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VECTOR BUNDLES OVER CLASSIFYING SPACES OF COMPACT LIE GROUPS

by Stefan Jackowski & Bob Oliver

In this paper, we describe the vector bundles over the classifying space BG of a compact Lie group G, up to stabilization by bundles coming from linear representations of G. In particular, the Grothendieck group of vector bundles over BG is expressed in terms of the representation rings of certain subgroups of G.

Define $\mathrm{Vect}(X)$, for any space X, and $\mathrm{Rep}(G)$, for any group G, to be the abelian monoids of isomorphism classes of complex vector bundles over X and complex finite dimensional G-representations, respectively; with addition in both cases defined by direct sum. Let

$$\alpha_G : \mathrm{Rep}(G) \longrightarrow \mathrm{Vect}(BG)$$

be the homomorphism defined by sending a complex G-representation V to the vector bundle $(EG\times_G V)\downarrow BG$ associated to the universal principal bundle $EG\downarrow BG$. This map is already known to be bijective when G is a finite p-group [DZ] or when G is p-toral [Nb]. (A group G is p-toral if its identity component G_0 is a torus and G/G_0 is a finite p-group.) But in general, α_G is neither surjective nor injective, and $\mathrm{Vect}(BG)$ can in fact even be uncountable. The situation does, however, become much simpler after passing to Grothendieck groups.

For each p-toral subgroup P of G, consider the composite

$$\mathrm{Vect}(BG) \xrightarrow{\text{restr}} \mathrm{Vect}(BP) \xrightarrow[\cong]{\alpha_P^{-1}} \mathrm{Rep}(P) \subseteq R(P),$$

where $R(P)$ is the complex representation ring of G. These maps define a homomorphism

$$r_G : \mathrm{Vect}(BG) \longrightarrow R_{\mathcal{P}}(G) \overset{\text{def}}{=} \varprojlim_{P} R(P),$$

where the inverse limit is taken over all p-toral subgroups of G (for all primes p) with respect to inclusion and conjugation of subgroups.

Now let $\mathbb{K}(X)$, for any space X, denote the Grothendieck group of the monoid $\mathrm{Vect}(X)$. Our main result (Theorem 1.8 below) is the following.

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Theorem. *For any compact Lie group G, r_G extends to an isomorphism of groups*

$$\bar{r}_G : \mathbb{K}(BG) \xrightarrow{\cong} R_{\mathcal{P}}(G).$$

Furthermore, any vector bundle over BG is a summand of a bundle $(EG\times_G V \downarrow BG)$ for some G-representation V; and thus $\mathbb{K}(BG)$ can be obtained from $\mathrm{Vect}(BG)$ by inverting only those vector bundles coming from G-representations.

In contrast to the situation for bundles over finite dimensional spaces, any subbundle of a trivial bundle over BG is itself trivial. For finite G, this follows from the version of the Sullivan conjecture proven by Miller [Mi, Theorem A], since any summand of a trivial bundle is classified by a map into a finite dimensional Grassmannian. When $\dim(G) > 0$, it follows, via a somewhat more complicated argument, from [FM, Theorem 3.1].

When G is connected, then $R_{\mathcal{P}}(G) \cong R(G)$, and the theorem implies that $\bar{\alpha}_G : R(G) \to \mathbb{K}(BG)$ is an isomorphism.

When X is compact or finite dimensional, $\mathbb{K}(X)$ is by definition equal to the K-theory ring $K(X)$. This is usually extended to a representable functor $K(X)$, defined for an arbitrary space X, by setting $K(X) \stackrel{\text{def}}{=} [X, \mathbb{Z} \times BU]$. Here, U denotes the infinite increasing union of the unitary groups $U(n)$. The obvious natural transformation

$$\beta_X : \mathbb{K}(X) \longrightarrow K(X)$$

need not be an isomorphism. In fact, the geometrically defined functor $\mathbb{K}(-)$ can behave very differently from $K(-)$. For example, $\mathbb{K}(-)$ is not exact, and it does not satisfy Bott periodicity in general (see the discussion after Theorem 1.1). This helps to explain why $\mathbb{K}(-)$ is more difficult to compute than $K(-)$.

The K-theory of classifying spaces of compact Lie groups has been computed by Atiyah and Segal [AS]. Their completion theorem says that the composite

$$R(G) \xrightarrow{\bar{\alpha}_G} \mathbb{K}(BG) \xrightarrow{\beta_{BG}} K(BG)$$

extends to an isomorphism $\widehat{\alpha}_G : R(G)\widehat{\ } \xrightarrow{\cong} K(BG)$, where $R(G)\widehat{\ }$ is the completion of the representation ring with respect to its augmentation ideal. We thus have the following commutative diagram

$$\begin{array}{ccc} R(G) & \xrightarrow{\lambda_G} & R(G)\widehat{\ } \\ {\scriptstyle \bar{\alpha}_G}\big\downarrow & & {\scriptstyle \widehat{\alpha}_G}\big\downarrow \\ \mathbb{K}(BG) & \xrightarrow{\beta_{BG}} & K(BG), \end{array}$$

where λ_G denotes the completion homomorphism.

One consequence of the above theorem is that β_{BG} is a monomorphism (Corollary 1.9). Its image can in fact be described internally, using the exterior power operations on $K(BG)$. Adams, in [Ad], defined and studied the subgroup $FF(BG) \subseteq K(BG)$ generated by the "formally finite dimensional elements"; i.e., those elements $x \in K(BG)$ such that $\lambda^k(x) = 0$ for k sufficiently large. Our results, when combined with his, imply that $FF(BG) = \mathrm{Im}(\beta_{BG})$.

The Atiyah-Segal completion theorem also describes the groups $K^{-i}(BG)$ for $i > 0$; i.e., the homotopy groups of the mapping space $\mathrm{map}(BG, \mathbb{Z}\times BU)$. Since $\coprod_{n=0}^{\infty} BU(n)$ is a topological monoid and commutative up to homotopy, the space of maps from X into it (the "topological monoid of vector bundles over X") is also a homotopy commutative topological monoid. Thus, we can consider its topological group completion $\mathfrak{K}^{\mathbb{C}}(X)$, where $\pi_0(\mathfrak{K}^{\mathbb{C}}(X)) \cong \mathbb{K}(X)$. When X is a finite complex, $\mathfrak{K}^{\mathbb{C}}(X)$ has the homotopy type of $\mathrm{map}(X, \mathbb{Z} \times BU)$. In Proposition 2.4 below, we show that when G is finite, the connected components of $\mathfrak{K}^{\mathbb{C}}(BG)$ have the same homotopy type as the components of $\mathrm{map}(BG, \mathbb{Z}\times BU)$. In contrast, even when G is a (nontrivial) torus, then the components of $\mathfrak{K}^{\mathbb{C}}(BG)$ are quite different from those of $\mathrm{map}(BG, \mathbb{Z}\times BU)$ (see Proposition 2.5), and in fact their homotopy groups are nonvanishing in odd degrees.

The above discussion has focused on the case of complex bundles, but all of these results (except for those in Proposition 2.5) are also shown to hold for real bundles.

This paper grew out of our earlier efforts to understand maps between the classifying spaces of compact Lie groups. The set of n-dimensional vector bundles over BG corresponds to that of homotopy classes of maps from BG to $BU(n)$. Thus the starting point for our computation of $\mathbb{K}(BG)$ are the theorems of Dwyer-Zabrodsky and Notbohm which describe up to p-completion the mapping space $\mathrm{map}(BP, BL)$ for a p-toral groups P and an arbitrary compact Lie group L (Theorem 1.1 below). A decomposition of BG at any prime p as a homotopy direct limit of classifying spaces of p-toral subgroups of G (Theorem 1.2) provides a tool for passing to more general groups. The key new element in the proof of the main theorem is provided by the vanishing of certain higher derived functors of inverse limits, which in turn depends on the equivariant Bott periodicity theorem.

In the course of the proof of vanishing higher limits, we also show the following extension of Smith theory (Proposition 3.3). If X is a finite dimensional G-complex with finitely many orbit types and all isotropy subgroups p-toral, then X^H is also $\mathbb{F}_p$-acyclic, not only for $H \subseteq G$ a p-toral subgroup (as follows from Smith theory), but also whenever H is a subgroup of any p-toral subgroup of G.

The main results about $\mathbb{K}(BG)$ and $\mathbb{KO}(BG)$ are shown in Section 1; and those about the spaces $\mathfrak{K}^{\mathbb{C}}(BG)$ and $\mathfrak{K}^{\mathbb{R}}(BG)$ in Section 2. Also, two other algebraic descriptions of $R_{\mathcal{P}}(G)$ are given in Section 1 (Proposition 1.12 and the following discussion). The vanishing theorem for higher inverse limits needed in the first two sections is shown in Section 3.

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Notation: All complex (real) representations of G are assumed to be equipped with G-invariant hermitian (inner) product. For any such representation V, we let $\mathrm{Aut}(V)$ denote the group of unitary (orthogonal) automorphisms of the vector space V, and $\rho_V : G \to \mathrm{Aut}(V)$ the homomorphism induced by the action. Also, for any $H \subseteq G$, $\mathrm{Aut}_H(V)$ denotes the subgroup of H-equivariant automorphisms.

It will be convenient to state some of the results simultaneously for real and complex vector bundles, or for orthogonal and unitary groups. In such situations, if $F = \mathbb{C}$ or $\mathbb{R}$, we write $U(n,F)$ for $U(n)$ or $O(n)$, respectively. Similarly, $KF(-) = K(-)$ or $KO(-)$ and $RF(-) = R(-)$ or $RO(-)$. And $\mathrm{Vect}^F(X)$ denotes the monoid of F-vector bundles over a space X, $\mathbb{K}F(X)$ ($= \mathbb{K}(X)$ or $\mathbb{KO}(X)$) its Grothendieck group, and $\mathrm{Rep}^F(G)$ the monoid of F-representations of the group G.

Throughout the paper, $Y\hat{_p}$ denotes the p-completion of a space Y in the sense of Bousfield & Kan [BK].

1. $\mathbb{K}(BG)$ for a compact Lie group G

Throughout this section, G denotes a fixed compact Lie group. Rather than working directly with vector bundles over BG, we work with their classifying maps $BG \to BU(n,F)$ ($F = \mathbb{C}$ or $\mathbb{R}$), via the isomorphism of monoids

$$\mathrm{Vect}^F(BG) \simeq \coprod_{n=0}^{\infty} [BG, BU(n,F)].$$

Maps $BG \to BU(n,F)$ are studied using our general strategy for studying maps between classifying spaces, as outlined in [JMO2, Section 3]. The first part of this section thus consists mostly of results already shown elsewhere, and needed in the proof of the main theorem (Theorem 1.8). The main exception to this is Proposition 1.5, where we prove the vanishing of the higher inverse limits which occur as obstructions, using also the results in Section 3.

The starting point for understanding vector bundles over BG is the following theorem, which in particular implies that $\mathbb{K}(BP) \cong R(P)$ for any p-toral group P. In fact, the theorem says that the underlying monoids are isomorphic in this case.

Theorem 1.1. (Dwyer-Zabrodsky and Notbohm) *Set $F = \mathbb{C}$ or $\mathbb{R}$. For any prime p and any p-toral group P, the homomorphism of monoids*

$$\alpha_P^F : \mathrm{Rep}^F(P) \xrightarrow{\quad\cong\quad} \mathrm{Vect}^F(BP) \cong \coprod\nolimits_{n=0}^{\infty} [BP, BU(n,F)],$$

which sends a representation V to the vector bundle $(EP \times_P V)\downarrow BP$ (or to the map $B\rho_V$), is an isomorphism. Also, for any P-representation V over F, which is odd *dimensional if* $F = \mathbb{R}$*, the homomorphism* $P \times Aut_P(V) \xrightarrow{(\rho_V,\mathrm{incl})} Aut(V)$ *induces (by adjointness) a homotopy equivalence*

$$BAut_P(V)^\wedge_p \xrightarrow{\;\simeq\;} \mathrm{map}\big(BP, BAut(V)^\wedge_p\big)_{B\rho_V}.$$

Proof. These are special cases of the following theorems of Dwyer & Zabrodsky [DZ] (when P is a finite p-group) and Notbohm [Nb] (in the general case). For any compact Lie group L, the map

$$\mathrm{Hom}(P,L)/\,\mathrm{Inn}(L) \xrightarrow[\cong]{(\rho\mapsto B\rho)} [BP, BL] \tag{1}$$

is a bijection. If L is connected, then for any $\rho : P \to L$, the homomorphism $P \times C_L(\rho(P)) \xrightarrow{(\rho,\mathrm{incl})} L$ induces a homotopy equivalence

$$BC_L(\rho(P))^\wedge_p \xrightarrow{\;\simeq\;} \mathrm{map}(BP, (BL)^\wedge_p)_{B\rho}. \tag{2}$$

Point (2) follows easily from results in [DZ] and [Nb], and is shown explicitly in [JMO, Theorem 3.2(iii)].

Theorem 1.1 is just the special case of (1) and (2) when $L = \mathrm{Aut}(V) \cong U(n)$ or $O(n)$. Note in particular that if $F = \mathbb{R}$ and $n = \dim(V)$ is odd, then $\mathrm{Aut}(V) \cong O(n) \cong SO(n)\times\{\pm I\}$. $\square$

Theorem 1.1 provides some simple examples of the exotic behavior of the functor $\mathbb{K}(-)$. For example, if C_p denotes the cyclic subgroup of order p, for any prime p, then the sequence

$$\widetilde{\mathbb{K}}(BS^1/BC_p) \xrightarrow{\;\mathrm{proj}\;} \mathbb{K}(BS^1) \xrightarrow{\;\mathrm{restr}\;} \mathbb{K}(BC_p)$$

is not exact. More precisely, if $\xi_n{\downarrow}BS^1$ (for $n \in \mathbb{Z}$) denotes the line bundle with Chern class n times some fixed generator of $H^2(BS^1)$, then $[\xi_1] - [\xi_{p+1}]$ lies in the kernel of the above restriction map, but not in the image of $\mathbb{K}(BS^1/BC_p)$. One can also show, using Theorem 1.1 again (and Propositions 2.3 and 2.5 below) that for any prime p and any finite p-group P,

$$\mathbb{K}(BP) \cong R(P) \cong \mathbb{Z}^r, \qquad (r = \mathrm{rk}(R(P)))$$

$$\widetilde{\mathbb{K}}(\Sigma^2(BP_+)) = \widetilde{\mathbb{K}}\big((S^2{\times}BP)/BP\big) \cong \mathbb{Z},$$

and

$$\mathbb{K}(S^2{\times}BP)/\mathbb{K}(BP) \cong \mathbb{Z} \times (\widehat{\mathbb{Z}}_p)^{r-1}.$$

In particular, these groups are pairwise nonisomorphic (if $P \neq 1$), and so Bott periodicity fails for $\mathbb{K}(-)$.

