THE COMPONENTS OF A VARIETY OF MATRICES WITH SQUARE ZERO AND SUBMAXIMAL RANK

DIKRAN KARAGUEUZIAN, BOB OLIVER, AND JOANA VENTURA

Abstract. The structure of the variety of upper-triangular square-zero matrices was investigated by Rothbach, who introduced techniques enabling him to determine its irreducible components.

In this paper, we fix a particular irreducible component of this variety and study the structure of the subvariety of matrices of submaximal rank in this component. We use Rothbach's techniques to determine the components of this variety. We also show that this subvariety contains the support variety for a certain universal homology module.

1. Introduction

A longstanding conjecture in algebraic topology describes the free rank of symmetry of a product of spheres. This conjecture states that if the elementary abelian group $(\mathbb{Z}/p)^r$ acts freely on $S^{n_1} \times \cdots \times S^{n_s}$, then $r \leq s$. A more ambitious generalization of this conjecture states that if $(\mathbb{Z}/p)^r$ acts freely on a manifold M, then $\sum_i \dim_{\mathbb{F}_p} H_i(M, \mathbb{F}_p) \geq 2^r$. For a survey of conjectures of this type, and partial results, the reader might consult Section 2 of [1].

In [2] G. Carlsson produced a functorial translation of the second conjecture (for $p = 2$) into the language of commutative algebra. Carlsson was able to obtain partial results [2, 3] on his version of the conjecture using techniques of commutative algebra; this yielded new results on the topological side as well.

The key point in the proofs in [3] is to show that an upper-triangular square-zero matrix over a polynomial ring can, through some specialization of variables, be forced to have submaximal rank. The matrix arises as the differential of a free differential graded module over a polynomial ring, and in the case of interest the module has even rank. The generic $2n \times 2n$ square-zero matrix has rank n; by a "matrix of submaximal rank" we mean a matrix of rank less than n. The structure of the variety of upper-triangular square-zero matrices was later investigated by Rothbach [5], who introduced techniques enabling him to determine its irreducible components.

In this paper, we fix a particular irreducible component of this variety and study the structure of the subvariety of matrices of submaximal rank in this component. We use Rothbach's techniques to determine the components of this variety. Also, following a suggestion of Carlsson, we show that this subvariety contains the support variety for a certain universal homology module. The hope is that this universal homology module and the component result will be useful for the commutative algebra version of the conjecture, but we have not yet made progress in that direction.

The structure of the rest of the paper is as follows: in Section 2 we introduce notation and terminology for the objects of study, restricting attention to an irreducible component Z of the

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variety of upper-triangular $2n \times 2n$ square-zero matrices. In Section 3 we relate our submaximal rank subvariety $Y \subset Z$ to the support variety for the universal homology module. In Section 4 we review Rothbach's techniques, and in Section 5 we determine the irreducible components of Y.

The work for this paper was done while the first and third authors were visiting the University of Paris 13; it is a pleasure to thank the faculty and staff there for their hospitality.

2. Definitions and Notation

In this paper, we work over an algebraically closed field k, which will be the ground field for all polynomial rings. We also regard all varieties as being defined over k. In view of the motivation mentioned in the introduction, the reader may wish to take $k = \overline{\mathbb{F}}_2$, but this restriction is not necessary for the results.

The following notation will be used throughout the rest of the paper.

- U_{2n} is the variety of strictly upper-triangular $2n \times 2n$ matrices over k.
- V_{2n} is the variety of square-zero matrices in U_{2n}.
- Z denotes a particular irreducible component of V_{2n}.
- R is the coordinate ring of Z.
- Y is the subvariety of matrices of rank less than n in Z.
- I denotes the ideal of R corresponding to Y.

The coordinate ring of U_{2n} is $R(U_{2n}) = k[x_{ij} \mid i < j]$. There are surjections of coordinate rings $R(U_{2n}) \to R(V_{2n}) \to R$ corresponding to the inclusions $Z \hookrightarrow V_{2n} \hookrightarrow U_{2n}$. Using these surjections, we can regard the images of the x_{ij} as elements of R. Let $M \in M_{2n}(R)$ be the $2n \times 2n$ upper triangular matrix whose (i,j)-entry is the image in R of x_{ij}. In particular, $M^2 = 0$. We regard M (the *universal matrix*) as a differential on the R-module R^{2n}.

