $S_i = N_S^{G_i}(T) = \{x \in S \mid c_x \in G_i\}$. Thus $S_i \trianglelefteq S$ and $\text{Aut}_{S_i}(T) \in \text{Syl}_p(G_i)$ for each i , $S_m = S$, and $S_0 = TC_S(T)$.

We claim that there are subsystems $\mathcal{F} = \mathcal{F}_m > \mathcal{F}_{m-1} > \cdots > \mathcal{F}_0 = \mathcal{E}$ in \mathcal{F} , each normal in \mathcal{F} and of p -power index or of index prime to p in the preceding one, where \mathcal{F}_i is over S_i , and such that $\text{Aut}_{\mathcal{F}_i}(T) = G_i$ for each i . If this holds, then $(\mathcal{F}_0)^\infty = \mathcal{F}^\infty$ by Lemma 1.13, and \mathcal{F}_0 satisfies all of the other conditions listed above.

Assume $\mathcal{F}_i \trianglelefteq \mathcal{F}$ has been constructed as claimed, with $i \geq 1$ and $\mathcal{F}_i \geq \mathcal{E}$. If G_i/G_{i-1} is a p' -group, then $S_{i-1} = S_i$. Let $\mathcal{F}_{i-1} \trianglelefteq \mathcal{F}_i$ be as in Proposition 1.8. In particular, \mathcal{F}_{i-1} has index prime to p in \mathcal{F}_i and contains \mathcal{E} . For each $\alpha \in \text{Aut}_{\mathcal{F}}(S_i)$, $\alpha|_T \in \text{Aut}_{\mathcal{F}}(T) = G_m$ normalizes $G_{i-1} = \text{Aut}_{\mathcal{F}_{i-1}}(T)$, and hence α normalizes \mathcal{F}_{i-1} by the uniqueness in Proposition 1.8. So $\mathcal{F}_{i-1} \trianglelefteq \mathcal{F}$ by Lemma 1.12.

If G_i/G_{i-1} is a p -group, then $\text{hnp}(\mathcal{F}_i) \leq S_{i-1}$ by Proposition 1.11, so there is a unique normal subsystem $\mathcal{F}_{i-1} \trianglelefteq \mathcal{F}_i$ over S_{i-1} of p -power index (see [AKO, Theorem I.7.4]), and $\mathcal{F}_{i-1} \geq \mathcal{E}$ by Proposition 1.11 again. For each $\alpha \in \text{Aut}_{\mathcal{F}}(S_i)$, $\alpha(S_{i-1}) = S_{i-1}$ since $S_{i-1} = N_S^{G_{i-1}}(T)$ and $\alpha|_T \in G_m$ normalizes G_{i-1} , so α normalizes \mathcal{F}_{i-1} by its uniqueness. Thus $\mathcal{F}_{i-1} \trianglelefteq \mathcal{F}$ by Lemma 1.12. \square

2. REDUCTIONS TO CENTRIC NORMAL SUBSYSTEMS

In order to get further results, we must also work with the linking system associated to a fusion system. When G is a group and $P, Q \leq G$, we set $T_G(P, Q) = \{g \in G \mid {}^gP \leq Q\}$ (the *transporter set*). When \mathcal{F} is a saturated fusion system over a p -group S , a *centric linking system* associated to \mathcal{F} is a category \mathcal{L} with $\text{Ob}(\mathcal{L}) = \mathcal{F}^c$, the set of \mathcal{F} -centric subgroups of S , together with a pair of functors

$$\mathcal{T}_{\mathcal{F}^c}(S) \xrightarrow{\delta} \mathcal{L} \xrightarrow{\pi} \mathcal{F}$$