To pass from p-toral groups to an arbitrary compact Lie group G, we use a decomposition of BG, at each prime p, as a homotopy direct limit of classifying spaces of p-toral subgroups of G. This decomposition is indexed by a certain orbit category $\mathcal{R}_p(G)$. The objects in $\mathcal{R}_p(G)$ are the orbits G/P such that (1) $P \subseteq G$ is p-toral, (2) $N(P)/P$ is finite, and (3) there is no normal p-subgroup $1 \neq Q \lhd N(P)/P$. The morphisms in $\mathcal{R}_p(G)$ are the G-maps between orbits. This is a discrete category, and is in fact equivalent to a finite category (see Lemma 1.7 below). It is also a "directed" or "EI" category, in that all endomorphisms of $\mathcal{R}_p(G)$ are isomorphisms, and hence there are morphisms in at most one direction between any pair of nonisomorphic objects. The importance of $\mathcal{R}_p(G)$ lies in the following theorem.

Theorem 1.2. *For any prime p, the map*

$$q_{G,p} : \underset{G/P \in \mathcal{R}_p(G)}{\underrightarrow{\text{hocolim}}} \big(EG/P\big) \xrightarrow{\qquad} BG,$$

induced by the projection $EG \to BG$, is an $\mathbb{F}_p$-homology equivalence. In particular, for any connected complex Y, $q_{G,p}$ induces a homotopy equivalence

$$(- \circ q_{G,p}) : \operatorname{map}(BG, Y^\wedge_p) \xrightarrow{\ \simeq\ } \operatorname{map}\Big(\underset{G/P \in \mathcal{R}_p(G)}{\underrightarrow{\text{hocolim}}} \big(EG/P\big), Y^\wedge_p \Big).$$

Proof. The first statement is shown in [JMO, Theorem 1.4]; and the second then follows from [BK, Proposition II.2.8]. □

Theorem 1.2 will be directly useful only to describe maps from BG to the $BU(n,F)^\wedge_p$; i.e., after p-completion. The next proposition will allow us to combine such maps, to get maps to $BU(n,F)$ itself.

Proposition 1.3. *Let $T \subseteq G$ be a maximal torus of G, and set $w = |N(T)/T|$. Then the following square is a pullback if $F = \mathbb{C}$, or if $F = \mathbb{R}$ and n is odd:*

$$\begin{array}{ccc} [BG, BU(n,F)] & \longrightarrow & \prod_{p|w} [BG, BU(n,F)^\wedge_p] \\ \Big\downarrow{\scriptstyle \text{restr}} & & \Big\downarrow{\scriptstyle \text{restr}} \\ [BT, BU(n,F)] & \longrightarrow & \prod_{p|w} [BT, BU(n,F)^\wedge_p]. \end{array}$$

Proof. This is a consequence of the arithmetic pullback square for the space $BU(n,F)$. For details, see [JMO3, Proposition 1.2]. □

In view of Theorem 1.2, we are faced with the problem of comparing maps defined on a homotopy direct limit with maps defined on its pieces. This will be studied via the obstruction theory described in the next proposition.

This obstruction theory is based on a skeletal decomposition of the homotopy colimit which generalizes the classical construction of obstructions related to the skeletal decomposition if the source space is a polyhedron (or CW-complex). For each $n \geq 0$, the "n-skeleton" of $\underrightarrow{\text{hocolim}}(F)$ is defined by setting

$$\underset{\mathcal{C}}{\underrightarrow{\text{hocolim}}}^{(n)}(F) = \Big(\coprod_{k=0}^{n}\Big(\coprod_{c_0 \to \dots \to c_k} F(c_0) \times \Delta^k\Big)\Big) \Big/ \sim,$$

where one divides out by face and degeneracy relations. The space $\underrightarrow{\text{hocolim}}(F)$ itself is just the case $n = \infty$. Note in particular that $\underrightarrow{\text{hocolim}}^{(0)}(F)$ is the disjoint union of the $F(c)$, taken over all $c \in \text{Ob}(\mathcal{C})$, and that $\underrightarrow{\text{hocolim}}^{(1)}(F)$ is the union of the mapping cylinders of all maps induced by morphisms in $\mathcal{C}$. The following result describes the obstructions we will need to consider.

Proposition 1.4. *Fix a discrete category $\mathcal{C}$, and a (covariant) functor $F : \mathcal{C} \to \mathsf{Top}$. Let Y be any other space, and fix maps $f_c : F(c) \to Y$ (for all $c \in \text{Ob}(\mathcal{C})$) whose homotopy classes define an element $\widehat{f} = ([f_c])_{c\in\mathcal{C}} \in \underleftarrow{\lim}[F(-), Y]$ (i.e., $\coprod f_c$ extends to a map $\underrightarrow{\text{hocolim}}^{(1)}(F) \to Y$). Set*

$$\alpha_n(c) = \pi_n\big(\text{map}(F(c), Y), f_c\big)$$

for all $c \in \text{Ob}(\mathcal{C})$. Then given a map $f_n : \underrightarrow{\text{hocolim}}_{\mathcal{C}}^{(n)}(F) \to Y$ (any $n \geq 1$) which extends $\widehat{f}$, the obstruction to constructing a map on $\underrightarrow{\text{hocolim}}_{\mathcal{C}}^{(n+1)}(F)$ which extends $f_n\big|\,\underrightarrow{\text{hocolim}}_{\mathcal{C}}^{(n-1)}(F)$ lies in $\underleftarrow{\lim}^{n+1}(\alpha_n)$. Also, given two maps $f, f' : \underrightarrow{\text{hocolim}}_{\mathcal{C}}(F) \to Y$ and a homotopy F_n defined on $\underrightarrow{\text{hocolim}}^{(n)}(F)$ (any $n \geq 0$), the obstruction to constructing a homotopy on $\underrightarrow{\text{hocolim}}_{\mathcal{C}}^{(n+1)}(F)\times I$ which extends $F_n\big|(\underrightarrow{\text{hocolim}}_{\mathcal{C}}^{(n-1)}(F)\times I)$ lies in $\underleftarrow{\lim}^{n+1}(\alpha_{n+1})$. Both of these obstructions are natural in Y.

Proof. This is shown by Wojtkowiak in [Wo]. Note in particular that for α_n to be well defined as a functor from $\mathcal{C}$ to groups or abelian groups, one must first choose a map $f_1 : \underrightarrow{\text{hocolim}}_{\mathcal{C}}^{(1)}(F) \to Y$ which can be extended to $\underrightarrow{\text{hocolim}}^{(2)}(F)$. This applies to all of the above situations except where one wants to extend a map f_1 from $\underrightarrow{\text{hocolim}}^{(1)}(F)$ to the 2-skeleton; and in this case the obstruction set $\underleftarrow{\lim}^2(\alpha_1)$ is defined for any functor from $\mathcal{C}$ to the category of groups with morphisms given by conjugacy classes of homomorphisms. □

When applying Proposition 1.4, we will need to deal with the higher limits of homotopy groups of mapping spaces $\text{map}\big(EG/P, B\text{Aut}(V)\hat{_p}\big)_{B\rho V} \simeq B\text{Aut}_P(V)\hat{_p}$ (Theorem 1.1), where $\text{Aut}_P(V)$ is a product of unitary, orthogonal, and symplectic groups. Since the higher homotopy groups of these spaces are unknown, we instead stabilize, by taking limits over all $V \in \text{Rep}^F(G)$. Such limits can be made more precise by taking them over some sequence $V_1 \subseteq V_2 \subseteq V_3 \subseteq \dots$ of G-representations (with G-invariant inner or hermitian product), such that each finite dimensional G-representation is contained in V_k for sufficiently large k. We can of course assume that each V_k is odd dimensional (in order to apply Theorem 1.1 or Proposition 1.3). Alternatively, one can choose a "universal" G-representation U (with G-invariant inner product), i.e., a representation which contains infinitely many copies of each irreducible G-representation, and then take the limit over all finite dimensional subrepresentations of U.

Proposition 1.5. *For each $i > 0$, let $\Pi_i^F : \mathcal{R}_p(G) \to \widehat{\mathbb{Z}}_p\text{-}\mathtt{mod}$ be the functor defined by setting*

$$\Pi_i^F(G/P) = \varinjlim_{V \in \mathrm{Rep}^F(G)} \pi_i\Big(\mathrm{map}(EG/P, BAut(V)^\wedge_p), B\rho_V\Big).$$

Then

$$\Pi_i^F \cong \widehat{\mathbb{Z}}_p \otimes KF_G^{-i}(-) \tag{1}$$

as functors on $\mathcal{R}_p(G)$, and

$$\varprojlim_{\mathcal{R}_p(G)}{}^j \, \Pi_i^F = 0 \tag{2}$$

for all $i, j > 0$.

Proof. By Corollary 3.6, $\varprojlim^j_{\mathcal{R}_p(G)}\big(\widehat{\mathbb{Z}}_p \times KF_G^{-i}(-)\big) = 0$ for all $i, j > 0$. So (2) follows immediately once point (1) has been shown.

Fix a G-representation V over F, such that $\dim(V)$ is odd if $F = \mathbb{R}$. By Theorem 1.1, for any p-toral subgroup $P \subseteq G$, the group homomorphism

$$P \times \mathrm{Aut}_P(V) \xrightarrow{(\rho_V, \mathrm{incl})} \mathrm{Aut}(V)$$

is, after taking classifying spaces, adjoint to a homotopy equivalence

$$c_{P,V} : B\mathrm{Aut}_P(V)^\wedge_p \xrightarrow{\simeq} \mathrm{map}(BP, B\mathrm{Aut}(V)^\wedge_p)_{B\rho_V}.$$

So for each $i > 0$, $c_{P,V}$ induces an isomorphism

$$c^i_{P,V} : \widehat{\mathbb{Z}}_p \otimes \pi_i\big(B\mathrm{Aut}_P(V)\big) \xrightarrow{\cong} \pi_i\Big(\mathrm{map}(BP, B\mathrm{Aut}(V)^\wedge_p), B\rho_V\Big).$$

By construction, the $c^i_{P,-}$ are natural with respect to inclusions of representations $V \subseteq W$ (where this means taking direct sums with the basepoints of the corresponding spaces in the orthogonal complement of V). Hence we can take the limit over all $V \in \mathrm{Rep}^F(G)$ to get an isomorphism

$$c^i_P : \widehat{\mathbb{Z}}_p \otimes \varinjlim_V \Big(\pi_i\big(B\mathrm{Aut}_P(V)\big)\Big) \xrightarrow{\cong} \varinjlim_V \Big(\pi_i\big(\mathrm{map}(BP, B\mathrm{Aut}(V)), B\rho_V\big)\Big) = \Pi_i^F(G/P).$$

For each P and V, $\mathrm{Aut}_P(V) \cong \mathrm{Aut}_G\big(G/P{\times}V \downarrow G/P\big)$: the group of G-bundle automorphisms covering the identity on G/P. Thus, for each $i > 0$,

$$\pi_i(B\mathrm{Aut}_P(V)) \cong \pi_{i-1}\Big(\mathrm{Aut}_G\big(G/P{\times}V \downarrow G/P\big)\Big)$$

can be identified with the set of isomorphism classes of G-vector bundles over

$$\Sigma^i(G/P_+) \cong \big(D^i{\wedge}(G/P_+)\big) \cup_{S^{i-1}\wedge(G/P_+)} \big(D^i{\wedge}(G/P_+)\big)$$

with a fixed identification of V with the fiber over the base point. Hence

$$\varinjlim_V \Big(\pi_i\big(B\mathrm{Aut}_P(V)\big)\Big) \cong \widetilde{KF}_G(\Sigma^i(G/P_+)) = KF_G^{-i}(G/P);$$

and this together with c_P^i defines the isomorphism $KF_G^{-i}(G/P) \cong \Pi_i^F(G/P)$.

It remains to show that this isomorphism is natural with respect to morphisms in $\mathcal{R}_p(G)$. We do this by replacing $c_{P,V}$, for each V, by a homotopy equivalent map which which is clearly functorial in G/P. Regard EG and $B\mathrm{Aut}(V)$ as the nerves of topological categories $\mathcal{E}G$ and $\mathcal{B}\mathrm{Aut}(V)$: $\mathcal{E}G$ has as objects the elements of G and has a unique morphism between each pair of objects; while $\mathcal{B}\mathrm{Aut}(V)$ has one object, and a morphism for each element of $\mathrm{Aut}(V)$. Let $\psi_V : EG{\times}_G B\mathrm{Aut}(V) \to B\mathrm{Aut}(V)$ (where G acts on $B\mathrm{Aut}(V)$ via ρ_V and conjugation) be the map induced by the functor $\mathcal{E}G \times \mathcal{B}\mathrm{Aut}(V) \to \mathcal{B}\mathrm{Aut}(V)$ which sends each morphism $\big([g{\to}h], a\big)$ to $\rho_V(g){\cdot}a{\cdot}\rho_V(h)^{-1}$.

For any G/P in $\mathcal{R}_p(G)$, let

$$\mathrm{ev} : G/P \times B\mathrm{Aut}_G(G/P{\times}V \downarrow G/P)^\wedge_p \longrightarrow B\mathrm{Aut}(V)^\wedge_p$$

be the evaluation map: for each $gP \in G/P$, $\mathrm{ev}(gP, -)$ is induced by restriction to the fiber over gP. Define

$$\begin{aligned}\psi_{P,V} = (\psi_V)^\wedge_p \circ (\mathrm{Id} \times \mathrm{ev}) : EG/P &\times B\mathrm{Aut}_G(G/P{\times}V \downarrow G/P)^\wedge_p \\ &= EG{\times}_G\Big(G/P \times B\mathrm{Aut}_G(G/P{\times}V \downarrow G/P)^\wedge_p\Big) \longrightarrow B\mathrm{Aut}(V)^\wedge_p;\end{aligned}$$

and let

$$c'_{P,V} : B\mathrm{Aut}_G(G/P{\times}V \downarrow G/P)^\wedge_p \longrightarrow \mathrm{map}\big(EG/P, B\mathrm{Aut}(V)^\wedge_p\big)$$

be its adjoint map. The restriction of $\psi_{P,V}$ to $BP \times B\mathrm{Aut}_P(V)$ is just the map induced by (ρ_V, incl) and multiplication. So $c'_{P,V}$ is homotopy equivalent to the map $c_{P,V}$ constructed earlier, and is natural in G/P by construction. □

In general, of course, direct and inverse limits cannot be switched. The following lemma describes one case where this can be done.