Note that I is the ideal generated by all $n \times n$ minors of the universal matrix M.

Definition 2.1. *The* universal homology *of Z, written $H(M)$, is the R-module* $\operatorname{Ker}(M)/\operatorname{Im}(M)$.

3. The Support Variety for Universal Homology

In this section, we show that Y contains the support variety for the universal homology module $H(M)$ of Z. We recall that by definition, the support variety of a module N is the variety corresponding to the annihiliator ideal of N. The two statements in the following proposition are thus equivalent.

Proposition 3.1. *$Y \supseteq \operatorname{supp} H(M)$ and $I \subseteq \operatorname{Ann} H(M)$.*

Proof. We must show, for each $x \in I$, that $x \cdot H(M) = 0$. We can restate this last condition as "for each $x \in I$ and for each $v \in \operatorname{Ker} M$, there is a $u \in R^{2n}$ such that $Mu = xv$". Of course, it is enough to show this for the $n \times n$ minors which generate I. We will show this one minor at a time, by constructing an explicit linear map $N\colon R^{2n} \to R^{2n}$, so that $M(Nv) = xv$ for all $v \in \operatorname{Ker}(M)$.

Let X be an $(n \times n)$ submatrix of M, and set $x = \det X$. We can assume $x \neq 0$; otherwise there is nothing to prove. Set $U = R^{2n}$, and write $U = V \oplus V' = W \oplus W'$, where the decompositions correspond to the choices of rows and columns used to define the submatrix X. Thus $M : U \longrightarrow U$ has components

$$\begin{array}{ccc} V & \xrightarrow{\alpha} & W \\ \oplus & \overset{\beta}{\swarrow}\ \ \overset{\gamma}{\nearrow} & \oplus \\ V' & \xrightarrow[\delta]{} & W' \end{array}$$

where α has matrix X and determinant x. These two decompositions of U thus correspond to "pushing X" to the upper left corner of the matrix M. Define $N : U \longrightarrow U$ to be the map

$$\begin{array}{ccc} W & \xrightarrow{\widehat{\alpha}} & V \\ \oplus & \overset{0}{\swarrow}\ \ \overset{0}{\nearrow} & \oplus \\ W' & \xrightarrow[0]{} & V' ; \end{array}$$

where $\widehat{\alpha}$ is the linear map whose matrix is the cofactor matrix of X (thus $\alpha\widehat{\alpha} = x \cdot \mathrm{Id}_W$).

Set $\bar{R} = R[1/x]$. Since R is an integral domain, and $x \neq 0$ by assumption, R is a subring of the localized ring $\bar{R}$. Since x is invertible in $\bar{R}$, the submatrix X of M chosen above (and corresponding to the minor x) is invertible over $\bar{R}$.

Set $\bar{U} = \bar{R} \otimes_R U$, $\bar{V} = \bar{R} \otimes_R V$, and similarly for $\bar{V}'$, $\bar{W}$, and $\bar{W}'$. Since $V \xrightarrow{\alpha} W$ is the map with matrix X, which is invertible over $\bar{R}$, α becomes invertible as a map from $\bar{V}$ to $\bar{W}$. Set $K = \mathrm{Ker}(M) \subseteq \bar{U}$, $I = \mathrm{Im}(M) \subseteq \bar{U}$ and

$$\begin{aligned} K' &= \{(\alpha^{-1}\gamma(v'), -v') \mid v' \in \bar{V}'\}, \\ I' &= \{(w, \beta\alpha^{-1}(w)) \mid w \in \bar{W}\}. \end{aligned}$$

Thus K' is the space of vectors in $\bar{V} \oplus \bar{V}'$ whose image under M lies in $\bar{W}'$ (has zero $\bar{W}$-component); while I' is the image of $\bar{V}$. Obviously $I' \subseteq I \subseteq K \subseteq K'$.