which satisfy certain axioms listed in [AKO, Definition III.4.1] and [BLO2, Definition 1.7]. Here, $\mathcal{T}_{\mathcal{F}^c}(S)$ is the *transporter category* of S : the category with object set \mathcal{F}^c , and with $\text{Mor}_{\mathcal{T}_{\mathcal{F}^c}(S)}(P, Q) = T_S(P, Q)$ (where composition is given by multiplication in S). Also, δ is the identity on objects and injective on morphism sets, π is the inclusion on objects and surjective on morphism sets, and $\pi \circ \delta$ sends $g \in T_S(P, Q)$ to $c_g \in \text{Hom}_{\mathcal{F}}(P, Q)$. The motivating example is the category $\mathcal{L}_S^c(G)$, when G is a finite group and $S \in \text{Syl}_p(G)$, where

$$\begin{aligned} \text{Ob}(\mathcal{L}_S^c(G)) &= \mathcal{F}_S(G)^c = \{P \leq S \mid C_G(P) = Z(P) \times O_{p'}(C_G(P))\} \\ \text{Mor}_{\mathcal{L}_S^c(G)}(P, Q) &= T_G(P, Q)/O_{p'}(C_G(P)). \end{aligned}$$

By comparison, note that $\text{Hom}_{\mathcal{F}_S(G)}(P, Q) \cong T_G(P, Q)/C_G(P)$.

Since we are working with extensions of fusion and linking systems, we also need to handle their automorphism groups. Automorphisms of fusion systems are straightforward. When \mathcal{F} is a saturated fusion system over a p -group S , an automorphism $\alpha \in \text{Aut}(S)$ is said to be *fusion preserving* if it induces an automorphism of the category \mathcal{F} , and we set

$$\begin{aligned} \text{Aut}(S, \mathcal{F}) &= \{\alpha \in \text{Aut}(S) \mid \alpha \text{ is fusion preserving}\} \\ \text{Out}(S, \mathcal{F}) &= \text{Aut}(S, \mathcal{F})/\text{Aut}_{\mathcal{F}}(S). \end{aligned}$$

Let \mathcal{L} be a centric linking system associated to \mathcal{F} , and let δ be the functor described above. For each $P \in \mathcal{F}^c = \text{Ob}(\mathcal{L})$, set $\iota_P = \delta_{P,S}(1) \in \text{Hom}_{\mathcal{L}}(P, S)$ (the ‘‘inclusion’’ of P in

S in the category \mathcal{L}), and set $\llbracket P \rrbracket = \delta_P(P) \leq \text{Aut}_{\mathcal{L}}(P)$. Define

$$\begin{aligned} \text{Aut}_{\text{typ}}^I(\mathcal{L}) &= \{ \beta \in \text{Aut}(\mathcal{L}) \mid \beta(\iota_P) = \iota_{\beta(P)}, \beta(\llbracket P \rrbracket) = \llbracket \beta(P) \rrbracket, \forall P \in \mathcal{F}^c \} \\ \text{Out}_{\text{typ}}^I(\mathcal{L}) &= \text{Aut}_{\text{typ}}^I(\mathcal{L}) / \langle c_\gamma \mid \gamma \in \text{Aut}_{\mathcal{L}}(S) \rangle. \end{aligned}$$

There is a natural homomorphism $\mu_{\mathcal{L}}: \text{Out}_{\text{typ}}^I(\mathcal{L}) \longrightarrow \text{Out}(S, \mathcal{F})$, defined by restriction to $\llbracket S \rrbracket \leq \text{Aut}_{\mathcal{L}}(S)$. We refer to [AKO, § III.4.3] or [AOV, § 1.3] for more details on these definitions.

When \mathcal{F} is a saturated fusion system over a p -group S , it is straightforward to define the centralizer fusion system $C_{\mathcal{F}}(P)$ of a subgroup $P \leq S$ (see [AKO, Definition I.5.3]): this is a fusion subsystem over $C_S(P)$ which is saturated if P is fully centralized in \mathcal{F} (i.e., receptive). One can also define $C_{\mathcal{F}}(\mathcal{E})$ when \mathcal{E} is a normal subsystem in \mathcal{F} (see [As, Chapter 6]), but this is much more complicated. For our purposes here, it will suffice to work with the following somewhat simpler definition. If $\mathcal{E} \trianglelefteq \mathcal{F}$ are saturated fusion systems over $T \trianglelefteq S$, then by [As, 6.7], there is a (unique) subgroup $C_S(\mathcal{E}) \leq C_S(T)$ with the property that for $P \leq C_S(T)$, $P \leq C_S(\mathcal{E})$ if and only if $\mathcal{E} \leq C_{\mathcal{F}}(P)$.