Lemma 1.6. *Fix a finite category $\mathcal{C}$ and a directed category $\mathcal{D}$, and let $M : \mathcal{C}\times\mathcal{D} \to \mathsf{Ab}$ be any functor to abelian groups. Then for any $i \geq 0$,*

$$\varprojlim_{\mathcal{C}}{}^{i}\Bigl(\varinjlim_{\mathcal{D}}(M)\Bigr) \cong \varinjlim_{\mathcal{D}}\Bigl(\varprojlim_{\mathcal{C}}{}^{i}(M)\Bigr).$$

Proof. For any functor $M' : \mathcal{C} \to \mathsf{Ab}$, the higher limits $\varprojlim_{\mathcal{C}}{}^{*} M'$ of M' are the homology groups of a cochain complex

$$0 \longrightarrow \prod_{c\in\mathcal{C}} M'(c) \longrightarrow \prod_{c_0\to c_1} M'(c_1) \longrightarrow \prod_{c_0\to c_1\to c_2} M'(c_2) \longrightarrow \dots$$

(cf. [BK, XI.6.2] or [O2, Lemma 2]). In particular, this applies when $M' = M(-,d)$ for any d in $\mathcal{D}$, and when $M' = \varinjlim_d M(-,d)$. Since $\mathcal{C}$ is a finite category, all of the products are finite, and hence commute with direct limits over the directed category $\mathcal{D}$. And since direct limits over $\mathcal{D}$ commute with taking homology, we now see that they commute with inverse limits over $\mathcal{C}$. □

Using Lemma 1.6, one can show that under certain conditions, homotopy direct limits and homotopy inverse limits commute for functors $\mathcal{C}\times\mathcal{D} \to \mathsf{Top}$. An example of this is shown in the next section (in the proof of Proposition 2.4), but the conditions for a general result seem too complicated to be worth stating here.

The following properties of the category $\mathcal{R}_p(G)$ will be needed in the proof of Theorem 1.8.

Lemma 1.7. *For any prime p, $\mathcal{R}_p(G)$ is equivalent to a finite category, in that it has finitely many isomorphism classes of objects and finite morphism sets. Also, there is an integer $d = d(G,p) \geq 0$, with the property that for any functor $F : \mathcal{R}_p(G) \to \widehat{\mathbb{Z}}_p\text{-}\mathsf{mod}$, $\varprojlim^{*}_{\mathcal{R}_p(G)}(F) = 0$ for all $* > d$.*

Proof. The finiteness of $\mathcal{R}_p(G)$ is shown in [JMO, Proposition 1.6], and the second statement in [JMO2, Proposition 4.11]. □

We are now ready to prove the main theorem. Recall that $RF_{\mathcal{P}}(G) = \varprojlim_P RF(P)$, where the inverse limit is taken over all p-toral subgroups of G (for all primes p) with respect to inclusion and conjugation of subgroups; and that

$$r_G^F : \mathrm{Vect}^F(BG) \longrightarrow RF_{\mathcal{P}}(G) \qquad \text{and} \qquad \bar{r}_G^F : \mathbb{K}F(BG) \longrightarrow RF_{\mathcal{P}}(G)$$

are defined to be the inverse limits of the restriction maps $\mathrm{Vect}^F(BG) \to \mathbb{K}F(BG) \to \mathbb{K}F(BP) \cong RF(P)$.

Theorem 1.8. *For any compact Lie group G,*

$$\bar{r}_G^{\mathbb{C}} : \mathbb{K}(BG) \xrightarrow{\ \cong\ } R_{\mathcal{P}}(G) \quad \textit{and} \quad \bar{r}_G^{\mathbb{R}} : \mathbb{KO}(BG) \xrightarrow{\ \cong\ } RO_{\mathcal{P}}(G)$$

are isomorphisms of groups. Furthermore, any vector bundle over BG can be embedded as a summand of a bundle $(EG\times_G V \downarrow BG)$ for some G-representation V; and $\mathbb{K}F(BG)$ ($F=\mathbb{C}$ or $\mathbb{R}$) can thus be obtained from $\mathrm{Vect}^F(BG)$ by inverting only those vector bundles coming from G-representations.

Proof. Upon applying the universality properties of the Grothendieck construction to the homomorphism $r_G^F : \mathrm{Vect}^F(BG) \to RF_{\mathcal{P}}(G)$ of monoids, we are reduced to proving the following two statements:

(1) For each $X \in RF_{\mathcal{P}}(G)$, there exist a vector bundle $\xi{\downarrow}BG$ and a G-representation V such that $r_G^F(\xi) = X + r_G^F(EG\times_G V)$.

(2) For each pair of bundles $\xi,\xi'{\downarrow}BG$ such that $r_G^F(\xi) = r_G^F(\xi')$, there exists a G-representation V such that $\xi \oplus (EG\times_G V) \cong \xi' \oplus (EG\times_G V)$.

To see that each bundle over BG can be embedded in one coming from a G-representation, fix $\eta{\downarrow}BG$, and apply (1) to the element $X = -r_G^F(\eta)$. Then $r_G^F(\eta) + r_G^F(\xi) = r_G^F(EG\times_G V)$ for some bundle $\xi{\downarrow}BG$ and some G-representation V; and by (2) $\eta\oplus\xi\oplus(EG\times_G V') \cong (EG\times_G(V\oplus V'))$ for some V'.

Throughout the proof, we will be dealing with maps $BG \to BU(n,F)$ rather than n-dimensional vector bundles over BG. For such a map f, it will be convenient to write ξ_f for the pullback via f of the universal vector bundle. The proof of points (1) and (2) will be given below for complex bundles. The proof in the real case is similar, except that one has to restrict to odd dimensional bundles when applying Proposition 1.3.

Fix a maximal torus $T \subseteq G$, and let $W_G = N(T)/T$ denote the Weyl group. For each prime $p\big||W_G|$, fix a subgroup $N_p(T) \subseteq G$ such that $N_p(T)/T$ is a Sylow p-subgroup of $N(T)/T$. This subgroup $N_p(T)$ is a *maximal* p-toral subgroup in the sense that any other p-toral subgroup of G is conjugate to a subgroup of $N_p(T)$ (cf. [JMO, Lemma A.1]).

Proof of point (1) Fix any element $X = \big(v_P\big)_{P\in\mathcal{S}_{\mathcal{P}}(G)} \in R_{\mathcal{P}}(G)$, where $\mathcal{S}_{\mathcal{P}}(G)$ denotes the set of p-toral subgroups of G for all primes p. For each prime $p\big||W_G|$, write $v_{N_p(T)} = [V_p'] - [V_p'']$, where V_p' and V_p'' are $N_p(T)$-representations. Since each V_p'' is contained in the restriction of some G-representation (cf. [Br, Theorem 0.4.2]), we can add a G-representation to X to arrange that each $v_{N_p(T)}$, and hence each v_P, is an actual representation. We are thus reduced to the case where

$$X = \big(V_P\big) \in \varprojlim_{P\in\mathcal{S}_{\mathcal{P}}(G)} \mathrm{Rep}(P).$$

Set $n = \dim(V_P)$ (and note that this is independent of P).

Fix a prime $p\big||W_G|$. Let $\Pi_i^{(X)} : \mathcal{R}_p(G) \to \widehat{\mathbb{Z}}_p\text{-}\mathtt{mod}$ be the functor

$$\Pi_i^{(X)}(G/P) = \pi_i\Big(\mathrm{map}(EG/P, BU(n)\hat{_p}), B\rho_{V_P}\Big).$$

By Proposition 1.4, the successive obstructions to the existence of a map

$$f_p \in [BG, BU(n)\hat{_p}] \cong \Big[\underset{G/P\in\mathcal{R}_p(G)}{\underrightarrow{\mathrm{hocolim}}} \big(EG/P\big)\,,\, BU(n)\hat{_p}\Big]$$

such that $f_p|BP \simeq B\rho_{V_P}$ for each p-toral $P \subseteq G$ lie in the groups $\varprojlim^{i+1}_{\mathcal{R}_p(G)} \Pi_i^{(X)}$ (for all $i > 0$), Also,

$$\varinjlim_{W \in \mathrm{Rep}(G)} \Pi_i^{(X\oplus W)} \cong \varinjlim_{W \in \mathrm{Rep}(G)} \Pi_i^{(W)} = \Pi_i^{\mathbb{C}}$$

(isomorphic as functors on $\mathcal{R}_p(G)$), since there is a G-representation which contains $V_{N_p(T)}$ (and hence each V_P) as a summand (by [Br, Theorem 0.4.2] again). Since $\mathcal{R}_p(G)$ is equivalent to a finite category (Lemma 1.7), Lemma 1.6 and Proposition 1.5 apply to show that

$$\varinjlim_{W \in \mathrm{Rep}(G)} \Big(\varprojlim^j_{\mathcal{R}_p(G)} \Pi_i^{(X\oplus W)} \Big) \cong \varprojlim^j_{\mathcal{R}_p(G)} \Big(\varinjlim_{W \in \mathrm{Rep}(G)} \Pi_i^{(X\oplus W)} \Big) \cong \varprojlim^j_{\mathcal{R}_p(G)} \big(\Pi_i^{\mathbb{C}}\big) = 0 \qquad (3)$$

for all $i, j > 0$. In other words, each successive obstruction vanishes after adding a sufficiently large G-representation to X. Also, by Lemma 1.7, all of the higher limits vanish in degrees above some fixed $d(G,p)$ (which is independent of X); and so there are only finitely many obstructions to constructing f_p.

After carrying out this procedure for each prime $p\big||W_G|$, we can arrange, by stabilizing further, that the same G-representation has been added for each prime p. In other words, we find a G-representation W, and maps $f_p : BG \to BU(m)^\wedge_p$ (for each prime $p\big||W_G|$), such that $f_p|BP \simeq (B\rho_{V_P\oplus W})^\wedge_p$ for each p-toral subgroup $P \subseteq G$. In particular, $f_p|BT \simeq (B\rho_{V_T\oplus W})^\wedge_p$ for each p. So by Proposition 1.3, there is a map $f : BG \to BU(m)$ whose completion at each prime p is homotopic to f_p. Since $[BP, BU(-)]$ injects into $[BP, BU(-)^\wedge_p]$ for each p-toral $P \subseteq G$ (by Proposition 1.3 again), this implies that $f|BP \simeq B\rho_{V_P\oplus W}$ for each P; and hence that

$$r_G(\xi_f) = X + r_G(EG \times_G W).$$

Proof of point (2) The proof of (2) is similar, but shorter. Fix maps $f, g : BG \to BU(n)$ such that $r_G(\xi_f) = r_G(\xi_g) = X$, where $X = \big(V_P\big)_{P \in \mathcal{S}_{\mathcal{P}}(G)}$. In other words, for each p and each p-toral subgroup $P \subseteq G$, $f|BP \simeq g|BP \simeq B\rho_{V_P}$. We must show that there is a G-representation W for which $f \oplus B\rho_W \simeq g \oplus B\rho_W$. The same argument as that used in Step 1 (but using the injectivity obstructions of Proposition 1.4) shows that there is a G-representation W such that

$$(f \oplus B\rho_W)^\wedge_p \simeq (g \oplus B\rho_W)^\wedge_p$$

for all $p\big||W_G|$. And then $f \oplus B\rho_W \simeq g \oplus B\rho_W$ by Proposition 1.3. □

As a first corollary to Theorem 1.8, we note:

Corollary 1.9. *For $F = \mathbb{C}$ or $\mathbb{R}$, $\beta_{BG} : \mathbb{K}F(BG) \to KF(BG)$ is injective.*

Proof. By [Seg, Proposition 3.10], $R(P)$ injects into $R(P)^\wedge$ (and hence $RO(P)$ injects into $RO(P)^\wedge$), for any p-toral group P (more generally, for any P such that $\pi_0(P)$ has prime power order). The composite

$$\mathbb{K}F(BG) \xrightarrow{\beta_{BG}} KF(BG) \xrightarrow{\text{restr}} \varprojlim_P (KF(BP)) \cong \varprojlim_P (RF(P)^\wedge)$$

is thus a monomorphism, since $\mathbb{K}F(BG) \cong \varprojlim_P(RF(P))$ by Theorem 1.8, and hence β_{BG} is a monomorphism. □

The next corollary summarizes what we know about the relationship between $RF(G)$, $\mathbb{K}F(BG)$, and $KF(BG)$. It is mostly a matter of combining Theorem 1.8 with the Atiyah-Segal completion theorem [AS], as well as with some results of Adams in [Ad].

Corollary 1.10. *Set $F = \mathbb{C}$ or $\mathbb{R}$, and consider the commutative diagram*

$$\begin{array}{ccccc} RF(G) & \xrightarrow{\overline{\alpha}_G} & \mathbb{K}F(BG) & \xrightarrow{\beta_{BG}} & KF(BG) \\ & & {\scriptstyle \bar{r}_G}\downarrow{\scriptstyle \cong} & & \uparrow{\scriptstyle \cong} \\ & & RF_{\mathcal{P}}(G) & \longrightarrow & RF(G)^{\wedge}. \end{array}$$

Here, $\mathrm{Ker}(\overline{\alpha}_G) \subseteq RF(G)$ is the subgroup of elements whose characters vanish on all connected components of G of prime power order in $\pi_0(G)$. Also, $\overline{\alpha}_G$ is surjective if G is finite or if $\pi_0(G)$ has prime power order.

Proof. The diagrams commute by construction, $\bar{r}_G$ is an isomorphism by Theorem 1.2, and $KF(BG) \cong RF(G)^{\wedge}$ by [AS, Theorems 2.1 & 7.1].

Since β_{BG} is injective, $\mathrm{Ker}(\overline{\alpha}_G) = \mathrm{Ker}\big[\mathrm{rs}_G : RF(G) \to RF_{\mathcal{P}}(G)\big]$, and this by definition is the set of elements whose characters vanish on the union of all p-toral subgroups (for all primes p) in G. Clearly, each p-toral subgroup is contained in the union of connected components of G of prime power order in $\pi_0(G)$. Conversely, assume that $x \in G$ is contained in a connected component of p-power order for some prime p, and let S be the subgroup generated by x and a maximal torus in the centralizer $C_G(x)$. This is a Cartan subgroup as defined by Segal [Seg, Section 1]. Let S_p be the maximal p-toral subgroup of S, choose $x' \in S_p$ in the same component of G as x, and let $S' \supseteq S_p$ be generated by x' and a maximal torus of $C_G(x')$. Then S' is p-toral and a Cartan subgroup, and x is conjugate to an element of S' by [Seg, Proposition 1.4]. This proves the description of $\mathrm{Ker}(\overline{\alpha}_G)$.

Adams, in [Ad, Theorems 1.8 & 1.10], showed (in the complex case) that $\beta_{BG}\overline{\alpha}_G(R(G)) = \beta_{BG}(\mathbb{K}(BG))$ whenever G is finite or $\pi_0(G)$ has prime power order; and the surjectivity of $\overline{\alpha}_G$ then follows from the injectivity of β_{BG}. Another proof of the surjectivity of $\overline{\alpha}_G$ for such G, for real as well as complex bundles, is given in [O3, Propositions 3.2 and 3.4]. Other conditions for $\overline{\alpha}_G^{\mathbb{C}}$ to be surjective are given in [O3, Corollary 3.11]. □

In general, if $\dim(G) > 0$ and $\pi_0(G)$ does not have prime power order, then the maps $\overline{\alpha}_G$ need not be surjective. One example of this was constructed by Adams [Ad, Example 1.4]: for the group $G = (S^1 \times_{C_2} Q(8)) \times C_3$, he constructed a 2-dimensional complex vector bundle $\xi{\downarrow}BG$ such that $[\xi] \notin \overline{\alpha}_G(R(G))$. The cokernel of $\overline{\alpha}_G^{\mathbb{C}}$ for arbitrary G is described in [O3, Lemma 3.8 and Theorem 3.9].

