SIMPLE FUSION SYSTEMS OVER p-GROUPS WITH ABELIAN SUBGROUP OF INDEX p : I

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ABSTRACT. For an odd prime p, we look at simple fusion systems over a finite nonabelian p-group S which has an abelian subgroup A of index p. When S has more than one such subgroup, we reduce this to a case already studied by Ruiz and Viruel. When A is the unique abelian subgroup of index p in S and is not essential (equivalently, is not radical) in the fusion system, we give a complete list of all possibilities which can occur. This includes several families of exotic fusion systems, including some which have proper strongly closed subgroups.

A *saturated fusion system* $\mathcal{F}$ over a finite p-group S is a category whose objects are the subgroups of S, whose morphisms are injective homomorphisms between subgroups, and whose morphism sets satisfy certain axioms first formulated by Puig and motivated by the properties of conjugacy relations among p-subgroups of a finite group. For example, for each finite group G and each $S \in \mathrm{Syl}_p(G)$, the category $\mathcal{F}_S(G)$, whose objects are the subgroups of G and whose morphisms are those homomorphisms induced by conjugation in G, is a saturated fusion system over S. We refer to Puig's paper [Pg], and to [AKO] and [Cr], for more background details on saturated fusion systems.

A saturated fusion system $\mathcal{F}$ over a finite p-group S is *realizable* if $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group G with $S \in \mathrm{Syl}_p(G)$; otherwise it will be called *exotic*. When p is odd, many examples have been constructed of exotic fusion systems over p-groups. We refer in particular to the classifications by Díaz, Ruiz, and Viruel of saturated fusion systems over extraspecial p-groups of order p^3 and exponent p [RV] and over p-groups of rank two [DRV]; and to a more general procedure described in [BLO4, § 5] for constructing such examples and checking that they are saturated.

A saturated fusion system $\mathcal{F}$ is *simple* if it contains no nontrivial proper normal subsystems (see [AKO, Definition I.6.1] or [Cr, §§ 5.4 & 8.1] for the precise definition of a normal subsystem). In this paper, we study simple fusion systems over nonabelian p-groups which have an abelian subgroup of index p. When $p = 2$, by [AOV1, Propositions 4.3–4.4] and [AOV2, Propositions 3.1 & 5.2(a)], each such fusion system is isomorphic to the fusion system of $PSL_2(q)$ or of $PSL_3(q)$ for some odd q, and the 2-group in question is dihedral, semidihedral, or a wreath product $C_{2^n} \wr C_2$.

When p is odd, it turns out that this class of p-groups supports a much richer collection of simple fusion systems, many of which are exotic. Let $\mathcal{F}$ be a simple fusion system over the nonabelian p-group S, with abelian subgroup A of index p. We split this into three different cases:

(1) S has more than one abelian subgroup of index p;

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(2) A is the unique abelian subgroup of index p in S, and A is not $\mathcal{F}$-essential (Definition 1.1); and

(3) A is the unique abelian subgroup of index p in S, and A is $\mathcal{F}$-essential.

In case (1), we show that S is extraspecial of order p^3 and exponent p (Theorem 2.1), and hence that we are in the situation considered by [RV]. In case (2), we give a complete list of all such fusion systems (Theorem 2.8), and determine which of them are exotic (most of them). This list includes many of the examples constructed earlier by Díaz, Ruiz, and Viruel [DRV] in a different context. Among the exotic examples constructed are several which are also noteworthy for having proper strongly closed subgroups (Definition 1.1).

Case (3) is more complicated, since it depends heavily on representation theory (via the action of $\mathrm{Aut}_{\mathcal{F}}(A)$ on A). This will be handled in a later paper together with David Craven.

1. Background

We first recall some of the terminology used for certain subgroups in a fusion system.

Definition 1.1. *Fix a prime p, a finite p-group S, and a saturated fusion system $\mathcal{F}$ over S. Let $P \leq S$ be any subgroup.*

- *$P^{\mathcal{F}}$ denotes the set of subgroups of S which are $\mathcal{F}$-conjugate (isomorphic in $\mathcal{F}$) to P. Also, $g^{\mathcal{F}}$ denotes the $\mathcal{F}$-conjugacy class of an element $g \in S$ (the set of images of g under morphisms in $\mathcal{F}$).*
- *P is* fully normalized *in $\mathcal{F}$ (*fully centralized *in $\mathcal{F}$) if $|N_S(P)| \geq |N_S(Q)|$ ($|C_S(P)| \geq |C_S(Q)|$) for each $Q \in P^{\mathcal{F}}$.*
- *P is $\mathcal{F}$-*centric *if $C_S(Q) = Z(Q)$ for each $Q \in P^{\mathcal{F}}$.*
- *P is $\mathcal{F}$-*essential *if $P < S$, P is $\mathcal{F}$-centric and fully normalized in $\mathcal{F}$, and $\mathrm{Out}_{\mathcal{F}}(P) \stackrel{\text{def}}{=} \mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Inn}(P)$ contains a strongly p-embedded subgroup. Here, a proper subgroup $H < G$ of a finite group G is* strongly p-embedded *if $p \big| |H|$, and $p \nmid |H \cap gHg^{-1}|$ for each $g \in G \smallsetminus H$.*
  Let $\mathbf{E}_{\mathcal{F}}$ denote the set of all $\mathcal{F}$-essential subgroups of S.
- *P is* normal in $\mathcal{F}$ *if each morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$ in $\mathcal{F}$ extends to a morphism $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(PQ,PR)$ such that $\overline{\varphi}(P) = P$. The maximal normal p-subgroup of a saturated fusion system $\mathcal{F}$ is denoted $O_p(\mathcal{F})$.*
- *P is* strongly closed in $\mathcal{F}$ *if for each $g \in P$, $g^{\mathcal{F}} \subseteq P$.*
- $\mathfrak{foc}(\mathcal{F}) = \langle gh^{-1} \,|\, g \in S,\ h \in g^{\mathcal{F}} \rangle$.

Let $O^p(\mathcal{F})$ and $O^{p'}(\mathcal{F})$ denote the smallest normal fusion subsystems of p-power index, and of index prime to p, respectively. Such normal subsystems are defined by analogy with finite groups, and we refer to [AKO, § I.7] or [Cr, § 7.5] for precise definitions and references.

Definition 1.2. *A saturated fusion system $\mathcal{F}$ is* reduced *if $O_p(\mathcal{F}) = 1$, and $O^p(\mathcal{F}) = \mathcal{F} = O^{p'}(\mathcal{F})$. A saturated fusion system is* simple *if it contains no nontrivial proper normal fusion subsystems, in the sense of* [AKO, Definition I.6.1] *or* [Cr, §§ 5.4 & 8.1].

For any saturated fusion system $\mathcal{F}$ over S, $O^p(\mathcal{F})$, $O^{p'}(\mathcal{F})$, and $\mathcal{F}_{O_p(\mathcal{F})}(O_p(\mathcal{F}))$ are all normal subsystems. Hence $\mathcal{F}$ is reduced if it is simple. If $\mathcal{E} \trianglelefteq \mathcal{F}$ is any normal subsystem over the subgroup $T \trianglelefteq S$, then by definition of normality, T is strongly closed in $\mathcal{F}$. Thus a reduced fusion system is simple if it has no proper nontrivial strongly closed subgroups.

There are reduced fusion systems which are not simple; constructed, for example, by taking direct products or wreath products. In this paper, our main interest in reduced fusion systems is as an intermediate step towards showing that they are simple. We refer to [AOV1, Theorems A–C] for the original motivation for defining them.

The next proposition lists some of the standard tools for handling $O_p(\mathcal{F})$ and $O^p(\mathcal{F})$.

Proposition 1.3. *The following hold for any saturated fusion system $\mathcal{F}$ over a finite p-group S.*

(a) *Each morphism in $\mathcal{F}$ is a composite of restrictions of elements of $\mathrm{Aut}_{\mathcal{F}}(P)$ for $P \in \mathbf{E}_{\mathcal{F}} \cup \{S\}$. Moreover, each morphism in $\mathcal{F}$ is a composite of restrictions of elements in $\mathrm{Aut}_{\mathcal{F}}(S)$, and in $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))$ for $P \in \mathbf{E}_{\mathcal{F}}$.*

(b) *Assume $Q \trianglelefteq S$ has the property that for each $P \in \mathbf{E}_{\mathcal{F}} \cup \{S\}$, $P \geq Q$ and $\mathrm{Aut}_{\mathcal{F}}(P)$ normalizes Q. Then $Q \trianglelefteq \mathcal{F}$.*

(c) $\mathfrak{foc}(\mathcal{F}) = \big\langle [\mathrm{Aut}_{\mathcal{F}}(P), P] \,\big|\, P \in \mathbf{E}_{\mathcal{F}} \cup \{S\} \big\rangle$.

(d) *$O^p(\mathcal{F}) = \mathcal{F}$ if and only if $\mathfrak{foc}(\mathcal{F}) = S$.*

Proof. The first statement in (a) is shown in [Pg, § 5], and also in [OV, Corollary 2.6], while the stronger statement follows from [O1, Proposition 1.10(a,b)]. Point (b) is shown in [AKO, Proposition I.4.5], and point (d) in [AKO, Corollary I.7.5]. Point (c) is an immediate consequence of (a) and the definition of $\mathfrak{foc}(\mathcal{F})$. □

Determining $O^{p'}(\mathcal{F})$ is more difficult, in general, but the following lemma suffices for our purposes.

Lemma 1.4. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. Assume that each $P \in \mathbf{E}_{\mathcal{F}}$ is minimal among all $\mathcal{F}$-centric subgroups. For each $P \in \mathbf{E}_{\mathcal{F}}$, set*

$$\mathrm{Aut}_{\mathcal{F}}^{(P)}(S) = \big\{\alpha \in \mathrm{Aut}_{\mathcal{F}}(S) \,\big|\, \alpha(P) = P,\ \alpha|_P \in O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))\big\}\ .$$

Then $O^{p'}(\mathcal{F}) = \mathcal{F}$ if and only if $\mathrm{Aut}_{\mathcal{F}}(S) = \big\langle \mathrm{Inn}(S), \mathrm{Aut}_{\mathcal{F}}^{(P)}(S) \,\big|\, P \in \mathbf{E}_{\mathcal{F}} \big\rangle$.

Proof. Set $H_P = \mathrm{Aut}_{\mathcal{F}}^{(P)}(S)$ and $H = \langle \mathrm{Inn}(S), H_P \,|\, P \in \mathbf{E}_{\mathcal{F}} \rangle$ for short. Let $\mathcal{F}^c \subseteq \mathcal{F}$ be the full subcategory with objects the $\mathcal{F}$-centric subgroups of S.

Since no $\mathcal{F}$-centric subgroup is properly contained in an essential subgroup by assumption, two $\mathcal{F}$-centric subgroups are $\mathcal{F}$-conjugate only if they are in the same $\mathrm{Aut}_{\mathcal{F}}(S)$-orbit by Proposition 1.3(a). Thus all $\mathcal{F}$-centric subgroups are fully normalized in $\mathcal{F}$. In particular, $P \in \mathbf{E}_{\mathcal{F}}$ and $Q \in P^{\mathcal{F}}$ imply $Q \in \mathbf{E}_{\mathcal{F}}$. For $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ and $P \in \mathbf{E}_{\mathcal{F}}$, $\alpha H_P \alpha^{-1} = H_{\alpha(P)}$; and thus $H \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(S)$.

