THE SIMPLE CONNECTIVITY OF $B\mathbf{Sol}(q)$

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ABSTRACT. A p-local finite group is an algebraic structure which includes two categories, a fusion system and a linking system, which mimic the fusion and linking categories of a finite group over one of its Sylow subgroups. The p-completion of the geometric realization of the linking system is the classifying space of the finite group. In this paper, we study the geometric realization, *without* completion, of linking systems of certain exotic 2-local finite groups whose existence was predicted by Solomon and Benson, and prove that they are all simply connected.

A p-local finite group consists of a finite p-group S together with a pair of categories $\mathcal{F}$ and $\mathcal{L}$ — the *fusion system* and the *centric linking system* — with auxiliary structures which relate $\mathcal{F}$ and $\mathcal{L}$. The idea is to mimic the structure of a finite group G having S as a Sylow p-subgroup, by first providing, by means of the fusion system $\mathcal{F}$, a collection of maps between subgroups of S which are consistent with the notion of conjugation by elements of G, and then, with the linking system $\mathcal{L}$, providing a collection of candidates for the G-normalizers of a large class of subgroups of S. The resulting object $(S, \mathcal{F}, \mathcal{L})$ should be indistinguishible from such a finite group G, at least from an algebraic point of view which takes only "p-local structure" into account. From the homotopy-theoretic viewpoint, the p-completion $|\mathcal{L}|^{\wedge}_p$ of the topological realization of $\mathcal{L}$ should be indistinguishible from the p-completion of a classifying space BG. In the case that these structures really do arise from a finite group G with Sylow p-subgroup S, we may denote the system $(S, \mathcal{F}, \mathcal{L})$ by $\mathcal{G}_S(G)$. If no such G exists, one says that $\mathcal{L}$ and the p-local finite group $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ are *exotic*.

This paper concerns the family $\mathrm{Sol}(q)$ of exotic 2-local finite groups – q an arbitrary odd prime power – constructed by Ran Levi and the second named author in [LO]. These objects were prefigured in a paper of David Benson [Be] and, earlier still, in work of Ron Solomon [So]; and they are the only exotic 2-local finite groups that are known to exist. They are called the "Solomon 2-local finite groups" in recognition that it was Solomon [So] who first discovered that there was a collection of group-like data which was internally consistent from a 2-local point of view, and which was not derivable from any finite group.

The classifying space of a p-local finite group $(S, \mathcal{F}, \mathcal{L})$ is defined to be the space $|\mathcal{L}|^{\wedge}_p$: the p-completion of the geometric realization of the category $\mathcal{L}$. This was originally motivated by the observation in [BLO1, Proposition 1.1] that when $\mathcal{L}$ is the linking system of a finite group G, then $|\mathcal{L}|^{\wedge}_p$ has the homotopy type of $BG^{\wedge}_p$; and also because whether or not $\mathcal{L}$ is associated to a group, $|\mathcal{L}|^{\wedge}_p$ shares many of the homotopy theoretic properties of p-completed spaces of finite groups. However, interest has recently been growing in the geometric realization $|\mathcal{L}|$ without p-completion, and in particular in its

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fundamental group, as an invariant of a p-local finite group $(S, \mathcal{F}, \mathcal{L})$. This has been spurred on by questions and conjectures formulated by Jesper Grodal.

Two general references for the geometric realization of a category are Segal's original paper [Se, §1-2], and the more recent book of Srinivas [Sr, Chapter 3]. In general, when $\mathcal{C}$ is a discrete, small category, $c_0 \in \mathrm{Ob}(\mathcal{C})$, and $\mathcal{I}$ is a set of morphisms in $\mathcal{C}$ which includes exactly one morphism between c_0 and each other object, then the fundamental group $\pi_1(|\mathcal{C}|)$ can be described algebraically as the group generated by $\mathrm{Mor}(\mathcal{C})$, modulo the relations given by composition, and modulo the relations given by setting morphisms in $\mathcal{I}$ equal to the identity. In the case of a linking system $\mathcal{L}$, we take c_0 to be the "Sylow subgroup" $S \in \mathrm{Ob}(\mathcal{C})$, and take $\mathcal{I}$ to be a set of "inclusion" morphisms to S.

When $\mathcal{L}$ is the linking system associated to a finite group G, then in many cases, $\pi_1(|\mathcal{L}|)$ is either isomorphic to G or surjects onto G. This is discussed briefly in Section 1, and several other examples will be given in the paper [GO] now in preparation. This connection with the underlying finite group, when there is one, made it natural to look at the fundamental groups of exotic linking systems.

The principal aim of this paper is to study the topological realizations of the linking systems of the Solomon 2-local groups, and to show that they are simply connected. More precisely, we show:

Theorem A. *For every odd prime power q, the geometric realization of the linking system $\mathcal{L}^c_{\mathrm{Sol}}(q)$ is simply connected.*

This will be proven as Theorem 5.1. These are the first (and only) examples we know of linking systems whose nerves are simply connected. In fact, these are the only examples we know where the automorphism groups in $\mathcal{L}$ do not all map injectively into $\pi_1(|\mathcal{L}|)$.

In [LO], an infinite "linking system" $\mathcal{L}^c_{\mathrm{Sol}}(p^\infty)$ was constructed for all odd primes p, roughly as the union of the $\mathcal{L}^c_{\mathrm{Sol}}(p^n)$ (taken over all n), and its 2-completed nerve was shown to have the homotopy type of the Dwyer-Wilkerson space $BDI(4)$ [DW]. One consequence of Theorem A is that $|\mathcal{L}^c_{\mathrm{Sol}}(p^\infty)|$ is also simply connected (Corollary 5.6).

When proving Theorem A, the first step is to show that if $|\mathcal{L}^c_{\mathrm{Sol}}(q)|$ is simply connected, then for all $n \geq 1$, $|\mathcal{L}^c_{\mathrm{Sol}}(q^n)|$ is also simply connected. This is fairly straightforward and simple. The following theorem then allows us to reduce the proof to showing that the topological realization of the linking system for Sol(3) is simply connected.

Theorem B. *Let q and q' be odd prime powers. Then the fusion systems $\mathcal{F}_{\mathrm{Sol}}(q)$ and $\mathcal{F}_{\mathrm{Sol}}(q')$, and also their associated linking systems $\mathcal{L}^c_{\mathrm{Sol}}(q)$ and $\mathcal{L}^c_{\mathrm{Sol}}(q')$, are isomorphic if and only if q^2-1 and q'^2-1 have the same 2-adic valuation.*

Theorem B will be shown below as Theorem 3.4, where we give a purely algebraic proof of the result. It also follows from a result of Broto and Møller [BM, Theorem C], when combined with [BLO2, Theorem A] which says that the homotopy type of the classifying space of a p-local finite group determines its homotopy type. However, Broto and Møller state this result only for odd fusion (the general result follows by the same argument and will appear in a later paper), and their proof uses some deep results in homotopy theory. Hence our decision to include a purely algebraic proof here.

An easy induction argument shows that if a is an odd integer such that $v_2(a \pm 1) = m \geq 2$, then $v_2(a^{2^k}-1) = m+k$ for all $k \geq 1$. Hence another consequence of Theorem B is that the methods in [AC] apply to construct all of the Solomon 2-local finite groups: since in that paper, the fusion and linking systems $\mathcal{F}_{\mathrm{Sol}}(q)$ and $\mathcal{L}^c_{\mathrm{Sol}}(q)$ are defined only when q is a power of a prime $p \equiv 3, 5 \pmod 8$.

As mentioned above, when $\mathcal{L}$ is a linking system, $\pi_1(|\mathcal{L}|)$ is the free group on the morphisms in $\mathcal{L}$ modulo certain relations given (roughly) by composition and inclusions. Thus the main problem when proving Theorem A is to find enough relations among the morphisms to show that they all vanish. In [AC], $\mathcal{L}^c_{\mathrm{Sol}}(3)$ (or its fundamental group) is shown to contain a certain amalgam of three maximal subgroups of the sporadic simple group Co_3. This allows us to reduce the proof of Theorem A to the following result, which is proven by using computer computations to show that a certain simplicial complex is simply connected:

Theorem C. *Let H_1, H_2, and H_3 be the three maximal overgroups of a fixed Sylow subgroup $S \in \mathrm{Syl}_2(Co_3)$, and let $\mathcal{G}$ be the amalgam formed by the H_i and their intersections. Then* $\mathrm{colim}(\mathcal{G}) \cong Co_3$.

Theorem C is proven as Proposition 4.1.

We would like to thank Jesper Grodal for first getting us interested in this question; this paper is in some sense an offshoot of the paper [GO] by Grodal and the second author. Particular thanks go to the mathematics department at Cal Tech, and especially Michael Aschbacher, for their hospitality in giving the first two authors, and later the first and third authors, a chance to meet and discuss these problems. Some of the key ideas in this paper were developped there. The second author would also like to thank the Mittag-Leffler Institute for providing ideal conditions for him to finish his share of the work on this paper.

1. Background

We first recall the definition of a (saturated) fusion system. This definition is originally due to [Pg], although it is presented here in the simpler, but equivalent, form given in [BLO2].

We first fix some general notation. For any group G, and any pair of subgroups $H, K \le G$, we set
$$N_G(H,K) = \{x \in G \,|\, xHx^{-1} \le K\},$$
let c_x denote conjugation by x *on the left* ($c_x(g) = xgx^{-1}$), and set
$$\mathrm{Hom}_G(H,K) = \big\{c_x \in \mathrm{Hom}(H,K) \,\big|\, x \in N_G(H,K)\big\} \cong N_G(H,K)/C_G(H).$$
By analogy, we also write $\mathrm{Aut}_G(H) = \mathrm{Hom}_G(H,H) \cong N_G(H)/C_G(H)$.

A *fusion system* over a finite p-group S is a category $\mathcal{F}$, where $\mathrm{Ob}(\mathcal{F})$ is the set of all subgroups of S, where each morphism set $\mathrm{Hom}_{\mathcal{F}}(P,Q)$ is a set of group monomorphisms from P to Q which contains $\mathrm{Hom}_S(P,Q)$, and where each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ is the composite of an isomorphism in $\mathcal{F}$ followed by an inclusion. Two subgroups $P, Q \le S$ are said to be *$\mathcal{F}$-conjugate* if they are isomorphic as objects of the category $\mathcal{F}$. A subgroup $P \le S$ is *fully centralized in* $\mathcal{F}$ if $|C_S(P)| \ge |C_S(P')|$ for all $P' \le S$ which is $\mathcal{F}$-conjugate to P. Similarly, a subgroup $P \le S$ is *fully normalized in* $\mathcal{F}$ if $|N_S(P)| \ge |N_S(P')|$ for all $P' \le S$ which is $\mathcal{F}$-conjugate to P.

A fusion system $\mathcal{F}$ is called *saturated* if the following two conditions hold:

(I) For each $P \leq S$ which is fully normalized in $\mathcal{F}$, P is fully centralized in $\mathcal{F}$ and $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$.

(II) If $P \leq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ are such that φP is fully centralized, and if we set

$$N_\varphi = \{g \in N_S(P) \mid \varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(\varphi P)\},$$

then there is $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ such that $\bar{\varphi}|_P = \varphi$.

If G is a finite group and $S \in \mathrm{Syl}_p(G)$, then by [BLO2, Proposition 1.3], the category $\mathcal{F}_S(G)$, defined by letting $\mathrm{Ob}(\mathcal{F}_S(G))$ be the set of all subgroups of S and setting $\mathrm{Mor}_{\mathcal{F}_S(G)}(P, Q) = \mathrm{Hom}_G(P, Q)$, is a saturated fusion system.

Again let $\mathcal{F}$ be an abstract saturated fusion system over a p-group S. A subgroup $P \leq S$ is *$\mathcal{F}$-centric* if $C_S(P') = Z(P')$ for all $P' \leq S$ which is $\mathcal{F}$-conjugate to P. A subgroup $P \leq S$ is *$\mathcal{F}$-radical* if $\mathrm{Out}_{\mathcal{F}}(P)$ is p-reduced; i.e., if $O_p(\mathrm{Out}_{\mathcal{F}}(P)) = 1$. Let $\mathcal{F}^c \subseteq \mathcal{F}$ denote the full subcategory whose objects are the $\mathcal{F}$-centric subgroups of S.

If $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group G, then $P \leq S$ is $\mathcal{F}$-centric if and only if P is p-centric in G (i.e., $Z(P) \in \mathrm{Syl}_p(C_G(P))$), and P is $\mathcal{F}$-radical if and only if $N_G(P)/(P{\cdot}C_G(P))$ is p-reduced. Thus in this situation, a subgroup being $\mathcal{F}$-radical is *not* the same as its being a radical p-subgroup of G.

Alperin's fusion theorem in a version for abstract saturated fusion systems was first formulated and proven by Puig [Pg]. Since we need to use it several times in what follows, we state the following version of the theorem, which is proven in [BLO2, Theorem A.10].

Theorem 1.1. *Let $\mathcal{F}$ be a saturated fusion system over a p-group S. Then each morphism in $\mathcal{F}$ is a composite of restrictions of morphisms between subgroups of S which are $\mathcal{F}$-centric, $\mathcal{F}$-radical, and fully normalized in $\mathcal{F}$. More precisely, for each $P, P' \leq S$ and each $\varphi \in \mathrm{Iso}_{\mathcal{F}}(P, P')$, there are subgroups $P = P_0, P_1, \ldots, P_k = P'$, subgroups $Q_i \geq \langle P_{i-1}, P_i \rangle$ ($i = 1, \ldots, k$) which are $\mathcal{F}$-centric, $\mathcal{F}$-radical, and fully normalized in $\mathcal{F}$, and automorphisms $\varphi_i \in \mathrm{Aut}_{\mathcal{F}}(Q_i)$, such that $\varphi_i(P_{i-1}) = P_i$ for all i and $\varphi = (\varphi_k|_{P_{k-1}}) \circ \cdots \circ (\varphi_1|_{P_0})$.* □

Again let $\mathcal{F}$ be a fusion system over the p-group S. A *centric linking system* associated to $\mathcal{F}$ is a category $\mathcal{L}$ whose objects are the $\mathcal{F}$-centric subgroups of S, together with a functor $\pi\colon \mathcal{L} \longrightarrow \mathcal{F}^c$, and "distinguished" monomorphisms $P \xrightarrow{\delta_P} \mathrm{Aut}_{\mathcal{L}}(P)$ for each $\mathcal{F}$-centric subgroup $P \leq S$, which satisfy the following conditions.

(A) π is the identity on objects. For each pair of objects P, Q in $\mathcal{L}$, $Z(P)$ acts freely on $\mathrm{Mor}_{\mathcal{L}}(P, Q)$ via composition and δ_P, and π induces a bijection

$$\mathrm{Mor}_{\mathcal{L}}(P, Q)/Z(P) \xrightarrow{\ \cong\ } \mathrm{Hom}_{\mathcal{F}}(P, Q).$$

(B) For each $\mathcal{F}$-centric subgroup $P \leq S$ and each $x \in P$, $\pi(\delta_P(x)) = c_x \in \mathrm{Aut}_{\mathcal{F}}(P)$.

(C) For each $f \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$ and each $x \in P$, the following square commutes in $\mathcal{L}$:

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "\delta_P(x)"'] & Q \arrow[d, "\delta_Q(\pi(f)(x))"] \\ P \arrow[r, "f"] & Q. \end{tikzcd}$$

A *p-local finite group* is defined to be a triple $(S,\mathcal{F},\mathcal{L})$, where S is a finite p-group, $\mathcal{F}$ is a saturated fusion system over S, and $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$. The *classifying space* of the triple $(S,\mathcal{F},\mathcal{L})$ is the p-completed nerve $|\mathcal{L}|^\wedge_p$.

For any finite group G with Sylow p-subgroup S, a category $\mathcal{L}^c_S(G)$ was defined in [BLO1], whose objects are the p-centric subgroups of G, and whose morphism sets are defined by

$$\mathrm{Mor}_{\mathcal{L}^c_S(G)}(P,Q)=N_G(P,Q)/O^p(C_G(P)).$$

Since $C_G(P)=Z(P)\times O^p(C_G(P))$ when P is p-centric in G, $\mathcal{L}^c_S(G)$ is easily seen to satisfy conditions (A), (B), and (C) above, and hence is a centric linking system associated to $\mathcal{F}_S(G)$. Thus $(S,\mathcal{F}_S(G),\mathcal{L}^c_S(G))$ is a p-local finite group, with classifying space $|\mathcal{L}^c_S(G)|^\wedge_p\simeq BG^\wedge_p$ (see [BLO1, Proposition 1.1]).

The following lifting lemma for linking systems helps to motivate some of the constructions made here.