Note that $\overline{\alpha}_G : RF(G) \to \mathbb{K}F(BG)$ is surjective if and only if bundles over BG have the following property: for each $\xi{\downarrow}BG$ there exist G-representations V, V' such that $\xi \oplus (EG\times_G V') \cong (EG\times_G V)$.

We now adopt the notation of Adams in [Ad], and let $FF(X) \subseteq K(X)$ (for any space X) denote the subgroup of "formally finite" elements; i.e., the subgroup generated by those elements $x \in K(X)$ such that $\lambda^n(x) = 0$ for n sufficiently large. Similarly, $FFO(X) \subseteq KO(X)$ will denote the subgroup of formally finite elements in the real K-theory of X.

Corollary 1.11. *For any G,*

$$\beta_{BG}(\mathbb{K}(BG)) = FF(BG) \qquad \text{and} \qquad \beta_{BG}(\mathbb{KO}(BG)) = FFO(BG).$$

Proof. For each prime p, let $G_p \subseteq G$ be a subgroup of finite index such that $\pi_0(G_p)$ is a Sylow p-subgroup of $\pi_0(G)$. By [Ad, Theorem 1.11], $FF(BG)$ is the set of those elements of $K(BG)$ whose restriction to $K(BG_p)$ (for any prime p) lies in the image of $R(G_p)$. By Theorem 1.2 (and the definition of $R_{\mathcal{P}}(G)$), $\beta_{BG}(\mathbb{K}(BG))$ is the set of elements of $K(BG)$ whose restriction to $K(BN_p(T))$ (for any prime p) lies in the image of $R(N_p(T))$. Thus, $\beta_{BG}(\mathbb{K}(BG)) \supseteq FF(BG)$; and the opposite inclusion follows immediately from the definitions.

To prove the corresponding result in the orthogonal case, the first step is to show that $KO(BG)$ is torsion free. This follows exactly the same lines as the proof in [Ad, Lemma 1.12] that $K(BG)$ is torsion free (by showing that $KO(BG)$ is detected by p-adic characters). Hence, since the composite

$$KO(BG) \xrightarrow{\mathbb{C}\otimes_{\mathbb{R}}} K(BG) \xrightarrow{\text{forget}} KO(BG)$$

is multiplication by 2, we can regard $KO(BG)$ as a subgroup of $K(BG)$. It is then clear that $FFO(BG) = FF(BG) \cap KO(BG)$. So given any element $x \in FFO(BG)$, we must show that $x|BP \in \mathrm{Im}(RO(P))$ for any p-toral subgroup $P \subseteq G$; and we know that $x|BP \in \mathrm{Im}(R(P)) \cap KO(BP)$. Also, any representation of a p-toral subgroup P whose restriction to all finite p-subgroups of P comes from real representations is itself a real representation. Thus, it remains only to show, for any finite p-group P, that the square

$$\begin{array}{ccc} RO(P) & \longrightarrow & KO(BP) \cong RO(P)\widehat{} \\ \downarrow & & \downarrow \\ R(P) & \longrightarrow & K(BP) \cong R(P)\widehat{} \end{array}$$

is a pullback square. And this follows since the only torsion in $R(P)/RO(P)$ is p-torsion (and only when $p = 2$); and since the I-adic completions are the same in this case as the p-adic completions of the augmentation ideals (cf. [AT, Proposition III.1.1]). $\square$

The groups $R_{\mathcal{P}}(G)$ and $RO_{\mathcal{P}}(G)$ have been defined rather abstractly. We note here a couple other simple characterizations of these groups. As before, let $T \subseteq G$

be a maximal torus, set $W_G = N(T)/T$, and for each prime $p\big||W_G|$ let $N_p(T)$ be the extension of T by a Sylow p-subgroup of W_G. Let $RF(N_p(T))^{G\text{-inv}}$ be the set of elements $v \in RF(N_p(T))$ such that $\chi_v(g) = \chi_v(g')$ if $g, g' \in N_p(T)$ are conjugate in G (the "G-invariant elements").

Proposition 1.12. *Set $w = |W_G|$. For $F = \mathbb{C}$ or $\mathbb{R}$, $RF_{\mathcal{P}}(G)$ sits in a pullback square*

$$\begin{array}{ccc} RF_{\mathcal{P}}(G) & \xrightarrow{\text{restr}} & \prod_{p|w} RF(N_p(T))^{G\text{-inv}} \\ {\scriptstyle\text{restr}}\big\downarrow & & {\scriptstyle\text{restr}}\big\downarrow \\ R(T) & \xrightarrow{\text{diag}} & \prod_{p|w} R(T). \end{array}$$

Proof. By [JMO, Lemma A.1], every p-toral subgroup of G is conjugate to a subgroup of $N_p(T)$. Hence the projection

$$\varprojlim_{P\ p\text{-toral}} RF(P) \longrightarrow RF(N_p(T))$$

is injective. Its image clearly contains $RF(N_p(T))^{G\text{-inv}}$, and is equal to this group by [JMO3, Proposition 1.6] (and the fact that elements of p-power order are dense in $N_p(T)$).

The formula for $RF_{\mathcal{P}}(G)$ now follows upon combining these inverse limits for all p. □

For another description of $RF_{\mathcal{P}}(G)$, let $G_{\mathcal{P}}$ denote the union of the connected components of G of prime power order in $\pi_0(G)$, and let $\text{Cl}(G_{\mathcal{P}})$ denote the space of continuous functions $G_{\mathcal{P}} \to \mathbb{C}$ which are constant on conjugacy classes. As seen in the proof of Corollary 1.10, $G_{\mathcal{P}}$ is the union of the p-toral subgroups of G (taken over all p). So given any element $\big(v_P\big)_{P\in\mathcal{S}_{\mathcal{P}}(G)} \in RF_{\mathcal{P}}(G)$, the characters of the v_P can be combined to give a conjugation invariant function $\chi : G_{\mathcal{P}} \to \mathbb{C}$, which is not hard to show is continuous. Thus,

$$RF_{\mathcal{P}}(G) \cong \Big\{ f \in \text{Cl}(G_{\mathcal{P}}) \,\Big|\, f|P \text{ is a character of } P, \text{ all } p\text{-toral } P \subseteq G, \text{ all } p \Big\}.$$

For more details, see [O3, Lemma 3.1].

If G is finite, then the relation between $\mathbb{K}(BG)$ and $K(BG)$ can be made even more explicit. For each $p\big||G|$, let r_p denote the number of conjugacy classes of elements $g{\in}G{\smallsetminus}1$ of p-power order. Then by Corollary 1.10 or Proposition 1.12,

$$\mathbb{K}(BG) \cong R(G)/\operatorname{Ker}(\overline{\alpha}_G) \cong \mathbb{Z}\times \prod_{p\,||G|} \mathbb{Z}^{r_p}.$$

In contrast,

$$K(BG) \cong R(G)\widehat{\ } \cong \mathbb{Z}\times \prod_{p\,||G|} (\widehat{\mathbb{Z}}_p)^{r_p} :$$

this follows from [Ja, proof of Theorem 2.2].

It is not hard to construct examples to show that the monoid $\mathrm{Vect}^{\mathbb{C}}(BG)$ is very far from having any general cancellation property. To finish this section, we note that in contrast, a complex bundle $\xi{\downarrow}BG$ splits off a trivial summand if and only if $r([\xi]) \in \varprojlim_{P}(\mathrm{Rep}^{\mathbb{C}}(P))$ does so; and that this splitting (if it exists) is unique up to isomorphism. Let ϵ denote the 1-dimensional trivial bundle over BG.

Proposition 1.13. *Two complex vector bundles $\xi_1, \xi_2{\downarrow}BG$ are isomorphic if $\xi_1{\oplus}\epsilon \cong \xi_2{\oplus}\epsilon$. A complex vector bundle $\xi{\downarrow}BG$ splits as a sum $\xi \cong \xi'{\oplus}\epsilon$ if for each prime p and each p-toral subgroup $P \subseteq G$, the P-representation V_P for which $\xi|BP \cong (EP{\times_P}V_P)$ contains a trivial summand.*

Proof. Let $\mathcal{S}_{\mathcal{P}}(G)$ denote the set of p-toral subgroups of G (for all p). The rough idea of the proof is to show, for any

$$X = \big(V_P\big)_{P\in\mathcal{S}_{\mathcal{P}}(G)} \in \varprojlim_{\mathcal{S}_{\mathcal{P}}(G)} \mathrm{Rep}^{\mathbb{C}}(-),$$

that the obstructions (from Proposition 1.4) to the existence and uniqueness of a map $f : BG \to BU(n,F)$ which "realizes X" (in the sense that $f|BP \simeq B\rho_{V_P}$ for each P) are the same as those for a map which realizes $X{\oplus}\epsilon$.

Fix $X = \big(V_P\big)_{P\in\mathcal{S}_{\mathcal{P}}(G)}$ as above. For each prime p, consider the functors

$$\Pi_i^{(X)}, \Pi_i^{(X\oplus\epsilon)} : \mathcal{R}_p(G) \longrightarrow \widehat{\mathbb{Z}}_p\text{-}\mathsf{mod};$$

where

$$\Pi_i^{(X)}(G/P) = \pi_i\Big(\mathrm{map}(EG/P, B\mathrm{Aut}(V_P)\hat{_p}), B\rho_{V_P}\Big) \cong \pi_i(B\mathrm{Aut}_P(V_P))\hat{_p}$$

(Theorem 1.1), and similarly for $\Pi_i^{(X\oplus\epsilon)}$. For each P, the automorphism groups $\mathrm{Aut}_P(V_P)$ and $\mathrm{Aut}_P(V_P{\oplus}\mathbb{C})$ are both products of unitary groups, and differ only in the factor corresponding to the trivial summand. In particular,

$$\Pi_1^{(X)} = \Pi_1^{(X\oplus\epsilon)} = 0. \tag{1}$$

By [JMO, Lemma 5.4 & Proposition 5.5], for any functor $F : \mathcal{R}_p(G) \to \widehat{\mathbb{Z}}_p$-$\mathsf{mod}$ such that $N(P)/P$ acts trivially on each $F(G/P)$, $\varprojlim^j_{\mathcal{R}_p(G)}(F) = 0$ for all $j > 0$. In particular, this applies to the kernel and cokernel of the maps $\Pi_i^{(X)}(G/P) \to \Pi_i^{(X\oplus\epsilon)}(G/P)$; and so

$$\varprojlim{}^j\big(\mathrm{Ker}[\Pi_i^{(X)} \to \Pi_i^{(X\oplus\epsilon)}]\big) = 0 = \varprojlim{}^j\big(\mathrm{Coker}[\Pi_i^{(X)} \to \Pi_i^{(X\oplus\epsilon)}]\big)$$

for all $j > 0$. It follows that

$$\varprojlim{}^j\Pi_i^{(X)} \cong \varprojlim{}^j\Pi_i^{(X\oplus\epsilon)} \;\; (\text{all } j \geq 2) \qquad \text{and} \qquad \varprojlim{}^1\Pi_i^{(X)} \twoheadrightarrow \varprojlim{}^1\Pi_i^{(X\oplus\epsilon)}. \tag{2}$$

Assume now that $f, g : BG \to BU(n)$ are maps such that $f \oplus B\epsilon \simeq g \oplus B\epsilon$. Set

$$X = \big(V_P\big)_{P \in \mathcal{S}_\mathcal{P}(G)} \in \varprojlim_{\mathcal{S}_\mathcal{P}(G)} \operatorname{Rep}^\mathbb{C}(-),$$

where the V_P are such that $f|BP \simeq g|BP \simeq B\rho_{V_P}$. By Proposition 1.3, it will suffice to show that $f_p^\wedge \simeq g_p^\wedge$ for each prime p. Fix p, and fix a homotopy $\overline{F} : BG \times I \to BU(n+1)_p^\wedge$. We will construct a homotopy $F : BG \times I \to BU(n)_p^\wedge$, and simultaneously a homotopy F' between $F \oplus B\epsilon$ and $\overline{F}$ which fixes the "edges" $f_p^\wedge$ and $g_p^\wedge$. Since

$$[BG, BU(n)_p^\wedge] \cong \Big[\underset{G/P \in \mathcal{R}_p(G)}{\underrightarrow{\operatorname{hocolim}}} \big(EG/P\big), BU(n)_p^\wedge\Big]$$

by Theorem 1.2 (and similarly for $[BG, BU(n+1)_p^\wedge]$), the obstructions to constructing F lie in $\varprojlim^i_{\mathcal{R}_p(G)} \Pi_i^{(X)}$ for $i \geq 1$ (Proposition 1.4); and the obstructions to constructing the homotopy F' between $F \oplus B\epsilon$ and $\overline{F}$ lie in $\varprojlim^i_{\mathcal{R}_p(G)} \Pi_{i+1}^{(X \oplus \epsilon)}$ for $i \geq 0$. At each stage in constructing F (i.e., on each successive skeleton of $\underrightarrow{\operatorname{hocolim}}\big(EG/P\big)$), the first obstruction vanishes by (1) or (2) and the existence of $\overline{F}$; and the second vanishes since $\varprojlim^i \Pi_{i+1}^{(X)}$ surjects onto $\varprojlim^i \Pi_{i+1}^{(X \oplus \epsilon)}$ by (1) or (2) again.

The proof of the second statement (that a bundle over BG can be destabilized under certain assumptions) is similar. In this case, we need to know that $\varprojlim^j \Pi_i^{(X)} \cong \varprojlim^j \Pi_i^{(X \oplus \epsilon)}$ whenever $j = i+1$ or $j = i$, and this follows from (1) or (2) again. □

2. The K-theory space of BG

Again, throughout this section, we fix a compact Lie group G with maximal torus T and Weyl group $W_G = N(T)/T$, and let $F = \mathbb{C}$ or $\mathbb{R}$. For any space X, we write

$$\mathfrak{Vect}_n^F(X) = \operatorname{map}(X, BU(n, F))$$

for each $n \geq 0$, and set

$$\mathfrak{Vect}^F(X) = \coprod_{n=0}^{\infty} \mathfrak{Vect}_n^F(X) \cong \operatorname{map}\Big(X, \coprod_{n=0}^{\infty} BU(n, F)\Big).$$

We think of $\mathfrak{Vect}^F(X)$ as the "space of F-vector bundles over X".