By definition, $(\bar{W}', I')$ and $(\bar{V}, K')$ are both pairs of complementary subspaces of $\bar{U}$, where $I' \subseteq K'$. Hence

$$\bar{R}^n \cong \bar{W}' \cong \bar{U}/I' \cong (\bar{V} \oplus K')/I' \cong \bar{V} \oplus (K'/I') \cong \bar{R}^n \oplus (K'/I').$$

Since $\bar{R}$ is noetherian, this implies that $K'/I' = 0$, and hence that $K' = I'$. Thus all four of the submodules I, K, I', K' are equal.

From the definitions, it follows easily that $MN(u) = x \cdot u$ for all $u \in I'$. Since $I' = K = \mathrm{Ker}(M)$ and R is a subring of $\bar{R}$, this completes the proof of Proposition 3.1. □

We conclude this section with the natural

Conjecture 3.2. *$I = \mathrm{Ann}\, H(M)$, or equivalently, $Y = \mathrm{supp}\, H(M)$.*

4. The Structure of V_{2n}

In this section, we review Rothbach's work on the structure of V_{2n} and its irreducible components; our decomposition of Y into irreducible components is obtained by similar methods. The reader familiar with [5] can safely skip this section. It should be noted that because of our motivation,

and to minimize technical difficulties, we have opted only to consider components of V_{2n}. However, the work in [5] applies to $(n \times n)$-matrices for odd n as well.

Rothbach's work is based on the decomposition of V_{2n} into Borel orbits.

Definition 4.1. *The* Borel orbits *in V_{2n} are the orbits of the conjugation action of the Borel group of all invertible upper-triangular matrices on V_{2n}.*

There is a one-to-one correspondence between the Borel orbits and *valid X^2 words*; the ordering induced on valid X^2 words can be described by *moves*. A move is a function from the set of valid X^2 words to itself; the ordering is described by moves in the sense that if $w < w'$ then there is a sequence of moves which transforms w' into w.

Definition 4.2. *A* partial permutation matrix *X is a matrix of 0's and 1's in which each row and each column contains at most one 1.*

To an upper-triangular partial permutation matrix we can associate a sequence of nonnegative integers $(a_1, a_2, \ldots, a_{2n})$ by setting

$$a_i = \begin{cases} j & \text{if } Xe_i = e_j, \\ 0 & \text{if } Xe_i = 0. \end{cases}$$

Definition 4.3. *A* valid X^2 word *is a sequence of nonnegative integers $(a_1, \ldots, a_{2n})$ associated to a partial permutation matrix X with $X^2 = 0$. The integers a_i are the* letters *of the word. If v is a valid X^2 word, we write* $\operatorname{rank}(v)$ *for the number of nonzero integers a_i in v, i.e. the rank of the partial permutation matrix associated to v.*

Remark 4.4. It follows from the definition of partial permutation matrix that the nonzero letters in a valid X^2 word are distinct.

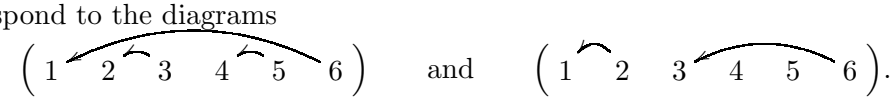
We can now describe the correspondence between Borel orbits and valid X^2 words: each Borel orbit contains a unique partial permutation matrix, and thus to each Borel orbit is associated a unique valid X^2 word. The closure of a Borel orbit is the closure of an image of the Borel group, which is an irreducible variety, so these closures are themselves irreducible varieties (cf. [4, Proposition I.8.1]). Clearly, the closure of a Borel orbit is itself a union of Borel orbits. Thus, V_{2n} is a finite union of irreducible varieties (closures of all Borel orbits), which are partially ordered by inclusion, and the components of V_{2n} are therefore the maximal elements of this poset. In this way, the problem of determining the components of V_{2n} is reduced to the combinatorics of the poset of valid X^2 words.