If $H = \mathrm{Aut}_{\mathcal{F}}(S)$, then $O^{p'}(\mathcal{F}) = \mathcal{F}$ by [BCGLO, Theorem 5.4] or [AKO, Theorem I.7.7], since $H \leq \mathrm{Aut}_{\mathcal{F}}^0(S)$ in the notation of those references.

Now assume $H < \mathrm{Aut}_{\mathcal{F}}(S)$. We claim that for each $P \leq S$ which is $\mathcal{F}$-centric,

$$\mathrm{Hom}_{\mathcal{F}}(P,S) = \begin{cases} \big\{\alpha|_P \,\big|\, \alpha \in \mathrm{Aut}_{\mathcal{F}}(S)\big\} & \text{if } P \notin \mathbf{E}_{\mathcal{F}} \\ \big\{\alpha|_P \,\big|\, \alpha \in \mathrm{Aut}_{\mathcal{F}}(S)\big\} \circ O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P)) & \text{if } P \in \mathbf{E}_{\mathcal{F}}\,. \end{cases} \tag{1}$$

If $P \notin \mathbf{E}_{\mathcal{F}}$, then since no subgroup in $P^{\mathcal{F}}$ is contained in an essential subgroup, each $\alpha \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ extends to some $\overline{\alpha} \in \mathrm{Aut}_{\mathcal{F}}(S)$ by Proposition 1.3(a), and (1) holds in this case. If $P \in \mathbf{E}_{\mathcal{F}}$, then each morphism in $\mathrm{Hom}_{\mathcal{F}}(P,S)$ is a composite of restrictions of elements in $\mathrm{Aut}_{\mathcal{F}}(S)$, and of elements in $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(Q))$ for $Q \in P^{\mathcal{F}}$. Via the relations $(\alpha|_Q)\mathrm{Aut}_{\mathcal{F}}(Q)(\alpha|_Q)^{-1} = \mathrm{Aut}_{\mathcal{F}}(\alpha(Q))$ for $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$, these morphisms can be arranged in the above form, proving the second case of (1).

Define a map $\theta\colon \mathrm{Mor}(\mathcal{F}^c) \longrightarrow \mathrm{Aut}_{\mathcal{F}}(S)/H$ as follows. For each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ in $\mathcal{F}^c$, set $\theta(\varphi) = \alpha H$ (for $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$) if $\varphi = \alpha|_P$, or if $P \in \mathbf{E}_{\mathcal{F}}$ and $\varphi = \alpha|_P \circ \beta$ where $\beta \in O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))$. If $\varphi = \alpha|_P = \alpha'|_P$, then $\alpha' = \alpha \circ c_g$ for some $g \in Z(P)$ (cf. [AKO, Lemma I.5.6]), so $\alpha' H = \alpha H$. If $P \in \mathbf{E}_{\mathcal{F}}$ and $\varphi = \alpha|_P \circ \beta = \alpha'|_P \circ \beta'$ (where $\beta, \beta' \in O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))$), then $\alpha = \alpha'\gamma$ for some $\gamma \in \mathrm{Inn}(S)H_P$ by the same lemma and the definition of H_P. Thus θ is well defined in all cases.

By construction, θ sends composites to products, and sends $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(Q))$ to the identity for each $Q \in \mathrm{Ob}(\mathcal{F}^c)$ since $p \nmid |\mathrm{Aut}_{\mathcal{F}}(S)/H|$. So $O^{p'}(\mathcal{F}) \subsetneqq \mathcal{F}$ by [AKO, Theorem I.7.7(c)]. More precisely, $\langle \theta^{-1}(1) \rangle$ is a proper normal subsystem of index prime to p in $\mathcal{F}$. □

The next two lemmas provide the tools we will need to determine which of the fusion systems we construct are realizable and which are exotic.

Lemma 1.5 ([DRV]). *Let $\mathcal{F}$ be a reduced fusion system over a p-group S. Assume, for each $1 \neq P \trianglelefteq S$ strongly closed in $\mathcal{F}$, that P is centric in S, is not elementary abelian, and does not factor as a product of two or more subgroups which are permuted transitively by* $\mathrm{Aut}_{\mathcal{F}}(P)$. *Then if $\mathcal{F}$ is realizable, it is the fusion system of a finite simple group.*

Proof. By [DRV, Proposition 2.19], $\mathcal{F} = \mathcal{F}_S(G)$ for some almost simple group G. We must show that when $\mathcal{F}$ is reduced, G can be chosen to be simple.

Assume G is not simple, and that no proper subgroup of G realizes $\mathcal{F}$. Since the outer automorphism group of every simple group is solvable (cf. [GLS3, Theorem 7.1.1(a)]), there is a proper normal subgroup $G_0 \trianglelefteq G$ of index p or of index prime to p. If $[G{:}G_0] = p$, then by the focal subgroup theorem [G, Theorem 7.3.4], $\mathfrak{foc}(\mathcal{F}) \leq S \cap G_0 < S$, which is impossible since $O^p(\mathcal{F}) = \mathcal{F}$ (Proposition 1.3(d)). If $[G{:}G_0]$ is prime to p, then $\mathcal{F}_S(G_0)$ is a normal subsystem of index prime to p in $\mathcal{F}$ by [BCGLO, Definition 3.1(b)]: it is saturated, and $\mathrm{Aut}_{G_0}(P) \geq O^{p'}(\mathrm{Aut}_G(P))$ for each $P \leq S$. Since $\mathcal{F}$ is reduced, it has no proper normal subsystems of index prime to p, so $\mathcal{F} = \mathcal{F}_S(G_0)$, which contradicts the minimality assumption on G. We thus have a contradiction in either case, and so G can be taken to be simple. □

Lemma 1.6. *Fix an odd prime p. Assume G is a finite simple group for which $S \in \mathrm{Syl}_p(G)$ is nonabelian and contains a unique abelian subgroup $A < S$ of index p. Assume also that $|Z(S)| = p$, $|S/[S,S]| = p^2$, and A is not essential in G. Then $p = 3$, and G is isomorphic to one of the groups $PSL_3(q)$ ($q \equiv 1 \pmod 3$), $PSU_3(q)$ ($q \equiv -1 \pmod 3$), $G_2(q)$ ($3 \nmid q$), or ${}^3D_4(q)$ ($3 \nmid q$).*

Proof. Since A is not essential, $\mathrm{Aut}_S(A) \trianglelefteq \mathrm{Aut}_G(A)$. Equivalently, $S \trianglelefteq N_G(A)$, and hence $N_G(S) = N_G(A)$. If $B < S$ is any other abelian subgroup, then either $B \leq A$, or $B \cap A \leq Z(S)$ and hence $|B| \leq p^2$.

If G is an alternating group, then $G \cong A_n$ for some $n = ap + b$ where $p \leq a < 2p$ and $0 \leq b < p$. Then $A \cong (C_p)^a$, and $\mathrm{Aut}_G(A)$ has index two in $\mathrm{Aut}_{\Sigma_n}(A) \cong C_{p-1} \wr \Sigma_a$. Since no such group has a normal subgroup of order p, G is not an alternating group.

If G is a sporadic group, then by the tables in [GL, § 1.5] or [GLS3, § 5.3], in almost all cases, either $|S| \leq p^3$, or S is abelian, or S contains an extraspecial group of type p^{1+2k} for $k \geq 2$, or S contains a special group of type 3^{2+4}. The exceptions are $(G,p) = (J_3, 3)$, where

$Z(S) \cong C_3^2$; $(Co_1, 5)$, where $J(S) \cong C_5^3$ and $N_G(J(S))/J(S) \cong C_4 \times \Sigma_5$; and $(F_3, 3)$, where G contains a subgroup $C_3^4 \rtimes SL_2(9)$. Thus G is not a sporadic group.

If G is a group of Lie type in characteristic p, then since S is nonabelian and $|S/[S,S]| = p^2$, G must be one of the groups $L_3(p)$, $Sp_4(p)$, $U_3(p)$, or $G_2(p)$ (see the description in [GLS3, Theorem 3.3.1] of the central series for S). By the commutator relations, any abelian subgroup of index p in S would have to be a parabolic subgroup, and hence essential in G.

Thus by the classification theorem, G is a group of Lie type in characteristic different from p. If $G \cong PSL_n(q)$ for some q and $|S| \geq p^4$, and k is the order of q in $\mathbb{F}_p^\times$, then S has a normal abelian subgroup A of order at least p^3 with $\mathrm{Aut}_G(A) \cong C_k \wr \Sigma_n$ for some n, and this has a normal Sylow subgroup of order p only when $p = n = 3$ and $k = 1$. Thus $G \cong PSL_3(q)$ for $q \equiv 1 \pmod 3$. A similar argument in the unitary case shows that $PSU_3(q)$ for $q \equiv 2$ (mod 3) is the only possibility. If G is symplectic or orthogonal, then $\mathrm{Aut}_G(A)$ always contains $C_2 \wr \Sigma_n$ or $(C_2)^{n-1} \rtimes \Sigma_n$, so A is not essential.

In all other cases, by the description in [GL, 10-1] of the Sylow subgroups, there is a normal abelian p-subgroup P_H in S, which must be contained in A by the above remarks, which is maximal abelian, and whose index in S is determined by the tables there. In particular, A has index p in S only in the following cases:

- $p = 3$, $G = G_2(q)$ or ${}^3D_4(q)$ ($q \equiv \pm 1 \pmod 3$), or $G = {}^2F_4(2^{2k+1})$;
- $p = 5$, $G = E_6(q)$ ($q \equiv 1 \pmod 5$), $G = {}^2E_6(q)$ ($q \equiv -1 \pmod 5$), $G = E_7(q)$ ($q \equiv \pm 1 \pmod 5$), $G = E_8(q)$ ($q \equiv \pm 2 \pmod 5$);
- $p = 7$, $G = E_7(q)$ or $E_8(q)$ ($q \equiv \pm 1 \pmod 7$).

When $p = 3$ and $G = {}^2F_4(2^{2k+1})$, then $\mathrm{Aut}_G(A) \cong GL_2(3)$ [Ma, Proposition 1.2]. When $p = 5$ and $G = E_8(q)$ for $q \equiv \pm 2 \pmod 5$, then $\mathrm{Aut}_G(A)/O_2(\mathrm{Aut}_G(A)) \cong \Sigma_6$ [LSS, Table 5.2]. In all other cases when $p = 5, 7$, $\mathrm{Aut}_G(A)$ is the Weyl group of E_m for $m = 6, 7, 8$, and contains a (quasi)simple subgroup of index two.

Thus the only cases where $\mathrm{Aut}_G(A)$ contains a normal Sylow p-subgroup of order p are those where $p = 3$ and $G = G_2(q)$ or ${}^3D_4(q)$. □

We finish the section with a few elementary group theoretic results.