Lemma 1.2. *Let $(S,\mathcal{F},\mathcal{L})$ be a p-local finite group. Fix $\mathcal{F}$-centric subgroups $P,Q,R\le S$, and let $\varphi\in\mathrm{Mor}_{\mathcal{L}}(P,R)$ and $\psi\in\mathrm{Mor}_{\mathcal{L}}(Q,R)$ be morphisms such that $\mathrm{Im}(\pi(\varphi))\le\mathrm{Im}(\pi(\psi))$. Then there is a unique morphism $\chi\in\mathrm{Mor}_{\mathcal{L}}(P,Q)$ such that $\varphi=\psi\circ\chi$.*

Proof. By definition of a fusion system, there is $f\in\mathrm{Hom}_{\mathcal{F}}(P,Q)$ such that $\pi(\varphi)=\pi(\psi)\circ f$ in $\mathrm{Hom}_{\mathcal{F}}(P,R)$. Fix any $\chi'\in\pi^{-1}(f)$. By (A), there is a unique $g\in Z(P)$ such that $\varphi=\psi\circ\chi'\circ\delta_P(g)$, and we set $\chi=\chi'\circ\delta_P(g)$. This proves existence, and the proof uniqueness is similar (again using (A)). (See [BLO2, Lemma 1.10].) □

When working with a p-local finite group $(S,\mathcal{F},\mathcal{L})$, we always assume we have chosen "inclusion morphisms" $\iota_P\in\mathrm{Mor}_{\mathcal{L}}(P,S)$ for each P; i.e., morphisms which are sent to the inclusion of P in S under the functor $\pi\colon\mathcal{L}\longrightarrow\mathcal{F}$ (and where $\iota_S=\mathrm{Id}_S$). Then by Lemma 1.2, for each $P\le Q\le S$ in $\mathcal{L}$, there is a unique "inclusion" morphism $\iota_{P,Q}\in\mathrm{Mor}_{\mathcal{L}}(P,Q)$ such that $\iota_P=\iota_Q\circ\iota_{P,Q}$. Moreover, for each $\varphi\in\mathrm{Mor}_{\mathcal{L}}(P,Q)$, and each $P_0\le P$ and $Q_0\le Q$ such that $\pi(\varphi)(P_0)\le Q_0$ and $P_0,Q_0\in\mathrm{Ob}(\mathcal{L})$, there is a unique "restriction" $\varphi|_{P_0,Q_0}\in\mathrm{Mor}_{\mathcal{L}}(P_0,Q_0)$ such that $\iota_{Q_0,Q}\circ\varphi|_{P_0,Q_0}=\varphi\circ\iota_{P_0,P}$.

Again fix $(S,\mathcal{F},\mathcal{L})$, let $|\mathcal{L}|$ be the nerve (geometric realization) of the category $\mathcal{L}$, and let $*\in|\mathcal{L}|$ be the vertex corresponding to the object S. Let

$$J=J_{\mathcal{L}}\colon\mathrm{Mor}(\mathcal{L})\longrightarrow\pi_1(|\mathcal{L}|,*)$$

be the map which sends $\varphi\in\mathrm{Mor}_{\mathcal{L}}(P,Q)$ to the loop in $|\mathcal{L}|$ formed by the edges $[\varphi]$, $[\iota_P]$, and $[\iota_Q]$. In particular, J sends each of the inclusions $[\iota_P]$ to the identity element in the fundamental group. Also, J sends composites to products, and hence can be thought of as a functor $J\colon\mathcal{L}\longrightarrow\mathcal{B}(\pi_1(|\mathcal{L}|,*))$.

Proposition 1.3. *Let $(S,\mathcal{F},\mathcal{L})$ be a p-local finite group. For any group Γ, and any map of sets*

$$\widehat{\Phi}\colon\mathrm{Mor}(\mathcal{L})\longrightarrow\Gamma$$

which sends composites to products and sends inclusion morphisms to the identity, there is a unique homomorphism $\Phi\colon\pi_1(|\mathcal{L}|,*)\longrightarrow\Gamma$ such that $\widehat{\Phi}=\Phi\circ J$. In other words, $\pi_1(|\mathcal{L}|,*)$ is the free group generated by the morphisms in $\mathcal{L}$, modulo relations defined by composition and inclusions.

Proof. Let $\mathcal{B}(\Gamma)$ be the category with one object $*$ and morphism group Γ. Then $\widehat{\Phi}$ extends to a functor $\Psi\colon\mathcal{L}\longrightarrow\mathcal{B}(\Gamma)$, and this in turn induces a map

$$|\Psi|\colon|\mathcal{L}|\longrightarrow|\mathcal{B}(\Gamma)|=B\Gamma$$

between the geometric realizations. Set

$$\Phi = \pi_1(|\Psi|)\colon \pi_1(|\mathcal{L}|,*) \longrightarrow \pi_1(|\mathcal{B}(\Gamma)|,*) = \Gamma.$$

The relation $\widehat{\Phi} = \Phi \circ J$ is clear by construction. The uniqueness of Φ holds since every element of $\pi_1(|\mathcal{L}|,*)$ can be represented by a loop which follows along the edges of $|\mathcal{L}|$ (corresponding to morphisms in $\mathcal{L}$), and any such loop can be factored as a composite of loops in $\mathrm{Im}(J)$. $\square$

Now, in the above situation, we let

$$\tau = \tau_{\mathcal{L}}\colon S \longrightarrow \pi_1(|\mathcal{L}|,*)$$

denote the composite $J \circ \delta_S$. If $g \in P \le S$, then by axiom (C) (applied with $Q = S$ and $f = \iota_P$), $\iota_P \circ \delta_P(g) = \delta_S(g) \circ \iota_P$. Thus $\tau(g) = J(\delta_S(g)) = J(\delta_P(g))$. In other words, $\tau(g)$ can be defined using any δ_P as long as $g \in P$.

Proposition 1.4. *Fix a p-local finite group $(S, \mathcal{F}, \mathcal{L})$, a (possibly infinite) group Γ, and an epimorphism*

$$\Phi\colon \pi_1(|\mathcal{L}|,*) \longrightarrow\!\!\!\!\rightarrow \Gamma.$$

Then the following hold.

(a) $\mathrm{Ker}(\Phi \circ \tau)$ *is strongly $\mathcal{F}$-closed in S.*

(b) *If $\Phi \circ \tau$ is the trivial homomorphism, then $\Phi \circ J$ restricts to a surjective homomorphism from $\mathrm{Aut}_{\mathcal{L}}(S)/\delta_S(S) \cong \mathrm{Out}_{\mathcal{F}}(S)$ onto Γ.*

Proof. For any isomorphism $\varphi \in \mathrm{Iso}_{\mathcal{F}}(P,Q)$ in $\mathcal{F}$ between $\mathcal{F}$-centric subgroups, and any $g \in P$, $\tau(g)$ and $\tau(\varphi(g))$ are conjugate in $\pi_1(|\mathcal{L}|,*)$ (since φ lifts to an isomorphism in $\mathcal{L}$); and hence either both lie in $\mathrm{Ker}(\Phi)$ or neither does. By Alperin's fusion theorem (Theorem 1.1), any pair of $\mathcal{F}$-conjugate elements of S is linked by a sequence of isomorphisms between $\mathcal{F}$-centric subgroups, and hence (a) holds.

Point (b) is basically a consequence of [BCGLO2, Lemma 3.4], but because it's hard to fit this situation precisely into that setting, we repeat the argument here. Assume $\Phi \circ \tau$ is the trivial homomorphism. In particular, $\Phi \circ J$ factors through a map

$$J'\colon \mathrm{Mor}(\mathcal{F}^c) \longrightarrow \Gamma$$

in this case, since $\Phi \circ J(Z(P)) = 1$ for all P. We must show that $J'|_{\mathrm{Aut}_{\mathcal{F}}(S)}$ is onto. Assume otherwise. By Alperin's fusion theorem again, Γ is generated by the subgroups $J'(\mathrm{Aut}_{\mathcal{F}}(P))$ for $P \le S$ $\mathcal{F}$-centric, $\mathcal{F}$-radical, and fully normalized; we fix such a subgroup $P \lneqq S$ which is maximal among all $P \le S$ such that $J'(\mathrm{Aut}_{\mathcal{F}}(P)) \nleq J'(\mathrm{Aut}_{\mathcal{F}}(S))$. Choose $\varphi \in \mathrm{Aut}_{\mathcal{F}}(P)$ such that $J'(\varphi) \notin J'(\mathrm{Aut}_{\mathcal{F}}(S))$.

Now, $J'(\mathrm{Aut}_S(P)) = 1$ and $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$; hence $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P)) \le \mathrm{Ker}(J')$. Set $K = \varphi \mathrm{Aut}_S(P) \varphi^{-1}$. Since K and $\mathrm{Aut}_S(P)$ are both Sylow p-subgroups of $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))$, there is $\chi \in O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))$ such that $\chi\varphi$ normalizes $\mathrm{Aut}_S(P)$. Thus $J'(\chi\varphi) = J'(\varphi)$, and by axiom (II) in the definition of a saturated fusion system, $\chi\varphi$ extends to an automorphism $\bar{\varphi} \in \mathrm{Aut}_{\mathcal{F}}(N_S(P))$. But $J'(\bar{\varphi}) = J'(\chi\varphi)$ (since J sends inclusions to the identity), $J'(\bar{\varphi}) \in J'(\mathrm{Aut}_{\mathcal{F}}(S))$ by the maximality of P, and this is a contradiction. This finishes the proof of (b). $\square$

For any $n \ge 0$, we write $\underline{\mathbf{n}} = \{1, 2, \ldots, n\}$. Let $\mathcal{C}(n)$ denote the category whose objects are the nonempty subsets $I \subseteq \underline{\mathbf{n}}$, with a unique morphism $I \to J$ whenever $I \subseteq J$. By an *amalgam* of groups of rank n, we mean a functor $\mathcal{A}$ from $\mathcal{C}(n)^{\mathrm{op}}$ to the category of groups and monomorphisms. A *faithful completion* of the amalgam $\mathcal{A}$ is a collection of monomorphisms $f_I\colon \mathcal{A}(I) \longrightarrow G$ for all $\varnothing \neq I \subseteq \underline{\mathbf{n}}$ which commute with the monomorphisms induced by $\mathcal{A}$, such that

$$G = \langle f_1(\mathcal{A}(1)), f_2(\mathcal{A}(2)), \dots, f_n(\mathcal{A}(n))\rangle.$$

The following properties of an amalgam of groups are well known; we include them here for ease of later reference.

Proposition 1.5. *Fix $n \geq 3$, let $\mathcal{A}$ be an amalgam of groups of rank n, and let G be a faithful completion of $\mathcal{A}$. Write $G_I = \mathcal{A}(I)$, $G_i = G_{\{i\}}$, $G_{ij} = G_{\{i,j\}}$, etc. for short, and regard these as subgroups of G for simplicity. Let X be the corresponding coset complex: the simplicial complex with vertex set $\coprod_{i=1}^n (G/G_i)$, with edges the union of the G/G_{ij}, etc. Then X is connected, and there is a short exact sequence of groups:*

$$1 \longrightarrow \pi_1(X) \longrightarrow \mathrm{colim}(\mathcal{A}) \longrightarrow G \longrightarrow 1.$$

In particular, the natural homomorphism from $\mathrm{colim}(\mathcal{A})$ to G is an isomorphism if and only if X is simply connected.

Proof. This follows from [T, Proposition 1]. Alternatively, it follows from the following argument which applies van Kampen's theorem to the Borel construction on X.

Consider the Borel construction on X:

$$X_{hG} \overset{\text{def}}{=} EG \times_G X = \big(EG \times X\big)/{\sim}.$$

Here, EG is a contractible space upon which G acts freely on the right, and we identify $(yg, x) \sim (y, gx)$ for all $y \in EG$, $g \in G$, and $x \in X$. Thus $EG \times X$ is a covering space of X_{hG}, and is also homotopy equivalent to X. By the standard properties of fundamental groups in covering spaces, this yields an exact sequence

$$1 \longrightarrow \pi_1(X) \longrightarrow \pi_1(X_{hG}) \longrightarrow G,$$

where the last homomorphism is surjective if and only if X is connected.

For each $i = 1, \dots, n$, let $X_i \subseteq X$ be the union of the orbit G/G_i together with all orbits of *open* simplices which have this orbit as a vertex. Thus $X = \bigcup_{i=1}^n X_i$. Also, X_i has one connected component for each vertex in G/G_i (and the components are contractible), and so $\pi_1((X_i)_{hG}) \cong G_i$. Similarly, for each $i \neq j$, $X_i \cap X_j$ has one connected component for each element of G/G_{ij}, and $\pi_1((X_i \cap X_j)_{hG}) \cong G_{ij}$. So by van Kampen's theorem, $\pi_1(X_{hG}) \cong \mathrm{colim}(\mathcal{A})$. Since G is generated by the G_i, this also proves that $\pi_1(X_{hG})$ surjects onto G, and hence that X is connected. □

Proposition 1.6. *Fix a finite group G, a prime p, and a Sylow subgroup $S \in \mathrm{Syl}_p(G)$. Assume $P_1, \dots, P_n \leq S$ are all centric in G (i.e., $C_G(P_i) \leq P_i$), and are all weakly closed in S with respect to G. Define, for each $I \subseteq \underline{\boldsymbol{n}}$,*

$$P_I = \langle P_i \,|\, i \in I\rangle \qquad \text{and} \qquad G_I = N_G(P_I) = \bigcap_{i \in I} N_G(P_i).$$

Let $\mathcal{L} \subseteq \mathcal{L}_S^c(G)$ be the full subcategory with $\mathrm{Ob}(\mathcal{L}) = \{P_I \,|\, \varnothing \neq I \subseteq \underline{\boldsymbol{n}}\}$. Then

$$\pi_1(|\mathcal{L}|) \cong \mathrm{colim}(\mathcal{A}),$$

where $\mathcal{A}$ denotes the amalgam of rank n defined by setting $\mathcal{A}(I) = G_I$.

Proof. Let X be as in Proposition 1.5: the simplicial complex with vertex set the disjoint union of the G/G_i, edges the disjoint union of the $G/(G_i \cap G_j)$, etc. Since the P_I are weakly closed and $G_I = N_G(P_I)$, X is equivalent (as a simplicial complex with

G-action) to the poset of all subgroups of G which are conjugate to some P_I. Hence by [BLO1, Lemma 1.2] (or its proof),

$$|\mathcal{L}| \simeq EG \times_G X.$$

So as in Proposition 1.5, $\pi_1(|\mathcal{L}|) \cong \operatorname{colim}(\mathcal{A})$. (Note, however, that in this case, $|\mathcal{L}|$ is connected only if $G = \langle G_1, \ldots, G_n \rangle$.) $\square$

The following examples will be needed later.

Proposition 1.7. *Fix a prime p, a finite group G, and a Sylow p-subgroup $S \leq G$.*

(a) *Assume G is a simple group of Lie type in characteristic p of Lie rank ≥ 3, or a quasisimple group in characteristic p of Lie rank ≥ 3 with center a p-group. Then $\pi_1(|\mathcal{L}_S^c(G)|) \cong G$. Also, for any $S \in \operatorname{Syl}_p(G)$, G is the colimit of the diagram of parabolic subgroups of G which contain S.*

(b) *Assume G is p-constrained. Then $\pi_1(|\mathcal{L}_S^c(G)|) \cong G/O_{p'}(G)$.*

Proof. **(a)** By the Borel-Tits theorem [GLS, Corollary 3.1.6], together with [Gr, Remark 4.3], there is a bijection of posets from the poset of parabolic subgroups of G to the opposite poset of the poset of radical p-centric subgroups of G, defined by sending $\mathfrak{P} \mapsto O_p(\mathfrak{P})$, and where $N_G(O_p(\mathfrak{P})) = \mathfrak{P}$.

We claim that for each $S \in \operatorname{Syl}_p(G)$ and each parabolic subgroup $\mathfrak{P} \geq S$, $O_p(\mathfrak{P})$ is weakly closed in S with respect to G. The following argument is taken from [AS, Lemma I.2.5]. Assume otherwise, and let $Q = O_p(\mathfrak{P})$ be maximal among subgroups of this form which are not weakly closed in S. By Alperin's fusion theorem (Theorem 1.1), there is a radical subgroup $Q' \leq S$ such that $Q' \gneqq Q$ — hence $Q' = O_p(\mathfrak{P}')$ for some other parabolic subgroup $\mathfrak{P}' \subsetneqq \mathfrak{P}$ — and an element $x \in N_G(Q') = \mathfrak{P}'$ such that $xQx^{-1} \neq Q$. But this is impossible, since $\mathfrak{P}' \leq \mathfrak{P} = N_G(Q)$.

Thus, by Proposition 1.6, $\pi_1(|\mathcal{L}_S^c(G)|)$ is isomorphic to the colimit of the amalgam $\mathcal{A}$ formed by the parabolic subgroups containing a given Sylow p-subgroup. By Proposition 1.5, there is a short exact sequence

$$1 \longrightarrow \pi_1(X) \longrightarrow \operatorname{colim}(\mathcal{A}) \longrightarrow G \longrightarrow 1,$$

where X is the geometric realization of the poset of parabolic subgroups.