Since $\coprod_{n=0}^{\infty} BU(n, F)$ is a topological monoid and commutative up to homotopy (via conjugation by an appropriate matrix $\left(\begin{smallmatrix} 0 & \pm I_m \\ \pm I_n & 0 \end{smallmatrix}\right)$ in $U(n+m)$ or $SO(n+m)$), $\mathfrak{Vect}^F(X)$ is also a homotopy commutative topological monoid. It thus has a classifying space $B\mathfrak{Vect}^F(X)$. So we can define the "K-theory space" of X by setting

$$\mathfrak{K}^F(X) \overset{\text{def}}{=} \Omega B\big(\mathfrak{Vect}^F(X)\big):$$

the topological group completion of $\mathfrak{Vect}^F(X)$ (cf. [MS]). By the group completion theorem of McDuff & Segal [MS], $\pi_0(\mathfrak{K}^F(X)) \cong \mathbb{K}F(X)$ for any X, and $\mathfrak{K}^F(X) \simeq \mathrm{map}(X, \mathbb{Z}\times BU(\infty,F))$ if X is a finite complex.

We now want to describe the individual components of the group-like topological monoids $\mathfrak{K}^F(BG)$, by comparing them with the direct limits of the mapping spaces $\mathrm{map}(BG, B\mathrm{Aut}(V))_{B\rho_V}$ taken over all $V \in \mathrm{Rep}^F(G)$. We will also have to consider the spaces of maps to localizations and completions of the $BU(n,F)$, and so the following notation will be useful.

Fix a G-space X. For any $V \in \mathrm{Rep}^F(G)$, set

$$\mathfrak{Vect}^F_G(X)_V \overset{\text{def}}{=} \mathrm{map}(EG\times_G X, B\mathrm{Aut}(V))_{B\rho_V}$$

(the connected component of the composite $EG\times_G X \xrightarrow{(X\to *)} BG \xrightarrow{B\rho_V} B\mathrm{Aut}(V)$),

$$\mathfrak{Vect}^F_G(X;\mathbb{Q})_V \overset{\text{def}}{=} \mathrm{map}(EG\times_G X, B\mathrm{Aut}(V)_{\mathbb{Q}})_{B\rho_V},$$

$$\mathfrak{Vect}^F_G(X;\widehat{\mathbb{Z}}_p)_V \overset{\text{def}}{=} \mathrm{map}(EG\times_G X, B\mathrm{Aut}(V)^\wedge_p)_{B\rho_V} \qquad (p \text{ any prime}),$$

$$\mathfrak{Vect}^F_G(X;\widehat{\mathbb{Z}})_V \overset{\text{def}}{=} \mathrm{map}(EG\times_G X, B\mathrm{Aut}(V)^\wedge)_{B\rho_V} \cong \prod_p \mathfrak{Vect}^F_G(X;\widehat{\mathbb{Z}}_p)_V$$

(where $B\mathrm{Aut}(V)^\wedge = \prod_p B\mathrm{Aut}(V)^\wedge_p$), and

$$\mathfrak{Vect}^F_G(X;\widehat{\mathbb{Q}})_V \overset{\text{def}}{=} \mathrm{map}\big(EG\times_G X, (B\mathrm{Aut}(V)^\wedge)_{\mathbb{Q}}\big)_{B\rho_V}.$$

Finally, define

$$\mathfrak{Vect}^F_G(X;-)_\infty \overset{\text{def}}{=} \underset{V\in\mathrm{Rep}^F(G)}{\underrightarrow{\mathrm{hocolim}}} \big(\mathfrak{Vect}^F_G(X;-)_V\big),$$

where (as in Proposition 1.5) the limit over $V \in \mathrm{Rep}^F(G)$ can be made more precise by taking it over a cofinal sequence $V_1 \subseteq V_2 \subseteq V_3 \subseteq \dots$ of representations. We will see in Proposition 2.3 that $\mathfrak{K}^F(BG)_0 \simeq \mathfrak{Vect}^F_G(\mathrm{pt})_\infty$. We first check that the obvious arithmetic pullback square holds for these spaces.

Proposition 2.1. *The space $\mathfrak{Vect}^F_G(\mathrm{pt})_\infty$ sits in a homotopy pullback square*

$$\begin{array}{ccc} \mathfrak{Vect}^F_G(\mathrm{pt})_\infty & \longrightarrow & \mathfrak{Vect}^F_G(\mathrm{pt};\widehat{\mathbb{Z}})_\infty \\ \big\downarrow & & \big\downarrow \\ \mathfrak{Vect}^F_G(\mathrm{pt};\mathbb{Q})_\infty & \longrightarrow & \mathfrak{Vect}^F_G(\mathrm{pt};\widehat{\mathbb{Q}})_\infty. \end{array} \tag{1}$$

Proof. We prove this in the orthogonal case. The unitary case is similar, but slightly simpler since $BU(n)$ is simply connected for each n.

If n is odd, then $BO(n) \simeq BSO(n) \times B\{\pm I\}$. Hence, the arithmetic pullback square for the simply connected space $BSO(n)$ (cf. [BK, VI.8.1]) induces a homotopy pullback square

$$\begin{array}{ccc} \operatorname{map}\bigl(BG, BO(n)\bigr) & \longrightarrow & \operatorname{map}\bigl(BG, BO(n)^{\wedge}\bigr) \\ \big\downarrow & & \big\downarrow \\ \operatorname{map}\bigl(BG, BO(n)_{\mathbb{Q}}\bigr) & \longrightarrow & \operatorname{map}\bigl(BG, (BO(n)^{\wedge})_{\mathbb{Q}}\bigr) \end{array} \tag{2}$$

of mapping spaces. Also, the top row in (2) induces an injection on $\pi_0(-)$ (cf. [JMO, Theorem 3.1]). Hence for any n-dimensional orthogonal G-representation V, (2) restricts to a homotopy pullback square involving the components of $B\rho_V$. Upon taking the homotopy direct limit over all (odd dimensional) representations, and using the exactness of direct limits (and the 5-lemma), the square in (1) is seen to be is a homotopy pullback. □

We next analyze the term $\mathfrak{Vect}_G^F(\mathrm{pt};\widehat{\mathbb{Z}})_\infty$ in the above pullback square. As in the proof of Theorem 1.8, this is done using the decomposition of BG at each prime p, switching direct and inverse limits, and using Proposition 1.5 to show that the appropriate higher inverse limits all vanish.

Proposition 2.2. *Set $w = |W_G|$. For each prime p,*

$$\mathfrak{Vect}_G^F(\mathrm{pt};\widehat{\mathbb{Z}}_p)_\infty \simeq \underset{\mathcal{R}_p(G)}{\underleftarrow{\operatorname{holim}}}\, \mathfrak{Vect}_G^F(-;\widehat{\mathbb{Z}}_p)_\infty. \tag{1}$$

And for each $i > 0$,

$$\pi_i\bigl(\mathfrak{Vect}_G^F(\mathrm{pt};\widehat{\mathbb{Z}}_p)_\infty\bigr) \cong \underset{\mathcal{R}_p(G)}{\underleftarrow{\lim}}\ \pi_i\bigl(\mathfrak{Vect}_G^F(-;\widehat{\mathbb{Z}}_p)_\infty\bigr) \cong \underset{\mathcal{R}_p(G)}{\underleftarrow{\lim}}\ \bigl(\widehat{\mathbb{Z}}_p \otimes KF_G^{-i}(-)\bigr) \tag{2}$$

and

$$\pi_i\bigl(\mathfrak{Vect}_G^F(\mathrm{pt};\widehat{\mathbb{Z}})_\infty\bigr) \cong \prod_{p|w} \pi_i\bigl(\mathfrak{Vect}_G^F(\mathrm{pt};\widehat{\mathbb{Z}}_p)_\infty\bigr) \times \Bigl[\Bigl(\prod_{p\nmid w} \widehat{\mathbb{Z}}_p\Bigr) \otimes KF_G^{-i}(G/T)^{W_G}\Bigr]. \tag{3}$$

Proof. We prove this in the orthogonal case. The unitary case is similar, but slightly simpler since $BU(n)$ is simply connected for each n.

By Theorem 1.2 (the approximation at p of BG as a homotopy direct limit),

$$\mathfrak{Vect}_G^{\mathbb{R}}(\mathrm{pt};\widehat{\mathbb{Z}}_p)_V \simeq \underset{\mathcal{R}_p(G)}{\underleftarrow{\operatorname{holim}}} \Bigl(\mathfrak{Vect}_G^{\mathbb{R}}(-;\widehat{\mathbb{Z}}_p)_V\Bigr)$$

for each V. By Lemma 1.6 and Proposition 1.5, for each $i, j > 0$,

$$\underset{V\in \mathrm{Rep}^{\mathbb{R}}(G)}{\underrightarrow{\lim}} \Bigl(\underset{\mathcal{R}_p(G)}{\underleftarrow{\lim}}^j\ \pi_i\bigl(\mathfrak{Vect}_G^{\mathbb{R}}(-;\widehat{\mathbb{Z}}_p)_V\bigr)\Bigr) \cong \underset{\mathcal{R}_p(G)}{\underleftarrow{\lim}}^j\ \pi_i\bigl(\mathfrak{Vect}_G^{\mathbb{R}}(-;\widehat{\mathbb{Z}}_p)_\infty\bigr) = 0. \tag{4}$$

By Lemma 1.7, there is an integer $d > 0$ such that $\varprojlim^j_{\mathcal{R}_p(G)} \pi_i\big(\mathfrak{Vect}^{\mathbb{R}}_G(-;\widehat{\mathbb{Z}}_p)_V\big) = 0$ for all V and all $j > d$. The obstruction theory of Wojtkowiak (Proposition 1.4) now applies to prove that

$$\begin{aligned}
\pi_i\big(\mathfrak{Vect}^{\mathbb{R}}_G(\mathrm{pt};\widehat{\mathbb{Z}}_p)_\infty\big) &\cong \varinjlim_{V\in\mathrm{Rep}^{\mathbb{R}}(G)} \pi_i\big(\mathfrak{Vect}^{\mathbb{R}}_G(\mathrm{pt};\widehat{\mathbb{Z}}_p)_V\big) \\
&\cong \varinjlim_{V\in\mathrm{Rep}^{\mathbb{R}}(G)} \Big(\varprojlim_{\mathcal{R}_p(G)} \pi_i\big(\mathfrak{Vect}^{\mathbb{R}}_G(-;\widehat{\mathbb{Z}}_p)_V\big)\Big) \\
&\cong \varprojlim_{\mathcal{R}_p(G)} \Big(\varinjlim_{V\in\mathrm{Rep}^{\mathbb{R}}(G)} \pi_i\big(\mathfrak{Vect}^{\mathbb{R}}_G(-;\widehat{\mathbb{Z}}_p)_V\big)\Big) \\
&\cong \varprojlim_{\mathcal{R}_p(G)} \pi_i\big(\mathfrak{Vect}^{\mathbb{R}}_G(-;\widehat{\mathbb{Z}}_p)_\infty\big).
\end{aligned}$$

This is the first isomorphism in (2), and the second follows from Proposition 1.5.

By (4), again, the space $\operatorname{holim}_{\longleftarrow \mathcal{R}_p(G)} \mathfrak{Vect}^{\mathbb{R}}_G(-;\widehat{\mathbb{Z}}_p)_\infty$ is connected, and

$$\pi_i\Big(\underset{\mathcal{R}_p(G)}{\underleftarrow{\operatorname{holim}}}\, \mathfrak{Vect}^{\mathbb{R}}_G(-;\widehat{\mathbb{Z}}_p)_\infty\Big) \cong \varprojlim_{\mathcal{R}_p(G)} \pi_i\big(\mathfrak{Vect}^{\mathbb{R}}_G(-;\widehat{\mathbb{Z}}_p)_\infty\big).$$

Together with (2), this proves point (1).

It remains to check formula (3). For each $p\nmid w=|W_G|$, $\mathcal{R}_p(G)$ contains only the one orbit type G/T. Hence by (2),

$$\pi_i\big(\mathfrak{Vect}^{\mathbb{R}}_G(\mathrm{pt};\widehat{\mathbb{Z}}_p)_\infty\big) \cong \widehat{\mathbb{Z}}_p \otimes \big(KO^{-i}_G(G/T)^{W_G}\big)$$

in this case. Also, for each V (and each $p \nmid w$),

$$\pi_i\big(\mathfrak{Vect}^{\mathbb{R}}_G(\mathrm{pt};\widehat{\mathbb{Z}}_p)_V\big) \cong \pi_i\big(\mathrm{map}(BT, B\mathrm{Aut}(V)^{\wedge}_p)_{B\rho V}\big)^{W_G} \cong \Big[\widehat{\mathbb{Z}}_p \otimes \pi_i\big(B\mathrm{Aut}_T(V)\big)\Big]^{W_G}$$

is finitely generated; and so

$$\varinjlim_{V\in\mathrm{Rep}^{\mathbb{R}}(G)} \Big(\prod_{p\nmid w} \pi_i\big(\mathfrak{Vect}^{\mathbb{R}}_G(\mathrm{pt};\widehat{\mathbb{Z}}_p)_V\big)\Big) \cong \Big(\prod_{p\nmid w} \widehat{\mathbb{Z}}_p\Big) \otimes \Big(K^{-i}_G(G/T)^{W_G}\Big).$$

Since direct limits commute with *finite* products, it now follows that

$$\begin{aligned}
\pi_i\big(\mathfrak{Vect}^{\mathbb{R}}_G(\mathrm{pt};\widehat{\mathbb{Z}})_\infty\big) &\cong \varinjlim_{V\in\mathrm{Rep}^{\mathbb{R}}(G)} \Big(\prod_p \pi_i\big(\mathfrak{Vect}^{\mathbb{R}}_G(\mathrm{pt};\widehat{\mathbb{Z}}_p)_V\big)\Big) \\
&\cong \prod_{p|w} \Big(\pi_i\big(\mathfrak{Vect}^{\mathbb{R}}_G(\mathrm{pt};\widehat{\mathbb{Z}}_p)_\infty\big)\Big) \times \Big(\Big(\prod_{p\nmid w} \widehat{\mathbb{Z}}_p\Big) \otimes K^{-i}_G(G/T)^{W_G}\Big);
\end{aligned}$$

and this finishes the proof. $\square$

It remains to describe the identity component of $\mathfrak{K}^F(BG)$ as a homotopy pullback of these spaces $\mathfrak{Vect}^F_G(\mathrm{pt},-)_\infty$.

Proposition 2.3. *The natural map*

$$\mathfrak{Vect}^F_G(\mathrm{pt})_\infty \xrightarrow{\quad\simeq\quad} [\Omega B\mathfrak{Vect}^F_G(\mathrm{pt})]_0 = \mathfrak{K}^F(BG)_0 \tag{1}$$

is a homotopy equivalence.

Proof. We prove this in the orthogonal case. The unitary case is similar, but slightly simpler since $BU(n)$ is simply connected for each n.

By the group completion theorem in [MS] (and since bundles coming from representations are cofinal by Theorem 1.8), the map in (1) is an equivalence of homology with any twisted coefficients. In particular, it induces a surjection on $\pi_1(-)$, whose kernel is perfect. But the pullback square of Proposition 2.1, together with points (2) and (3) in Proposition 2.2 (and the fact that $BO(n,F)_\mathbb{Q}$ is a product of Eilenberg-Maclane spaces when $F = \mathbb{C}$ or n is odd), show that $\pi_1(\mathfrak{Vect}^F_G(\mathrm{pt})_\infty)$ is solvable. Hence $\pi_1(\mathfrak{Vect}^F_G(\mathrm{pt})_\infty)$ is abelian, and so (1) is a homotopy equivalence. □

The pullback square of Proposition 2.1 will first be applied to study the components of $\mathfrak{K}^F(BG)$ when G is finite.