Lemma 1.7. *Fix a prime p, a finite p-group P, and a group $G \leq \mathrm{Aut}(P)$ of automorphisms of P. Let $P_0 \trianglelefteq P_1 \trianglelefteq \cdots \trianglelefteq P_m = P$ be a sequence of subgroups, all normal in P and normalized by G, such that $P_0 \leq \mathrm{Fr}(P)$. Let $H \leq G$ be the subgroup of those $g \in G$ which act via the identity on P_i/P_{i-1} for each $1 \leq i \leq m$. Then H is a normal p-subgroup of G.*

Proof. See, e.g., [G, Theorems 5.3.2 & 5.1.4]. □

Lemma 1.8. *Assume P is a nonabelian group of order p^3, for some odd prime p. Then either P has exponent p and $\mathrm{Out}(P) \cong GL_2(p)$, or P has exponent p^2 and $O_p(\mathrm{Out}(P)) \in \mathrm{Syl}_p(\mathrm{Out}(P))$.*

Proof. If P has exponent p, then each automorphism of $P/[P,P] \cong C_p^2$ lifts to an automorphism of P. Also, each automorphism of P which induces the identity on $P/[P,P]$ is inner, so $\mathrm{Out}(P) \cong \mathrm{Aut}(P/[P,P]) \cong GL_2(p)$.

If P has exponent p^2, then it contains a unique subgroup $Q < P$ with $Q \cong C_p^2$. So by Lemma 1.7, there is a homomorphism from $\mathrm{Aut}(P)$ to $\mathrm{Aut}(P/Q) \times \mathrm{Aut}(Q/[P,P]) \cong C_{p-1} \times C_{p-1}$ whose kernel is $O_p(\mathrm{Aut}(P))$. □

We will adopt the usual notation, and write p_+^{1+2} and p_-^{1+2} for nonabelian groups of order p^3 and of exponent p or p^2, respectively.

Lemma 1.9. *Let S be a nonabelian p-group, and assume $A \trianglelefteq S$ is an abelian subgroup of index p. Then either*

(a) $\big|[S,S]\big| \geq p^2$, $\big|S/Z(S)\big| \geq p^3$, *and A is the unique abelian subgroup of index p in S; or*

(b) $\big|[S,S]\big| = p$, $S/Z(S) \cong C_p^2$, *and S contains exactly $p+1$ abelian subgroups of index p.*

Proof. Fix any $x \in S \smallsetminus A$. Then $[S,S] = [x,A]$ is the image of $(\mathrm{Id} - c_x)$ as a homomorphism from A to itself, and $Z(S) = C_A(x)$ is its kernel. Hence $|S/Z(S)| = p{\cdot}|A/Z(S)| = p{\cdot}\big|[S,S]\big|$. Since S is nonabelian, $S/Z(S)$ is not cyclic.

If B is a second abelian subgroup of index p in S, then $S = AB$, so $A \cap B \leq Z(S)$, and $|S/Z(S)| \leq p^2$. Thus $\big|[S,S]\big| = p$ in this case. Conversely, if $|S/Z(S)| \leq p^2$, then $S/Z(S) \cong C_p^2$ since it is noncyclic, and each subgroup of index p in S containing $Z(S)$ is abelian (generated by $Z(S)$ and one more element). Since each abelian subgroup of index p in S contains $Z(S)$, S has exactly $p+1$ abelian subgroups of index p. □

We finish the section with two lemmas which deal with actions on finite abelian p-groups.

Lemma 1.10. *Fix a finite abelian p-group A and a subgroup $G \leq \mathrm{Aut}(A)$. Assume, for $S \in \mathrm{Syl}_p(G)$, that $S \ntrianglelefteq G$ and $|A/C_A(S)| = p$. Then $|S| = p$.*

Proof. Let $S_1, S_2 \in \mathrm{Syl}_p(G)$ be two distinct Sylow subgroups, and set $A_i = C_A(S_i)$. For $i = 1,2$, S_i induces the trivial action on $A/A_i \cong C_p$, and hence $[S_i, A] \leq A_i$. Also, $A_1 \neq A_2$, since otherwise $\langle S_1, S_2\rangle$ would act trivially on A_1 and on A/A_1, and hence would be a p-group by Lemma 1.7.

If $|S_i| > p$, then $\big|[S_i,A]\big| > p$, so there are elements $1 \neq x_i \in S_i$ such that $[x_i, A] \leq A_1 \cap A_2$. Set $T = \langle x_1, x_2\rangle$. Then T acts trivially on $A_1 \cap A_2$ and on $A/(A_1 \cap A_2)$, so T is a p-group by Lemma 1.7, while $C_A(T) = C_A(x_1) \cap C_A(x_2) = A_1 \cap A_2$. Since this contradicts the assumption that $|A/C_A(S)| = p$ for each $S \in \mathrm{Syl}_p(G)$, we conclude that $|S_i| = p$. □

Lemma 1.11. *Fix a finite abelian p-group A, and a subgroup $G \leq \mathrm{Aut}(A)$. Assume the following.*

(i) *Each Sylow p-subgroup of G has order p and is not normal in G.*

(ii) *For each $x \in G$ of order p, $[x,A]$ has order p, and hence $C_A(x)$ has index p.*

Set $H = O^{p'}(G)$, $A_1 = C_A(H)$, and $A_2 = [H,A]$. Then G normalizes A_1 and A_2, $A = A_1 \times A_2$, and $H \cong SL_2(p)$ acts faithfully on $A_2 \cong C_p^2$. There are groups of automorphisms $G_i \leq \mathrm{Aut}(A_i)$ $(i = 1,2)$, such that $p \nmid |G_1|$, $G_2 \geq \mathrm{Aut}_H(A_2) \cong SL_2(p)$, and $G \trianglelefteq G_1 \times G_2$ (as a subgroup of $\mathrm{Aut}(A)$) with index dividing $p-1$.

Proof. For each $B \leq A$, let $\mathbb{V}_1(B)$ be the set of subgroups of B of order p. Define

$$\theta\colon \mathrm{Syl}_p(G) \longrightarrow \mathbb{V}_1(A)$$

by setting $\theta(S) = [S,A]$. If $[S,A] = [T,A]$ for $S,T \in \mathrm{Syl}_p(G)$, then $\langle S,T\rangle$ acts via the identity on $[S,A]$ and on $A/[S,A]$, so $\langle S,T\rangle$ is a p-group by Lemma 1.7, and $S = T$. Thus θ is injective.

Assume $S_1, S_2 \in \mathrm{Syl}_p(G)$ are distinct, set $K = \langle S_1, S_2\rangle$, and consider the action of K on $[K,A] = \theta(S_1)\theta(S_2) \cong C_p^2$. Since K acts trivially on $A/[K,A]$, the subgroup $K_0 =$

$C_K([K,A]) \trianglelefteq K$ is a normal p-subgroup by Lemma 1.7, hence contained in all Sylow p-subgroups of K, and hence $K_0 = 1$ by assumption. Hence K acts faithfully on $[K,A] \cong C_p^2$. Under an appropriate choice of basis for $[K,A]$, S_1 and S_2 are the groups of (strict) upper and lower triangular matrices in $GL_2(p)$, and thus generate $SL_2(p)$. So $K \cong SL_2(p)$. Also, $\theta(\mathrm{Syl}_p(K)) = \mathbb{V}_1([K,A]) = \mathbb{V}_1(\theta(S_1)\theta(S_2))$.

Thus $\theta(\mathrm{Syl}_p(G)) = \mathbb{V}_1(A_2)$ for some elementary abelian subgroup $A_2 \le A$. If $\mathrm{rk}(A_2) \ge 3$, then there are distinct subgroups $T_1, T_2 \in \mathrm{Syl}_p(G)$ such that $[T_1, A] \le C_A(T_2)$, $\langle T_1, T_2 \rangle$ induces the identity on $[T_1, A]$, $C_A(T_2)/[T_1, A]$, and $A/C_A(T_2)$, so $\langle T_1, T_2\rangle$ is a p-group by Lemma 1.7 again, which is impossible. We conclude that $\mathrm{rk}(A_2) = 2$, and hence that $O^{p'}(G) = \langle \mathrm{Syl}_p(G) \rangle \cong SL_2(p)$.

Set $H = O^{p'}(G)$, and set $A_1 = C_A(H)$. Then $|A/A_1| = p^2$, $A_1 \cap A_2 = 1$ since $H \cong SL_2(p)$ acts faitfully on A_2, and hence $A = A_1 \times A_2$. Since $H \trianglelefteq G$, the subgroups $A_1 = C_A(H)$ and $A_2 = [H,A]$ are both G-invariant. Set $G_i = \mathrm{Aut}_G(A_i)$ $(i = 1, 2)$. Thus G_1 and G_2 are quotient groups of G, G_1 has order prime to p, $G_2 \le \mathrm{Aut}(A_2) \cong GL_2(p)$ and contains $SL_2(p)$, and $G \trianglelefteq G_1 \times G_2$ with index dividing $p-1 = [GL_2(p){:}SL_2(p)]$. □

2. Reduced fusion systems over nonabelian p-groups with index p abelian subgroup

Throughout this section, p is an odd prime. We want to describe all simple fusion systems over nonabelian p-groups which contain an abelian subgroup of index p. We begin by showing that if S has more than one abelian subgroup of index p, and there is a simple fusion system over S, then S must be extraspecial of order p^3 and exponent p (Theorem 2.1). This is the case already handled by Ruiz and Viruel [RV]. Afterwards, we develop the tools needed to study simple or reduced fusion systems over a p-group S which contains a unique abelian subgroup $A < S$ of index p. Our main result is Theorem 2.8, which lists simple fusion systems over such S when A is not essential. The more complicated case, that where the unique abelian subgroup of index p is essential, will be handled in a later paper.

Theorem 2.1. *Assume p is odd, and let S be a nonabelian p-group containing more than one abelian subgroup of index p. If there is a simple (or reduced) fusion system over S, then S is extraspecial of order p^3 and exponent p.*

Proof. Assume $\mathcal{F}$ is a reduced fusion system over S. Set $Z = Z(S)$ and $S' = [S,S]$ for short. By Lemma 1.9, $|S'| = p$ and $S/Z \cong C_p^2$. In particular, $S' \le Z$. The only proper subgroups centric in S are the abelian subgroups of index p, and hence they are the only possible $\mathcal{F}$-essential subgroups.

Fix some $A \in \mathbf{E}_{\mathcal{F}}$, and set $G = \mathrm{Aut}_{\mathcal{F}}(A)$. Then $\mathrm{Aut}_S(A) \cong S/A$ has order p, it is a Sylow p-subgroup of G, but is not the only Sylow p-subgroup since $A \in \mathbf{E}_{\mathcal{F}}$. Also, $[S,A] = S'$ has order p.

The hypotheses of Lemma 1.11 thus hold. So if we set $H = O^{p'}(G)$, $A_1 = C_A(H)$, and $A_2 = [H,A]$, then $A = A_1 \times A_2$, A_1 and A_2 are normalized by G, $A_2 \cong C_p^2$, and $H \cong SL_2(p)$ acts trivially on A_1 and faithfully on A_2. In particular, the subgroup $N_H(\mathrm{Aut}_S(A))/\mathrm{Aut}_S(A)$ acts trivially on A_1 and nontrivially on $S' = Z \cap A_2 \cong C_p$.

Thus $Z = S' \times A_1$, and A_1 is the unique subgroup of Z which is complementary to S' and normalized by $N_H(\mathrm{Aut}_S(A))/\mathrm{Aut}_S(A)$. Hence A_1 is also the unique subgroup of Z which is complementary to S' and normalized by $\mathrm{Aut}_{\mathcal{F}}(Z)$ (there is at least one such subgroup since $|\mathrm{Aut}_{\mathcal{F}}(Z)|$ is prime to p). It follows that A_1 is $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant, and (by the same argument) is also normalized by $\mathrm{Aut}_{\mathcal{F}}(P)$ for each $P \in \mathbf{E}_{\mathcal{F}}$. By Proposition 1.3(b), $A_1 \trianglelefteq \mathcal{F}$. Since $\mathcal{F}$ is reduced, this implies that $A_1 = 1$, and so $Z = Z(S) = S'$ has order p.