If G has Lie rank n, then by [Bw, §V.3], the geometric realization of the poset of its parabolic subgroups is a building of rank n, and hence by [Bw, Theorem IV.5.2] has the homotopy type of a bouquet of $(n-1)$-spheres. Thus if $n \geq 3$, the geometric realization is simply connected, and $\operatorname{colim}(\mathcal{A}) \cong G$.

(b) Assume G is p-constrained, and set $\bar{G} = G/O_{p'}(G)$ and $Q = O_p(\bar{G})$. Thus $C_{\bar{G}}(Q) = Z(Q)$, and $\operatorname{Aut}_{\mathcal{L}}(Q) \cong \bar{G}$. Let $\mathcal{L}_S^{rc}(G) \subseteq \mathcal{L}_S^c(G)$ be the full subcategory with objects the centric radical subgroups of $\bar{G}$; then $|\mathcal{L}_S^{rc}(G)|$ and $|\mathcal{L}_S^c(G)|$ have the same homotopy type by [BCGLO1, Theorem B]. Since each centric radical subgroup of $\bar{G}$ contains Q, one easily sees that $|\mathcal{L}_S^{rc}(\bar{G})|$ contains as deformation retract the nerve of the subcategory with unique object Q. Thus

$$|\mathcal{L}_S^c(G)| = |\mathcal{L}_S^c(\bar{G})| \simeq |\mathcal{L}_S^{rc}(\bar{G})| \simeq B\operatorname{Aut}_{\mathcal{L}}(Q) \simeq B\bar{G}.$$

In particular, $\pi_1(|\mathcal{L}_S^c(G)|) \cong \bar{G}$. $\square$

2. The linking system of $\mathrm{Spin}_7(q)$

Let q be any prime power such that $q \equiv \pm 3 \pmod 8$. In this section, we describe the fundamental group of $\mathcal{L}_{\mathrm{Sol}}(q)$ as the colimit of a certain triangle of groups. Before doing this, we first need to look at the linking system of $\mathrm{Spin}_7(q)$.

Set $H = \mathrm{Spin}_7(q)$ for short, and fix $S \in \mathrm{Syl}_2(H)$. By [BCGLO1, Theorem B], $|\mathcal{L}_S^{cr}(H)| \simeq |\mathcal{L}_S^{c}(H)|$ (the inclusion is a homotopy equivalence), and thus these two spaces have the same fundamental group. By [LO, Proposition A.12], every 2-subgroup $P \le H$ which is centric and radical in the fusion system $\mathcal{F}_S(H)$ is in fact centric in H; i.e., $C_H(P) = Z(P)$. (This also follows from the proof of Proposition 2.1 below.) Hence the linking system $\mathcal{L}_S^{rc}(H)$ is a full subcategory of the transporter category of H: $\mathrm{Mor}_{\mathcal{L}_S^{rc}(H)}(P,Q)$ is the set of elements of H which conjugate P into Q. Thus there is a functor from $\mathcal{L}_S^{rc}(H)$ to $\mathcal{B}(H)$ — the category with one object and morphism group H — which sends a morphism to the corresponding element in H, and in particular sends inclusions to the identity. Upon taking fundamental groups of the geometric realizations of these categories, this defines a homomorphism

$$\mu\colon \pi_1(|\mathcal{L}_S^{c}(H)|) \cong \pi_1(|\mathcal{L}_S^{rc}(H)|) \longrightarrow H.$$

Proposition 2.1. *For any prime power $q \equiv \pm 3 \pmod 8$, there is an isomorphism*

$$\pi_1(|\mathcal{L}_2^{c}(\mathrm{Spin}_7(q))|) \cong \mathrm{Spin}_7(\mathbb{Z}[\tfrac{1}{2}])$$

which commutes with the natural homomorphisms

$$\pi_1(|\mathcal{L}_2^{c}(\mathrm{Spin}_7(q))|) \xrightarrow{\ \mu\ } \mathrm{Spin}_7(q) \longleftarrow \mathrm{Spin}_7(\mathbb{Z}[\tfrac{1}{2}]).$$

Proof. By [BCGLO2, Theorem 6.8], this is equivalent to showing that

$$\pi_1(|\mathcal{L}_2^{c}(\Omega_7(q))|) \cong \Omega_7(\mathbb{Z}[\tfrac{1}{2}]).$$

We work with $\Omega_7(q)$ for simplicity.

Set $V = \mathbb{F}_q{}^7$, let $\mathfrak{q}$ be its standard quadratic form, and fix an orthonormal basis $\{u_1, \dots, u_n\}$ of V. For each $i = 1,2,3$, set $v_{2i-1} = u_{2i-1} + u_{2i}$ and $v_{2i} = u_{2i-1} - u_{2i}$. Thus $\{v_1, \dots, v_6, u_7\}$ is an orthogonal basis of V, and $\mathfrak{q}(v_j) = 2$ for all $j = 1, \dots, 6$. Set $W_i = \langle u_{2i-1}, u_{2i}\rangle = \langle v_{2i-1}, v_{2i}\rangle$ $(i = 1,2,3)$. Set $\widehat{G} = GO(V,\mathfrak{q}) \cong GO_7(q)$ and $G = \Omega(V,\mathfrak{q}) \cong \Omega_7(q)$.

Let $\Gamma_4 \le \Omega(W_1 \oplus W_2, \mathfrak{q})$ be the subgroup of those automorphisms α of the form $\alpha(u_i) = \epsilon_i u_{\sigma(i)}$, where $\epsilon_i = \pm 1$, $\epsilon_1\epsilon_2\epsilon_3\epsilon_4 = 1$, and $\sigma \in \mathrm{Alt}_4$ lies in the normal subgroup of order 4. Thus $\Gamma_4 \cong D_8 \times_{C_2} D_8$ is an extraspecial 2-group of order 2^5. Consider the following subgroups of G:

$$\begin{aligned} R &= \{\alpha \in G \mid \alpha(u_i) = \pm u_i \text{ for all } i = 1, \dots, 7\} \\ R^* &= \{\alpha \in G \mid \alpha(v_i) = \pm v_i \text{ for all } i = 1, \dots, 6\} \\ Q &= \{\alpha \in G \mid \alpha|_{W_1 \oplus W_2} \in \Gamma_4,\ \alpha(u_i) = \pm u_i \text{ for } i = 5,6,7\}. \end{aligned}$$

Set $S = RR^*Q$. By [GO, Proposition 10.1], $S \in \mathrm{Syl}_2(G)$, and the seven subgroups R, R^*, Q, RR^*, RQ, R^*Q, and $S = RR^*Q$ are representatives for the distinct conjugacy classes of 2-subgroups of $G = \Omega_7(q)$ which are centric and radical in the fusion system $\mathcal{F}_2(G)$. Also, R, R^*, and Q are all weakly closed in $S = RR^*Q \in \mathrm{Syl}_2(G)$, and hence $N_G(P_1P_2) = N_G(P_1) \cap N_G(P_2)$ for any pair P_1, P_2 of such subgroups.

Let $\mathcal{L} \subseteq \mathcal{L}_2^c(G)$ be the full subcategory whose objects are the subgroups of G which are centric and radical in $\mathcal{F}_2(G)$. By [BCGLO1, Theorem 3.5], $|\mathcal{L}| \simeq |\mathcal{L}_2^c(G)|$, and hence they have the same fundamental group. By Proposition 1.6, $\pi_1(|\mathcal{L}|)$ is the colimit of the triangle of groups $\mathcal{A}$ with vertices $N_G(R)$, $N_G(R^*)$, and $N_G(Q)$ and with edges their pairwise intersections.

Now set $\Gamma = \Omega_7(\mathbb{Z}[\frac{1}{2}])$, and define bases $\{u_1, \ldots, u_7\}$ and $\{v_1, \ldots, v_6, u_7\}$ of $\mathbb{Z}[\frac{1}{2}]^7$ analogous to the above elements. Via these bases, we can lift R, R^*, and Q to Γ, and check directly that $N_\Gamma(P) \cong N_G(P)$ for P any of these three subgroups.

We want to compare $\operatorname{colim}(\mathcal{A})$ to a similar colimit of subgroups of Γ studied by Kantor in [Ka, §§5,7]. He constructs a certain 3-dimensional complex Δ_7, together with an action of Γ which is transitive on 3-simplices. This action has four orbits of vertices

$$\Gamma/N_\Gamma(R),\ \Gamma/N_\Gamma(Q),\ \Gamma/W^+,\ \Gamma/W^-,$$

where $W^+ \cong W^-$ are representatives of the two conjugacy classes of $\Omega_7(2)$ in Γ, and $W^+ \cap W^- = N_\Gamma(R^*)$. (See [Ka, p.213].) By [Ka, Corollary 7.4], Δ_7 is equivalent to the Euclidean building for $\Omega_7(\mathbb{Q}_2)$, and hence contractible. So by Proposition 1.5 again, if we let $\mathcal{A}_7$ denote the rank four amalgam consisting of the four stabilizer subgroups of a 3-simplex and their intersections, then $\operatorname{colim}(\mathcal{A}_7) \cong \Gamma$.

We now construct group homomorphisms

$$\operatorname{colim}(\mathcal{A}) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \operatorname{colim}(\mathcal{A}_7),$$

which will be inverses to each other. The first one is clear: Φ is defined by sending $N_G(P)$ to $N_\Gamma(P)$ for $P = R$ and Q, and $N_G(R^*)$ to $W^+ \cap W^-$. To define Ψ, we first note that by [Ka, p.213] again, $W^\pm \cap N_\Gamma(R)$, $W^\pm \cap N_\Gamma(Q)$, and $N_\Gamma(R^*)$ are the three maximal parabolic subgroups of $W^\pm \cong 2Sp_6(2)$ containing $S \in \operatorname{Syl}_2(W^\pm)$, and the colimit of these groups (together with their intersections) is $W^\pm$ by Proposition 1.7(a). This defines homomorphisms from $W^\pm$ to $\operatorname{colim}(\mathcal{A})$, and together with the canonical isomorphisms $N_\Gamma(P) \cong N_G(P)$ for $P = R$ and Q these induce a homomorphism Ψ. It is clear by construction that Φ and Ψ are inverses, and thus $\operatorname{colim}(\mathcal{A}) \cong \operatorname{colim}(\mathcal{A}_7) \cong \operatorname{Spin}_7(\mathbb{Z}[\frac{1}{2}])$.

See also [GO, Theorem 10.2] for a slightly different argument. □

We now set up some notation which will be used in this section and the next. For any odd prime power q, there is a homomorphism

$$\omega\colon SL_2(\bar{\mathbb{F}}_q)^3 \longrightarrow \operatorname{Spin}_7(\bar{\mathbb{F}}_q),$$

with $\operatorname{Ker}(\omega) = \langle(-I, -I, -I)\rangle$, which arises from identifications $\operatorname{Spin}_3(\bar{\mathbb{F}}_q) \cong SL_2(\bar{\mathbb{F}}_q)$ and $\operatorname{Spin}_4(\bar{\mathbb{F}}_q) \cong SL_2(\bar{\mathbb{F}}_q)^2$. (See [LO, Definition 2.2] for more details.) The three factors are ordered so that $Z(\operatorname{Spin}_7(\bar{\mathbb{F}}_q)) = \langle\omega(-I, -I, I)\rangle$. We write $[\![X_1, X_2, X_3]\!] = \omega(X_1, X_2, X_3)$ for short, and set $U = \langle[\![\pm I, \pm I, \pm I]\!]\rangle \cong C_2^2$. By [LO, Proposition 2.5] or [AC, Lemma 4.4(c)],

$$C_{\operatorname{Spin}_7(\bar{\mathbb{F}}_q)}(U) = \omega(SL_2(\bar{\mathbb{F}}_q)^3) \qquad \text{and} \qquad N_{\operatorname{Spin}_7(\bar{\mathbb{F}}_q)}(U) = \omega(SL_2(\bar{\mathbb{F}}_q)^3)\cdot\langle\tau\rangle$$

where $\tau^2 = 1$ and $\tau[\![X_1, X_2, X_3]\!]\tau^{-1} = [\![X_2, X_1, X_3]\!]$. Finally, $\operatorname{Im}(\omega) \cap \operatorname{Spin}_7(q)$ is generated by $\omega(SL_2(q)^3)$, together with an element $[\![Y, Y, Y]\!]$ for $Y \in N_{SL_2(q^2)}(SL_2(q))$ but not in $SL_2(q)$. This will be described in more detail in the next section, in the proof of Lemma 3.1.

We now restrict to the case $q = 3$. Let $\overline{SL}_2(3)$ be the normalizer in $SL_2(\bar{\mathbb{F}}_3)$ of $SL_2(3)$. Thus $\overline{SL}_2(3)$ contains $SL_2(3)$ with index 2, and is the 2-fold central extension of Sym(4) whose Sylow 2-subgroup is quaternion of order 16. Set

$$\widehat{K} = (\overline{SL}_2(3))^3/\langle(z,z,z)\rangle \rtimes \mathrm{Sym}(3) \qquad \text{and} \qquad \widehat{B} = (\overline{SL}_2(3))^3/\langle(z,z,z)\rangle \cdot \langle\tau\rangle \le \widehat{K},$$

where $\tau = (1\,2) \in \mathrm{Sym}(3)$ acts by switching the first two coordinates. Let $[X_1, X_2, X_3]$ denote the class of a triple (X_1, X_2, X_3). Choose any $Y \in \overline{SL}_2(3) \smallsetminus SL_2(3)$, and set

$$\begin{aligned} B_1 &= \bigl(SL_2(3)^3/\langle(z,z,z)\rangle\bigr) \cdot \bigl\langle [Y,Y,Y] \bigr\rangle \\ &= \bigl\{ [X_1,X_2,X_3] \in (\overline{SL}_2(3))^3/\langle(z,z,z)\rangle \bigm| X_1 \equiv X_2 \equiv X_3 \pmod{SL_2(3)} \bigr\}. \end{aligned}$$

Finally, define

$$K = B_1 \rtimes \mathrm{Sym}(3) \le \widehat{K} \qquad \text{and} \qquad B = \widehat{B} \cap K = B_1\langle\tau\rangle,$$

and let $\bar{\omega}\colon B \longrightarrow \mathrm{Spin}_7(3)$ be the homomorphism induced by ω.

The following two propositions hold, in fact, for $\mathcal{L}^c_{\mathrm{Sol}}(q)$ for any $q \equiv \pm 3 \pmod 8$. We state them here only for $q = 3$, since that simplifies somewhat the proofs, and suffices for our later applications.

Proposition 2.2. *Set $H = \mathrm{Spin}_7(\mathbb{Z}[\frac{1}{2}])$. Then $\bar{\omega}$ lifts to an embedding $\lambda\colon B \longrightarrow H$; and there is an epimorphism*

$$\chi\colon H \mathop{*}_{B} K \twoheadrightarrow \pi_1(|\mathcal{L}^c_{\mathrm{Sol}}(3)|),$$

where $H *_B K$ is the amalgamated free product defined by the amalgam

$$\bigl(H \xleftarrow{\ \lambda\ } B \xrightarrow{\ \mathrm{incl}\ } K\bigr).$$

Proof. Fix $S \in \mathrm{Syl}_2(B) \subseteq \mathrm{Syl}_2(\mathrm{Spin}_7(3))$. By the constructions in [LO] and [AC], $\mathcal{L}^c_{\mathrm{Sol}}(q)$ is generated by its two subcategories $\mathcal{L}^c_S(\mathrm{Spin}_7(3))$ and $\mathcal{L}^c_S(K)$, which intersect in $\mathcal{L}^c_S(B)$. Also, $\pi_1(|\mathcal{L}^c_S(K)|) \cong K$ and $\pi_1(|\mathcal{L}^c_S(B)|) \cong B$ by Proposition 1.7(b) (K and B are 2-constrained and $O_{2'}(K) = O_{2'}(B) = 1$). The inclusion of $\mathcal{L}^c_S(B)$ into $\mathcal{L}^c_S(\mathrm{Spin}_7(3))$ induced by $\bar{\omega}$ now induces an inclusion of B into $H \cong \pi_1(|\mathcal{L}^c_S(\mathrm{Spin}_7(3))|)$, together with a homomorphism

$$\chi\colon H \mathop{*}_{B} K \longrightarrow \pi_1(|\mathcal{L}^c_{\mathrm{Sol}}(3)|);$$

and χ is surjective since by construction, all morphisms in $\mathcal{L}^c_{\mathrm{Sol}}(3)$ are composites of morphisms in these subcategories. □

The following proposition will be needed in Section 5, in the proof that $|\mathcal{L}^c_{\mathrm{Sol}}(3)|$ is simply connected.