In order to determine which Borel orbits are contained in the closure of a given Borel orbit, in terms of the corresponding valid X^2 words, Rothbach defined certain "moves" which give an order relation on the valid X^2 words. To explain this, we introduce the following terminology. Let $(a_1, \ldots, a_{2n})$ be a valid X^2 word.

Definition 4.5. *We say that the i-th letter a_i is* bound *if $a_i = 0$ and there exists a j such that $a_j = i$. If the letter a_i is not bound, then it is* free.

It is helpful to regard valid X^2 words as "partial permutations" of the set $\{1, \ldots, 2n\}$. The word $(a_1, \ldots, a_{2n})$ is thought of as the partial permutation with domain $\{i \mid a_i \neq 0\}$, which sends i to a_i. The $X^2 = 0$ condition translates to saying that the domain and range of the permutation are

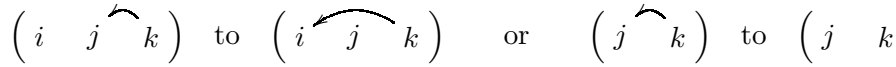
disjoint. These can be illustrated by diagrams with arrows. For example, the words 002041 and 010003 correspond to the diagrams



In the following descriptions, whenever we show a "subdiagram" of a partial permutation by restricting to some subset of indices $I \subseteq \{1, \ldots, 2n\}$, it is understood that no index $i \notin I$ is sent to any index $j \in I$, and no index $i \in I$ is sent to any nonzero index $j \notin I$.

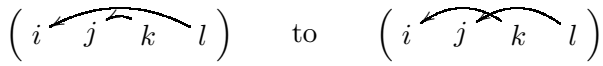
The three moves are the following:

- A *move of type 1* takes a nonzero letter a_k and replaces it with a_k^*, the largest integer less than a_k such that the replacement yields a new valid X^2 word. (Note that a_k^* always exists since replacement with 0 always yields a valid X^2 word.) In other words, if we set $j = a_k$ and $i = a_k^*$ (so $i < j < k$), then this move sends

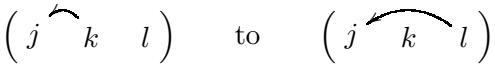


  when $i \neq 0$ or $i = 0$, respectively.

- A *move of type 2* takes two free letters a_k, a_l such that $k < l$ and $a_k > a_l$, and swaps their locations. In other words, it either sends

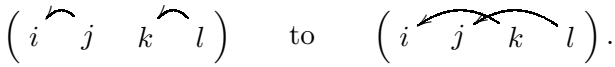


  if $a_k = j$ and $a_l = i \neq 0$ (and thus $i < j < k < l$), or else it sends



  if $a_k = j$ and $a_l = 0$ (thus $j < k < l$).

- A *move of type 3* is defined whenever there are indices $i < j < k < l$ such that $i = a_j$ and $k = a_l$ (hence $a_i = a_k = 0$), and replaces a_l by j, a_k by i, and a_j by 0. Pictorially, it sends



Observe that a move of type 2 or 3 preserves the rank of words. In fact, the only way of getting a word of smaller rank is to replace a letter by zero. This corresponds to applying move 1 one or more times. A sequence of moves of type 1 which results in a letter being replaced by zero will be called a move of type 1′.

The order relation on valid X^2 words is defined by letting $w \geq w'$ if and only if w can be transformed into w' by a (possibly empty) finite sequence of moves. The maximal valid X^2 words are thus those which are not the result of any of the three types of moves.

Example 4.6. The word $(0, 1, 0, 3)$ is transformed to $(0, 0, 1, 2)$ by a move of type 3, so in the ordering defined above, $(0, 0, 1, 2) < (0, 1, 0, 3)$.

Finally, the maximal valid X^2 words are also called *bracket words* because there is a one-to-one correspondence between maximal valid X^2 words and sequences of left and right parentheses of length $2n$ which form valid LISP expressions. A bracket word corresponds to the valid X^2 word $(a_1, \ldots, a_{2n})$ where $a_i = 0$ if the i-th parenthesis in the bracket word is a left parenthesis, and $a_i = j$ if the i-th parenthesis is a right parenthesis which closes the j-th parenthesis.