Thus S is a nonabelian group of order p^3. If $S \cong p_-^{1+2}$, then there is a unique subgroup $Q \le S$ isomorphic to C_p^2, this is the only possible $\mathcal{F}$-essential subgroup, so $Q \trianglelefteq \mathcal{F}$, again contradicting the assumption $\mathcal{F}$ is reduced. We conclude that $S \cong p_+^{1+2}$. □

Fusion systems over extraspecial groups of order p^3 and exponent p have already been classified by Ruiz and Viruel [RV]. In particular, they showed that there are exactly three distinct exotic fusion systems over such groups, all for $p = 7$, of which two are simple.

We now turn to the case where S has a unique abelian subgroup of index p. We first fix some notation.

Notation 2.2. *Fix a nonabelian p-group S with unique abelian subgroup A of index p, and a saturated fusion system $\mathcal{F}$ over S. Define*

$$S' = [S, S], \quad Z = Z(S), \quad Z_0 = Z \cap S', \quad Z_2 = Z_2(S), \quad A_0 = ZS'.$$

Thus $Z_0 \le S' \le A_0 \le A$ and $Z_0 \le Z \le A_0$. Also, set

$$\mathcal{H} = \bigl\{ Z\langle x\rangle \bigm| x \in S \smallsetminus A \bigr\} \qquad \text{and (when } Z_2 \le A) \qquad \mathcal{B} = \bigl\{ Z_2\langle x\rangle \bigm| x \in S \smallsetminus A \bigr\}.$$

Recall that a p-group P of order p^n has *maximal class* if its lower (or upper) central series has length $n-1$.

Lemma 2.3. *Assume the notation and hypotheses of 2.2.*

(a) *For each $P \in \mathbf{E}_{\mathcal{F}}$, $P \in \{A\} \cup \mathcal{H} \cup \mathcal{B}$ and $|N_S(P)/P| = p$.*

(b) *If $\mathbf{E}_{\mathcal{F}} \nsubseteq \{A\}$, then $Z_2 \le A$ and $|Z_0| = p = |A/A_0| = |Z_2/Z|$. Also, S/Z has maximal class.*

(c) *If $Z_2\langle x\rangle \in \mathbf{E}_{\mathcal{F}}$ for some $x \in S \smallsetminus A$, then $Z\langle x\rangle$ is not $\mathcal{F}$-centric and $Z\langle x\rangle \notin \mathbf{E}_{\mathcal{F}}$.*

(d) *In the situation of (b), there is $x \in S \smallsetminus A$ such that $A_0\langle x\rangle$ is normalized by $\mathrm{Aut}_{\mathcal{F}}(S)$.*

(e) *For each $P \in \mathbf{E}_{\mathcal{F}}$ and each $\alpha \in N_{\mathrm{Aut}_{\mathcal{F}}(P)}(\mathrm{Aut}_S(P))$, α extends to some $\bar{\alpha} \in \mathrm{Aut}_{\mathcal{F}}(S)$.*

Proof. **(a)** Fix some $P \in \mathbf{E}_{\mathcal{F}}$ where $P \ne A$. Then $P \nleq A$ since P is $\mathcal{F}$-centric. Set $P_0 = P \cap A$, and fix some element $x \in P \smallsetminus P_0$. Since $\mathrm{Out}_{\mathcal{F}}(P)$ contains a strongly p-embedded subgroup, $O_p(\mathrm{Out}_{\mathcal{F}}(P)) = 1$ (cf. [AKO, Proposition A.7(c)]).

(a1) Assume P is nonabelian. Since $Z \le P$ (P is $\mathcal{F}$-centric), $Z(P) = C_{P_0}(x) = Z$. For each $g \in N_A(P) \smallsetminus P$, c_g is the identity on P_0 and on P/P_0. If P_0 is characteristic in p, then $c_g \in O_p(\mathrm{Aut}_{\mathcal{F}}(P))$ by Lemma 1.7, which is impossible since $O_p(\mathrm{Out}_{\mathcal{F}}(P)) = 1$. Thus P_0 is not characteristic in P, and hence is not the unique abelian subgroup of index p in P. By Lemma 1.9, $|P_0/Z| = p$, $P/Z(P) \cong P/Z \cong C_p^2$, and $[P, P] = [x, P_0] \cong C_p$.

Now, $\mathrm{Out}_{\mathcal{F}}(P)$ maps injectively to $\mathrm{Aut}(Z(P)) \times \mathrm{Aut}(P/Z(P))$ (since the kernel is a p-group and $O_p(\mathrm{Out}_{\mathcal{F}}(P)) = 1$); $\mathrm{Aut}_S(P)$ is sent trivially to $\mathrm{Aut}(Z(P))$, and so $|N_S(P)/P| = p$ since p^2 does not divide the order of $\mathrm{Aut}(P/Z(P)) \cong GL_2(p)$. For $g \in N_A(P) \smallsetminus P$, $[g, P] \nleq Z$ (otherwise c_g induces the identity on $P/Z(P)$ and on $Z(P) = Z$, which would imply $c_g \in O_p(\mathrm{Aut}_{\mathcal{F}}(P)) = \mathrm{Inn}(P)$), so $g \notin Z_2$. In particular, S/Z is nonabelian, so $[x, A] \nleq Z$ and $Z_2 \le A$. Thus $Z_2 \le P_0$, and $Z_2 = P_0$ since $Z_2 > Z$ and $|P_0/Z| = p$. So $P = Z_2\langle x\rangle \in \mathcal{B}$.

(a2) Assume $P \in \mathbf{E}_{\mathcal{F}}$ is abelian (and $P \ne A$). Then $P_0 = Z$: it contains Z since P is centric, and cannot be larger since then P would be nonabelian. Hence $P = Z\langle x\rangle \in \mathcal{H}$. Also, $\mathrm{Aut}_A(P) = \mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$ centralizes P_0. The conditions of Lemma 1.10 thus hold (with P and $\mathrm{Aut}_{\mathcal{F}}(P)$ in the roles of A and G), so $|N_S(P)/P| = |\mathrm{Aut}_S(P)| = p$.

Since $[S{:}P] > p$ (A is the unique abelian subgroup of index p), this implies that S/Z is nonabelian, so $[x, A] \nleq Z$, and $Z_2 \leq A$.

For each $g \in A$, $g \in N_S(P)$ if and only if $[g, x] \in P_0 = Z$, if and only if $gZ \in C_{A/Z}(x) = Z(S/Z) = Z_2/Z$. Thus $N_A(P) = Z_2$, $N_S(P) = Z_2\langle x\rangle = Z_2P$, and $|Z_2/Z| = |N_S(P)/P| = p$.

(b) Fix $x \in S{\smallsetminus}A$. Since $\mathbf{E}_{\mathcal{F}} \nsubseteq \{A\}$, $(\mathcal{H} \cup \mathcal{B}) \cap \mathbf{E}_{\mathcal{F}} \neq \varnothing$ by (a). Hence $Z_2 \leq A$ and $|Z_2/Z| = p$ by the proofs of (a1) and (a2). Also, $S' = [x, A] = \mathrm{Im}(c_x - \mathrm{Id}_A)$ and $Z = C_A(x) = \mathrm{Ker}(c_x - \mathrm{Id}_A)$, so $|S'|{\cdot}|Z| = |A|$. Since $Z_2/Z = C_{A/Z}(x)$, $Z_2 = (c_x - \mathrm{Id}_A)^{-1}(Z)$, so $(c_x - \mathrm{Id}_A)$ sends Z_2/Z isomorphically to $Z_0 = Z \cap S'$. Hence $|Z_0| = p$, and $A_0 = ZS'$ has index p in A.

Define inductively $A_1 = [x, A] = S'$, and $A_n = [x, A_{n-1}] = [S, A_{n-1}]$ for $n \geq 2$. If $n \geq 1$ and $A_n \neq 1$, then $A_{n+1} = (c_x - \mathrm{Id})(A_n)$, $\mathrm{Ker}\big((c_x - \mathrm{Id})|_{A_n}\big) = C_{A_n}(x) = Z_0$ since $C_{A_n}(x) \neq 1$ and $|Z_0| = p$, and so $|A_n/A_{n+1}| = |Z_0| = p$. Since A_nZ/Z is the n-th term in the lower central series for S/Z, and since $|S/A_1Z| = |S/A_0| = p^2$, this proves that S/Z has maximal class.

(c) Assume $x \in S{\smallsetminus}A$ is such that $Z_2\langle x\rangle$ is $\mathcal{F}$-essential. There are $p+1$ subgroups of index p in $Z_2\langle x\rangle$ which contain $Z(Z_2\langle x\rangle) = Z$, and by the proof of (a1), $\mathrm{Aut}_S(Z_2\langle x\rangle)$ permutes all of them except Z_2 transitively. Since $Z_2\langle x\rangle \in \mathbf{E}_{\mathcal{F}}$, Z_2 is not normalized by $\mathrm{Aut}_{\mathcal{F}}(Z_2\langle x\rangle)$, and there is $\alpha \in \mathrm{Aut}_{\mathcal{F}}(Z_2\langle x\rangle)$ such that $\alpha(Z\langle x\rangle) = Z_2 < A$. Thus $Z\langle x\rangle$ is not $\mathcal{F}$-centric, and hence cannot be $\mathcal{F}$-essential.

(d) For each $x \in S{\smallsetminus}A$, $x^p \in C_A(x) = Z \leq A_0$. Hence $S/A_0 \cong C_p^2$ since $|A/A_0| = p$ by (b). Also, the $\mathrm{Out}_{\mathcal{F}}(S)$-action on S/A_0 normalizes A/A_0 since A is characterisitic in S. Since $\mathrm{Out}_{\mathcal{F}}(S)$ has order prime to p, there is an $\mathrm{Out}_{\mathcal{F}}(S)$-invariant splitting of $A/A_0 < S/A_0$. If xA_0 generates such a splitting (where $x \in S{\smallsetminus}A$), then $A_0\langle x\rangle$ is normalized by $\mathrm{Aut}_{\mathcal{F}}(S)$.

(e) Fix $P \in \mathbf{E}_{\mathcal{F}}$ and $\alpha \in N_{\mathrm{Aut}_{\mathcal{F}}(P)}(\mathrm{Aut}_S(P))$. By the extension axiom, α extends to some $\widehat{\alpha} \in \mathrm{Aut}_{\mathcal{F}}(N_S(P))$. By (a,b), P is maximal among all $\mathcal{F}$-essential subgroups: either $P = A$, or $P \in \mathcal{B}$, or $P = Z\langle x\rangle \in \mathcal{H}$ for some $x \in S{\smallsetminus}A$ and $Z_2\langle x\rangle \notin \mathbf{E}_{\mathcal{F}}$. So $\widehat{\alpha}$ extends to an element $\overline{\alpha} \in \mathrm{Aut}_{\mathcal{F}}(S)$ by Proposition 1.3(a). □

Lemma 2.4. *Assume the notation and hypotheses of 2.2, and also that $O_p(\mathcal{F}) = 1$ and $A \notin \mathbf{E}_{\mathcal{F}}$. Then $\mathcal{H} \cap \mathbf{E}_{\mathcal{F}} \neq \varnothing$, $Z = Z_0$, $S' = A_0$, and S has maximal class.*

Proof. Since $A \notin \mathbf{E}_{\mathcal{F}}$, $\mathbf{E}_{\mathcal{F}} \subseteq \mathcal{H} \cup \mathcal{B}$ by Lemma 2.3(a). If $\mathbf{E}_{\mathcal{F}} \subseteq \mathcal{B}$, then $Z = Z(P)$ for each $P \in \mathbf{E}_{\mathcal{F}}$, so $Z \trianglelefteq \mathcal{F}$ by Proposition 1.3(b), which contradicts our assumption. Thus there is $Q \in \mathcal{H} \cap \mathbf{E}_{\mathcal{F}}$.