Proposition 2.3. *Again set $H = \mathrm{Spin}_7(\mathbb{Z}[\frac{1}{2}])$, and*

$$\chi\colon H \mathop{*}_{B} K \twoheadrightarrow \pi_1(|\mathcal{L}^c_{\mathrm{Sol}}(3)|).$$

be as in Proposition 2.2. Then there are subgroups $H_0 \le H$, $K_0 \le K$, and $B_0 = H_0 \cap K_0 \le B$ such that $H_0/Z \cong Sp_6(2)$, $[K{:}K_0] = 3$, and $(H_0 \ge B_0 \le K_0)$ is an amalgam of maximal subgroups of Co_3. Furthermore, if $\omega \ne 1$ (is not the trivial homomorphism), then $\omega|_{\langle H_0,K_0\rangle} \ne 1$.

Proof. The inclusions of linking systems $\mathcal{L}^c_S(H_0) \subseteq \mathcal{L}^c_S(\mathrm{Spin}_7(q))$ and $\mathcal{L}^c_S(K_0) \subseteq \mathcal{L}^c_S(K)$ (where $S \in \mathrm{Syl}_2(\mathrm{Spin}_7(q))$) were constructed in [AC, Theorem B], in a way so that they intersect in $\mathcal{L}^c_S(B_0)$. Also, $H_0 \cong \pi_1(|\mathcal{L}^c_S(H_0)|)$ by Proposition 1.7(a), and the analogous result for K_0 and B_0 holds by Proposition 1.7(b). The inclusions $H_0 \le H$ and $K_0 \le K$ now follow upon taking fundamental groups.

Now let $N \trianglelefteq H *_B K$ be the normal closure of $\langle H_0, K_0\rangle$. To prove the last statement, we must show that $N = H *_B K$. Set $G = \mathrm{Spin}_7(3)$, and fix $S \in \mathrm{Syl}_2(G)$, also regarded as a subgroup of B. Since $[B{:}B_0] = 3$, B_0 contains S (up to conjugacy), and hence $N \geq S$. By [Ka, Corollary 7.4], $H \cong \mathrm{Spin}_7(\mathbb{Z}[\frac{1}{2}])$ is generated by two of the subgroups $W^{\pm} \cong 2Sp_6(2)$ described in the proof of Proposition 2.1, which contain S by construction. (Note that in [Ka], G_7 is defined to be the subgroup of $SO_7(\mathbb{Q})$ generated by these two subgroups). Since both of these are quasisimple, N contains W^+ and W^- since it contains S, and thus $N \geq H \geq B$. Since the normal closure of B in K is K, this shows that $N = H *_B K$. □

3. Identifying $\mathcal{F}_{\mathrm{Sol}}(q)$ from its Sylow 2-subgroup

For any odd prime power q, let $\mathcal{F}_{\mathrm{Sol}}(q)$ be the exotic fusion system constructed in [LO], over a 2-group $S(q)$. Our aim in this section is to prove Theorem 3.4, which states that $\mathcal{F}_{\mathrm{Sol}}(q)$ and $\mathcal{F}_{\mathrm{Sol}}(q')$ are isomorphic if and only if $S(q)$ and $S(q')$ have the same order.

The results in this section will be used to reduce our main theorem — the simple connectivity of $|\mathcal{L}^c_{\mathrm{Sol}}(q)|$ for all odd prime powers q — to the case where q is a power of 3.

We first need some concrete information about the structure of the Sylow subgroups of these groups, and of their fusion.

Lemma 3.1. *Let q be an odd prime power, set $H = \mathrm{Spin}_7(q)$, and let $S \in \mathrm{Syl}_2(H)$. Set $\mathcal{F} = \mathcal{F}_{\mathrm{Sol}}(q)$. Set $n = v_2(q^2-1)$ (i.e., 2^n is the highest power of 2 dividing q^2-1), and let Q_{2^n} be a generalized quaternion group of order 2^n. Then the following hold.*

(a) *There are unique normal subgroups $U \trianglelefteq S$ and $E \trianglelefteq S$ which are elementary abelian of rank two and three, respectively.*

(b) *There is a unique abelian subgroup $T \leq S_0$ which is homocyclic of rank three and exponent 2^{n-1}.*

(c) *There are exactly six normal subgroups of $C_S(U)$ which are isomorphic to Q_{2^n}. They can be labelled R_1, R_2, R_3, $\overline{R}_1$, $\overline{R}_2$, $\overline{R}_3$ so as to have the following properties:*

  (1) *For each $i = 1,2,3$, $UR_i = U\overline{R}_i$, and $R_i \cap \overline{R}_i$ is cyclic of order 2^{n-1} and contained in T.*

  (2) *Of the R_i and $\overline{R}_i$, R_3 and $\overline{R}_3$ are the only ones which are normal in S.*

  (3) *If $P \leq R_i$ is quaternion of order 8, then $\mathrm{Aut}_{N_H(U)}(P) = \mathrm{Aut}(P)$. If $P \leq \overline{R}_i$ is quaternion of order 8, then $\mathrm{Aut}_{N_H(U)}(P) = \mathrm{Aut}_S(P)$.*

  (4) *The three subgroups R_1, R_2, and R_3 are $N_{\mathcal{F}}(E)$-conjugate.*

(d) *Let q' be any other odd prime power such that $v_2(q^2-1) = v_2(q'^2-1)$. Set $H' = \mathrm{Spin}_7(q')$, fix $S' \in \mathrm{Syl}_2(H')$, set $\mathcal{F}'$, and let $U' \leq E' \trianglelefteq S'$ be as in (a). Let $R'_i, \overline{R}'_i \leq S'$ be the subgroups which have the same properties as the $R_i, \overline{R}_i \leq S$ described in (c). Then any isomorphism $\varphi\colon S \xrightarrow{\cong} S'$ which induces an isomorphism of categories $N_{\mathcal{F}}(E) \cong N_{\mathcal{F}'}(E')$ and sends the R_i to the R'_i also induces an isomorphism of fusion categories $\mathcal{F}_S(N_H(U)) \cong \mathcal{F}_{S'}(N_{H'}(U'))$.*

Proof. We recall the notation used in Section 2. There is a homomorphism

$$\omega\colon SL_2(\bar{\mathbb{F}}_q)^3 \longrightarrow \mathrm{Spin}_7(\bar{\mathbb{F}}_q)$$

with $\mathrm{Ker}(\omega) = \langle(-I,-I,-I)\rangle$, and we write $[\![X_1,X_2,X_3]\!] = \omega(X_1,X_2,X_3)$. Set $U = \{[\![\pm I,\pm I,\pm I]\!]\}$, and set $B = N_H(U)$, $B_0 = C_H(U) = H\cap \mathrm{Im}(\omega)$, and $S_0 = C_S(U)$. Set $L = SL_2(q)$, and let $\widehat{L} \le SL_2(q^2)$ be the subgroup generated by L together with the matrix $\mathrm{diag}(\sqrt{a}, 1/\sqrt{a})$ for any $a\in\mathbb{F}_q^\times$ which is not a square (so $[\widehat{L}{:}L]=2$). Then

$$B_0 = \bigl\{[\![X_1,X_2,X_3]\!] \bigm| X_i\in\widehat{L}, X_1\equiv X_2\equiv X_3 \pmod{L}\bigr\} \le \omega(\widehat{L}^3) \cong \widehat{L}^3\big/\bigl\langle(-I,-I,-I)\bigr\rangle;$$

and $B = B_0\langle\tau\rangle$, where $\tau^2=1$ and $\tau[\![X_1,X_2,X_3]\!]\tau^{-1} = [X_2,X_1,X_3]$.

Fix Sylow subgroups $\widehat{Q}\in\mathrm{Syl}_2(\widehat{L})$ and $Q\in\mathrm{Syl}_2(L)$; then $Q\cong Q_{2^n}$ and $\widehat{Q}\cong Q_{2^{n+1}}$. Fix a pair of generators $y,b\in\widehat{Q}$, where $|y|=2^n$ and $|b|=4$, and set $a=y^{2^{n-2}}$ and $z=a^2(=-I)$. Thus $\langle a,b\rangle\cong Q_8$, and $\langle z\rangle = Z(\widehat{Q})$. Since $n\ge 3$, $\langle y\rangle$ is the unique cyclic subgroup of $\widehat{Q}$ of order 2^n. Thus

$$\begin{aligned} S_0 &= \bigl\{[\![X_1,X_2,X_3]\!] \bigm| X_i\in\widehat{Q}, X_1\equiv X_2\equiv X_3 \pmod{Q}\bigr\} \\ &= \bigl\{[\![X_1,X_2,X_3]\!] \bigm| X_i\in Q\bigr\}\cdot\bigl\langle[\![y,y,y]\!]\bigr\rangle, \end{aligned}$$

and $S = S_0\langle\tau\rangle$.

Set $y_1=[\![y,1,1]\!]$, $y_2=[\![1,y,1]\!]$, $y_3=[\![1,1,y]\!]$; and similarly for b_i, a_i, and z_i. Also, set $\widehat{y} = [\![y,y,y]\!] = y_1y_2y_3$, and similarly for $\widehat{b}$ and $\widehat{a}$. (By definition, $[\![z,z,z]\!]=1$.) We defined $U=\langle z_1,z_2\rangle\cong C_2^2$, and now set

$$E = U\langle\widehat{a}\rangle = \langle z_1,z_2,\widehat{a}\rangle \cong C_2^3.$$

Let $T\le S_0$ be the "toral" subgroup:

$$T = \{[\![y^i,y^j,y^k]\!] \,|\, i\equiv j\equiv k \pmod 2\}.$$

Then $T\cong (C_{2^{n-1}})^3$. If $T'\le S_0$ is any subgroup such that $T'\cong T$, then $T'/(T\cap T')\le S_0/T$ is elementary abelian, so $T'\ge E$ (the 2-torsion subgroup of T), $T'\le C_{S_0}(E) = T\cdot\langle\widehat{b}\rangle$; and since $\widehat{b}=[\![b,b,b]\!]$ inverts T it follows that $T'=T$. This proves (b). (In fact, T is the unique subgroup of S of its isomorphism type: this was shown in the proof of [LO, Proposition 2.9], and was shown in [AC, Lemma 4.9(c)] when $n=3$.)

If $V\trianglelefteq S$ is a normal elementary abelian subgroup, then $[V,T]\le V\cap T$ is an elementary abelian subgroup of T. Fix $v\in V$. If $v\notin T\langle\tau,\widehat{b}\rangle$, then $[v,\widehat{y}]$ has order $2^{n-1}\ge 4$; while if $v\in\tau\cdot\langle T,\widehat{b}\rangle$, then $[v,y_1^2]$ has order 2^{n-1}. Also, if $v\in\widehat{b}\cdot T$, then $[v,T]\ge E$. Thus if $\mathrm{rk}(V)\le 3$, then $V\le T$, and hence $V\le E$ (the 2-torsion subgroup of T). This shows that E is the unique such normal subgroup of rank 3. Also, since the four elements $[\![z^ia,z^ja,z^ka]\!]$ of $E\smallsetminus U$ are all S-conjugate to each other, U is the unique such subgroup of rank 2. This proves (a).

For $i=1,2,3$, set $R_i=\langle y_i^2,b_i\rangle\cong Q_{2^n}$. Thus R_1 is the image in S_0 of $Q\times 1\times 1$, R_2 is the image of $1\times Q\times 1$, etc. Also, for each i, $R_iU\cong Q_{2^n}\times C_2$. Let $\overline{R}_i\le R_iU$ be the unique subgroup isomorphic to R_i such that $R_i\cap\overline{R}_i\le T$ and is cyclic of order 2^{n-1}. All six of these subgroups R_i and $\overline{R}_i$ are normal in S_0.

Now, $S_0/T\cong C_2^3$, with coset representatives the elements $b_1^ib_2^jb_3^k$ for $i,j,k\in\{0,1\}$. Also, $[b_i,T]$ is cyclic of order 2^{n-1} for each $i=1,2,3$; while for any $x\in S_0$ such that $xT\notin\{T,b_iT\}$, $U\le[x,T]$ and hence $[x,T]$ is not cyclic. If $R\trianglelefteq S_0$ is isomorphic to Q_{2^n}, then since R and T are both normal, $[R,T]\le R\cap T$ must be cyclic; and thus $[R,T]\le R\le T\langle b_i\rangle$ for some i. Hence $R=[R,T]\langle gb_i\rangle$ for some $g\in T$ such that

$(gb_i)^2 = b_i^2$; i.e., such that $b_i g b_i^{-1} = g^{-1}$; and this implies that $g \in [R,T]U$. Hence $R = R_i$ or $R = \overline{R}_i$, and this finishes the proof that the R_i and $\overline{R}_i$ are the unique normal subgroups of S_0 isomorphic to Q_{2^n}.

Points (c1), (c2), and (c3) now follow easily from the above descriptions of these subgroups of S. By the construction of $\mathcal{F} = \mathcal{F}_{\mathrm{Sol}}(q)$ in [LO] or [AC], there is an element $\beta \in \mathrm{Aut}_{\mathcal{F}}(S_0)$ which permutes the subgroups R_1, R_2, and R_3 cyclically. Also, $\beta(T) = T$ by (b) (the uniqueness of T), so β normalizes E (the 2-torsion subgroup of T). Thus the three subgroups R_i are conjugate in $N_{\mathcal{F}}(E)$. This proves (c4), and hence finishes the proof of (b) and (c).

It remains to prove (d). Let q' be any other odd prime power such that $v_2(q^2-1) = v_2(q'^2-1)$, and let $H' = \mathrm{Spin}_7(q')$, $S' \in \mathrm{Syl}_2(H')$, and $\mathcal{F}' = \mathcal{F}_{\mathrm{Sol}}(q')$. Let $U' \le E' \le S'$ be the unique normal subgroups with $U' \cong U$ and $E' \cong E$, and let R_i' be the subgroups of $C_{S'}(U')$ with the same properties as the $R_i \trianglelefteq C_S(U)$. Let $\varphi\colon S \xrightarrow{\cong} S'$ be an isomorphism which induces an isomorphism of categories $N_{\mathcal{F}}(E) \cong N_{\mathcal{F}'}(E')$, and which sends each R_i to some R_{j_i}'. In particular, $\varphi(R_3) = R_3'$ by (c2), and hence φ sends $\{R_1, R_2\}$ to $\{R_1', R_2'\}$. Upon composing with conjugation by τ, if necessary, we can assume that $\varphi(R_i) = R_i'$ for all i. By (c3), the R_i and R_i' are contained in the factors $SL_2(q) \le H, H'$.

Now, the only subgroups of $\widehat{Q} \cong Q_{2^{n+1}}$ whose automorphism groups are not 2-groups are the quaternion subgroups of order 8. Hence $\mathcal{F}_Q(L)$ is generated by $\mathcal{F}_Q(Q)$ together with the groups $\mathrm{Aut}(P)$ for all $P \le Q$ quaternion of order 8. Also, if $P \le \widehat{Q}$ is quaternion of order 8 but not contained in Q, then $\mathrm{Aut}_{\widehat{L}}(P) = \mathrm{Aut}_{\widehat{Q}}(P)$ since any automorphism leaves $P \cap Q$ invariant. This shows that $\mathcal{F}_{S_0}(B_0)$ is generated by $\mathcal{F}_S(S)$, together with those automorphisms $\alpha \in \mathrm{Aut}(P_iR_jR_k)$ (where $\{i,j,k\} = \{1,2,3\}$ and $Q_8 \cong P_i \le R_i$) such that $\alpha|_{R_jR_k} = \mathrm{Id}$ and $\alpha|_{P_i}$ has order 3. Hence $\mathcal{F}_S(B)$ is generated by $\mathcal{F}_{S_0}(B_0)$ and $\mathcal{F}_S(S)$ together with all automorphisms of the form $\beta \in \mathrm{Aut}(P_1P_2R_3\langle\tau'\rangle)$ for $P_1 \le R_1$ and $P_2 \le R_2$ both quaternion of order 8 and exchanged by $\tau' \in S_0\tau$, where $\beta(\tau) = \tau$, $\beta|_{R_3} = \mathrm{Id}$, and $\beta|_{P_1P_2}$ has order 3. This proves that φ sends $B = N_H(U)$-fusion to $N_{H'}(U')$-fusion, and thus induces an isomorphism of fusion categories. $\square$

Recall that a subgroup H of a group G is *strongly embedded* in G (at the prime 2) if H is a proper subgroup of even order such that $|H \cap H^g|$ is odd for all $g \in G \smallsetminus H$. A 2-subgroup $P \le G$ is *essential* if $Z(P) \in \mathrm{Syl}_2(C_G(P))$ and $\mathrm{Out}_G(P)$ has a strongly embedded subgroup. In particular, if $S \in \mathrm{Syl}_2(G)$ and $P \le S$ an essential 2-subgroup of G, then P is centric and radical in the fusion system $\mathcal{F}_S(G)$.

By the Alperin-Goldschmidt fusion theorem [Go], every morphism in $\mathcal{F}_S(G)$ is a composite of restrictions of automorphisms of S, and of essential subgroups $P \le S$ such that $N_S(P) \in \mathrm{Syl}_2(N_G(P))$ (i.e., are fully normalized in $\mathcal{F}_S(G)$). For this reason, we need information about the essential 2-subgroups of $\mathrm{Spin}_7(q)$, which means information about the essential 2-subgroups of $\Omega_7(q)$.