Proposition 2.4. *If G is finite, then the natural map*

$$\mathfrak{K}^F(BG)_0 \longrightarrow \mathit{map}(BG, BU(\infty, F))_0$$

is a homotopy equivalence.

Proof. We prove this in the unitary case; the orthogonal case is identical. Upon comparing the pullback square in Proposition 2.1 with the arithmetic pullback square for $\mathrm{map}(BG, BU)$, we see that it will suffice to prove that

$$\mathfrak{Vect}^\mathbb{C}_G(\mathrm{pt};\widehat{\mathbb{Z}})_\infty \simeq \mathrm{map}(BG, BU\widehat{\ })_0, \qquad \mathfrak{Vect}^\mathbb{C}_G(\mathrm{pt};\mathbb{Q})_\infty \simeq \mathrm{map}(BG, BU_\mathbb{Q})_0,$$

and similarly for $\mathfrak{Vect}^\mathbb{C}_G(\mathrm{pt};\widehat{\mathbb{Q}})_\infty$. This is clear for the $\mathbb{Q}$-local terms, since BG is rationally a point.

Since G is finite, $K^{-i}_G(G/T) = K^{-i}(T) = K^{-i}(\mathrm{pt})$ is finitely generated for all i, and hence $\mathfrak{Vect}^\mathbb{C}_G(\mathrm{pt};\widehat{\mathbb{Z}})_\infty$ is the product of the $\mathfrak{Vect}^\mathbb{C}_G(\mathrm{pt};\widehat{\mathbb{Z}}_p)_\infty$ by Proposition 2.2(3). So it remains only to show that

$$\mathfrak{Vect}^\mathbb{C}_G(\mathrm{pt};\widehat{\mathbb{Z}}_p)_\infty \longrightarrow \mathrm{map}(BG, BU\hat{_p})_0 \tag{1}$$

is a homotopy equivalence for each p. By Proposition 2.2(1) and Theorem 1.2, it will suffice to show this when G is a p-group.

By the Atiyah-Hirzebruch spectral sequence for K-theory (and the fact that $K^{-i}(BG^{(m)})$ is finitely generated for each i and each finite skeleton $BG^{(m)}$),

$$\pi_i\big(\mathrm{map}(BG, BU\hat{_p})_0\big) \cong \big(K^{-i}(BG)\big)\hat{_p}.$$

Also, by Proposition 2.2(2),

$$\pi_i\big(\mathfrak{Vect}_G^{\mathbb{C}}(\mathrm{pt};\widehat{\mathbb{Z}}_p)_\infty\big) \cong \widehat{\mathbb{Z}}_p \otimes K_G^{-i}(\mathrm{pt}).$$

And by the Atiyah-Segal theorem [AS], $K^{-i}(BG) \cong K_G^{-i}(\mathrm{pt})\widehat{\ }$: the completion with respect to the augmentation ideal $IR(G) \subseteq R(G)$.

It remains only to check that $\widehat{\mathbb{Z}}_p \otimes R(G) = R(G)\hat{_p}$ is complete with respect to its augmentation ideal; or equivalently that $IR(G)^m \subseteq pIR(G)$ for some m. And this is shown in [AT, Proposition III.1.1]. (The corresponding result for $IRO(G)$ is not shown directly in [AT], but it follows by the same proof.) □

We now turn to the case $\dim(G) > 0$, and consider only the complex case. As will be seen, the components of $\mathfrak{K}^{\mathbb{C}}(BG)$ are in this case very different from the components of $\mathrm{map}(BG, BU)$.

As in Proposition 1.12, for each prime p, we let $N_p(T)$ be the extension of T by some Sylow p-subgroup of $W_G = N(T)/T$, and let $R(N_p(T))^{G\text{-inv}} \subseteq R(N_p(T))$ be the subgroup of elements whose characters are constant on G-conjugacy classes. The arithmetic pullback square of Proposition 2.1 (and 2.3) for $\pi_{2i}\big(\mathfrak{K}^{\mathbb{C}}(BG)\big)$ can be simplified in a way which parallels that for $R_{\mathcal{P}}(G) \cong \mathbb{K}(BG)$ in Proposition 1.12.

Proposition 2.5. *Set $w = |W_G|$. For each $i > 0$, $\pi_{2i}\big(\mathfrak{K}^{\mathbb{C}}(BG)\big)$ sits in a pullback square*

$$\begin{array}{ccc} \pi_{2i}\big(\mathfrak{K}^{\mathbb{C}}(BG)\big) & \longrightarrow & \prod_{p|w} \widehat{\mathbb{Z}}_p \otimes RF(N_p(T))^{G\text{-inv}} \\ \downarrow & & \downarrow \scriptstyle\text{restr} \\ R(T) & \xrightarrow{\text{diag}} & \prod_{p|w} \widehat{\mathbb{Z}}_p \otimes R(T). \end{array}$$

Also, $\pi_{2i-1}\big(\mathfrak{K}^{\mathbb{C}}(BG)\big)$ is an infinite dimensional rational vector space if $\dim(G) > 0$.

Proof. The proof will be carried out in three steps. The homotopy groups of $\mathfrak{Vect}_G^{\mathbb{C}}(\mathrm{pt};\widehat{\mathbb{Z}})_\infty$ and of $\mathfrak{Vect}_G^{\mathbb{C}}(\mathrm{pt};\mathbb{Q})_\infty$ will be computed in Steps 1 and 2, respectively. The homotopy groups of $\mathfrak{K}^{\mathbb{C}}(BG)$ will then be computed in Step 3, using the homotopy pullback square of Proposition 2.1.

Step 1 Set $w = |W_G|$ for short. By Proposition 2.2,

$$\pi_i\big(\mathfrak{Vect}_G^{\mathbb{C}}(\mathrm{pt};\widehat{\mathbb{Z}})_\infty\big) \simeq \prod_{p|w} \pi_i\big(\mathfrak{Vect}_G^{\mathbb{C}}(\mathrm{pt};\widehat{\mathbb{Z}}_p)_\infty\big) \times \left(\Big(\prod_{p\nmid w} \widehat{\mathbb{Z}}_p\Big) \otimes K_G^{-i}(G/T)^{W_G}\right);$$

where for each prime $p|w$,

$$\pi_i\big(\mathfrak{Vect}_G^{\mathbb{C}}(\mathrm{pt};\widehat{\mathbb{Z}}_p)_\infty\big) \cong \varprojlim_{\mathcal{R}_p(G)} \big(\widehat{\mathbb{Z}}_p \otimes K_G^{-i}(-)\big) \cong \begin{cases} \widehat{\mathbb{Z}}_p \otimes R(N_p(T))^{G\text{-inv}} & \text{if } i \text{ is even} \\ 0 & \text{if } i \text{ is odd} \end{cases}$$

by Proposition 1.5 (and the finiteness of $\mathcal{R}_p(G)$). In other words,

$$\pi_i\big(\mathfrak{Vect}_G^{\mathbb{C}}(\mathrm{pt};\widehat{\mathbb{Z}})_\infty\big) \cong \begin{cases} \Big[\prod_{p|w} \widehat{\mathbb{Z}}_p \otimes R(N_p(T))^{G\text{-inv}}\Big] \oplus \Big[\Big(\prod_{p\nmid w} \widehat{\mathbb{Z}}_p\Big) \otimes R(T)^W\Big] & \text{if } i \text{ is even} \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

Step 2 We now consider the homotopy groups $\pi_{2i}\big(\mathfrak{Vect}^{\mathbb{C}}_G(\mathrm{pt};\mathbb{Q})_\infty\big)$. For any $V\in\mathrm{Rep}(G)$ and any $\varphi\in\pi_{2i}(\mathfrak{Vect}^{\mathbb{C}}_G(\mathrm{pt};\mathbb{Q})_V)$, let $\widetilde{\varphi}: S^{2i}\times BG\to B\mathrm{Aut}(V)_{\mathbb{Q}}$ denote its adjoint map, and set

$$\Theta_i(\varphi)=[\widetilde{\varphi}]-[B\rho_V]\in\big[S^{2i}\times BG\,,\,BU_{\mathbb{Q}}\big].$$

Since $B\mathrm{Aut}(V)_{\mathbb{Q}}\xrightarrow{\prod c_j}\prod_{j=1}^{\dim(V)}K(\mathbb{Q},2j)$ is a homotopy equivalence, there is an isomorphism

$$\widetilde{\delta}_{i,V}:\pi_{2i}\big(\mathfrak{Vect}^{\mathbb{C}}_G(\mathrm{pt};\mathbb{Q})_V\big)\xrightarrow{\ \cong\ }H^*(BG;\mathbb{Q})^{\le 2(\dim(V)-i)},$$

defined via the formula $C(\widetilde{\varphi})=C(V)+\langle S^{2i}\rangle\cdot\widetilde{\delta}_{i,V}(\varphi)$. Here, $\langle S^{2i}\rangle\in H^{2i}(S^{2i})$ is the dual orientation class, and $C(-)$ denotes the total Chern class of a bundle or representation. Furthermore, $\Theta_i(\varphi)$ is invariant under stabilization by adding representations. So if we set

$$\delta_{i,G}(\varphi)=\widetilde{\delta}_{i,V}(\varphi)/C(V)\qquad\text{so that}\qquad C(\Theta_i(\varphi))=1+\langle S^{2i}\rangle\cdot\delta_{i,G}(\varphi),$$

then this defines an isomorphism

$$\begin{aligned}\delta_{i,G}:\pi_{2i}\big(\mathfrak{Vect}^{\mathbb{C}}_G(\mathrm{pt};\mathbb{Q})_\infty\big)\xrightarrow{\ \cong\ }&\bigcup_{V\in\mathrm{Rep}(G)}\frac{1}{C(V)}H^*(BG;\mathbb{Q})^{\le 2(\dim(V)-i)}\\ &=\bigcup_{V\in\mathrm{Rep}(G)}\frac{1}{C(V)}H^*(BG;\mathbb{Q})\overset{\mathrm{def}}{=}\Delta(G).\end{aligned}$$

(The degree restrictions disappear since the Chern class of any representation with trivial action is 1.) We regard $\Delta(G)$ as a subring of $\prod_{k=0}^{\infty}H^{2k}(BG;\mathbb{Q})\cong\pi_{2i}\big(\mathrm{map}(BG,BU_{\mathbb{Q}})\big)$. Note that $\Delta(G)\cong\Delta(T)^{W_G}$, since $H^*(BG;\mathbb{Q})\cong H^*(BT;\mathbb{Q})^{W_G}$, and the restrictions of G-representations are cofinal among T-representations. There is an analogous isomorphism of $\pi_{2i}\big(\mathfrak{Vect}^{\mathbb{C}}_G(\mathrm{pt};\widehat{\mathbb{Q}})_\infty\big)$ with $\widehat{\mathbb{Q}}\otimes_{\mathbb{Q}}\Delta(G)$.

We defined $\delta_{i,G}$ using the Chern class $C(\Theta_i(\varphi))=C(\widetilde{\varphi})/C(V)$, in order to more easily identify the image of $\pi_{2i}\big(\mathfrak{Vect}^{\mathbb{C}}_G(\mathrm{pt};\mathbb{Q})_\infty\big)$, but of course it is much easier in general to work with the Chern character. The relation between the two is quite simple in this case. Since $c_j(\Theta_i(\varphi))c_k(\Theta_i(\varphi))=0$ for all $j,k>0$,

$$\mathrm{ch}(\Theta_i(\varphi))=\sum_{k=1}^{\infty}\frac{s_k(c_1,c_2,c_3,\dots)}{k!}=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}kc_k(\Theta_i(\varphi))}{k!}=\langle S^{2i}\rangle\cdot E_i\big(\delta_{i,G}(\varphi)\big)\tag{1}$$

by Newton's formula (cf. [MSt, Problem 16-A]); where

$$E_i=\prod\nolimits_{k=0}^{\infty}\left(\frac{(-1)^{i+k-1}}{(i+k-1)!}\cdot\right):\prod_{k=0}^{\infty}H^{2k}(BG;\mathbb{Q})\longrightarrow\prod_{k=0}^{\infty}H^{2k}(BG;\mathbb{Q}).$$

It remains to describe $\pi_{2i}\big(\mathfrak{Vect}_G^{\mathbb{C}}(\mathrm{pt};\widehat{\mathbb{Z}})_\infty\big) \to \pi_{2i}\big(\mathfrak{Vect}_G^{\mathbb{C}}(\mathrm{pt};\widehat{\mathbb{Q}})_\infty\big)$ in terms of these identifications. Since $\Delta(G) = \Delta(T)^{W_G}$, it will suffice to describe the composite homomorphism

$$\gamma_i : R(T) \cong K_T^{-2i}(\mathrm{pt}) \xrightarrow[\cong]{\Phi_T} \pi_{2i}\big(\mathfrak{Vect}_T^{\mathbb{C}}(\mathrm{pt};\mathbb{Q})_\infty\big) \xrightarrow{\delta_{i,T}} \Delta(T),$$

where Φ_T is the isomorphism of Proposition 1.5. Let $\eta \in \widetilde{K}(S^2)$ be the generator with $\mathrm{ch}(\eta) = \langle S^2\rangle$. Then $\mathrm{ch}(\eta^{\wedge i}) = \langle S^2\rangle^{\otimes i} = \langle S^{2i}\rangle \in H^{2i}(S^{2i})$. For any T-representation V, let $[V]_{(i)} \in K_T^{-2i}(\mathrm{pt})$ be the element induced by V via Bott periodicity. Then $\Theta_i\big(\Phi_T([V]_{(i)})\big) = [\eta^{\wedge i}] \otimes [EG\otimes_G V]$ (in $K(S^{2i}\times BG)$), and this element has Chern character $\langle S^{2i}\rangle\cdot \mathrm{ch}(V)$. So by (1),

$$\gamma_i([V]) = \delta_{i,T}\big(\Phi_T([V]_{(i)})\big) = E_i^{-1}(\mathrm{ch}(V)).$$

In particular, if V is irreducible (so that $\mathrm{ch}(V) = \exp(c_1(V))$), this takes the form

$$\gamma_i([V]) = (-1)^{i-1}(i-1)!\cdot \frac{1}{C(V)^i}. \tag{2}$$

These elements are linearly independent, so $\gamma_i : \mathbb{Q}\otimes R(T) \to \Delta(T)$ is injective.