Proposition 2.1. *Let $\mathcal{E} \trianglelefteq \mathcal{F}$ be a pair of saturated fusion systems over the p -groups $T \trianglelefteq S$ such that $C_S(\mathcal{E}) \leq T$. Let \mathcal{L} be a centric linking system associated to \mathcal{E} . Then the natural homomorphism $S \longrightarrow \text{Aut}(T, \mathcal{E})$ lifts to a homomorphism $\tilde{\omega}: S \longrightarrow \text{Aut}_{\text{typ}}^I(\mathcal{L})$, and this factors through an injective homomorphism $\omega: S/T \longrightarrow \text{Out}_{\text{typ}}^I(\mathcal{L})$.*

Proof. Since the conclusion of the proposition involves only S and \mathcal{E} , we can assume that $\mathcal{F} = S\mathcal{E}$ as defined in [As, Theorem 5, Chapter 8]. In particular, $\mathcal{E} \geq O^p(\mathcal{F})$. So by [AOV, Proposition 1.31(a)], there is a pair of linking systems $\mathcal{L} \trianglelefteq \mathcal{L}^*$ associated to $\mathcal{E} \trianglelefteq \mathcal{F}$, where $\text{Ob}(\mathcal{L}) = \mathcal{E}^c$, and $\text{Ob}(\mathcal{L}^*)$ is the set of all $P \leq S$ such that $P \cap T \in \text{Ob}(\mathcal{L})$. (This was shown in [AOV] only when $\mathcal{E} = O^p(\mathcal{F})$, but the same argument applies in this situation.)

This inclusion $\mathcal{L} \trianglelefteq \mathcal{L}^*$ induces a natural homomorphism $\tilde{\omega}$ from S to $\text{Aut}_{\text{typ}}^I(\mathcal{L})$, defined by conjugation in \mathcal{L}^* , and which factors through a homomorphism

$$\omega: S/T \longrightarrow \text{Out}_{\text{typ}}^I(\mathcal{L}) = \text{Aut}_{\text{typ}}^I(\mathcal{L}) / \{ c_\gamma \mid \gamma \in \text{Aut}_{\mathcal{L}}(T) \}.$$

Assume ω is not injective, and let $x \in S \setminus T$ be such that $xT \in \text{Ker}(\omega)$. Thus $\tilde{\omega}(x) = c_{\delta_T(x)}$ is conjugation by some element $\gamma \in \text{Aut}_{\mathcal{L}}(T)$. Since $\tilde{\omega}(x)$ has p -power order, we can assume that γ has p -power order, and hence $\gamma = \delta_T(y)$ for some $y \in T$. Upon replacing x by xy^{-1} , we can arrange that $c_{\delta_T(x)} = \text{Id}_{\mathcal{L}}$, and hence that $\delta_T(x)$ and its restrictions commute with all morphisms in \mathcal{L} . In particular, $x \in C_S(T)$.

Fix $P, Q \leq T$ and $\psi \in \text{Iso}_{\mathcal{L}}(P, Q)$. We just showed that $\delta_Q(x)\psi = \psi\delta_P(x)$. So by [Ol, Proposition 4.e], ψ extends to an isomorphism $\bar{\psi} \in \text{Iso}_{\mathcal{L}^*}(P\langle x \rangle, Q\langle x \rangle)$. Set $y = \pi(\bar{\psi})(x)$, where $\pi(\bar{\psi}) \in \text{Hom}_{\mathcal{F}}(P\langle x \rangle, Q\langle x \rangle)$. By axiom (C) for a linking system [AKO, Definition III.4.1],

$$\bar{\psi} \circ \delta_{P\langle x \rangle}(x) = \delta_{Q\langle x \rangle}(y) \circ \bar{\psi}.$$

But this is also equal to $\delta_{Q\langle x \rangle}(x) \circ \bar{\psi}$ since extensions are unique in a linking system [Ol, Proposition 4.e or 4.f], and so $\delta_{Q\langle x \rangle}(x) = \delta_{Q\langle x \rangle}(y)$. Since δ is injective by [Ol, Proposition 4.c], we have $x = y = \pi(\bar{\psi})(x)$.