Lemma 3.2. *Fix an odd prime power q, set $G = \mathrm{Spin}_7(q)$, and fix $S \in \mathrm{Syl}_2(G)$. Let $U \le E \trianglelefteq S$ be the unique elementary abelian subgroups which are normal in S and of rank two and three, respectively (see Lemma 3.1). If $P \le S$ is an essential 2-subgroup of G, then P is G-conjugate to a subgroup $P' \le S$ such that either*

(1) *$U \trianglelefteq P'$ is an $\mathrm{Aut}_G(P')$-invariant subgroup; or*

(2) *$E \trianglelefteq P'$ is an $\mathrm{Aut}_G(P')$-invariant subgroup.*

Proof. Set $V = \mathbb{F}_q^7$, and let $\mathfrak{q}$ be a quadratic form on V with orthonormal basis. We identify $G = \mathrm{Spin}(V, \mathfrak{q})$. Set $Z = Z(G)$, $\bar{G} = G/Z = \Omega(V, \mathfrak{q})$, and $\bar{S} = S/Z$, and let u be a generator of U/Z.

Let $P \leq S$ be an essential 2-subgroup of $G = \mathrm{Spin}(V, \mathfrak{q})$. Then $\bar{P} = P/Z$ is an essential 2-subgroup of $\bar{G}$ (cf. [LO, Lemma A.11(e)]). Let $V = V_1 \oplus \cdots \oplus V_m$ be a decomposition of V as a sum of pairwise orthogonal $\bar{P}$-invariant subspaces, chosen so that m is as large as possible. This decomposition can be chosen such that for each i, either V_i is irreducible as a $\bar{P}$-representation, or it is a sum of two irreducible $\bar{P}$-representations neither of which supports a nondegenerate quadratic form (cf. [O1, Lemma 7.1]). In particular, each element of $N_{\bar{G}}(\bar{P})$ leaves invariant the sum of all of the V_i of any given dimension.

Set $d_i = dim(V_i)$, and assume the summands are ordered so that the sequence $\Delta = (d_1, \cdots, d_m)$ is non-increasing. This sequence may be written in abbreviated fashion, using exponents to indicate repeated dimensions. For example, $(4, 1^3)$ is an abbreviation for one such sequence. By [LO, Lemma A.6], each d_i is a power of 2, and the discriminant of V_i is a square in $\mathbb{F}_q^\times$ if $d_i > 1$.

Assume first that there is an $N_{\bar{G}}(\bar{P})$-invariant orthogonal decomposition $V = V' \oplus V''$, where $\dim(V') = 4$ and $\dim(V'') = 3$. Let u' be the involution $(-\mathrm{Id})_{V'} \oplus \mathrm{Id}_{V''}$. Then u' centralizes $\bar{P}$, so $u' \in \bar{P}$ since $\bar{P}$ is 2-centric, and $N_{\bar{G}}(\bar{P}) \leq C_{\bar{G}}(u')$. Also, u' is $\bar{G}$-conjugate to u. Since $u \in Z(\bar{S})$, there is $\bar{P}' \leq \bar{S}$ which is $\bar{G}$-conjugate to $\bar{P}$ and such that $u \in Z(\bar{P}')$ and $N_{\bar{G}}(\bar{P}') \leq C_{\bar{G}}(u)$, and we are thus in the situation of case (1).

Next assume $\Delta = (2^3, 1)$. Let $Q \leq \bar{G}$ be the group of elements which are $\pm\mathrm{Id}$ on each of V_1, V_2, and V_3 (i.e., on the 2-dimensional summands), are the identity on V_4, and which negate an even number of summands. Thus Q is a fours group, and is $\bar{G}$-conjugate to E/Z. Also, Q centralizes $\bar{P}$, so $Q \leq Z(\bar{P})$ since $\bar{P}$ is 2-centric in $\bar{G}$, and every element of $\mathrm{Aut}_{\bar{G}}(\bar{P})$ leaves Q invariant. So by the same reasoning as in the last paragraph, we are in the situation of case (2).

By inspection, we are now left only with the cases where $\Delta = (2, 1^5)$ or (1^7). Let n_+ be the number of 1-dimensional summands $V_i = \langle v \rangle$ such that $\mathfrak{q}(v)$ is a square, and let n_- be the number of $V_i = \langle v \rangle$ such that $\mathfrak{q}(v)$ is not a square. Then n_- is even (since V itself has square discriminant), and n_+ is odd.

If $\Delta = (2, 1^5)$, then $\bar{P} \leq O_2(q) \times C_2^5$, and $O_2(q)$ is a dihedral group. Also, since $\bar{P}$ is 2-centric in $\bar{G}$, it contains every involution which negates four of the 1-dimensional summands. So the V_i are pairwise distinct as $\bar{P}$-representations, and thus are permuted by $\mathrm{Aut}_{\bar{G}}(\bar{P})$. Also, $N_{\bar{G}}(\bar{P})$ contains elements whose projections to $O(V_1, \mathfrak{q})$ represent all cosets of $\bar{G} = \Omega(V, \mathfrak{q})$ in $O(V, \mathfrak{q})$, so $\mathrm{Out}_{\bar{G}}(\bar{P}) = \mathrm{Out}_{O(V,\mathfrak{q})}(\bar{P})$. Thus $\mathrm{Out}_{\bar{G}}(\bar{P}) \cong A \times \mathrm{Sym}(n_+) \times \mathrm{Sym}(n_-)$, where $|A| \leq 2$. Since $O_2(\mathrm{Out}_{\bar{G}}(\bar{P})) = 1$, this implies that $A = 1$, $n_+ = 5$, and $\mathrm{Out}_{\bar{G}}(\bar{P}) \cong \mathrm{Sym}(5)$, which is impossible since $\mathrm{Sym}(5)$ does not contain a strongly embedded subgroup.

Finally, if $\Delta = (1^7)$, then similar (but simpler) arguments show that $(n_+, n_-) = (7, 0)$ or $(1, 6)$, and that $\mathrm{Out}_{\bar{G}}(\bar{P})$ is one of the groups $\mathrm{Sym}(7)$, $\mathrm{Sym}(6)$, $\mathrm{Alt}(7)$, or $\mathrm{Alt}(6)$ — none of which contains a strongly embedded subgroup. □

Recall that if $\mathcal{F}$ is a fusion system over a p-group S, and Φ is a set of $\mathcal{F}$-morphisms, then one says that $\mathcal{F}$ is *generated by* Φ, and writes $\mathcal{F} = \langle \Phi \rangle$, if every morphism ϕ in $\mathcal{F}$ is a composite of restrictions of morphisms in Φ. That is, there is no fusion system over S whose set of morphisms contains Φ, and which is properly contained in $\mathcal{F}$.

Proposition 3.3. *Fix an odd prime power q, set $H = \mathrm{Spin}_7(q)$, let $S \in \mathrm{Syl}_2(H)$, and let $U \le E \le S$ be the unique normal elementary abelian subgroups of ranks two and three, respectively. Let $\mathcal{F}_0 \subseteq \mathcal{F}$ be the fusion systems over S: $\mathcal{F}_0 = \mathcal{F}_S(H)$ and $\mathcal{F} = \mathcal{F}_{\mathrm{Sol}}(q)$. Then*

$$\mathcal{F}_0 = \langle N_{\mathcal{F}_0}(U), N_{\mathcal{F}_0}(E) \rangle \qquad \text{and} \qquad \mathcal{F} = \langle N_{\mathcal{F}_0}(U), N_{\mathcal{F}}(E) \rangle.$$

Proof. By the Alperin-Goldschmidt fusion theorem [Go], for any finite group G, any prime p, and any $S \in \mathrm{Syl}_p(G)$, the fusion system $\mathcal{F}_S(G)$ is generated by the automorphism groups $\mathrm{Aut}_G(P)$ for $P = S$, and for subgroups $P \le S$ which are essential in $\mathcal{F}_S(G)$.

Now set $G = \mathrm{Spin}_7(q)$ and $Z = Z(G)$, and set $\mathcal{F}_0' = \langle N_{\mathcal{F}_0}(U), N_{\mathcal{F}_0}(E) \rangle$. Assume the lemma is false for $\mathcal{F}_0$; i.e., that $\mathcal{F}_0' \subsetneqq \mathcal{F}_0$. Let P be a maximal essential subgroup for which $\mathrm{Aut}_G(P)$ is not contained in $\mathcal{F}_0'$. By Lemma 3.2, P is G-conjugate to some P' such that either U or E is contained in P' and is $\mathrm{Aut}_G(P')$-invariant. Thus $\mathrm{Aut}_G(P')$ is in $\mathcal{F}_0'$; while by the maximality assumption, P is conjugate to P' by an isomorphism in $\mathcal{F}_0'$. It follows that $\mathrm{Aut}_G(P)$ is in $\mathcal{F}_0'$, a contradiction; and thus $\mathcal{F}_0 = \mathcal{F}_0'$.

By construction in [LO], $\mathcal{F}_{\mathrm{Sol}}(q)$ is generated by $\mathcal{F}_S(\mathrm{Spin}_7(q))$ together with one morphism of order three which normalizes U, and which also can be chosen to normalize E. So the result for $\mathcal{F}$ follows from that for $\mathcal{F}_0$. $\square$

Recall that $v_2(-)$ denotes the 2-adic valuation of an integer: $v_2(n) = k$ if k is the largest integer such that $2^k | n$.

Theorem 3.4. *For any pair of odd prime powers q and q', $\mathcal{F}_{\mathrm{Sol}}(q) \cong \mathcal{F}_{\mathrm{Sol}}(q')$ — and hence $\mathcal{L}^c_{\mathrm{Sol}}(q) \cong \mathcal{L}^c_{\mathrm{Sol}}(q')$ — if and only if $v_2(q^2-1) = v_2(q'^2-1)$.*

Proof. By [LO, Lemma 3.2], together with the obstruction theory in [BLO2, Proposition 3.1], $\mathcal{F}_{\mathrm{Sol}}(q)$ has a unique associated linking system. Hence the equivalence of linking systems follows from the equivalence of fusion systems.

Set $H = \mathrm{Spin}_7(q)$ and $H' = \mathrm{Spin}_7(q')$, fix $S \in \mathrm{Syl}_2(H)$ and $S' \in \mathrm{Syl}_2(H')$, and set $\mathcal{F} = \mathcal{F}_{\mathrm{Sol}}(q)$ and $\mathcal{F}' = \mathcal{F}_{\mathrm{Sol}}(q')$. If $v_2(q^2-1) \ne v_2(q'^2-1)$, then $|S| \ne |S'|$, and hence $\mathcal{F} \ncong \mathcal{F}'$.

Now assume $v_2(q^2-1) = v_2(q'^2-1)$; we prove that $\mathcal{F} \cong \mathcal{F}'$. Let $U \le E \le S$ and $U' \le E' \le S'$ be the normal subgroups of ranks two and three (Lemma 3.1(a)). Set $B = N_H(U)$ and $B' = N_{H'}(U')$. Then $S \cong S'$ by Lemma 3.1(d). We use the notation of the proof of Lemma 3.1, when needed, to describe elements of S.

Set $n = v_2(q^2-1)$. By Lemma 3.1(b), there is a unique homocyclic subgroup $T \le S_0 = C_S(U)$ of rank 3 and exponent 2^{n-1}. In particular, T is weakly closed and centric in any fusion system over S. Hence by [BLO1, Proposition A6], $N_{\mathcal{F}}(T)$ is a 2-constrained, saturated fusion system; so by [BCGLO2, Proposition 4.3], there is a group L such that $S \in \mathrm{Syl}_2(L)$, $F^*(L) = O_2(L)$, and $\mathcal{F}_L = N_{\mathcal{F}}(T)$. By construction [LO, Section 2], we have $C_L(E) = C_S(E) = T\langle \widehat{b} \rangle$, where $\widehat{b}$ acts on T by inverting; and $L/C_L(E) \cong GL_3(2)$ with the obvious action on E.

Now let $T' \trianglelefteq S'$ and $L' \ge S'$ be the corresponding groups for the fusion system $\mathcal{F}'$. We next show that $L \cong L'$ via an isomorphism which sends S onto S'. Basically, this is done by showing that L is the unique extension of T by $L/T \cong GL_3(2) \times C_2$ for which $GL_3(2)$ has the standard action on E while the C_2 factor acts on T by inverting, and $GL_3(2)$ splits over T while L/T does not split.

Let $L_0 \trianglelefteq L$ be the subgroup of index two such that $L_0/T \cong GL_3(2)$. Set $S_0 = S \cap L_0$, and let $S_0' \le L_0'$ be the corresponding subgroups of L'. Choose isomorphisms

$$\varphi_0 \colon E \xrightarrow{\;\cong\;} E' \qquad \text{and} \qquad \psi \colon L_0/T \xrightarrow{\;\cong\;} L_0'/T'$$

such that $\psi(S_0/T) = S_0'/T'$, and φ_0 commutes with the conjugation actions of $L_0/T \cong L_0'/T'$ when identified via ψ. By [G, Proposition 6.4], the action of $L_0/T \cong GL_3(2)$ on $E \cong (\mathbb{Z}/2)^3$ has a unique lifting to $T \cong (\mathbb{Z}/2^{n-1})^3$: unique up to an automorphism of T. Thus φ_0 extends to an isomorphism $\varphi_1 \colon T \xrightarrow{\;\cong\;} T'$ which still commutes with the conjugation actions of $L_0/T \cong L_0'/T'$.

We next claim that L_0 splits over T, and similarly for L_0'. Let $T^2 \le T$ be the subgroup of squares in T. Then $C_L(E)/T^2 = T/T^2 \times \langle \widehat{b} \rangle \cong C_2^4$, and the quotient group S/T^2 splits over $C_L(E)/T^2$ via (for example) the subgroup

$$\langle b_1, b_2, \tau \rangle {\cdot} T^2 \cong S/C_L(E) \cong D_8.$$

Hence by Gaschütz's theorem (i.e., since $H^2(GL_3(2); C_2^4)$ injects into $H^2(D_8; C_2^4)$), L/T^2 splits as a semidirect product $(C_L(E)/T^2){\cdot}R'$ for some subgroup $R' \cong GL_3(2)$. In particular, L_0/T^2 is a semidirect product of T/T^2 by $GL_3(2)$. By [G, Theorem 6.5], the surjection of T onto T/T^2 induces an isomorphism in group cohomology from $H^2(L_0/T; T)$ to $H^2(L_0/T; T/T^2)$, and thus L_0 also splits as a semidirect product over T.

Fix subgroups $L_1 \le L_0$ and $L_1' \le L_0'$, both isomorphic to $GL_3(2)$. Let $\varphi_2 \colon L_0 \xrightarrow{\cong} L_0'$ be the isomorphism which extends φ_1 by sending L_1 to L_1' via ψ.

Now fix elements $d \in C_S(E) \smallsetminus T$ and $d' \in C_{S'}(E') \smallsetminus T'$. Thus d inverts T and $L = L_0\langle d \rangle$; and similarly for d'. Since $H^1(L_0/T; T) \cong \mathbb{Z}/2$ [G, Theorem 6.5], L_0 contains two T-conjugacy classes of subgroups isomorphic to $GL_3(2)$. If $(L_1)^d$ is T-conjugate to L_1, then some element of dT centralizes L_1, and L would be split over T. Since $U \le T$ is centralized by $\langle b_1, b_2, b_3 \rangle \le L/T$, this would imply that S contains some C_2^5, which is impossible since S has rank four (cf. [LO, Proposition A.8] or [AC, Theorem A]).

Thus conjugation by d switches the two T-conjugacy classes of subgroups $GL_3(2) \le L_0$, and similarly for d'. So there is some $t \in T'$ such that $\varphi_2((L_1)^d) = (L_1')^{td'}$; and we can now extend φ_2 to an isomorphism $\varphi_3 \colon L \xrightarrow{\;\cong\;} L'$ by setting $\varphi_3(d) = td'$. By construction, $\varphi_3(S) = S'$.

Set $\varphi = \varphi_3|_S$. Since φ extends to L, and $\mathcal{F}_S(L) = N_{\mathcal{F}}(E)$ by construction, φ defines an isomorphism from $N_{\mathcal{F}}(E)$ to $N_{\mathcal{F}'}(E')$. Set $\mathcal{F}_U = \mathcal{F}_S(N_H(U))$ and $\mathcal{F}_U' = \mathcal{F}_{S'}(N_{H'}(U'))$. Since $\mathcal{F} = \langle N_{\mathcal{F}}(E), \mathcal{F}_U \rangle$ by Proposition 3.3, and similarly for $\mathcal{F}'$, we will be done if we can show that φ induces an isomorphism $\mathcal{F}_U \cong \mathcal{F}_U'$. By Lemma 3.1(d), this means showing that φ sends the set $\mathcal{R} = \{R_1, R_2, R_3\}$ to $\mathcal{R}' = \{R_1', R_2', R_3'\}$.