Step 3 Since the odd dimensional homotopy groups of $\mathfrak{Vect}_G^{\mathbb{C}}(\mathrm{pt};\mathbb{Q})_\infty$, $\mathfrak{Vect}_G^{\mathbb{C}}(\mathrm{pt};\widehat{\mathbb{Z}})_\infty$, and $\mathfrak{Vect}_G^{\mathbb{C}}(\mathrm{pt};\widehat{\mathbb{Q}})_\infty$ all vanish, the homotopy pullback square of Proposition 2.1 induces for each $i > 0$ an exact sequence

$$\begin{aligned} 0 \longrightarrow \pi_{2i}\big(\mathfrak{K}^{\mathbb{C}}(BG)\big) &\longrightarrow \pi_{2i}\big(\mathfrak{Vect}_G^{\mathbb{C}}(\mathrm{pt};\widehat{\mathbb{Z}})_\infty\big) \oplus \pi_{2i}\big(\mathfrak{Vect}_G^{\mathbb{C}}(\mathrm{pt};\mathbb{Q})_\infty\big) \\ &\longrightarrow \pi_{2i}\big(\mathfrak{Vect}_G^{\mathbb{C}}(\mathrm{pt};\widehat{\mathbb{Q}})_\infty\big) \longrightarrow \pi_{2i-1}\big(\mathfrak{K}^{\mathbb{C}}(BG)\big) \longrightarrow 0. \end{aligned}$$

After substituting the groups computed in Steps 1 and 2, this takes the form

$$\begin{aligned} 0 \longrightarrow \pi_{2i}\big(\mathfrak{K}^{\mathbb{C}}(BG)\big) \longrightarrow \Big[\prod_{p|w}\widehat{\mathbb{Z}}_p\otimes R(N_p(T))^{G\text{-inv}}\Big]\oplus\Big[\Big(\prod_{p\nmid w}\widehat{\mathbb{Z}}_p\Big)\otimes R(T)^{W_G}\Big]\oplus\Delta(T)^{W_G} \\ \xrightarrow{(\oplus\gamma_i,\mathrm{incl})} \widehat{\mathbb{Q}}\otimes_{\mathbb{Q}}\Delta(T)^{W_G} \longrightarrow \pi_{2i-1}(\mathfrak{K}^{\mathbb{C}}(BG)) \longrightarrow 0. \end{aligned} \tag{3}$$

The short exact sequence $0 \to \mathbb{Z} \to \widehat{\mathbb{Z}}\oplus\mathbb{Q} \to \widehat{\mathbb{Q}} \to 0$ is still exact after tensoring with the free $\mathbb{Z}$-module $R(T)^{W_G}$, and hence induces an exact sequence

$$\begin{aligned} 0 \longrightarrow R(T)^{W_G} \xrightarrow{(\mathrm{incl},\gamma_i)} \big(\widehat{\mathbb{Z}}\otimes R(T)^{W_G}\big)\oplus\Delta(T)^{W_G} \xrightarrow{\gamma_i-\mathrm{incl}} \widehat{\mathbb{Q}}\otimes_{\mathbb{Q}}\Delta(T)^{W_G} \\ \longrightarrow (\widehat{\mathbb{Q}}/\mathbb{Q})\otimes_{\mathbb{Q}}\big(\Delta(T)/\gamma_i(\mathbb{Q}\otimes R(T))\big)^{W_G} \longrightarrow 0. \end{aligned}$$

Upon comparing this with sequence (3), we see that

$$\pi_{2i-1}\big(\mathfrak{K}^{\mathbb{C}}(BG)\big) \cong (\widehat{\mathbb{Q}}/\mathbb{Q})\otimes_{\mathbb{Q}}\Big(\Delta(T)/\gamma_i(\mathbb{Q}\otimes R(T))\Big)^{W_G}; \tag{4}$$

and that the sequence

$$0 \longrightarrow \pi_{2i}\big(\mathfrak{K}^{\mathbb{C}}(BG)\big) \longrightarrow \Big(\prod_{p|w} \widehat{\mathbb{Z}}_p \otimes R(N_p(T))^{G\text{-inv}}\Big) \oplus R(T)^{W_G} \longrightarrow \prod_{p|w} \widehat{\mathbb{Z}}_p \otimes R(T)^{W_G} \longrightarrow 0$$

is exact. This proves the formula for $\pi_{2i}(\mathfrak{K}^{\mathbb{C}}(BG))$. If $\dim(G) > 0$ (and hence $T \neq 1$), then by (2), $1/C(V)^{i+1} \notin \gamma_i\big(\mathbb{Q} \otimes R(T)\big)$ for any T-representation V with nontrivial action; and so $\pi_{2i-1}(\mathfrak{K}^{\mathbb{C}}(BG))$ is an infinite dimensional $\mathbb{Q}$-vector space by (4). $\square$

For example, when $G = T$ is a positive dimensional torus, then for $i > 0$, $\pi_{2i}(\mathfrak{K}^{\mathbb{C}}(BT)_0) \cong R(T)$, and $\pi_{2i-1}(\mathfrak{K}^{\mathbb{C}}(BT)_0)$ is a nonzero $\mathbb{Q}$-vector space. In contrast, $\pi_{2i}(\mathrm{map}(BG, BU)_0) \cong R(T)^{\wedge}$, a power series ring; and $\pi_{2i-1}(\mathrm{map}(BG, BU)_0) = 0$. More generally, whenever G is connected, since $R(G) \cong R(T)^{W_G}$, Proposition 2.5 takes the form

$$\pi_{2i}\big(\mathfrak{K}^{\mathbb{C}}(BG)\big) \cong R(T)^{W_G} \oplus \bigoplus_{p|w} \Big(\widehat{\mathbb{Z}}_p \otimes \mathrm{Ker}\big[R(N_p(T)) \twoheadrightarrow R(T)^{W_G}\big]\Big). \qquad (w = |W_G|)$$

In the remarks after Theorem 1.1, we noted some counterexamples to Bott periodicity for the functor $\mathbb{K}(-)$, which in part came from the difference between $\pi_0(\mathfrak{K}^{\mathbb{C}}(BG))$ and $\pi_2(\mathfrak{K}^{\mathbb{C}}(BG))$. Proposition 2.5 shows a more subtle failure of Bott periodicity, in that the natural map

$$\pi_i\big(\mathfrak{K}^{\mathbb{C}}(BG)\big) \xrightarrow{\ \eta\wedge\ } \pi_{i+2}\big(\mathfrak{K}^{\mathbb{C}}(BG)\big),$$

induced by multiplication by a generator $\eta \in K^{-2}(\mathrm{pt}) = \widetilde{K}(S^2)$, is *not* an isomorphism for odd $i > 0$. To see this, note that this "periodicity" map commutes with the isomorphisms involving $\pi_{2i}\big(\mathfrak{Vect}^{\mathbb{C}}_G(\mathrm{pt}; \widehat{\mathbb{Z}})_\infty\big)$ in Step 1 of the proof of Proposition 2.5. But Bott periodicity does not commute with the maps $\delta_{i,G}$ defined in Step 2. Instead, we get a commutative diagram

$$\begin{array}{ccccc}
\pi_{2i}\big(\mathfrak{Vect}^{\mathbb{C}}_G(\mathrm{pt};\mathbb{Q})_\infty\big) & \xrightarrow[\cong]{\delta_{i,G}} & \Delta(G) & \xrightarrow{E_i} & \prod_{k=0}^{\infty} H^{2k}(BG;\mathbb{Q}) \\
\downarrow{\scriptstyle \eta\wedge} & & \downarrow{\scriptstyle -\prod_{k=0}^{\infty}(i+k)\cdot\, =E_{i+1}^{-1}\circ E_i} & & \downarrow{\scriptstyle \mathrm{Id}} \\
\pi_{2i+2}\big(\mathfrak{Vect}^{\mathbb{C}}_G(\mathrm{pt};\mathbb{Q})_\infty\big) & \xrightarrow[\cong]{\delta_{i+1,G}} & \Delta(G) & \xrightarrow{E_{i+1}} & \prod_{k=0}^{\infty} H^{2k}(BG;\mathbb{Q}),
\end{array}$$

where, for example, $\frac{1}{C(V)} \in \Delta(G)$ (for irreducible V) is not in the image of $E_{i+1}^{-1} \circ E_i$. Formula (4) in the proof now shows that the periodicity map is not an isomorphism.

3. Acyclicity of Mackey functors via Smith theory

It remains to show the vanishing result for certain higher inverse limits which was needed in the proof of Proposition 1.5. A general vanishing result, for higher limits of Mackey functors on $\mathcal{R}_p(G)$, is given in Theorem 3.5; and then the special case used in Section 1 is shown in Corollary 3.6. The proof is based on the interpretation, in [JM1] and [JMO, Theorem 1.7], of higher limits of functors defined on orbit categories as equivariant cohomology groups of certain spaces with group action. For this reason, we begin by considering results related to Smith theory and equivariant cohomology.

Throughout this section, we fix a compact Lie group G and a prime p. One source of difficulties when working with actions of positive dimensional groups is that (in contrast to the situation for finite groups) not all subgroups of a p-toral group are p-toral. Smith theory implies, among other things, that if G acts on a finite dimensional $\mathbb{F}_p$-acyclic space X with finitely many orbit types, then for any p-toral subgroup $P \subseteq G$, the fixed point set X^P is also $\mathbb{F}_p$-acyclic. Here, we show (Proposition 3.3) that under the additional assumption that all isotropy subgroups of the G-action are p-toral, then X^Q is $\mathbb{F}_p$-acyclic for any subgroup $Q \subseteq G$ which is *contained in* a p-toral subgroup of G, whether or not Q is itself p-toral. Simple examples of circle actions make it clear that this extra assumption about the isotropy subgroups is necessary.

For convenience, a *sub-p-toral subgroup* of a group G is defined here to be a subgroup of G which is contained in a p-toral subgroup. In the following lemma, we collect for later use various results about such subgroups.

Lemma 3.1. *The following hold for any sub-p-toral subgroup $Q \subseteq G$.*

(a) There is a unique "singular" subgroup $Q_s \lhd Q$ with the properties that $[Q : Q_s]$ is a power of p, and that Q_s is a product of a torus with a finite abelian group of order prime to p.

(b) For any p-toral subgroup $P \supsetneq Q$ in G, $N_P(Q)/Q$ is p-toral and nontrivial.

(c) Let $Q' \subseteq G$ be any subgroup such that $Q \lhd Q'$ and $[Q' : Q]$ is a power of p. Then Q' is also sub-p-toral.

(d) If G acts smoothly on a compact manifold M, then $\chi(M^Q) \equiv \chi(M) \pmod p$.

Proof. Fix a p-toral subgroup $P \subseteq G$ such that $Q \subseteq P$. Let $S = P_0$ be the identity component of P (a torus).

(a) Set $A = Q \cap S$, a normal abelian subgroup of p-power index in Q, and let Q_s be the minimal subgroup of p-power index in A. Then Q_s is abelian, $p \nmid |\pi_0(Q_s)|$, and $[Q : Q_s]$ is a power of p. And, Q_s is the unique such subgroup, since it is the intersection of all subgroups p-power index in Q.

(d) By (a), $Q_s \lhd S$, Q/Q_s is a finite p-group, and S/Q_s is a torus. So for any compact manifold M with smooth G-action,

$$\chi(M^Q) = \chi((M^{Q_s})^{Q/Q_s}) \equiv \chi(M^{Q_s}) = \chi((M^{Q_s})^{S/Q_s}) = \chi(M^S) = \chi(M) \pmod p.$$

(b) Note that $N_P(Q_s)$ is p-toral since it contains the identity component of P, and that $N_P(Q_s) \supseteq N_P(Q)$ by the uniqueness of Q_s. Hence

$$N_P(Q)/Q \cong N_{N_P(Q_s)/Q_s}(Q/Q_s)/(Q/Q_s)$$

is p-toral since the normalizer of one p-toral subgroup in another is p-toral [JMO, Lemma A.3]. Finally, if $Q \subsetneqq P$, then

$$\chi(N_P(Q)/Q) = \chi((P/Q)^Q) \equiv \chi(P/Q) \equiv 0 \pmod p$$

by (d), and so $N_P(Q)/Q \neq 1$.

(c) Let $N_p(T) \subseteq G$ be a maximal p-toral subgroup; i.e., an extension of a maximal torus T by a Sylow p-subgroup of $N(T)/T$. Then a subgroup $Q' \subseteq G$ is sub-p-toral if and only if $(G/N_p(T))^{Q'}$ is non-empty. So by (d), Q' is sub-p-toral if and only if

$$\chi((G/N_p(T))^{Q'}) \equiv \chi(G/N_p(T)) \not\equiv 0 \pmod p.$$

Hence, if $Q \lhd Q'$ and Q'/Q is a p-group, then

$$\chi((G/N_p(T))^{Q}) \equiv \chi((G/N_p(T))^{Q'}) \pmod p,$$

and Q is sub-p-toral if and only if Q' is sub-p-toral. □

It will be convenient here to write $O_p(\Gamma)$, for a finite group Γ and a prime p, to denote the intersection of the Sylow p-subgroups of Γ (or equivalently, the maximal normal p-subgroup of Γ). As in [JMO], a subgroup $P \subseteq G$ will be called *p-stubborn* if P is p-toral, $N(P)/P$ is finite, and $O_p(N(P)/P) = 1$. In other words, P is a p-stubborn subgroup of G if and only if the orbit G/P lies in the category $\mathcal{R}_p(G)$.

Lemma 3.2. *Let $Q \subseteq G$ be a sub-p-toral subgroup such that $N(Q)/Q$ is finite and $O_p(N(Q)/Q) = 1$. Then Q is p-toral.*

Proof. Let $Q_s \lhd Q$ be the singular subgroup of Lemma 3.1(a). Set $H = \big(C_G(Q_s)\big)_0$ (the identity component of the centralizer), and $Q' = Q \cap H \lhd Q$.

Recall that Q_s is characterized by the property that $p \nmid [Q_s{:}Q_0]$, while Q/Q_s is a p-group. Since Q is sub-p-toral, this means that Q_s is contained in some torus $S \subseteq G$, and

$$Q_s \subseteq S \subseteq \big(C_G(Q_s)\big)_0 = H. \tag{1}$$

In particular, $Q' \supseteq Q_s$, and so Q/Q' is a finite p-group.

Consider the group

$$K = \{g \in H \,|\, [g, Q] \subseteq Q'\} \lhd N(Q).$$

Here, K/Q' is finite (since $|N(Q)/Q| < \infty$); and so $K/Q' = C_{H/Q'}(Q/Q')$ is a p-group by [JMO, Proposition A.4] (the group of components of the centralizer of a

p-toral subgroup is a p-group). Thus, $K/Q' \subseteq O_p(N(Q)/Q') = Q/Q'$ by assumption. It follows that $K = Q' = Q \cap H$; and so

$$\begin{aligned} \chi(H/Q') &\equiv \chi\big((H/Q')^{Q/Q'}\big) \qquad \text{(where } Q/Q' \text{ acts by conjugation)} \\ &= \chi(K/Q') = 1 \pmod p. \end{aligned}$$

In particular, $\chi(H/Q') \neq 0$, and hence $\dim(H/Q') = 0$. By (1), this implies that $Q_s = S$ is a torus, and hence that Q is p-toral. □

We are now ready to look at fixed point sets of sub-p-toral subgroups.