By Lemma 2.3(a), $\mathrm{Aut}_S(Q) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(Q))$ has order p. Also, for each $g \in N_S(Q){\smallsetminus}Q$, $[g, Q] \leq Q \cap S' = Z \cap S' = Z_0$, where $|Z_0| = p$ by Lemma 2.3(b). Hence by Lemma 1.11, if we set $\Gamma_Q = O^{p'}(\mathrm{Aut}_{\mathcal{F}}(Q))$, $Q_1 = C_Q(\Gamma_Q) \leq Z(S)$, and $Q_2 = [\Gamma_Q, Q] \geq Z_0$, then Q_1 and Q_2 are both normalized by $\mathrm{Aut}_{\mathcal{F}}(Q)$ and $Q = Q_1 \times Q_2$. Also, $\Gamma_Q \cong SL_2(p)$ acts faithfully on $Q_2 \cong C_p^2$ and trivially on Q_1.

In particular, there is a subgroup $H \leq N_{\Gamma_Q}(\mathrm{Aut}_S(Q))$ of order $p-1$ which acts as the full group of automorphisms of Z_0 and of Q_2/Z_0, and acts trivially on Q_1. Thus $Z = Q_1 \times Z_0$, and Q_1 is the unique complement to Z_0 in Z which is normalized by H. Since H restricts to a subgroup of $\mathrm{Aut}_{\mathcal{F}}(Z)$, this shows that Q_1 is also the unique complement to Z_0 in Z which is normalized by $\mathrm{Aut}_{\mathcal{F}}(Z)$ (there is at least one such subgroup since $|\mathrm{Aut}_{\mathcal{F}}(Z)|$ is prime to p).

By a similar argument, Q_1 is normalized by $\mathrm{Aut}_{\mathcal{F}}(P)$ for each $P \in \mathcal{H} \cap \mathbf{E}_{\mathcal{F}}$. If $P \in \mathcal{B} \cap \mathbf{E}_{\mathcal{F}}$ or $P = S$, then for each $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$, $\alpha(Z) = Z$ since $Z = Z(P)$, so $\alpha|_Z = \mathrm{Aut}_{\mathcal{F}}(Z)$, and $\alpha(Q_1) = Q_1$. Thus $Q_1 \trianglelefteq \mathcal{F}$ by Proposition 1.3(b), and $Q_1 = 1$ (hence $Z = Z_0$) since $O_p(\mathcal{F}) = 1$. Also, $A_0 = ZS' = S'$ since $Z = Z_0 \leq S'$.

Since $|Z(S)| = p$ and $S/Z(S)$ has maximal class, S also has maximal class. □

We now need some more notation.

Notation 2.5. *Assume the notation and hypotheses of 2.2, and also that* $|Z_0| = |A/A_0| = |Z_2/Z| = p$. *Fix* $\mathbf{a} \in A \smallsetminus A_0$ *and* $\mathbf{x} \in S \smallsetminus A$, *where* $\mathbf{x}$ *is chosen so that* $\langle A_0, \mathbf{x} \rangle$ *is normalized by* $\mathrm{Aut}_{\mathcal{F}}(S)$ *(Lemma 2.3(d)). For each* $i = 0, 1, \ldots, p-1$, *define*

$$H_i = \langle Z, \mathbf{x}\mathbf{a}^i \rangle \in \mathcal{H} \qquad \text{and} \qquad B_i = \langle Z_2, \mathbf{x}\mathbf{a}^i \rangle \in \mathcal{B}\,.$$

Let $\mathcal{H}_i$ *and* $\mathcal{B}_i$ *denote the* S*-conjugacy classes of* H_i *and* B_i, *respectively, and set*

$$\mathcal{H}_* = \mathcal{H}_1 \cup \cdots \cup \mathcal{H}_{p-1} \quad \text{and} \quad \mathcal{B}_* = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_{p-1}.$$

Thus $\mathcal{H} = \mathcal{H}_0 \cup \mathcal{H}_*$ *and* $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_*$, *since* $|A/A_0| = p$. *Also, for* $P \leq S$, *set*

$$\mathrm{Aut}_{\mathcal{F}}^{(P)}(S) = \bigl\{ \alpha \in \mathrm{Aut}_{\mathcal{F}}(S) \bigm| \alpha(P) = P,\ \alpha|_P \in O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P)) \bigr\} \leq \mathrm{Aut}_{\mathcal{F}}(S)\,.$$

Set

$$\Delta = (\mathbb{Z}/p)^\times \times (\mathbb{Z}/p)^\times\,, \qquad \text{and} \qquad \Delta_i = \{(r, r^i) \,|\, r \in (\mathbb{Z}/p)^\times\} \leq \Delta \quad (\text{for } i \in \mathbb{Z}).$$

Define

$$\mu\colon \mathrm{Aut}(S) \longrightarrow \Delta \qquad \text{and} \qquad \widehat{\mu}\colon \mathrm{Out}(S) \longrightarrow \Delta$$

by setting, for $\alpha \in \mathrm{Aut}(S)$,

$$\mu(\alpha) = \widehat{\mu}([\alpha]) = (r, s) \quad \text{if} \quad \begin{cases} \alpha(x) \in x^r A & \text{for } x \in S \smallsetminus A \\ \alpha(g) = g^s & \text{for } g \in Z_0\,. \end{cases}$$

The next two lemmas describe the role played by μ and Δ in controlling these fusion systems.

Lemma 2.6. *Assume the notation and hypotheses of 2.2 and 2.5, and let* $m \geq 3$ *be such that* $|A/Z| = p^{m-1}$. *Then the following hold for each* $\alpha \in \mathrm{Aut}(S)$.

(a) *Set* $(r, s) = \mu(\alpha)$, *and let* u *be such that* $\alpha(g) \in g^u A_0$ *for each* $g \in A \smallsetminus A_0$. *Then* $s \equiv ur^{m-1}$ *(mod p).*

(b) *Either* $\mu(\alpha) \in \Delta_m$, *and* α *normalizes each of the* S*-conjugacy classes* $\mathcal{H}_i$ *and* $\mathcal{B}_i$ *(*$0 \leq i \leq p-1$*); or* $\mu(\alpha) \notin \Delta_m$, *and* α *normalizes only the classes* $\mathcal{H}_0$ *and* $\mathcal{B}_0$. *Also,* α *acts via the identity on* A/A_0 *if and only if* $\mu(\alpha) \in \Delta_{m-1}$.

Proof. **(a)** Define inductively $A_1 = [\mathbf{x}, A] = S'$, and $A_n = [\mathbf{x}, A_{n-1}]$ for $n \geq 2$. If $n \geq 1$ and $A_n \neq 1$, then $A_{n+1} = (c_{\mathbf{x}} - \mathrm{Id})(A_n)$, $\mathrm{Ker}\bigl((c_{\mathbf{x}} - \mathrm{Id})|_{A_n}\bigr) = Z_0$, and so $|A_n/A_{n+1}| = |Z_0| = p$. Since $|A_1| = |A|\big/|Z| = p^{m-1}$, this shows that $A_{m-1} = Z_0$ and $A_m = 1$.

Fix $\alpha \in \mathrm{Aut}(S)$, set $\mu(\alpha) = (r, s)$, and let $u \in (\mathbb{Z}/p)^\times$ be such that $\alpha(g) \in g^u A_0$ for each $g \in A \smallsetminus A_0$. Then for each k, α acts on A_k/A_{k-1} via $g \mapsto g^{ur^k}$. In particular, α acts on Z_0 via $g \mapsto g^{ur^{m-1}}$, and hence $s \equiv ur^{m-1}$ (mod p).

(b) Fix $\alpha \in \mathrm{Aut}(S)$, and let r, s, u be as in (a). Then α acts via the identity on A/A_0 if and only if $u = 1$, which by (b) holds if and only if $\mu(\alpha) \in \Delta_{m-1}$. Similarly, α normalizes each subgroup $\langle A_0, \mathbf{x}\mathbf{a}^j \rangle$ if and only if $r \equiv u$ (mod p); i.e., $s \equiv r^m$, and $\mu(\alpha) \in \Delta_m$. □

Lemma 2.7. *Assume the notation and hypotheses of 2.2 and 2.5. Assume also that* $Z = Z_0$. *Let* m *be such that* $|A/Z| = p^{m-1}$ *(hence* $|A| = p^m$*). Fix* $P \in \mathcal{H} \cup \mathcal{B}$, *and set* $t = -1$ *if* $P \in \mathcal{H}$, $t = 0$ *if* $P \in \mathcal{B}$.

(a) *Assume $P \in \mathbf{E}_{\mathcal{F}}$. Then $P \cong C_p^2$ or p_+^{1+2}, and $\mu(\mathrm{Aut}_{\mathcal{F}}^{(P)}(S)) = \Delta_t$. If $P \in \mathcal{H}_*$ or $P \in \mathcal{B}_*$, then $m \equiv t$ (mod $p-1$).*

(b) *Conversely, assume that $P \cong C_p^2$ or p_+^{1+2}, and also that $\mu(H_P) \geq \Delta_t$ where*

$$H_P = \{\alpha \in \mathrm{Aut}_{\mathcal{F}}(S) \mid \alpha(P) = P\} \qquad \text{and} \qquad \widehat{H}_P = \{\alpha|_P \mid \alpha \in H_P\}.$$

Then there is a subgroup $\Theta \leq \mathrm{Out}(P)$ such that $O_p(\Theta) = 1$, $\mathrm{Out}_S(P) \in \mathrm{Syl}_p(\Theta)$, and $N_\Theta(\mathrm{Out}_S(P)) = \widehat{H}_P/\mathrm{Inn}(P)$.

(c) *In the situation of (b), or in the situation of (a) when $\Theta = \mathrm{Out}_{\mathcal{F}}(P)$, $O^{p'}(\Theta) \cong SL_2(p)$, and*

$$\Theta \cong \begin{cases} SL_2(p) & \text{if } \mu(H_P) = \Delta_t \\ GL_2(p) & \text{if } \mu(H_P) = \Delta\,. \end{cases}$$

Proof. Set $P_0 = P \cap A$, and $P_1 = [S, P_0] = [\mathbf{x}, P_0]$. Thus $|P/P_0| = p$ and $|P/P_1| = p^2$ in all cases. Set $\Gamma_P = O^{p'}(\mathrm{Out}(P))$.

(a) Assume $P \in \mathbf{E}_{\mathcal{F}}$. If $P \in \mathcal{H}$, then $|P| = p^2$ since $|Z| = p$, and if $P \in \mathcal{B}$, then P is nonabelian of order p^3. In either case, by Lemma 1.11, applied to the action of $\mathrm{Out}_{\mathcal{F}}(P)$ on $P/[P,P]$, $P/[P,P] \cong C_p^2$ and the action contains that of $SL_2(p)$. Hence by Lemma 1.8, $P \cong C_p^2$ or p_+^{1+2}, $\mathrm{Out}(P) \cong GL_2(p)$, and $\mathrm{Out}_{\mathcal{F}}(P)$ contains $\Gamma_P \cong SL_2(p)$.