Assume otherwise. Then by Lemma 3.1(c4), φ sends $\overline{\mathcal{R}} = \{\overline{R}_1, \overline{R}_2, \overline{R}_3\}$ to $\mathcal{R}'$. We claim there is an automorphism $\alpha \in \mathrm{Aut}(L)$ such that $\alpha(S) = S$ and α exchanges $\mathcal{R}$ with $\overline{\mathcal{R}}$. Once we have shown this, then we can replace φ by $\varphi \circ \alpha|_S$, and we are done.

As seen above, there are two T-conjugacy classes of subgroups $GL_3(2)$ in L_0, and the two classes are exchanged by elements of the coset dT. Furthermore, by [G, Theorem 6.5] again, the inclusion of E into T induces an isomorphism from $H^1(L_0/T; E)$ to $H^1(L_0/T; T)$; and hence the two classes are both represented in the subgroup EL_1.

We can thus choose $d \in C_S(E) \smallsetminus T$ such that $[d, L_1] \le E$. Let $\alpha \in \mathrm{Aut}(L)$ be the automorphism such that $\alpha|_{EL_1}$ is conjugation by d, and $\alpha|_{C_S(E)}$ is the identity. Set $V = E\langle d\rangle$, and regard it as a 4-dimensional L_1-representation.

Clearly, $C_E(L_1) = 1$, and hence $C_V(L_1) = 1$ since otherwise L would split over T — which we already know is not the case. Since L_1 is generated by three involutions (and all of its involutions are conjugate), this means that $|C_V(g)| = 4$ for each involution $g \in L_1$. Also, $[V, g] \le C_V(g)$ (since $g^2 = 1$), and hence $[V, g] = C_V(g)$ also has order 4.

Recall the notation set up in Section 2 for elements of S. In particular, $S = T{\cdot}\langle b_1, b_2, b_3, \tau\rangle$, $C_S(E) = T{\cdot}\langle b_1b_2b_3\rangle$, and $R_i = (R_i \cap T)\langle b_i\rangle$. For each $i = 1, 2, 3$, let s_i be the unique element of L_1 in the coset $b_iC_S(T)$ (an involution). Since $s_i \in b_iC_S(E) \le C_S(U)$, $[V, s_i] = C_V(s_i) \ge U$, and thus $[V, s_i] = U$ since it has order 4. Recall that $U^\# = \{z_1, z_2, z_3\}$, where $\langle z_i\rangle = Z(R_i) = Z(\bar{R}_i)$. Since $s_iC_S(E) = b_iC_S(E)$, we have $[E, s_i] = [E, b_i] = \langle[a_1a_2a_3, b_i]\rangle = \langle z_i\rangle$, and so $[\alpha, s_i] = [d, s_i] \in U \smallsetminus \langle z_i\rangle$. Since $\alpha|_{C_S(E)} = \mathrm{Id}$ (and since $C_S(E)s_i = C_S(E)b_i$), we now get $[\alpha, b_i] \in U \smallsetminus \{z_i\}$. As z_i is the unique involution in $R_i \cong Q_{2^n}$, we conclude that $(R_i)^\alpha = \bar{R}_i$ for all i. This completes the proof. □

4. The Co_3 geometry

Let $\mathcal{G}$ be the rank three 2-local geometry of $G = Co_3$ constructed in [A]. It can be described as follows. There are two conjugacy classes of involutions in G, of which 2A denotes the class of central involutions (those in centers of Sylow 2-subgroups). The elements of $\mathcal{G}$ are the 2A-pure elementary abelian subgroups of G of rank 1, 2, or 4, and incidence is given by symmetrized containment. By [Fi, Lemmas 5.8 & 5.9] (where the conjugacy class 2A is denoted 2_1), G acts transitively by conjugation on the set of such subgroups of a fixed order. Furthermore, if $E \in \mathcal{G}$ has rank 4, then $\mathrm{Aut}_G(E)$ is the full automorphism group $GL_4(2)$. It follows that G acts flag transitively on $\mathcal{G}$; i.e., it acts transitively on the set of all maximal flags $X \le Y \le E$ (where $\mathrm{rk}(X) = 1$ and $\mathrm{rk}(Y) = 2$).

Fix such a maximal flag $X \lneqq Y \lneqq E$ in $\mathcal{G}$. The maximal parabolics corresponding to this flag are the three maximal subgroups of G containing a given Sylow subgroup S: $L = N(E) \cong 2^4{\cdot}GL_4(2)$, $M = N(Y) \cong 2^{2+6}.3^2.D_{12}$, and $N = N(X) \cong 2{\cdot}Sp_6(2)$ (see [A]). Notice that S has index three in the Borel subgroup $B = L \cap M \cap N$ of order $2^{10}{\cdot}3$.

We will identify the elements of $\mathcal{G}$ as follows. We will call the conjugates of X *points*, the conjugates of Y *lines*, and the conjugates of E *3-spaces* (for the lack of a better name; note that 3 here represents the projective dimension).

Let $|\mathcal{G}|$ be the flag complex of the geometry $\mathcal{G}$: the simplicial complex with one vertex for each element of $\mathcal{G}$ (each point, line, and 3-space), and a simplex for each flag in $\mathcal{G}$ (each set of elements of $\mathcal{G}$ which are pairwise incident). A geometry is called simply connected if it has no (proper) covering geometries, and this is the case if and only if its flag complex is simply connected as a space. We refer to [Pn, §8.3] for more details about coverings of geometries.

Equivalently, $|\mathcal{G}|$ is the coset complex for the three orbits G/L, G/M, and G/N. Since G is generated by L, M, and N, the geometry $\mathcal{G}$ is connected; and in fact residually connected (the link of each vertex in $|\mathcal{G}|$ is connected) since each of L, M, and N is generated by its intersections with the other two subgroups.

The following proposition is the main result to be proven in this section.

Theorem 4.1. *The geometry $\mathcal{G}$ (or its realization $|\mathcal{G}|$) is simply connected. Hence for any complete flag $X \le Y \le E$ in $\mathcal{G}$, the colimit of the triangle of groups involving $N_G(X)$, $N_G(Y)$, $N_G(E)$ and their intersections is isomorphic to $G = Co_3$.*

The last statement in Proposition 4.1 follows from the simple connectivity of $|\mathcal{G}|$ together with Proposition 1.5 (the standard argument involving Tits' Lemma).

Let Γ be the graph whose vertex set is the set of points in $\mathcal{G}$ (i.e., the central involutions in Co_3), and where two vertices are adjacent whenever they are colinear in $\mathcal{G}$ (whenever their product is also a point in $\mathcal{G}$). Since the product of two commuting central involutions in Co_3 is again a central involution (this follows from [Fi, Lemma 4.7]), two vertices of Γ are adjacent if and only if the corresponding involutions commute. Thus, Γ coincides with the commutation graph on the central involutions of G.

A cycle in Γ (i.e., a loop) is called *geometric* if all of its vertices are incident to a common 3-space.

Proposition 4.2. *Assume every cycle in Γ can be decomposed as a product of geometric cycles. Then Theorem 4.1 holds.*

Proof. This is a standard argument in diagram geometry (cf. [Pn, §12.6]), but we repeat it here. We regard a cycle γ in Γ as a sequence $\gamma = (x_0, x_1, \dots, x_n = x_0)$ of vertices (points in $\mathcal{G}$) which are pairwise adjacent; i.e., such that $\langle x_{i-1}, x_i \rangle$ is a line in $\mathcal{G}$ (or a point if $x_{i-1} = x_i$) for each i. For each such cycle γ, set $y_i = \langle x_{i-1}, x_i \rangle$, and let $\widehat{\gamma}$ be the cycle in $|\mathcal{G}|$ defined by the sequence $(x_0, y_1, x_1, y_2, x_2, \dots, y_n, x_n)$. If γ decomposes as a product of cycles δ_1 and δ_2, then $\widehat{\gamma}$ decomposes as the product of the cycles $\widehat{\delta_1}$ and $\widehat{\delta_2}$. If γ is geometric — if the x_i ($i = 0, \dots, n$) are all contained in some 3-space V — then every vertex in $\widehat{\gamma}$ is adjacent to the vertex V in $|\mathcal{G}|$, and so $\widehat{\gamma}$ is homotopic to a trivial loop.

Thus, under the hypotheses of the proposition, for every cycle γ in Γ, $\widehat{\gamma}$ is homotopic to the trivial loop. It remains to check that every cycle in $|\mathcal{G}|$ is homotopic to one of this form.

Fix a cycle in $|\mathcal{G}|$, regarded as a sequence $(V_0, V_1, \dots, V_n = V_0)$ of elements of $\mathcal{G}$ such that each pair (V_i, V_{i+1}) is incident. For each i, let x_i be a point which is incident to V_i (and set $x_n = x_0$). For each $i = 1, \dots, n$, $\langle x_{i-1}, x_i \rangle$ is contained in V_{i-1} or V_i (whichever is larger), and hence x_{i-1} and x_i are adjacent in Γ. Set $\gamma = (x_0, x_1, \dots, x_n = x_0)$, a cycle in Γ, and set $y_i = \langle x_{i-1}, x_i \rangle$. For each i, the paths (x_{i-1}, y_i, x_i) and $(x_{i-1}, V_{i-1}, V_i, x_i)$ are homotopic in $|\mathcal{G}|$ (relative to endpoints) since all of the vertices involved are adjacent to V_{i-1} or V_i. Thus $\widehat{\gamma}$ is homotopic to the loop $(V_0, V_1, \dots, V_n = V_0)$ we started with, and this is what we had to prove. □

The next lemma shows that it suffices to decompose each cycle in Γ as a product of cycles of length three.

Lemma 4.3. *Every 3-cycle in Γ (i.e., every cycle of length three) is geometric.*

Proof. Fix a 3-cycle in Γ; i.e., a sequence of points (x, y, z) in $\mathcal{G}$ any two of which are incident to a line. Thus, if we regard x, y, and z as central involutions in G, they generate an elementary abelian subgroup of rank two or three, all involutions in which are still central (see [Fi, Lemma 4.7] again). Since each 2A-pure elementary abelian subgroup of G is contained in one of rank four [Fi, Lemma 5.9], $\langle x, y, z\rangle$ is contained in some 3-space in the geometry $\mathcal{G}$, and so the cycle (x, y, z) is geometric.

This also follows directly from the analysis given below of all pairs and triples of involutions of class 2A in G (see Figure 1). For example, Figure 1 classifies pairs of central involutions by the conjugacy class of their product, and shows that if the product is an involution then it must again be central. □

It remains to show that every cycle in Γ is a product of 3-cycles. This has been shown computationally, using the computer algebra system GAP [GAP]. We realize G in GAP in its primitive action of length 276. This action can be found in a standard library of GAP, namely, in the library of primitive permutation groups. Below, we provide an account of the computation.

The first task is to classify the orbits of G on the pairs of central involutions. Equivalently, we need the orbits of the centralizer $C = C(s)$ of a fixed central involution s on the set of central involutions (that is, the orbits of the stabilizer of the vertex s on the vertex set of Γ). Every group in GAP comes with a distinguished set of generators. As it turns out, the first generator of our copy of G is an element of order 4 and its square is a central involution, which we choose to be s. By taking random conjugates s^g of s and by computing the double stabilizers $C(\langle s, s^g\rangle) = C_C(s^g)$ we soon find the following orbits:

- $O_2 = s_2^C$ of size 630; s and s_2 commute and ss_2 is again a central involution;
- $O_3 = s_3^C$ of size 1920; ss_3 is of order 3 (class $3C$);
- $O_{3'} = s_{3'}^C$ of size 8960; $ss_{3'}$ is of order 3 (class $3B$);
- $O_4 = s_4^C$ of size 30240; ss_4 is of order 4;
- $O_5 = s_5^C$ of size 48384; ss_5 is of order 5; and
- $O_6 = s_6^C$ of size 80640; ss_6 is of order 6.

We notice that the lengths of these orbits sum to $170774 = [G{:}C]-1$ (where the missing 1 clearly represents s itself), and so our count of orbits is complete. We also remark that every representative s_i comes with the conjugating element g_i, such that $s_i = s^{g_i}$. We store the elements g_i for future use, alongside s_i.

It follows that the orbits of G on the set of pairs (a, b) of central involutions can be distinguished by the order of ab, except when $|ab| = 3$. In the latter situation the two orbits with $|ab| = 3$ can be distinguished by the order of the centralizer $C(\langle a, b\rangle)$ equal to 1512 if ab is in class $3C$ and equal to 324 if ab is in class $3B$. Since the edges of Γ correspond to the pairs of commuting involutions, we conclude that Γ has degree 630 (each vertex is adjacent to 630 other vertices).

With this information, it is now easy, for each $t \in \{s, s_2, s_3, s_{3'}, s_4, s_5, s_6\}$, to find the 630 neighbors of t and then determine the orbits of the double stabilizer $C(\langle s, t\rangle) = C_C(t)$ on the edges starting from t. Indeed, the set of neighbors of s coincides with O_2, while the set of neighbors of each s_i is $O_2^{g_i} = \{x^{g_i} \mid x \in O_2\}$. Once the orbits of $C_C(t)$, $t \in \{s, s_2, s_3, s_{3'}, s_4, s_5, s_6\}$, on the neighbors of t are determined, we can place each of these orbits in a particular O_j by checking the order of sx, where x is a representative of the orbit, as described above. The results of this computation are presented in the distance distribution diagram of Γ shown in Figure 1.

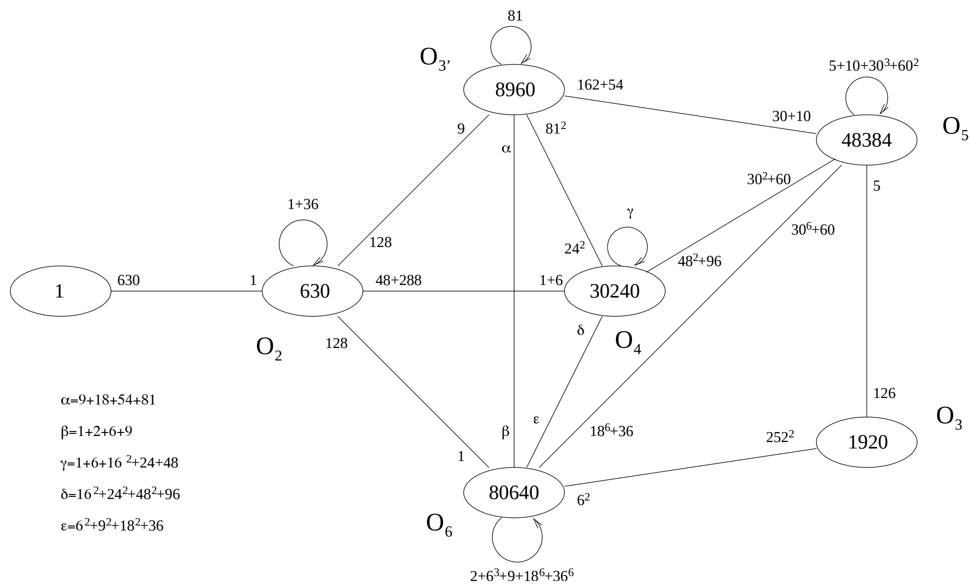

FIGURE 1. Distance distribution diagram of Γ

For example, this diagram indicates that each element of O_2 is incident to 37 other elements of O_2, in two $C(\langle s, s_2\rangle)$-orbits of orders 1 and 36. Hence G has two orbits on the set of 3-cycles. One of the orbits consists of triples of points incident to a common line (i.e., the three involutions in a subgroup of rank 2). The other consists of noncollinear triples of points which generate an elementary abelian subgroup of rank three.

We now start decomposing cycles. An *n-cycle* in Γ means a cycle of length n. A cycle is called *isometric* if the distance between two vertices of the cycle is the same when it is computed in the cycle and in Γ. If a cycle is not isometric then it can be decomposed as a product of two shorter cycles. Thus, we only need to deal with isometric cycles. Since the diameter of Γ is 3, there are no isometric cycles of length more than 7.

We start with 4-cycles. Suppose $abcd$ is an isometric 4-cycle. Clearly, $d(a, c) = 2$, and b and d are common neighbors of a and c. Since s and s_6 have only one common neighbor, the pair (a, c) is conjugate to $(s, s_{3'})$ or (s, s_4). We start with the second case. In this case $i = (ac)^2$ is a central involution which commutes with all four involutions a, b, c, and d. Thus, the 4-cycle can be decomposed as a product of four 3-cycles.

Now suppose that (a, c) is conjugate to $(s, s_{3'})$. Without loss of generality $a = s$ and $c = s_{3'}$. Let $X = X(a, c)$ be the set of common neighbors of a and c. Then $|X| = 9$ and the double stabilizer $C_{3'} = C(\langle a, c\rangle)$ acts on X transitively. Clearly, $b, d \in X$. Checking the orders of xy for $x, y \in X$ we see that all pairs (x, y) are conjugates of $(s, s_{3'})$. So the graph induced on X has no edges, and we need a new idea if we want to decompose these 4-cycles.