Proposition 3.3. *Let X be any finite dimensional $\mathbb{F}_p$-acyclic G-complex, having finitely many orbit types, and such that all isotropy subgroups are p-toral. Then for any sub-p-toral subgroup $Q \subseteq G$, X^Q is $\mathbb{F}_p$-acyclic.*

Proof. Let $\mathcal{S}(G)$ be the compact space of all closed subgroups of G with the Hausdorff topology (cf. [tD, Proposition IV.3.2(i)]). Let $\mathcal{T} \subseteq \mathcal{S}(G)$ be the subset of all sub-p-toral subgroups $Q \subseteq G$ such that X^Q is not $\mathbb{F}_p$-acyclic. Assume that $\mathcal{T} \neq \emptyset$. We first show (Step 1) that $\mathcal{T}$ contains a maximal element, and then apply that in Step 2 to get a contradiction.

Step 1 Set $k = \max\big\{\dim(Q) \,|\, Q \in \mathcal{T}\big\}$. If $\mathcal{T}$ does not contain a maximal element, then there exists an infinite chain

$$Q_1 \subsetneqq Q_2 \subsetneqq Q_3 \subsetneqq \cdots$$

of k-dimensional sub-p-toral subgroups for which X^{Q_i} is not $\mathbb{F}_p$-acyclic. Let Q be the closure of the union of the Q_i. Then $\dim(Q) > k$, and we will get a contradiction upon showing that $Q \in \mathcal{T}$. Since X has only finitely many orbit types, $X^Q = X^{Q_i}$ for i sufficiently large [tD, Proposition IV.3.4], and hence X^Q is not $\mathbb{F}_p$-acyclic. It remains to show that Q is sub-p-toral; then $Q \in \mathcal{T}$, and this contradicts the definition of k.

Choose p-toral subgroups $P_i \supseteq Q_i$. Since the space $\mathcal{S}(G)$ of closed subgroups of G is compact, as noted above, we can assume (after restricting to a subsequence if necessary) that the P_i converge to a closed subgroup $P \supseteq Q$. Then $\pi_0(P)$ is a p-group, since $\pi_0(P_i)$ surjects onto it for i sufficiently large (by definition of the Hausdorff topology). Also, by a theorem of Jordan (cf. [tD, Proposition IV.6.4]), there exists some integer j such that each finite (hence each p-toral) subgroup of P contains a normal abelian subgroup of index $< j$. In particular, P contains a normal abelian subgroup of finite index, and hence has torus identity component. Thus, P is p-toral, and so Q is sub-p-toral.

Step 2 Now let $Q \in \mathcal{T}$ be a maximal element. In other words, Q is sub-p-toral in G and X^Q is not $\mathbb{F}_p$-acyclic, but $X^{Q'}$ is $\mathbb{F}_p$-acyclic for any sub-p-toral subgroup $Q' \supsetneqq Q$. Also, Q is not p-toral, since otherwise X^Q would be $\mathbb{F}_p$-acyclic by Smith theory (cf. [Br, Chapter III]). So by Lemma 3.2, either $\dim(N(Q)/Q) > 0$, or $O_p(N(Q)/Q) \neq 1$.

Now, for each isotropy subgroup $H \supseteq Q$ of the action on X, H is p-toral by assumption, and hence X^H is $\mathbb{F}_p$-acyclic. So by [JMO, Lemmas 2.4 & 2.3], there is a finite dimensional G-complex $X' \supseteq X$ such that $(X')^Q$ is $\mathbb{F}_p$-acyclic, and all orbits of $X' \smallsetminus X$ are of type G/Q. The $N(Q)/Q$-action on $(X')^Q$ has finitely many orbit types, since the action of $N(Q)$ on any G-orbit has finitely many orbit types. Also, for any $x \in X^Q$, its isotropy subgroup $\bigl(N(Q)/Q\bigr)_x = N_{G_x}(Q)/Q$ is a nontrivial p-toral subgroup by Lemma 3.1(b). In other words, X^Q is the singular set (the set of elements in nonfree orbits) of the finite dimensional $(N(Q)/Q)$-complex $(X')^Q$ with finitely many orbit types and p-toral isotropy subgroups. Thus, X^Q is itself $\mathbb{F}_p$-acyclic: by [JMO, Lemma 2.12] if $\dim(N(Q)/Q) > 0$, or by [JMO, Lemma 2.13] if $O_p(N(Q)/Q) \neq 1$. $\square$

In order to translate this to a result about higher limits over orbit categories, we consider, for any full subcategory $\mathcal{C}$ of the orbit category $\mathcal{O}(G)$, the G-space

$$E\mathcal{C} \overset{\text{def}}{=} \underset{G/H\in\mathcal{C}}{\underrightarrow{\text{hocolim}}}(G/H).$$

By [JMO, Theorem 1.7], the higher inverse limits of a functor $F : \mathcal{C} \to \mathsf{Ab}$ can be interpreted as equivariant ordinary cohomology groups $H^*_G(E\mathcal{C}; F)$. See, e.g., [JMO, appendix] for more details on equivariant ordinary cohomology.

Proposition 3.4. *For any contravariant functor $F : \mathcal{O}(G) \to \mathbb{Z}_{(p)}$-$\mathsf{mod}$, and any maximal p-toral subgroup $N_p(T)$ of G,*

$$H^i_{N_p(T)}(E\mathcal{R}_p(G); F) = \begin{cases} F(G/N_p(T)) & \text{if } i = 0 \\ 0 & \text{if } i > 0. \end{cases} \tag{1}$$

Proof. Let X be a finite dimensional $\mathbb{F}_p$-acyclic G-complex, all of whose isotropy subgroups are p-stubborn: as constructed in [JMO, Section 2]. By [JMO, Proposition 1.1], $E\mathcal{R}_p(G)$ is characterized by the properties that all isotropy subgroups are p-stubborn, and that $E\mathcal{R}_p(G)^P$ is contractible for each p-stubborn subgroup $P \subseteq G$. So there is a G-map $f : X \to E\mathcal{R}_p(G)$ which induces an $\mathbb{F}_p$-homology equivalence on the fixed point set of any isotropy subgroup. Then by [JMO, Lemma A.10], f^H is an $\mathbb{F}_p$-homology equivalence for all subgroups $H \subseteq G$. For each $Q \subseteq N_p(T)$, X^Q is $\mathbb{F}_p$-acyclic by Proposition 3.3, so $E\mathcal{R}_p(G)^Q$ is also $\mathbb{F}_p$-acyclic, and hence is $\mathbb{Z}_{(p)}$-acyclic since its homology is finitely generated in each dimension (cf. [JMO, Proposition 1.1]). Formula (1) now follows from [JMO, Lemma A.13]. $\square$

We now consider the stable orbit category $\mathcal{O}^{\text{st}}(G)$: the category whose objects are the orbits of G, and where

$$\text{Mor}_{\mathcal{O}^{\text{st}}(G)}(G/H, G/K) = \underset{V\in\text{Rep}^{\mathbb{R}}(G)}{\underrightarrow{\lim}} \bigl[S^V \wedge (G/H_+), S^V \wedge (G/K_+)\bigr]^G_*.$$

Here, $[-,-]^G_*$ means pointed homotopy classes of pointed G-maps. Following the notation of Lewis, May, and McClure [LMM], the term "Mackey functor" will be used here to denote an additive contravariant functor defined on $\mathcal{O}^{\rm st}(G)$. We want to show that p-local Mackey functors (i.e., Mackey functors which take values in $\mathbb{Z}_{(p)}$-modules) are acyclic over $\mathcal{R}_p(G)$. For finite G, this follows from [JM2, Proposition 5.14].

Theorem 3.5. *Let $F : \mathcal{O}^{\rm st}(G)^{\rm op} \longrightarrow \mathbb{Z}_{(p)}$-$\mathtt{mod}$ be any p-local Mackey functor. Then*

$$\varprojlim_{\mathcal{R}_p(G)}{}^i (F) = 0 \qquad \text{for all } i > 0.$$

Proof. Set $N = N_p(T)$, for short. By [JMO, Theorem 1.7],

$$\varprojlim_{\mathcal{R}_p(G)}{}^* (F) \cong H^*_G(E\mathcal{R}_p(G); F),$$

where $H^*_G(-;F)$ denotes ordinary equivariant cohomology with coefficients in F. Also,

$$H^i_G((G/N)\times E\mathcal{R}_p(G); F) \cong H^i_G(G\times_N E\mathcal{R}_p(G); F) \cong H^i_N(E\mathcal{R}_p(G); F) = 0$$

for $i > 0$ by Proposition 3.4. So the theorem follows from [LMM], which says that the homomorphism

$$H^*_G(E\mathcal{R}_p(G); F) \longrightarrow H^*_G\big((G/N)\times E\mathcal{R}_p(G); F\big)$$

(induced by projection) is a split monomorphism.

We sketch here one simple way to see this. Choose an embedding $G/N \hookrightarrow V$, for some G-representation V, and let

$$f : S^V \longrightarrow S^V \wedge (G/N_+)$$

be the map given by the Pontrjagin-Thom construction. By [O1, Lemma], $f \wedge E\mathcal{R}_p(G)_+$ is homotopic to a map

$$g : S^V \wedge E\mathcal{R}_p(G)_+ \longrightarrow S^V \wedge (G/N \times E\mathcal{R}_p(G))_+$$

which preserves the skeleta of the spaces (under any G-CW-structure on the product space $G/N\times E\mathcal{R}_p(G)$). Thus, since F is defined on the stable category, any map $S^V(G/H_+) \to S^V(G/K_+)$ induces a homomorphism $F(G/K) \to F(G/H)$, and so g induces a chain homomorphism

$$g^* : C^*_G(G/N\times E\mathcal{R}_p(G); F) \longrightarrow C^*_G(E\mathcal{R}_p(G); F).$$

Here, $C^*_G(-;F)$ denotes the cellular cochain complex, which in degree k is the product of one copy of $F(G/H)$ for each orbit $G/H\times D^k$ of cells in the complex. The composite

$$H^*_G(E\mathcal{R}_p(G);F) \xrightarrow{(pr_2)^*} H^*_G((G/N)\times E\mathcal{R}_p(G);F) \xrightarrow{H(g^*)} H^*_G(E\mathcal{R}_p(G);F) \quad (1)$$

is induced by $\varphi\times$ Id, where $\varphi : S^V \to S^V$ is the composite

$$S^V \xrightarrow{\quad f \quad} S^V\wedge(G/N_+) \xrightarrow{\text{proj}} S^V.$$

We claim that for any p-toral subgroup $P \subseteq G$, $\varphi|P$ is invertible in the localized P-equivariant stable homotopy ring $\left(\omega_0^P\right)_{(p)}$. To see this, note that φ corresponds to the class of the orbit G/N under the isomorphism of $\omega_0^P \overset{\text{def}}{=} \varinjlim_V [S^V, S^V]_*^P$ with the Burnside ring $A(P)$ (see [tD, §II.8]). Here, $A(P)$ is the ring of equivalence classes of finite P-complexes X, where $[X]=[Y]$ if $\chi(X^H)=\chi(Y^H)$ for all $H \subseteq P$. By [tD, Theorem IV.4.2], any prime ideal of $A(P)$ has the form

$$q(Q,\mathfrak{p}) = \{[X]\,|\,\chi(X^Q)\in\mathfrak{p}\}$$

for some prime ideal $\mathfrak{p}\subseteq\mathbb{Z}$ and some $Q \subseteq P$. Also, by Lemma 3.1(d), $q(Q,p\mathbb{Z}) = q(1,p\mathbb{Z})$ for all $Q\subseteq G$. So $A(P)_{(p)}$ is a local ring with maximal ideal $q(1,p\mathbb{Z})$, and (since $p\nmid\chi(G/N)$) $[G/N]$ is invertible in $A(P)_{(p)}$.

Hence, for each p-toral subgroup $P\subseteq G$,

$$\varphi\wedge \text{Id} : S^V\wedge(G/P_+) \longrightarrow S^V\wedge(G/P_+) \cong (G\times_P S^V)/(G\times_P *)$$

is an isomorphism in $\text{Mor}_{\mathcal{O}^{\text{st}}(G)}(G/P,G/P)_{(p)}$. In particular, this applies to each orbit in $E\mathcal{R}_p(G)$, and so the composite in (1) is an isomorphism. □

Note that it does not necessarily follow, under the conditions in Theorem 3.5, that $\varprojlim^0_{\mathcal{R}_p(G)}(F)\cong F(G/G)$. The functor $K_G(-)$ provides a simple counterexample.

The vanishing result needed in Section 1 is the following:

Corollary 3.6. *For all $i\in\mathbb{Z}$ and all $j>0$,*

$$\varprojlim_{\mathcal{R}_p(G)}{}^j\left(\widehat{\mathbb{Z}}_p\otimes K^i_G(-)\right) = \varprojlim_{\mathcal{R}_p(G)}{}^j\left(\widehat{\mathbb{Z}}_p\otimes KO^i_G(-)\right) = 0.$$

Proof. By the Bott periodicity theorem for equivariant K-theory [At, Theorems 4.3 and 6.1], for any compact G-space X,

$$K_G(X)\cong K_G(V\times X) = \widetilde{K}_G(\Sigma^V(X_+))$$

for any complex G-representation V, and

$$KO_G(X) \cong KO_G(V\times X) = \widetilde{KO}_G(\Sigma^V(X_+))$$

for any real Spin G-representation V of dimension a multiple of 8. Since any real representation is a summand of a Spin representation, this shows that for any G-representation V, a continuous G-map $f : \Sigma^V(X_+) \to \Sigma^V(Y_+)$ induces homomorphisms $K_G^*(Y) \to K_G^*(X)$ any $KO_G^*(Y) \to KO_G^*(X)$. The functors $K_G^*(-)$ and $KO_G^*(-)$ are thus defined on the stable category, and the result follows from Theorem 3.5. □

We also note the following, second corollary of Theorem 3.5:

Corollary 3.7. *Let $h^*(-)$ be any p-local cohomology theory. Then for all $i > 0$,*

$$\varprojlim_{G/P\in\mathcal{R}_p(G)}{}^{i} \; h^*(EG/P) = 0.$$

Proof. For any G-representation V, and any pair of finite G-complexes X and Y, a G-map $\varphi : \Sigma^V(X_+) \to \Sigma^V(Y_+)$ induces a map

$$\mathrm{Id}_{EG_+} \wedge_G \varphi : S^V \wedge_G \big((EG\times X)_+\big) \longrightarrow S^V \wedge_G \big((EG\times Y)_+\big)$$

between the Thom spaces of bundles over $EG\times_G X$ and $EG\times_G Y$. Upon restricting to finite skeleta of EG and adding complementary bundles, we get a map $\Sigma^\infty((EG\times_G X)_+) \to \Sigma^\infty((EG\times_G Y)_+)$ between the suspension spectra, which in turn induces a homomorphism $h^*(EG\times_G Y) \to h^*(EG\times_G X)$. The functor $h^*(EG\times_G -)$ is thus defined on the stable category, and so its higher limits vanish by Theorem 3.5. □

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