Remark 4.7. For a bracket word w of length $2n$, we have $\operatorname{rank}(w) = n$.

The key theorem of Rothbach's paper is

Theorem 4.8 (Rothbach)**.** *For any pair of valid X^2 words v, w, the Borel orbit $\mathcal{O}_v$ associated to v is contained in the closure of the Borel orbit $\mathcal{O}_w$ associated to w if and only if $v \leq w$. The irreducible components of V_{2n} are thus the closures of the Borel orbits associated to the maximal valid X^2 words; and the irreducible component of V_{2n} associated to a maximal valid X^2 word w is the union of the Borel orbits associated to the valid X^2 words which are less than or equal to w.*

Since Rothbach's paper is not generally available, we give a very brief sketch here of his techniques. For each $i \leq 2n$, let $k^i \subseteq k^{2n}$ be the subspace of elements $(x_1, \ldots, x_i, 0, \ldots, 0)$ for $x_1, \ldots, x_i \in k$. These are the subspaces of k^{2n} which are invariant under the action of all elements in the Borel group. For any $X \in V_{2n}$ and any $0 \leq j < i$, define $r(i, j, X) = \dim_k(X(k^i) + k^j)$. One easily sees that $r(i, j, X) = r(i, j, Y)$ if X and Y are in the same Borel orbit. For any valid X^2 word v, associated to a partial permutation matrix X, set $v_{ij} = r(i, j, X)$. Rothbach then shows:

- Two matrices $X, Y \in V_{2n}$ are in the same Borel orbit if and only if $r(i, j, X) = r(i, j, Y)$ for all i, j. The Borel orbit associated to v is thus the set $\{X \in V_{2n} \mid r(i, j, X) = v_{ij}\ \forall i, j\}$.
- For any two valid X^2 words v, w, $v \leq w$ (as defined above via moves) if and only if $v_{ij} \leq w_{ij}$ for all i, j.
- If v is obtained from w by a move of one of the above types, then the Borel orbit $\mathcal{O}_v$ is in the closure of the Borel orbit $\mathcal{O}_w$.

For any given valid X^2 word w, the union of the Borel orbits associated to words $v \leq w$ is just the set

$$\{X \in V_{2n} \mid r(i, j, X) \leq w_{ij}\ \forall i, j\}.$$

This is an algebraic set (hence closed), since it is defined by requiring determinants of certain submatrices to vanish. So together with the three points above, this proves that it is the closure of the Borel orbit associated to w.

Rothbach's theorem says that the irreducible components of V_{2n} are determined by the poset of all valid X^2 words. In the next section, we will study the subposet of words associated to orbits contained in Y, and thus determine the irreducible components of Y.

5. The Irreducible Components of Y

In this section, we will identify the irreducible components of Y. More specifically, if Z is an irreducible component of V_{2n} corresponding to a valid X^2 word w, we will describe the components of the subvariety $Y \subseteq Z$ in terms of the structure of w.

Definition 5.1. *We say that a bracket word is* irreducible *if it cannot be expressed as the concatenation of bracket words of smaller length.*

Example 5.2. The bracket word (( )( )) is irreducible; the bracket word ( )(( )) is expressible as a concatenation of the irreducible bracket words ( ) and (( )).

Let w be a bracket word of length $2n$. Observe that w can be expressed as a concatenation of irreducible bracket words $w = w_1 \cdots w_m$. (If w is irreducible then $m = 1$.) We shall use w, w_i to

denote not only the irreducible bracket words in this factorization, but also the corresponding valid X^2 words. For a bracket word w factored in this way we make the following

Definition 5.3. *For each $i \in \{1, \dots, m\}$, let $w^{(i)}$ be the valid X^2 word obtained from w by replacing the last letter of w_i by a zero.*

Notice that the words $w^{(i)}$ defined in 5.3 are all obtained from w by a move of type $1'$, and so all have rank $n-1$. Notice also that, in general, there are other words obtained by a move $1'$. For example, if $w = 002041$ then 002001 is one such word.