According to Figure 1, $C_{3'}$ has two orbits (cf. 81^2; *i.e.* two orbits of length 81) on the neighbors of c in O_4. Checking representatives of these two orbits, we find that one of them (call it e) has the following properties:

- $|ex| \in \{4, 6\}$ for all $x \in X$; and
- 5 elements of X have neighbors in Y, where $Y = X(a, e)$ is the set of common neighbors of $a = s$ and e.

If b and d are among these 5 elements of X then $abcd$ can be decomposed. Indeed, let b be adjacent to $f \in Y$ and d be adjacent to $h \in Y$. Then $abcd$ is a product of abf, $fbce$, adh, $hdce$, and (if $f \neq h$) $afeh$. Notice that the 4-cycles used in this decomposition have a pair of opposite vertices with product 4, hence these 4-cycles are decomposable.

Consider the equivalence relation on X defined by setting $x \sim y$ if if $axcy$ is decomposable. Since $C_{3'}$ acts transitively on X, this splits X as a union of equivalence classes of the same order (which must divide 9). We have just shown that there is an equivalence class of order at least 5, and hence the relation is transitive. Thus, we have verified the following.

Lemma 4.4. *All* 4*-cycles in* Γ *are decomposable.* □

We now turn to 5-cycles. Suppose $abcde$ is an isometric 5-cycle. For x and y at distance two from each other let $X(x, y)$ denote, as above, the set of common neighbors of x and y (the so-called *μ-graph* of x and y). Notice that $b \in X(a, c)$ and $e \in X(a, d)$. If we substitute b by any other vertex $b' \in X(a, c)$ then the new 5-cycle $ab'cde$ differs from $abcde$ by a 4-cycle. Hence, by Lemma 4.4, $abcde$ is decomposable if and only if $ab'cde$ is. Similarly, e can be substituted by any other vertex $e' \in X(a, d)$. It means that we can only keep track of one vertex, a, and of the edge, cd, opposite that vertex. Without loss of generality, we can assume that $a = s$, in which case cd is an edge between two vertices at distance two from s. According to Figure 1, there are 50 C-orbits of such edges, and so we have 50 cases to consider. The representative of all these 50 orbits were collected and stored, when the orbits of $C_i = C_C(s_i)$ on the neighbors of s_i, $i \in \{3', 4, 6\}$, were determined.

Suppose an edge cd represents one of the 50 cases. We will call this case *easy* if $X(a, c)$ and $X(a, d)$ either intersect, or have an edge connecting them. If this is the case then all 5-cycles containing a and cd are decomposable as a product of 3- and 4-cycles. It turns out that 30 of the 50 cases are easy.

Most of the remaining 20 cases can be handled using an additional trick. Suppose the distance between $X(a, c)$ and $X(a, d)$ is two, but there is a choice of $b \in X(a, c)$ such that the edge $a^g b^g$ (where g is selected to satisfy $s = a = d^g$) represents an easy case (or more generally, a previously handled case of 5-cycles). Then, for any $e \in X(a, d)$, the cycle $d^g e^g a^g b^g c^g$ is decomposable and hence $abcde$ is decomposable, too. This trick can be used iteratively, as more and more cases are settled, and eventually it helps decompose 5-cycles in 18 out of 20 "hard" case.

The remaining 2 orbits have been disposed of via a further trick, which probably applies in many other cases, as well. Namely, suppose we find a vertex f among the common neighbors of c and d, such that f is at distance 2 from a and, furthermore, cf and df both fall into the previously decomposed cases. Then, clearly, we can decompose $abcde$ as a product of cdf, $abcfg$, and $aedfg$, where g is an arbitrary vertex from $X(a, f)$. Thus, $abcde$ is also decomposable.

This concludes the verification of the following statement.

Lemma 4.5. *All* 5*-cycles in* Γ *are decomposable.* □

Once all cycles up to length 5 are decomposed, the 6-cycles and 7-cycles are an easy gain. For $t = s_3, s_5$ construct the set $X = \{x \mid d(t, x) = 1 \text{ and } d(a, x) = 2\}$ by selecting among the neighbors x of t the involutions belonging to $O_{3'}$, O_4, and O_6. Using the package GRAPE [GAP], we then define a graph on X via commutation (so it is the subgraph of Γ induced on X) and check that this graph is connected for both choices of t. Connectivity means that all 6-cycles can be decomposed as products of 3-cycles and 5-cycles.

Finally, according to Figure 1, there are 9 cases of isometric 7-cycles. (As was the case for 5-cycles, we only need to keep track of one vertex, say $a = s$, and the edge, say de, opposite it.) In each of these case d and e have a common neighbor that is at distance 2 from a, and so the 7-cycle can be decomposed as a product of a 3-cycle and two 6-cycles. So the following is true.

Lemma 4.6. *All* 6- *and* 7-*cycles in* Γ *are decomposable.* □

Thus, all isometric cycles in Γ are decomposable, and this finishes the proof of Theorem 4.1.

5. The fundamental group of $B\mathrm{Sol}(q)$ and $BDI(4)$

In this section, we prove the following theorem.

Theorem 5.1. *For each odd prime power* q*, the geometric realization of the linking system* $\mathcal{L}^c_{\mathrm{Sol}}(q)$ *is simply connected.*

Theorem 5.1 will follow fairly easily from results in the first two sections, once we have shown the special case $q = 3$. So we first set up notation which will be used to prove this case.

Set $H = \mathrm{Spin}_7(\mathbb{Z}[\frac{1}{2}])$ for short, and let

$$\omega\colon H \underset{B}{*} K \longrightarrow\!\!\!\!\!\rightarrow \pi_1(|\mathcal{L}^c_{\mathrm{Sol}}(3)|)$$

be the surjective homomorphism of Proposition 2.2. Fix $S \in \mathrm{Syl}_2(B)$ (thus also a Sylow 2-subgroup of $\mathrm{Spin}_7(3)$), and let $U \trianglelefteq S$ be the unique normal subgroup of order 4. Then B is a finite subgroup of order $2^{10}{\cdot}3^3$, and has index 3 in K.

Set $G = H *_B K$ and $\overline{G} = \omega(G)$ for short. Also, for any subgroup $R \leq G$, we write $\overline{R} = \omega(R) \leq \overline{G}$. Since ω is surjective, $\overline{G} \cong \pi_1(|\mathcal{L}^c_{\mathrm{Sol}}(3)|)$.

Lemma 5.2. *Set* $Z = Z(H) \cong C_2$*. Then* $\overline{C_G(Z)} = \overline{H}$*.*

Proof. Since $H \leq C_G(Z)$, we need only show that $\overline{C_G(Z)} \leq \overline{H}$. Fix $g \in C_G(Z)$; we must show that $\bar{g} = \omega(g) \in \overline{H}$.

Let Λ be the standard tree for G, and set $\alpha = H$ and $\beta = K$ as vertices of Λ. Thus $G_\alpha = H$, $G_\beta = K$, each vertex of Λ is in the orbit of α or of β, and G acts transitively on the set of edges of Λ. In particular, H acts transitively on the set of vertices adjacent to α, and K acts transitively on the set of vertices adjacent to β.

Let

$$\big(\alpha = \alpha_0, \beta_1, \alpha_1, \ldots, \beta_k, \alpha_k = g(\alpha)\big)$$

be the geodesic in Λ from α to $g(\alpha)$, where each α_i is in the G-orbit of α and each β_i in the G-orbit of β. Since H acts transitively on the set of vertices adjacent to α, $\beta_1 = g_1(\beta)$ for some $g_1 \in H$. Then $g_1^{-1}(\alpha_1)$ is adjacent to β, and hence there is $g_2 \in K$ such that $g_1^{-1}(\alpha_1) = g_2(\alpha)$ and thus $\alpha_1 = g_1g_2(\alpha)$. Upon continuing in this way, we obtain a sequence of elements g_i for $i = 1, \ldots, 2k$, where $g_i \in H$ for i odd and $g_i \in K$ for i even, and such that $\beta_i = g_1 \cdots g_{2i-1}(\beta)$ and $\alpha_i = g_1 \cdots g_{2i}(\alpha)$ for each i. Set

$\widehat{g}_i = g_1 \cdots g_i$ for each i. Then $g^{-1}\widehat{g}_{2k}(\alpha) = \alpha$, so $\widehat{g}_{2k} \in gH$, it suffices to prove that $\omega(\widehat{g}_{2k}) \in \overline{H}$, and we can thus assume that $g = \widehat{g}_{2k} = g_1 \cdots g_{2k}$.

Now, $Z \leq H = G_\alpha$, and $Z \leq gHg^{-1} = G_{g(\alpha)}$. Since the fixed point set of the Z action is a tree, this means that Z fixes the entire geodesic from α to $g(\alpha)$. Thus for each i, $Z \leq G_{\beta_i} = \widehat{g}_{2i-1}K\widehat{g}_{2i-1}^{-1}$ and $Z \leq G_{\alpha_i} = \widehat{g}_{2i}H\widehat{g}_{2i}^{-1}$. So if we set $Z_i = \widehat{g}_i^{-1}Z\widehat{g}_i$, then for each $i = 1, \ldots, 2k$, $Z_i \leq K$ (if i is odd) or $Z_i \leq H$ (if i is even), and $Z_i = g_i^{-1}Z_{i-1}g_i \in H \cap K = B$.

Now, each Z_i is H-conjugate to a subgroup of U (this follows from [LO, Proposition A.8], since B is the same as a subgroup of $H = \mathrm{Spin}_7(\mathbb{Z}[\frac{1}{2}])$ or of $\mathrm{Spin}_7(3)$); and each subgroup of order 2 in U is K-conjugate to Z. Thus there is $t_i \in HK$ such that $t_i^{-1}Z_it_i = Z$. Using this, we can write g as a product of elements in $KHKHK$, each of which centralizes Z. So it suffices to prove the lemma for such g. In other words, we are reduced to the case where $k = 3$ and $g_1 = 1$. We can regard this situation schematically as follows.

$$Z = Z_1 \xrightarrow[(K)]{g_2} Z_2 \xrightarrow[(H)]{g_3} Z_3 \xrightarrow[(K)]{g_4} Z_4 \xrightarrow[(H)]{g_5} Z_5 \xrightarrow[(K)]{g_6} Z_6 = Z.$$

Assume first that $g_i \in B = H \cap K$ for some i. If $i = 3, 4, 5$, then $g \in KHK$. If $g_2 \in B$, then $g_2g_3 \in H$, $Z_3 = Z$, and we need only consider the product $g_4g_5g_6$. Similarly, if $g_6 \in B$, then we need only consider the product $g_2g_3g_4$. Thus in all cases, we can relabel the elements and assume that $g_5 = g_6 = 1$ (and $Z_4 = Z$). Also, $Z_2 = Z$ if and only if $Z_3 = Z$, since $Z_3 = g_3^{-1}Z_2g_3$ and $g_3 \in H$. If $Z_2 = Z_3 = Z$, then $g_2, g_4 \in C_K(Z) = B$, so $g \in H$, and the result follows. If $Z_2 \neq Z \neq Z_3$, then $U = ZZ_2 = ZZ_3$, so $g_3 \in N_H(U) = B$, $g \in K$ and centralizes Z, so $g \in H$.

Now assume that none of the g_i lies in B. Thus $g_2, g_6 \notin H$, so $Z_2, Z_5 \leq U$ and are distinct from Z. Hence $U = ZZ_2 = ZZ_5$. Also, $g_3 \in H \smallsetminus K$ implies $ZZ_3 = g_3^{-1}ZZ_2g_3 \neq U$, and hence that $Z_3 \nleq U$. Similarly, $Z_4 \nleq U$.

Let $E_i \leq C_H(U)$ (all $1 \leq i \leq 6$) be the rank three elementary abelian subgroups defined by the requirements that $E_3 = UZ_3$, $E_4 = UZ_4$, and $g_i^{-1}E_{i-1}g_i = E_i$. Thus $U = ZZ_5 \leq g_5^{-1}E_4g_5 = E_5$ since $[g_5, Z] = 1$, and $U \leq E_6$ since $g_6 \in K$ normalizes U. Via similar considerations for E_1 and E_2, we see that $U \leq E_i$ for all $1 \leq i \leq 6$, and hence that $E_i \leq C_H(U)$.

Set $R = C_H(U)$ for short. Then $C_S(U) \in \mathrm{Syl}_2(R)$, so each E_i is R-conjugate to a subgroup E_i' such that $C_S(E_i') \in \mathrm{Syl}_2(C_R(E_i'))$. Hence after composing with appropriate elements of $R \leq B$, we can assume that $C_S(E_i) \in \mathrm{Syl}_2(C_R(E_i))$ for each i, and that $g_i^{-1}C_S(E_{i-1})g_i = C_S(E_i)$ for each i. The subgroups $C_S(E_i)$ are all $\mathcal{F}_{\mathrm{Sol}}(3)$-centric, and thus g defines an isomorphism in $C_{\mathcal{L}^c_{\mathrm{Sol}}(3)}(Z)$ from $C_S(E_1)$ to $C_S(E_6)$.

Now, $C_S(E)$ is centric in both H and K. The easiest way to see this is to note that it contains a subgroup C_2^4 which is self-centralizing in K, and also in $H = \mathrm{Spin}_7(Z[1/2])$ since its eigenspaces in $(\mathbb{Z}[1/2])^7$ are all 1-dimensional.

Let $\mathcal{L} = \mathcal{L}^c_{\mathrm{Sol}}(3)$ for short, and set $\mathcal{L}_H = C_{\mathcal{L}}(Z)$ and $\mathcal{L}_K = N_{\mathcal{L}}(U)$. Let

$$\begin{aligned} J_H \colon \mathrm{Mor}(\mathcal{L}_H) &\longrightarrow H \cong \pi_1(|\mathcal{L}_H|) \\ J_K \colon \mathrm{Mor}(\mathcal{L}_K) &\longrightarrow K \cong \pi_1(|\mathcal{L}_K|) \\ J_{\mathcal{L}} \colon \mathrm{Mor}(\mathcal{L}) &\longrightarrow \pi_1(|\mathcal{L}|) \end{aligned}$$

be the maps defined in Section 1. For each i, $g_i \in X$ where $X = H$ or $X = K$ depending on the parity of i, and c_{g_i} lifts to some morphism $f_i \in \mathrm{Iso}_{\mathcal{L}_X}(C_S(E_{i-1}), C_S(E_i))$. Then $g_i^{-1}J_X(f_i) \in C_X(C_S(E_{i-1})) = E_{i-1}$ since $C_S(E_{i-1})$ is centric in X, and we can thus choose f_i such that $g_i = J_X(f_i)$. Hence

$$\omega(g) = \omega(g_6)\cdots\omega(g_2) = \omega(J_K(f_6))\cdot\omega(J_H(f_5))\cdots\omega(J_K(f_2)) = J_{\mathcal{L}}(f) \in \pi_1(|\mathcal{L}|)$$

where $f \in \mathrm{Iso}_{\mathcal{L}}(C_S(E_1), C_S(E_6))$ is the composite of the f_i. Since f centralizes Z, it is a morphism in $\mathcal{L}_H$, and thus $\omega(g) = \omega(J_H(f))$ where $J_H(f) \in H$. □

By Proposition 2.3, there are subgroups $H_0 \le H$ and $K_0 \le K$ such that $H_0/Z \cong Sp_6(2)$, $[K{:}K_0] = 3$, and $(H_0 \ge B_0 \le K_0)$ is an amalgam of maximal subgroups of Co_3. In the terminology of Section 4, H_0 is the stabilizer of a point in the geometry $\mathcal{G}$, and K_0 is the stabilizer of a line. Set $G_0 = \langle H_0, K_0\rangle \le G$.

Lemma 5.3. *If $\overline{G} \ne 1$, then $\overline{H_0} \cong H_0$, $\overline{K_0} \cong K_0$, and $\overline{G_0} \cong Co_3$.*

Proof. The normalizer N_0 in $\mathcal{L}^c_{\mathrm{Sol}}(3)$ of a rank four subgroup in B_0 is an extension of C_2^4 by $GL_4(2)$, the stabilizer of a 3-space in $\mathcal{G}$. In other words, ω defines a homomorphism from the amalgam $\{H_0, K_0, N_0\}$ of stabilizers of a complete flag in $\mathcal{G}$ to $\overline{G}$, and the images of these subgroups generate $\overline{G}_0$. Since the colimit of this amalgam is isomorphic to Co_3 by Proposition 4.1, this defines a surjection of Co_3 onto $\overline{G}_0$. Since Co_3 is simple, and $\overline{G}_0 \ne 1$ by Proposition 2.3 again, we have $\overline{G}_0 \cong Co_3$. □

We are now ready to prove a special case of the main theorem.

Proposition 5.4. *$|\mathcal{L}^c_{\mathrm{Sol}}(3)|$ is simply connected.*

Proof. As we have already noted, ω is onto, and hence $\overline{G} \cong \pi_1(|\mathcal{L}^c_{\mathrm{Sol}}(3)|)$. Assume by way of contradiction that $\overline{G} \ne 1$. In particular, by Lemma 5.3, $\overline{G_0} \cong Co_3$, and $\overline{S} \le \overline{B}$ is a Sylow 2-subgroup of $\overline{G_0}$. We also identify U and Z as subgroups of $\overline{G_0} \le \overline{G}$.