Thus all isomorphisms in \mathcal{E} between objects in \mathcal{L} extend to morphisms in \mathcal{F} which send x to itself. Since $\text{Ob}(\mathcal{L}) = \mathcal{E}^c$, this statement holds for all morphisms in \mathcal{E} by Alperin's fusion theorem. So $\mathcal{E} \leq C_{\mathcal{F}}(x)$, hence $x \in C_S(\mathcal{E})$, which contradicts the assumption that $C_S(\mathcal{E}) \leq T$. We conclude that ω is injective. \square

We saw in Theorem 1.14 the importance of getting control of the quotient group $TC_S(T)/T$, when $\mathcal{E} \trianglelefteq \mathcal{F}$ are saturated fusion systems over $T \trianglelefteq S$.

Corollary 2.2. *Let $\mathcal{E} \trianglelefteq \mathcal{F}$ be saturated fusion systems over $T \trianglelefteq S$ such that $C_S(\mathcal{E}) \leq T$. Then $TC_S(T)/T$ is abelian, and $C_S(T) \leq T$ if p is odd.*

Proof. Let \mathcal{L} be a centric linking system associated to \mathcal{E} . Consider the homomorphisms

$$S/T \xrightarrow{\omega} \text{Out}_{\text{typ}}(\mathcal{L}) \xrightarrow{\mu=\mu_{\mathcal{L}}} \text{Out}(T, \mathcal{E}) \leq \text{Aut}(T)/\text{Aut}_{\mathcal{E}}(T)$$

where ω is injective by Proposition 2.1 and $\mu(\omega(xT)) = [c_x]$ for $x \in S$. Thus $TC_S(T)/T = \text{Ker}(\mu \circ \omega)$ injects into $\text{Ker}(\mu)$. In particular, $C_S(T) \leq T$ if μ is injective, and this always holds if p is odd by [O2, Theorem C] and [GL]. Otherwise, $TC_S(T)/T$ is abelian since $\text{Ker}(\mu)$ is abelian (see [AKO, Proposition III.5.12]). \square

We are now ready to prove our main result, which says that under appropriate conditions on $\mathcal{E} \trianglelefteq \mathcal{F}$, \mathcal{F} reduces down to \mathcal{E} in the sense which we have been studying.

Theorem 2.3. *Let $\mathcal{E} \trianglelefteq \mathcal{F}$ be saturated fusion systems over $T \trianglelefteq S$ such that $C_S(\mathcal{E}) \leq T$. Assume either*

- (a) $\text{Aut}_{\mathcal{F}}(T)/\text{Aut}_{\mathcal{E}}(T)$ is p -solvable; or
- (b) $\text{Out}(T, \mathcal{E})$ is p -solvable.

Then $\mathcal{F}^{\infty} = \mathcal{E}^{\infty}$.

Proof. Since

$$\text{Aut}_{\mathcal{F}}(T)/\text{Aut}_{\mathcal{E}}(T) \leq \text{Aut}(T, \mathcal{E})/\text{Aut}_{\mathcal{E}}(T) = \text{Out}(T, \mathcal{E}),$$

$\text{Aut}_{\mathcal{F}}(T)/\text{Aut}_{\mathcal{E}}(T)$ is p -solvable if $\text{Out}(T, \mathcal{E})$ is p -solvable, and thus (b) implies (a). So from now on, we assume $\text{Aut}_{\mathcal{F}}(T)/\text{Aut}_{\mathcal{E}}(T)$ is p -solvable. By Theorem 1.14, it suffices to prove this when $S = TC_S(T)$ and $\text{Aut}_{\mathcal{F}}(T) = \text{Aut}_{\mathcal{E}}(T)$. So by Corollary 2.2, S/T is abelian.