Now we can describe the components of Y:

Theorem 5.4. *The irreducible components of Y are the closures of the Borel orbits corresponding to the words $w^{(i)}$. Alternatively, the irreducible component of Y corresponding to $w^{(i)}$ is the union of the Borel orbits corresponding to the valid X^2 words which are less than or equal to $w^{(i)}$.*

Example 5.5. Let $n = 3$ and let Z be the component corresponding to the bracket word $(\,)((\,))$. Then the corresponding valid X^2 word is $w = 010043$. In this case, we express w as the concatenation of $(\,)$ and $((\,))$. Thus, in the notation of the paragraph after Example 5.2, $m = 2$, $w_1 = (\,)$, and $w_2 = ((\,))$. Writing this decomposition in terms of valid X^2 words, we have $w_1 = 01$, $w_2 = 0043$ and so $w^{(1)} = 000043$ and $w^{(2)} = 010040$. Thus the subvariety $Y \subseteq Z$ has two components.

In this case, we can describe these varieties in a simple way in terms of matrices. The component Z consists of all 6×6 matrices of the form

$$\begin{bmatrix} 0 & a & b & c & d & e \\ 0 & 0 & 0 & 0 & f & g \\ 0 & 0 & 0 & 0 & h & i \\ 0 & 0 & 0 & 0 & j & k \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{where} \quad \begin{bmatrix} a & b & c \end{bmatrix} \cdot \begin{bmatrix} f & g \\ h & i \\ j & k \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

The subvariety Y has two components: one where

$$\operatorname{rank} \begin{bmatrix} f & g \\ h & i \\ j & k \end{bmatrix} \leq 1, \quad \text{and another where} \quad \begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}.$$

Unfortunately it is not always possible to describe the components in this fashion.

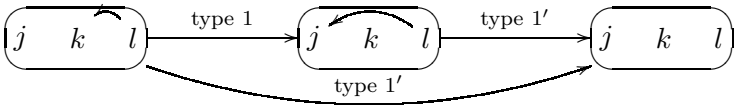
Proof of Theorem 5.4. The closure of each Borel orbit in Y is the closure of a continuous image of the Borel group of upper triangular matrices, and hence is irreducible (cf. [4, Proposition I.8.1]). Clearly, the closure of any Borel orbit is a union of Borel orbits, and hence the components of Y are just the maximal closures of Borel orbits. So by Rothbach's theorem (Theorem 4.8), the components of Y are the closures of the orbits associated to valid X^2 words which are maximal among those in Y.

It thus suffices to prove that if w is a maximal valid X^2 word, then any v with $\operatorname{rank}(v) < n$ and $v \leq w$ satisfies $v \leq w^{(i)}$ for some i. For any such v, there is a sequence of moves that we can apply which transforms w into v, and one of these moves must be of type $1'$ (since that is the only type of move which decreases the rank). We must show two things: that we can always make a move of type $1'$ first, and that the words $w^{(i)}$ are maximal among those obtained from w by a move of type $1'$. The first statement is proved in Lemma 5.6, and the second in Lemma 5.7. □

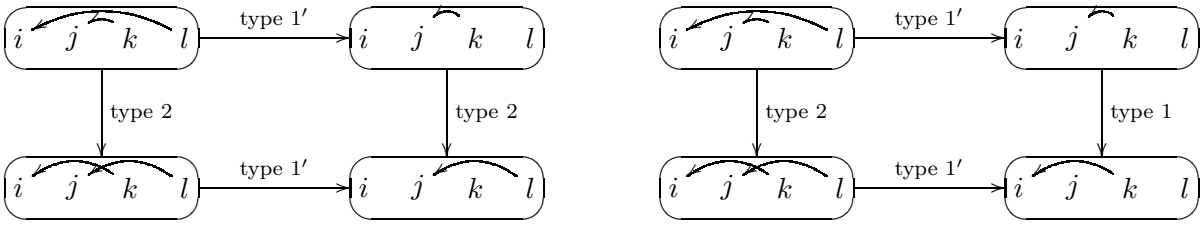
Lemma 5.6. *Let w be a bracket word of length $2n$ and let v be a valid X^2 word such that $\operatorname{rank}(v) < n$ and $v \leq w$. Then there is a valid X^2 word u obtained from w by a move of type $1'$ such that $v \leq u \leq w$.*