Now, $N_{\Gamma_P}(\mathrm{Out}_S(P)) = \mathrm{Out}_S(P) \rtimes \langle\alpha\rangle \cong C_p \rtimes C_{p-1}$, where for some generator $r \in (\mathbb{Z}/p)^\times$, α acts on P/P_0 via $g \mapsto g^r$, and on P_0/P_1 via $g \mapsto g^{1/r}$. By Lemma 2.3(e), α extends to $\bar{\alpha} \in \mathrm{Aut}_{\mathcal{F}}(S)$. Hence $\mathrm{Aut}_{\mathcal{F}}^{(P)}(S) \geq \mathrm{Aut}_{N_S(P)}(S)\langle\bar{\alpha}\rangle$, with equality since restrictions of elements in $\mathrm{Aut}_{\mathcal{F}}^{(P)}(S)$ must be contained in the normalizer of a Sylow p-subgroup in $SL_2(p)$. If $P \cong C_p^2$, then $\mu(\bar{\alpha}) = (r, r^{-1})$, while if $P \cong p_+^{1+2}$, then $\alpha|_Z = \mathrm{Id}$ and hence $\mu(\bar{\alpha}) = (r, 1)$. Thus in either case, $\mu(\bar{\alpha})$ generates Δ_t, so $\mu(\mathrm{Aut}_{\mathcal{F}}^{(P)}(S)) = \Delta_t$.

If $P \in \mathcal{H}_* \cup \mathcal{B}_*$, then $\mu(\bar{\alpha}) \in \Delta_m$ by Lemma 2.6(b), since $\bar{\alpha}$ normalizes $P \in \mathcal{H}_i$. So $\Delta_m = \Delta_t$, and $m \equiv t$ (mod $p-1$).

(b) By assumption, $P \cong C_p^2$ or p_+^{1+2}, so $\Gamma_P = O^{p'}(\mathrm{Out}(P)) \cong SL_2(p)$. Choose $\alpha \in H_P$ such that $\mu(\alpha)$ generates Δ_t. Then for some generator $r \in (\mathbb{Z}/p)^\times$, α induces $x \mapsto x^r$ on P/P_0 and induces $x \mapsto x^{1/r}$ on P_0/P_1. Thus $\alpha|_P \in \Gamma_P$ and $\mathrm{Aut}_S(P)\langle\alpha|_P\rangle = N_{\Gamma_P}(\mathrm{Aut}_S(P))$.

Set $\Theta = \Gamma_P{\cdot}(\widehat{H}_P/\mathrm{Inn}(P))$. Then $O_p(\Theta) = 1$ since $O_p(SL_2(p)) = 1$ and $p \nmid [\Theta{:}\Gamma_P]$, $\mathrm{Out}_S(P) \in \mathrm{Syl}_p(\Theta)$ since $p \nmid [\Theta{:}\Gamma_P]$, and $N_\Theta(\mathrm{Aut}_S(P)) = \widehat{H}_P$ since $\mathrm{Aut}_S(P) \trianglelefteq \widehat{H}_P$.

(c) In either case (a) or (b), Θ acts faithfully on $P/P_1 \cong C_p^2$, and contains $O^{p'}(\mathrm{Out}(P)) \cong SL_2(p)$ by Lemma 1.11. Also, $N_\Theta(\mathrm{Out}_S(P)) = \widehat{H}_P/\mathrm{Inn}(P)$: by assumption in case (b), and by the extension axiom in case (a). The last statement now follows since $[\Theta{:}O^{p'}(\mathrm{Out}(P))] = [\mu(H_P){:}\Delta_t]$. □

We are now ready to state and prove our main theorem: a description of all simple fusion systems in the situation of Notation 2.2 for which A is not essential. In the statement of the theorem, we set

$$\zeta = \zeta_p = e^{2\pi i/p}, \quad R = \mathbb{Z}[\zeta], \quad \text{and} \quad \mathfrak{p} = (1-\zeta)R\,.$$

We also set $\mathbf{U} = \mathrm{Aut}_S(A)$, $\mathbf{u} = c_{\mathbf{x}} \in \mathbf{U}$, and $\sigma = \sum_{i=0}^{p-1} \mathbf{u}^i \in \mathbb{Z}[\mathbf{U}]$. We regard A as a $\mathbb{Z}[\mathbf{U}]$-module, and also when possible as an R-module by setting $\zeta{\cdot}a = \mathbf{u}(a)$ for $a \in A$.

Theorem 2.8. *Fix an odd prime p, a finite nonabelian p-group S with a unique abelian subgroup $A \leq S$ of index p, and a simple fusion system $\mathcal{F}$ over S for which A is not* *essential. Assume the notation and hypotheses of 2.2 and 2.5. Let* $m \geq 3$ *be such that* $|A| = p^m$. *Then* $|Z| = p$, $\widehat{\mu}|_{\mathrm{Out}_{\mathcal{F}}(S)}$ *is injective,* $\mathbf{x} \in S \smallsetminus A$ *can be chosen so that* $\mathbf{x}^p = 1$ *and* $A_0\langle \mathbf{x}\rangle$ *is normalized by* $\mathrm{Aut}_{\mathcal{F}}(S)$, *and one of the following holds.*

(a) $S \cong (R/\mathfrak{p}^m) \rtimes \langle \mathbf{x}\rangle$, *and either*

- (i) $m \equiv -1$ *(mod* $p{-}1$*),* $\widehat{\mu}(\mathrm{Out}_{\mathcal{F}}(S)) = \Delta_{-1}$, *and* $\mathbf{E}_{\mathcal{F}}$ *is the union of between* 1 *and* p *of the* S*-conjugacy classes* $\mathcal{H}_i$, *with* $\mathrm{Aut}_{\mathcal{F}}(H_i) \cong SL_2(p)$ *when* $H_i \in \mathbf{E}_{\mathcal{F}}$*; or*
- (ii) $m \equiv -1$ *(mod* $p{-}1$*),* $\widehat{\mu}(\mathrm{Out}_{\mathcal{F}}(S)) = \Delta$, *and* $\mathbf{E}_{\mathcal{F}} = \mathcal{B}_0 \cup \mathcal{H}_*$, *where* $\mathrm{Out}_{\mathcal{F}}(B_0) \cong GL_2(p)$, *the subgroups in* $\mathcal{H}_*$ *are all* $\mathcal{F}$*-conjugate, and* $\mathrm{Out}_{\mathcal{F}}(H_i) \cong SL_2(p)$*; or*
- (iii) $m \equiv 0$ *(mod* $p{-}1$*),* $\widehat{\mu}(\mathrm{Out}_{\mathcal{F}}(S)) = \Delta$, *and* $\mathbf{E}_{\mathcal{F}} = \mathcal{H}_0 \cup \mathcal{B}_*$, *where* $\mathrm{Out}_{\mathcal{F}}(H_0) \cong GL_2(p)$, *the subgroups in* $\mathcal{B}_*$ *are all* $\mathcal{F}$*-conjugate, and* $\mathrm{Out}_{\mathcal{F}}(B_i) \cong SL_2(p)$*; or*
- (iv) $m \not\equiv 0, -1$ *(mod* $p{-}1$*),* $\widehat{\mu}(\mathrm{Out}_{\mathcal{F}}(S)) = \Delta_{-1}$, *and* $\mathbf{E}_{\mathcal{F}} = \mathcal{H}_0$ *with* $\mathrm{Out}_{\mathcal{F}}(H_0) \cong SL_2(p)$.

(b) *As a* $\mathbb{Z}[\mathbf{U}]$*-module,*

$$A \cong \mathbb{Z}[\mathbf{U}]/\big\langle p\sigma, p^k + \ell\sigma\big\rangle \quad \text{where } k \geq 1,\ \ell \in \mathbb{Z}, \text{ and } \begin{cases} p \nmid \ell & \text{if } k \geq 2 \\ p \nmid \ell{+}1 & \text{if } k = 1. \end{cases}$$

Also, $m = k(p-1)+1 \equiv 1$ *(mod* $p{-}1$*),* $S \cong A \rtimes \langle \mathbf{x}\rangle$, $\widehat{\mu}(\mathrm{Out}_{\mathcal{F}}(S)) = \Delta_{-1}$, $\mathbf{E}_{\mathcal{F}} = \mathcal{H}_0$, *and* $\mathrm{Out}_{\mathcal{F}}(H_0) \cong SL_2(p)$.

Conversely, in each of these cases, there is up to isomorphism a unique simple fusion system $\mathcal{F}$ *which satisfies the given description of* S, $\widehat{\mu}(\mathrm{Out}_{\mathcal{F}}(S))$, $\mathbf{E}_{\mathcal{F}}$, *and the groups* $\mathrm{Out}_{\mathcal{F}}(P)$ *for* $P \in \mathbf{E}_{\mathcal{F}}$. *Furthermore,*

- $A_0H_0 < S$ *is strongly closed in* $\mathcal{F}$ *in cases* (a.iv) *and* (b), A_0H_i *is strongly closed in* $\mathcal{F}$ *in case* (a.i) *if* $\mathbf{E}_{\mathcal{F}} = \mathcal{H}_i$, *and these are the only occurrences of proper strongly closed subgroups in these fusion systems; and*
- *all of these fusion systems are exotic, with the following exceptions when* $p = 3$.
  - *Case* (a.i)*, when* $\mathbf{E}_{\mathcal{F}} = \mathcal{H} = \mathcal{H}_0 \cup \mathcal{H}_*$: $\mathcal{F}$ *is the* 3*-fusion system of* $PSL_3(q)$ *for appropriate* $q \equiv 1$ *(mod* 3*), and also of* $PSU_3(q)$ *for appropriate* $q \equiv -1$ *(mod* 3*).*
  - *Case* (a.ii)*:* $\mathcal{F}$ *is the* 3*-fusion system of* ${}^3D_4(q)$ *for appropriate* q *prime to* 3.

Proof. Let $\mathcal{F}$ be any reduced fusion system over S such that $A \notin \mathbf{E}_{\mathcal{F}}$. By Lemmas 2.3(b) and 2.4, $Z = Z_0$, $S' = A_0$, and $|Z| = |A/A_0| = p$.

For each $\alpha \in \mathrm{Ker}(\mu)$, α induces the identity on S/A, and induces the identity on A/A_0 by Lemma 2.6(a). Thus $\mathrm{Ker}(\mu)$ is a p-group by Lemma 1.7, and $\mathrm{Ker}(\mu) = O_p(\mathrm{Aut}(S))$ since $\mathrm{Im}(\mu)$ has order prime to p. In particular, $\widehat{\mu}|_{\mathrm{Out}_{\mathcal{F}}(S)}$ is injective since $p \nmid |\mathrm{Out}_{\mathcal{F}}(S)|$.