We refer to [Fi, §4] for information about the involutions of Co_3 and their normalizers. In particular, Co_3 has two classes of involutions, of which those of type 2A are in centers of Sylow subgroups. Fix an involution $\tau' \in Co_3$ of type 2B. Then $C_{Co_3}(\tau') = L' \times \langle\tau'\rangle$ where $L' \cong M_{12}$. By well known properties of M_{12} (see Lemma 5.5 below), there are elementary abelian subgroups $Z' \le U' \le L'$ of rank one and two, such that $\mathrm{Aut}_{L'}(U') = \mathrm{Aut}(U')$ and $L' = \langle N_{L'}(Z'), N_{L'}(U')\rangle$. By [Fi, Lemma 5.1], $\mathrm{Aut}_{Co_3}(V)$ has order three for any $2B$-pure fours subgroup $V \le Co_3$, so the involutions in U' must have type $2A$. Since Co_3 contains a unique conjugacy class of $2A$-pure subgroup of rank 2 [Fi, Lemma 5.8], there is an isomorphism $\gamma\colon Co_3 \xrightarrow{\cong} \overline{G_0}$ such that $\gamma(U') = U$ and $\gamma(Z') = Z$. Furthermore, since $C_{\overline{S}}(U)$ is a Sylow subgroup of $C_{\overline{G_0}}(U)$, we can choose γ to send $\langle\tau', U'\rangle$ into $C_{\overline{S}}(U)$. Set $\bar\tau = \gamma(\tau') \in \overline{S}$, the image of some $\tau \in S \le G$, and set $L = \gamma(L')$. Thus $C_{\overline{G_0}}(\tau) = L \times \langle\tau\rangle$, $L \cong M_{12}$, and $L = \langle N_L(Z), N_L(U)\rangle$.

We now have

$$L = \langle N_L(Z), N_L(U)\rangle \le \langle C_{\overline{H_0}}(\tau), C_{\overline{K_0}}(\tau)\rangle = \overline{\langle C_{H_0}(\tau), C_{K_0}(\tau)\rangle} \le \overline{C_G(\tau)}.$$

where the second equality holds since H_0 and K_0 are sent isomorphically to $\overline{H_0}$ and $\overline{K_0}$. Since $\langle\tau\rangle$ is G-conjugate to Z (all involutions in S are conjugate in G), $C_G(\tau)$ is G-conjugate to $C_G(Z)$, and hence $\overline{C_G(\tau)}$ is $\overline{G}$-conjugate to $\overline{C_G(Z)} = \overline{H}$ by Lemma 5.2. In particular, M_{12} is contained in H/N for some subgroup N normal in $H \cong \mathrm{Spin}_7(\mathbb{Z}[\frac{1}{2}])$.

We claim that this is impossible. By a theorem of Margulis [M, Theorem 2.4.6], the only normal subgroups of H are those which contain congruence subgroups, and those which are contained in $Z(H)$. By a theorem of Kneser [Kn, 11.1] (see also the "Zusatz bei der Korrektur"), the "congruence kernel" of $H = \mathrm{Spin}_7(\mathbb{Z}[\frac{1}{2}])$ is central, which implies that every normal subgroup of finite index contains the commutator subgroup of a congruence subgroup. If $N \leq Z(H)$, then clearly M_{12} is contained in $\Omega_7(\mathbb{Z}/n)$ for some odd n. If H/N is finite, and N contains the commutator subgroup of the congruence subgroup for $n\mathbb{Z}[\frac{1}{2}]$, then since M_{12} is not abelian, it must be contained in some quotient group of $\Omega_7(\mathbb{Z}/n)$. From this, using the simplicity of M_{12} again, and also the simplicity of the groups $\Omega_7(p)$, one sees that M_{12} is isomorphic to a subgroup of $\Omega_7(p)$ for some odd prime p.

Since M_{12} has no faithful irreducible (complex) characters of degree less than 8 (cf. [Frb, §5]), p must divide $|M_{12}|$. The odd primes dividing $|M_{12}|$ are 3,5, and 11. For $p = 3$ and $p = 5$, one finds that $|\Omega_7(p)|$ is not divisible by 11. Suppose $p = 11$. We note that $\mathrm{Alt}(6)$ is a subgroup of M_{12}, and that the only irreducible complex character degrees for $\mathrm{Alt}(6)$ which are less than 8 are 1 and 5. Thus $\mathrm{Alt}(6)$ centralizes a 2-space in any orthogonal representation of M_{12} on a space V of dimension 7 over $\mathbb{F}_{11}$. A Sylow 3-subgroup of M_{12} is extraspecial of order 3^3, so $\mathrm{Alt}(6)$ contains a central 3-element r from M_{12}. Then $[V, r]$ admits a faithful action by a group of order 27. Since 27 doesn't divide $|\Omega_5(11)|$, we have a contradiction; and this completes the proof of Proposition 5.4. □

The following lemma was needed in the above proof.

Lemma 5.5. *Set $L \cong M_{12}$. Then there are elementary abelian subgroups $Z \leq U \leq L$ of ranks one and two, such that* $\mathrm{Aut}_L(U) = \mathrm{Aut}(U)$ *and* $L = \langle N_L(Z), N_L(U)\rangle$.

Proof. It is very well known (see [Co, p. 235]) that Z and U can be chosen such that both normalizers are maximal subgroups in L. However, since we know of no published proof of this, we give the following short argument (where in fact, the subgroup U which we take is *not* in the same conjugacy class as the one whose normalizer is maximal).

Let X be a set of order 12 upon which L acts 5-transitively [G2, Theorem 6.18], and let $Y \subseteq X$ be any subset of order 10. By [G2, Exercise 6.25.2], the subgroup $L_0 \leq L$ of elements which stabilize Y is isomorphic to $\mathrm{Aut}(\mathrm{Alt}(6))$ — an extension of $\mathrm{Sym}(6)$ by an outer automorphism of order 2. Let $Z \leq U \leq L_0' = [L_0, L_0] \cong A_6$ be elementary abelian 2-subgroups of rank one and two. (Note that $\mathrm{Aut}_L(U) = \mathrm{Aut}_{L_0'}(U) = \mathrm{Aut}(U)$.) The two subgroups $U, U' \leq N_{L_0'}(Z) \cong D_8$ isomorphic to C_2^2 are conjugate in L_0, and L_0' is generated by their normalizers. From this, it is clear that $L_0 \leq \langle N_L(Z), N_L(U)\rangle$.

By 5-transitivity, L_0 is a maximal subgroup of L, and it remains only to show that $N_L(Z)$ or $N_L(U)$ contains elements of $L \smallsetminus L_0$. Since a Sylow 2-subgroup S_0 of L is not elementary abelian, $Z(S_0)$ contain elements which are squares in $S_0 \leq L \leq A_{12}$. Since a product of three 4-cycles is an odd permutation, this shows that $Z(S_0)$ contains involutions which have fixed points on X; and thus that M_{10} contains involutions which are central in Sylow subgroups of L. Since $M_{10} \smallsetminus A_6$ contains no involutions, and A_6 contains a unique class of involutions, this shows that for the subgroups Z constructed above, $C_L(Z)$ contains a Sylow 2-subgroup of L, and thus (by counting) is not contained in L_0. This finishes the proof that $L = \langle N_L(Z), N_L(U)\rangle$. □

We can now prove the main theorem.

Proof of Theorem 5.1. By Theorem 3.4, for any odd prime power q, $|\mathcal{L}^c_{\mathrm{Sol}}(q)|$ is homotopy equivalent to $|\mathcal{L}^c_{\mathrm{Sol}}(3^m)|$ for some $m \geq 1$. So it suffices to prove the theorem when $q = 3^m$. When $m = 1$, this is Proposition 5.4.

Let $S(3) \le S(3^m)$ be the Sylow subgroups of the linking systems $\mathcal{L}^c_{\mathrm{Sol}}(3)$ and $\mathcal{L}^c_{\mathrm{Sol}}(3^m)$. Let $\tau\colon S(3^m) \longrightarrow \pi_1(|\mathcal{L}^c_{\mathrm{Sol}}(3^m)|)$ be the homomorphism of Proposition 1.4, and let τ_0 be the corresponding homomorphism defined on $S(3)$. We claim there is a homomorphism from $\pi_1(|\mathcal{L}^c_{\mathrm{Sol}}(3)|)$ to $\pi_1(|\mathcal{L}^c_{\mathrm{Sol}}(3^m)|)$ which makes the following square commute:

$$\begin{tikzcd}
S(3) \arrow[r, "\mathrm{incl}"] \arrow[d, "\tau_0"'] & S(3^m) \arrow[d, "\tau)"'] \\
1{=}\pi_1(|\mathcal{L}^c_{\mathrm{Sol}}(3)|) \arrow[r] & \pi_1(|\mathcal{L}^c_{\mathrm{Sol}}(3^m)|)\,.
\end{tikzcd}$$

This follows from [LO, Lemma 4.1] and from [AC, Theorem C], using two very different approaches. Hence

$$S(3) \le K \overset{\mathrm{def}}{=} \mathrm{Ker}\bigl[S(3^m) \xrightarrow{\ \tau\ } \pi_1(|\mathcal{L}^c_{\mathrm{Sol}}(3^m)|)\bigr],$$

and K is strongly closed in $\mathcal{F}_{\mathrm{Sol}}(3^m)$ by Proposition 1.4(a). From the description in Lemma 3.1 of $S(3^m)$ and its fusion, this implies that K contains the subgroups R_i in $S(3^m)$, hence the subgroup $T \le S(3^m)$ (since $R_1R_2R_3 \cap T$ has index 2 in T), and hence that $K = S(3^m)$.

Thus τ is the trivial homomorphism. So by Proposition 1.4(b), $\mathrm{Out}_{\mathcal{F}_{\mathrm{Sol}(3^m)}}(S(3^m))$ surjects onto $\pi_1(|\mathcal{L}^c_{\mathrm{Sol}}(3^m)|)$. Also,

$$\mathrm{Out}_{\mathcal{F}_{\mathrm{Sol}(3^m)}}(S(3^m)) = \mathrm{Out}_{\mathrm{Spin}_7(3^m)}(S(3^m)),$$

since $\mathcal{F}_{\mathrm{Spin}_7(3^m)}(S(3^m))$ is the centralizer of an involution. By [LO, Proposition 1.9] (or by its proof), $S(3^m)$ contains a unique subgroup $R_0 \cong (C_{2^k})^3$ (where 2^k is the largest power dividing $3^m \pm 1$), $C_{S(3^m)}(R_0) = R_0$, and $\mathrm{Aut}_{\mathrm{Spin}_7(3^m)}(R_0) \cong C_2 \times \mathrm{Sym}_4$. So every element in $N_{\mathrm{Spin}_7(3^m)}(S(3^m))$ acts on R_0 and on $S(3^m)/R_0$ with 2-power order; this implies that $\mathrm{Out}_{\mathrm{Spin}_7(3^m)}(S(3^m))$ is a 2-group (hence trivial), and thus that $|\mathcal{L}^c_{\mathrm{Sol}}(3^m)|$ is simply connected. □

For any odd prime p, let $\mathcal{L}^c_{\mathrm{Sol}}(p^\infty)$ be the category constructed in [LO, Section 4], as a "linking system" associated to the union $\mathcal{F}_{\mathrm{Sol}}(p^\infty)$ of the fusion systems $\mathcal{F}_{\mathrm{Sol}}(p^m)$. By [LO, Proposition 4.3], $|\mathcal{L}^c_{\mathrm{Sol}}(p^\infty)|^\wedge_2 \simeq BDI(4)$: the classifying space of the exotic 2-compact group constructed by Dwyer and Wilkerson. We can now show:

Corollary 5.6. *For any odd prime p, $|\mathcal{L}^c_{\mathrm{Sol}}(p^\infty)|$ is simply connected.*

Proof. By the construction in [LO, Section 4], the linking category $\mathcal{L}^c_{\mathrm{Sol}}(p^\infty)$ is the union of subcategories $\mathcal{L}^{cc}_{\mathrm{Sol}}(p^m)$, whose nerves have the homotopy type of $|\mathcal{L}^c_{\mathrm{Sol}}(p^m)|$ [LO, Lemma 4.1], and hence are simply connected by Theorem 5.1. Thus $|\mathcal{L}^c_{\mathrm{Sol}}(p^\infty)|$ is simply connected. □

References

[A] M. Aschbacher, Overgroups of Sylow subgroups in sporadic groups, Memoirs of the Amer. Math. Soc. 60 (1986)

[AC] M. Aschbacher & A. Chermak, A group-theoretic approach to a family of 2-local finite groups constructed by Levi and Oliver, preprint

[AS] M. Aschbacher & S. Smith, The classification of quasithin groups: I. Structure of strongly quasithin $\mathcal{K}$-groups, Amer. Math. Soc. (2004)

[Be] D. Benson, Cohomology of sporadic groups, finite loop spaces, and the Dickson invariants, Geometry and cohomology in group theory, London Math. Soc. Lecture notes ser. 252, Cambridge Univ. Press (1998), 10–23

[BCGLO1] C. Broto, N. Castellana, J. Grodal, R. Levi, & B. Oliver, Subgroup families controlling p-local finite groups, Proc. London Math. Soc. (to appear)

[BCGLO2] C. Broto, N. Castellana, J. Grodal, R. Levi, & B. Oliver, Extensions of p-local finite groups, Trans. Amer. Math. Soc. (to appear)

[BLO1] C. Broto, R. Levi, & B. Oliver, Homotopy equivalences of p-completed classifying spaces of finite groups, Invent. math. 151 (2003), 611–664

[BLO2] C. Broto, R. Levi, & B. Oliver, The homotopy theory of fusion systems, Journal Amer. Math. Soc. 16 (2003), 779–856

[BM] C. Broto & J. Møller, Finite Chevalley versions of p-compact groups (preprint)

[Bw] K. Brown, Buildings, Springer-Verlag (1989)

[Co] J. Conway, Three lectures on exceptional groups, Higman–Powell: Finite simple groups, Academic press (1971)

[DW] W. Dwyer & C. Wilkerson, A new finite loop space at the prime two, J. Amer. Math. Soc. 6 (1993), 37-64

[Fi] L. Finkelstein, The maximal subgroups of Conway's group C_3 and McLaughlin's group, J. Algebra 25 (1973), 58–89

[Frb] F. G. Frobenius, Über de Charaktere der mehrfach transitiven Gruppen, Sitzungsberichte König. Preuß. Akad. Wiss. zu Berlin (1904), 558–571; Gesammelte Abhandlungen III, 335–348

[GAP] The GAP Group, GAP — Groups, Algorithms, and Programming, Version 4.4 (2006); package GRAPE (`http://www.gap-system.org`)

[Go] D. Goldschmidt, A conjugation family for finite groups, J. Algebra 16 (1970), 138-142

[GLS] D. Gorenstein, R. Lyons, & R. Solomon, The classification of the finite simple groups, nr. 3, Amer. Math. Soc. (1998)

[G] R. Griess, Sporadic groups, code loops, and nonvanishing cohomology, J. Pure and Applied Alg. 44 (1987), 191–214

[G2] R. Griess, Twelve sporadic groups, Springer-Verlag (1998)

[Gr] J. Grodal, Higher limits via subgroup complexes, Annals of Math. 155 (2002), 405–457

[GO] J. Grodal & B. Oliver, Fundamental groups of linking systems, in preparation

[Ka] W. Kantor, Some exceptional 2-adic buildings, J. Algebra 92 (1985), 208–223

[Kn] M. Kneser, Normalteiler ganzzahliger Spingruppen, J. Reine Angew. Math. 311/312 (1979), 191–214

[LO] R. Levi & B. Oliver, Construction of 2-local finite groups of a type studied by Solomon and Benson, Geometry and Topology 6 (2002), 917–990

[M] G. Margulis, Finiteness of quotient groups of discrete subgroups, Functional Analysis and Its Applications 13 (1979), 178-187

[O1] B. Oliver, Equivalences of classifying spaces completed at the prime two, Memoirs Amer. Math. Soc. (to appear)

[Pn] A. Pasini, Diagram geometries, Oxford (1994)

[Pg] Ll. Puig, Unpublished notes (ca. 1990)

[Se] G. Segal, Classifying spaces and spectral sequences, Publ. Math. I.H.E.S. 34 (1968), 105–112

[So] R. Solomon, Finite groups with Sylow 2-subgroups of type .3, J. Algebra 28 (1974), 182–198

[Sr] V. Srinivas, Algebraic K-theory, 2nd ed., Birkhäuser (1996)

[T] J. Tits, Ensembles ordonnés, immeubles et sommes amalgamées, *Bull. Soc. Math. Belg. Sér. A*, **38** (1986), 367–387.

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