Set $\mathcal{H} = \{P \leq S \mid P \geq C_S(T)\}$. If $P \in \mathcal{F}^c$ and $P \notin \mathcal{H}$, then $PC_S(T) > P$, so $N_{PC_S(T)}(P) > P$, and there is $x \in N_{PC_S(T)}(P) \setminus P$. Then $c_x \in \text{Aut}_S(P)$ induces the identity on $P \cap T$ and on $P/(P \cap T)$, so $c_x \in O_p(\text{Aut}_{\mathcal{F}}(P))$. Also, $c_x \notin \text{Inn}(P)$ since $x \notin P$ and $P \in \mathcal{F}^c$, so $1 \neq [c_x] \in \text{Out}_S(P) \cap O_p(\text{Out}_{\mathcal{F}}(P))$.

Let $(\Psi, \widehat{\Psi}): \mathcal{F} \rightarrow \mathcal{F}/T$ be the morphism of fusion systems which sends P to PT/T and $\varphi \in \text{Mor}(\mathcal{F})$ to the induced homomorphism between quotient groups [AKO, Theorem II.5.12]. Let $\mathcal{F}^* \subseteq \mathcal{F}^c$ be the full subcategory with objects in $\mathcal{H} \cap \mathcal{F}^c$. By definition, for each $P \in \mathcal{H}$, $PT = TC_S(T) = S$ since $P \geq C_S(T)$. Define $\chi: \text{Mor}(\mathcal{F}^*) \rightarrow \text{Aut}_{\mathcal{F}/T}(S/T)$ by sending $\varphi \in \text{Mor}(\mathcal{F}^*)$ to $\widehat{\Psi}(\varphi) \in \text{Aut}_{\mathcal{F}/T}(S/T)$. This clearly satisfies conditions (iii)–(v) in Lemma 1.6. Since we just checked condition (ii) in the lemma, and (i) is clear, we conclude that there is a normal fusion subsystem $\mathcal{F}_0 \trianglelefteq \mathcal{F}$ containing $O^{p'}(\mathcal{F})$ such that $\text{Aut}_{\mathcal{F}_0}(S) = \text{Ker}(\chi|_{\text{Aut}_{\mathcal{F}}(S)})$ and hence $\text{Aut}_{\mathcal{F}_0/T}(S/T) = 1$.

Since S/T is abelian and $\text{Aut}_{\mathcal{F}_0/T}(S/T) = 1$, we have $\mathcal{F}_0/T = \mathcal{F}_{S/T}(S/T)$. So $\text{hnp}(\mathcal{F}_0) \leq T$ by Lemma 1.10(a). By [AKO, Theorem I.7.4], there is a unique fusion subsystem $\mathcal{F}_1 \trianglelefteq \mathcal{F}_0$ over T of p -power index. Also, $\mathcal{E} \trianglelefteq \mathcal{F}_1$ by Proposition 1.11, so \mathcal{E} has index prime to p in \mathcal{F}_1 by [AOV, Lemma 1.26]. Thus $\mathcal{F}^{\infty} = (\mathcal{F}_0)^{\infty} = (\mathcal{F}_1)^{\infty} = \mathcal{E}^{\infty}$ by Lemma 1.13. \square

To explain the motivation for Theorem 2.3, we recall some definitions in [AOV]. When $\mathcal{F} = \mathcal{F}_S(G)$ and $\mathcal{L} = \mathcal{L}_S^c(G)$ for some finite group G with $S \in \text{Syl}_p(G)$, there is a natural

homomorphism

$$\mathrm{Out}(G) \cong N_{\mathrm{Aut}(G)}(S) / \mathrm{Aut}_{N_G(S)}(G) \xrightarrow{\kappa_G} \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}),$$

defined by sending $\alpha \in N_{\mathrm{Aut}(G)}(S)$ to the automorphism of \mathcal{L} induced by α . See [AOV, §2.2] for more details. The fusion system \mathcal{F} is *tamely realized* by G if $\mathcal{F} \cong \mathcal{F}_S(G)$ and κ_G is split surjective, and \mathcal{F} is tame if it is tamely realized by some finite group.