By Proposition 1.3(c,d), and since $[\mathrm{Aut}_{\mathcal{F}}(P), P] = P$ for $P \in \mathbf{E}_{\mathcal{F}}$, $O^p(\mathcal{F}) = \mathcal{F}$ if and only if $\langle [\mathrm{Aut}_{\mathcal{F}}(S), S], \mathbf{E}_{\mathcal{F}}\rangle = S$. Since $S' = A_0$ has index p^2 in S, $[\mathrm{Aut}_{\mathcal{F}}(S), S] = A_0\langle \mathbf{x}\rangle$ if $\mu(\mathrm{Aut}_{\mathcal{F}}(S)) \leq \Delta_{m-1}$ and $[\mathrm{Aut}_{\mathcal{F}}(S), S] = S$ otherwise (Lemma 2.6(b)). Thus

$$O^p(\mathcal{F}) = \mathcal{F} \quad \Longleftrightarrow \quad \mu(\mathrm{Aut}_{\mathcal{F}}(S)) \not\leq \Delta_{m-1} \quad \text{or} \quad (\mathcal{H}_* \cup \mathcal{B}_*) \cap \mathbf{E}_{\mathcal{F}} \neq \varnothing\,. \tag{1}$$

By Lemma 1.4, $O^{p'}(\mathcal{F}) = \mathcal{F}$ if and only if $\mu(\mathrm{Aut}_{\mathcal{F}}(S))$ is generated by the subgroups $\mu(\mathrm{Aut}_{\mathcal{F}}^{(P)}(S))$ for $P \in \mathbf{E}_{\mathcal{F}}$. Together with Lemma 2.7(a) (and since $\mathcal{H} \cap \mathbf{E}_{\mathcal{F}} \neq \varnothing$ in all cases by Lemma 2.4), this implies that

$$O^{p'}(\mathcal{F}) = \mathcal{F} \quad \Longleftrightarrow \quad \mathcal{B} \cap \mathbf{E}_{\mathcal{F}} \neq \varnothing \quad \text{or} \quad \widehat{\mu}(\mathrm{Out}_{\mathcal{F}}(S)) = \Delta_{-1}\,. \tag{2}$$

Step 1: We first check that each of the above choices of $S = A \rtimes \langle \mathbf{x} \rangle$, $\mu(\mathrm{Aut}_{\mathcal{F}}(S))$, $\mathbf{E}_{\mathcal{F}}$, and $\mathrm{Out}_{\mathcal{F}}(P)$ for $P \in \mathbf{E}_{\mathcal{F}}$, determines a reduced fusion system which is unique up to isomorphism. In each case, we fix an identification of A with a quotient group of R or of $\mathbb{Z}[\mathbf{U}]$, as described, and let $\mathbf{a} \in A$ be the element identified with $1 \in R$ or $1 \in \mathbb{Z}[\mathbf{U}]$.

For each $P \in \mathbf{E}_{\mathcal{F}}$, either $P \in \mathcal{H}$ and $P \cong C_p^2$ or $P \in \mathcal{B}$ and $P \cong p_+^{1+2}$, so $\mathrm{Out}(P) \cong GL_2(p)$ in either case, and there is a unique subgroup of $\mathrm{Out}(P)$ isomorphic to $SL_2(p)$. Also, $\mathrm{Out}(S)$ is isomorphic to a semidirect product of $\mathrm{Ker}(\widehat{\mu}) = O_p(\mathrm{Out}(S))$ with $\mathrm{Im}(\widehat{\mu}) \leq \Delta$, so by the Schur-Zassenhaus theorem (see [G, Theorem 6.2.1]), the statements about $\mu(\mathrm{Aut}_{\mathcal{F}}(S))$ determine $\mathrm{Aut}_{\mathcal{F}}(S)$ up to conjugacy in $\mathrm{Aut}(S)$ (hence up to an isomorphism of fusion systems). Since $\mathcal{F}$ is generated by $\mathrm{Aut}_{\mathcal{F}}(S)$, $\mathrm{Aut}_{\mathcal{F}}(P)$ for $P \in \mathbf{E}_{\mathcal{F}}$, and restrictions of those automorphisms, it is uniquely determined by such data.

For each $i \in \mathbb{Z}$ prime to p, define $\lambda_i \in \mathrm{Aut}(S)$ by setting $\lambda_i(a) = a^i$ for $a \in A$ and $\lambda_i(\mathbf{x}) = \mathbf{x}$. For each $j \in (\mathbb{Z}/p)^\times$, define $\nu_j \in \mathrm{Aut}(S)$ by setting $\nu_j(\mathbf{x}) = \mathbf{x}^j$ and $\nu_j(\mathbf{a}) = \mathbf{a}$. Then $\nu_j(\mathbf{u}^k(\mathbf{a})) = \mathbf{u}^{jk}(\mathbf{a})$ for each k (recall $\mathbf{u} = c_{\mathbf{x}}$), so ν_j acts on A via a Galois automorphism on R in case (a), or via an automorphism of the group $\mathbf{U}$ in case (b). In all cases, ν_j is well defined by the description of A as a quotient of R or of $\mathbb{Z}[\mathbf{U}]$. Let $\Lambda < \mathrm{Aut}(S)$ be the subgroup generated by those λ_i of order prime to p (i.e., such that $i^{p-1} \equiv 1$ modulo the exponent of A), and by all ν_j for $j \in (\mathbb{Z}/p)^\times$. Thus $\Lambda \cong C_{p-1} \times C_{p-1}$. By Lemma 2.6(a), $\mu(\Lambda) = \Delta$. By construction, $\Lambda(H_0) = H_0$, $\Lambda(B_0) = B_0$, and Λ permutes each of the sets $\{H_1, \ldots, H_{p-1}\}$ and $\{B_1, \ldots, B_{p-1}\}$.

Assume we are in the situation of one of cases (a.i)–(a.iv) or (b). Let $\Lambda_0 \leq \Lambda$ be such that $\mu(\Lambda_0)$ is the given subgroup, and set $G = S \rtimes \Lambda_0$. Let $Q_1, \ldots, Q_k$ be representatives for the G-conjugacy classes in $\mathbf{E}_{\mathcal{F}}$ as listed, chosen among the H_i and B_i for $0 \leq i \leq p-1$. By Lemma 2.6(b), Λ_0 sends the S-conjugacy class of each Q_i to itself, except in (a.ii) and (a.iii) when $Q \in \mathcal{H}_* \cup \mathcal{B}_*$, in which cases $\Lambda_0 = \Lambda$ contains a subgroup of order $p-1$ which sends each class to itself. Thus Λ_0, or a subgroup of order $p-1$ in Λ_0, normalizes each Q_i by the above remarks. By Lemma 2.7(b), for each $i = 1, \ldots, k$, there is a subgroup $\Theta_i \leq \mathrm{Out}(Q_i) \cong GL_2(p)$ such that $\mathrm{Out}_S(Q_i) \in \mathrm{Syl}_p(\Theta_i)$, $\mathrm{Out}_G(Q_i) = N_{\Theta_i}(\mathrm{Out}_S(Q_i))$, and $O_p(\Theta_i) = 1$; and Θ_i is uniquely determined because it contains $SL_2(p)$ and the normalizer of its Sylow p-subgroup is given.

Set $\mathcal{F} = \langle \mathcal{F}_S(G), \Theta_1, \ldots, \Theta_k \rangle$, and set $K_i = \mathrm{Out}_G(Q_i)$. No Q_i is G-conjugate to a subgroup of Q_j for $j \neq i$. For each i,

(1) $p \nmid [\Theta_i : K_i]$,

(2) Q_i is p-centric in G but no proper subgroup of Q_i is $\mathcal{F}$-centric or essential in G, and

(3) K_i is strongly p-embedded in Θ_i.

Hence $\mathcal{F}$ is saturated by [BLO4, Proposition 5.1].

Any normal p-subgroup of $\mathcal{F}$ must be contained in all essential subgroups, hence is contained in $Z \cong C_p$; and $Z \ntrianglelefteq \mathcal{F}$ since some $P \in \mathbf{E}_{\mathcal{F}}$ is abelian in each case. Thus $O_p(\mathcal{F}) = 1$. By inspection, the conditions on the right-hand side of (1) and (2) hold, so $O^p(\mathcal{F}) = \mathcal{F} = O^{p'}(\mathcal{F})$.

Step 2: We next list the proper strongly closed subgroups in $\mathcal{F}$, and use that to determine whether $\mathcal{F}$ is realizable and to prove that $\mathcal{F}$ is simple in all cases.

If $1 \neq Q \trianglelefteq S$ is strongly closed in $\mathcal{F}$, then it contains $Z_0 = Z(S) \cong C_p$ (each nontrivial normal subgroup intersects nontrivially with the center), and hence contains each abelian subgroup in $\mathbf{E}_{\mathcal{F}}$. Thus $\mathcal{H}_i \subseteq \mathbf{E}_{\mathcal{F}}$ implies that $Q \geq \langle \mathcal{H}_i \rangle = A_0\langle \mathbf{x}\mathbf{a}^i \rangle$. In particular, $Q \geq A_0 \geq Z_2$, so Q also contains all nonabelian subgroups in $\mathbf{E}_{\mathcal{F}}$. Hence if $Q < S$, then $\mathbf{E}_{\mathcal{F}} = \mathcal{H}_i$

and $Q = A_0H_i$ for some i. By inspection, the only cases where this occurs are (a.iv) and (b), with $i = 0$, and sometimes in case (a.i). Conversely, A_0H_i is strongly closed in each of these cases, since it contains $\langle \mathbf{E}_\mathcal{F} \rangle$ and is normalized by $\mathrm{Aut}_\mathcal{F}(S)$ (by Lemma 2.6(b) and the assumptions in (a.i) when $i \neq 0$).

If A_0H_i is strongly closed, then it is centric in S, is nonabelian, and does not split as a product. So in all cases, by Lemma 1.5, if $\mathcal{F}$ is realizable, it is realizable by a finite simple group. Hence by Lemma 1.6, $p = 3$, and $\mathcal{F}$ is the fusion system of one of the simple groups $G \cong PSL_3(q)$, $PSU_3(q)$, $G_2(q)$, or ${}^3D_4(q)$ for some q prime to 3. When G is a Chevalley group $\mathbb{G}(q)$, then all of the classes $\mathcal{H}_i$ are conjugate in the Sylow 3-subgroup of $\mathbb{G}(q^3)$, so all of them are essential in $\mathcal{F}$ since at least one of them is. Thus we must be in case (a.i), and since $m \equiv 1 \pmod 2$, we have $G \cong PSL_3(q)$ for some $q \equiv 1 \pmod 3$. The groups $PSU_3(q)$ are handled in a similar way. The 3-fusion system of ${}^3D_4(q)$ has type (a.ii) by the description of its maximal subgroups in [Kl].

If $\mathcal{F}$ is not simple, then since it is reduced, A_0H_i must be strongly closed for some i, and there is a normal fusion subsystem $\mathcal{E} \trianglelefteq \mathcal{F}$ over $T = A_0H_i$. Then $\mathcal{H}_i$ splits into p T-conjugacy classes. For each $P \in \mathcal{H}_i$, $N_S(P) = Z_2P \leq T$ since $Z_2 \leq A_0$, so $\mathrm{Aut}_\mathcal{E}(P)$ contains the normal closure $O^{p'}(\mathrm{Aut}_\mathcal{F}(P))$ of $\mathrm{Aut}_S(P)$, and hence $P \in \mathbf{E}_\mathcal{E}$. So if $m \geq 4$ (if $|T| \geq p^4$), then by Lemma 2.7(a) (and since $|A_0| = p^{m-1}$), $m - 1 \equiv -1 \pmod{p-1}$, which does not hold in the cases we are considering. If $m = 3$ (if T is extraspecial of order p^3), then by [RV, Tables 1.1 & 1.2], there is no saturated fusion system over T with exactly p essential (or radical) subgroups of order p^2. Thus $\mathcal{F}$ is simple.

Step 3: It remains to show that the fusion systems of cases (a) and (b) are the only simple fusion systems satisfying our hypotheses. By Lemma 2.3(a,c), each $P \in \mathbf{E}_\mathcal{F}$ lies in exactly one of the classes $\mathcal{H}_i$ or $\mathcal{B}_i$ for $0 \leq i \leq p-1$, and $\mathcal{H}_i$ and $\mathcal{B}_i$ cannot both be in $\mathbf{E}_\mathcal{F}$.

Recall that $\mathbf{U} = \mathrm{Aut}_S(A)$, $\mathbf{u} = c_\mathbf{x} \in \mathbf{U}$, and $\sigma = \sum_{i=0}^{p-1} \mathbf{u}^i \in \mathbb{Z}[\mathbf{U}]$. Define

$$\Psi \colon \mathbb{Z}[\mathbf{U}] \longrightarrow A$$

by setting $\Psi(\mathbf{u}^k) = \mathbf{u}^k(\mathbf{a})$ for all k. Then $\mathrm{Im}(\Psi)$ is normalized by $\mathbf{x}$, $\mathrm{Im}(\Psi)\langle \mathbf{x} \rangle \geq \langle \mathbf{a}, \mathbf{x} \rangle = S$ since $A_0 = S'$, so $\mathrm{Im}(\Psi) = A$ and Ψ is onto.

For each $i \in \mathbb{Z}$, $(\mathbf{x}\mathbf{a}^i)^p = \mathbf{u}(\mathbf{a}^i)\mathbf{u}^2(\mathbf{a}^i) \cdots \mathbf{u}^p(\mathbf{a}^i)\mathbf{x}^p = \Psi(i\sigma)\mathbf{x}^p$. Hence

$$\begin{aligned} \Psi(\sigma) = 1 \quad &\Longleftrightarrow \quad (\mathbf{x}\mathbf{a}^i)^p = \mathbf{x}^p \text{ for each } i \in \mathbb{Z} \\ &\Longleftrightarrow \quad (\mathbf{x}\mathbf{a}^i)^p = (\mathbf{x}\mathbf{a}^j)^p \text{ for some } 0 \leq i < j \leq p-1. \end{aligned} \tag{3}$$

Since $(\mathbf{x}\mathbf{a}^i)^p = 1$ if $H_i \in \mathbf{E}_\mathcal{F}$ or $B_i \in \mathbf{E}_\mathcal{F}$, this shows that $\mathbf{x}^p = 1$ if $\mathcal{H}_0 \subseteq \mathbf{E}_\mathcal{F}$, or if $\mathcal{B}_0 \subseteq \mathbf{E}_\mathcal{F}$, or if $\Psi(\sigma) = 1$. We will see that at least one of these holds in each case.

Note that $R \cong \mathbb{Z}[\mathbf{U}]/\langle \sigma \rangle$, and that $\mathfrak{p}$ is the image in R of $(1 - \mathbf{u})\mathbb{Z}[\mathbf{U}]$. Also, $\mathfrak{p}^{p-1} = pR$: this follows, for example, from the congruence $(1 - \mathbf{u})^{p-1} \equiv \sigma \pmod{p\mathbb{Z}[\mathbf{U}]}$. Thus each ideal of p-power index in R is a power of $\mathfrak{p}$.

By Lemma 2.4, $\mathbf{E}_\mathcal{F}$ contains at least one abelian subgroup. There are three cases to consider:

Case 1: Assume $\mathbf{E}_\mathcal{F} \cap \mathcal{B} \neq \varnothing$. By Lemma 2.7(a), and since $\mathbf{E}_\mathcal{F} \cap \mathcal{H} \neq \varnothing$, $\mu(\mathrm{Aut}_\mathcal{F}(S)) \geq \Delta_0\Delta_{-1} = \Delta$. Hence all subgroups in $\mathcal{H}_*$ and in $\mathcal{B}_*$ are $\mathcal{F}$-conjugate (Lemma 2.6(b)), and $\mathbf{E}_\mathcal{F} = \mathcal{H}_0 \cup \mathcal{B}_*$ or $\mathcal{B}_0 \cup \mathcal{H}_*$.

By Lemma 2.7(a), $m \equiv -1 \pmod{p-1}$ if $\mathcal{H}_* \subseteq \mathbf{E}_\mathcal{F}$, while $m \equiv 0 \pmod{p-1}$ if $\mathcal{B}_* \subseteq \mathbf{E}_\mathcal{F}$. By Lemma 2.7(c), for $P \in \mathbf{E}_\mathcal{F}$, $\mathrm{Out}_\mathcal{F}(P) \cong GL_2(p)$ if $P \in \mathcal{H}_0 \cup \mathcal{B}_0$, while $\mathrm{Out}_\mathcal{F}(P) \cong SL_2(p)$ if $P \in \mathcal{H}_* \cup \mathcal{B}_*$.

Since all subgroups in $\mathbf{E}_{\mathcal{F}}$ have exponent p, $(\mathbf{xa}^i)^p = 1$ for each i. Hence $\Psi(\sigma) = 1$ by (3), so Ψ factors through $\mathbb{Z}[\mathbf{U}]/\langle\sigma\rangle \cong R$. We can thus regard A as an R-module, and $A \cong R/\mathfrak{p}^m$ (recall $p^m = |A|$) by the above remarks. Thus we are in case (a.ii) or (a.iii).

Case 2: Assume $\mathbf{E}_{\mathcal{F}}$ contains only abelian subgroups, and also that $H_i, H_j \in \mathbf{E}_{\mathcal{F}}$ for some $0 \le i < j \le p-1$. Then $(\mathbf{xa}^i)^p = (\mathbf{xa}^j)^p = 1$, so $\sigma \in \mathrm{Ker}(\Psi)$ by (3), and $A \cong R/\mathfrak{p}^m$ by the above remarks.

Since $H_j \in \mathbf{E}_{\mathcal{F}}$ where $j \ne 0$, $m \equiv -1 \pmod{p-1}$ by Lemma 2.7(a), and $\mu(\mathrm{Aut}_{\mathcal{F}}(S)) \ge \Delta_m = \Delta_{-1}$. Since $\mathcal{F}$ is reduced, $O^{p'}(\mathcal{F}) = \mathcal{F}$, so $\mu(\mathrm{Aut}_{\mathcal{F}}(S)) = \Delta_{-1} = \Delta_m$ by (2). Thus no two of the H_i are $\mathcal{F}$-conjugate (Lemma 2.6(b)), $\mathrm{Aut}_{\mathcal{F}}(H) \cong SL_2(p)$ for each $H \in \mathbf{E}_{\mathcal{F}}$ by Lemma 2.7(c), and we are in case (a.i).

Case 3: Assume $\mathbf{E}_{\mathcal{F}} = \mathcal{H}_i$ for some i. Then $\mu(\mathrm{Aut}_{\mathcal{F}}(S)) = \Delta_{-1}$ by (2), and $\mathrm{Out}_{\mathcal{F}}(H_i) \cong SL_2(p)$ by Lemma 2.7(c). Also, by Lemma 2.7(a), either $i = 0$ or $m \equiv -1 \pmod{p-1}$. If $i = 0$, then by (1), $\mu(\mathrm{Aut}_{\mathcal{F}}(S)) \nleq \Delta_{m-1}$, and hence $m \not\equiv 0 \pmod{p-1}$.

Thus if $\Psi(\sigma) = 1$ (so that $A \cong R/\mathfrak{p}^m$), then either $m \equiv -1 \pmod{p-1}$ and we are in the situation of (a.i); or $i = 0$, $m \not\equiv 0, -1 \pmod{p-1}$, and we are in the situation of (a.iv).

Now assume $\Psi(\sigma) \ne 1$. By (3), $(\mathbf{xa}^j)^p = 1$ for a unique $0 \le j \le p-1$, and $j = i$ since subgroups in $\mathbf{E}_{\mathcal{F}} = \mathcal{H}_i$ have exponent p. Also, $A_0\langle \mathbf{xa}^i\rangle$ is characteristic in S since it splits over A_0 while $A_0\langle \mathbf{xa}^\ell\rangle$ does not split for $\ell \ne i$. So we can assume that $\mathbf{x}$ was chosen with $\mathbf{x} = \mathbf{xa}^i$. Thus $i = 0$, and $\mathbf{E}_{\mathcal{F}} = \mathcal{H}_0$.

Set $I = \mathrm{Ker}(\Psi)$, $\tau = 1-\mathbf{u}$ and $J = \sigma\mathbb{Z}[\mathbf{U}]$; we identify $R = \mathbb{Z}[\mathbf{U}]/J$ and $\mathfrak{p} = (\tau\mathbb{Z}[\mathbf{U}]+J)/J$. Then $\Psi(J) = \langle\Psi(\sigma)\rangle \ne 1$, $\Psi(\sigma) \in Z$, so $\Psi(J) = Z$ since $|Z| = p$. Thus $J \cap I = pJ$. Also,

$$A/Z \cong \mathbb{Z}[\mathbf{U}]/(I+J) \cong \frac{\mathbb{Z}[\mathbf{U}]/J}{(I+J)/J} \cong R/\mathfrak{p}^{m-1},$$

since $|A/Z| = p^{m-1}$ and $\mathfrak{p}$ is the unique maximal ideal in R which contains p. Hence $I+J = \langle\sigma, \tau^{m-1}\rangle$, $\Psi(\tau^{m-1}) \in Z$, and so $I = \langle p\sigma, \tau^{m-1} + t\sigma\rangle$ for some $t \in \mathbb{Z}$. Also, $\tau^m \in I$ since $\tau\sigma = 0$.

Define $\varepsilon\colon \mathbb{Z}[\mathbf{U}] \longrightarrow \mathbb{Z}$ by setting $\varepsilon\bigl(\sum n_i\mathbf{u}^i\bigr) = \sum n_i$. Then $\mathrm{Ker}(\varepsilon) = \tau\mathbb{Z}[\mathbf{U}]$. Since $|\mathbb{Z}[\mathbf{U}]/(I + \tau\mathbb{Z}[\mathbf{U}])| = |A/A_0| = p$, we have $\varepsilon(I) = p\mathbb{Z}$, and hence $p \nmid t$.

Since $\mu(\mathrm{Aut}_{\mathcal{F}}(S)) = \Delta_{-1}$, I is invariant under all automorphisms of $\mathbf{U}$. So $(1-\mathbf{u}^r)^{m-1} + t\sigma \in I$ for each $1 \le r \le p-1$. Since $\tau^m \in I$, this implies that

$$0 \equiv \tau^{m-1}(1 + \mathbf{u} + \ldots + \mathbf{u}^{r-1})^{m-1} + t\sigma \equiv r^{m-1}\tau^{m-1} + t\sigma \pmod{I}.$$

Sice $p \nmid t$, this implies $r^{m-1} \equiv 1 \pmod{p}$ for each r, and hence $m \equiv 1 \pmod{p-1}$.

Set $k = (m-1)/(p-1)$. Then $\mathfrak{p}^{m-1} = p^kR$, so $I = \langle p\sigma, p^k + \ell\sigma\rangle$ for some ℓ, and the condition $\varepsilon(I) = p\mathbb{Z}$ implies that $p \nmid \ell$ if $k \ge 2$, $p \nmid (\ell+1)$ if $k = 1$. We are thus in the situation of case (b). □

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