Finally, a saturated fusion system \mathcal{F} is *reduced* if $O_p(\mathcal{F}) = 1$ and $O^p(\mathcal{F}) = \mathcal{F} = O^{p'}(\mathcal{F})$. If \mathcal{F} is any fusion system, and $Q = O_p(\mathcal{F})$, set $\mathrm{red}(\mathcal{F}) = (C_{\mathcal{F}}(Q)/Z(Q))^\infty$: the *reduction* of \mathcal{F} . By [AOV, Proposition 2.2], $\mathrm{red}(\mathcal{F})$ is reduced for any saturated fusion system \mathcal{F} . By [AOV, Theorem A], \mathcal{F} is tame if its reduction is tame.

Corollary 2.4. *Let $\mathcal{E} \trianglelefteq \mathcal{F}$ be saturated fusion systems over $T \trianglelefteq S$, where $C_S(\mathcal{E}) \leq T$, and where \mathcal{E} is simple and is tamely realized by a known simple group K . Then $\mathcal{F}^\infty = \mathcal{E}$, and \mathcal{F} is tamely realized by an extension of K .*

Proof. By the Schreier conjecture (see [GLS3, Theorem 7.1.1]), $\mathrm{Out}(K)$ is solvable. Set $\mathcal{L} = \mathcal{L}_T^c(K)$. Then $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$ is solvable since κ_K is surjective, and $\mathrm{Out}(T, \mathcal{E})$ is solvable since $\mu_{\mathcal{L}}$ is surjective by [O2, Theorem C] and [GL]. Hence $\mathcal{F}^\infty = \mathcal{E}^\infty$ by Theorem 2.3, $\mathrm{red}(\mathcal{F}) = \mathcal{E}^\infty = \mathcal{E}$ since \mathcal{E} is simple, and so \mathcal{F} is tame by [AOV, Theorem 2.20]. More precisely, \mathcal{F} is tamely realized by an extension of K by successive applications of [AOV, Proposition 2.16], together with the existence of “compatible” linking systems as made precise in the proof of [AOV, Theorem 2.20]. \square

One can take this further by stating it in terms of the generalized Fitting subsystem $F^*(\mathcal{F})$ of a saturated fusion system \mathcal{F} [As, 9.9].

Corollary 2.5. *Let \mathcal{F} be a saturated fusion system. Assume that $F^*(\mathcal{F}) = O_p(\mathcal{F})\mathcal{E}$ (a central product), where $\mathcal{E} \trianglelefteq \mathcal{F}$ is quasisimple, and where $\mathcal{E}/Z(\mathcal{E})$ is tamely realized by a known simple group K . Then $\mathrm{red}(\mathcal{F}) \cong \mathcal{E}/Z(\mathcal{E})$, and \mathcal{F} is tamely realized by a finite group G such that $F^*(G) = O_p(G)H$, where H is quasisimple and $H/Z(H) \cong K$.*

Proof. Set $Q = O_p(\mathcal{F})$. Then $C_{F^*(\mathcal{F})}(Q)/Z(Q) \cong \mathcal{E}/Z(\mathcal{E})$ is simple, and hence the pair $C_{F^*(\mathcal{F})}(Q)/Z(Q) \trianglelefteq C_{\mathcal{F}}(Q)/Z(Q)$ satisfies the hypotheses of Corollary 2.4. So by that corollary, $\mathrm{red}(\mathcal{F}) \cong \mathcal{E}/Z(\mathcal{E})$, and $C_{\mathcal{F}}(Q)/Z(Q)$ is tamely realized by an extension of K . Together with [AOV, Theorem 2.20] (and its proof), this implies that \mathcal{F} itself is tamely realized by a finite group G of the form described above. \square

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