Proof. By induction on the number of moves applied on w to get v, it's enough to show that a move of any type followed by a move of type $1'$ is the same as a move of type $1'$ followed by some other move. Note, however, that the letters involved in each of the moves $1'$ may not be the same and that the type of the other move may change as we "commute" it past the move of type $1'$. We prove this in cases, according to the type of move which is being composed with the move of type $1'$. In what follows we will assume that the move of type $1'$ is applied to one of the letters involved in the other move; if this is not the case the two moves commute and the conclusion of the lemma follows immediately.

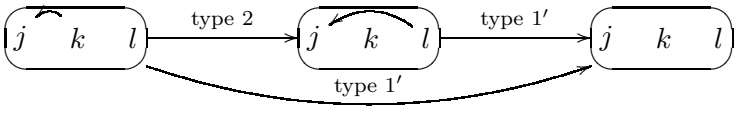
Case I: move 1 followed by move $1'$. A move of type 1 followed by a move of type $1'$ applied to the same index is equal to the move of type $1'$ applied to that index. This is illustrated by the following diagram:



Case II: move 2 followed by move $1'$. Suppose we have a valid X^2 word $a_1 \cdots a_{2n}$, and indices $i < j < k < l$ such that $a_l = i$ and $a_k = j$ (and thus $a_i = a_j = 0$). So we can apply move 2 to the pair a_k, a_l and then apply move $1'$ to either of these letters. The following two commutative squares of diagrams of moves show that the composite of these two moves is always a type $1'$ move followed by a move of type 2 or 1.

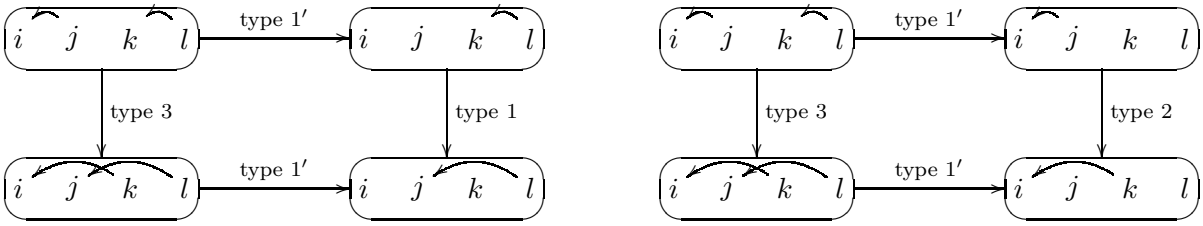


It remains to consider the possibility of a type 2 move which switches two letters of which one is zero. The composite of such a move followed by a type $1'$ move is itself a type $1'$ move, as illustrated by the following diagram.



Case III: move 3 followed by move $1'$. Suppose we are given a valid X^2 word $a_1 \cdots a_{2n}$ to which we can apply a move of type 3. This means that there are indices $i < j < k < l$ such that $a_j = i$ and $a_l = k$ (and hence $a_k = a_i = 0$). After applying move 3 to this word, we can then apply move $1'$ to the letter in the l-th or k-th position, as illustrated in the bottom side of the following two squares. The first square illustrates the subcase where we apply the move of type $1'$ to the

index k, and the second the subcase where we apply the move of type $1'$ to the index l.



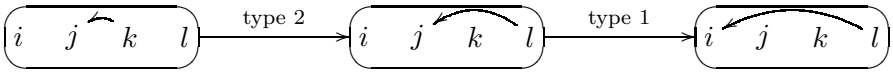
Thus both composites of moves can also be described as a type $1'$ move followed by a move of type 1 or 2. □

Lemma 5.7. *If u is a valid X^2 word with rank $n-1$ obtained from a bracket word w by a move of type $1'$, then $u \leq w^{(s)}$ for some s.*

Proof. Write $w = (a_1, \ldots, a_{2n})$. Let $j < k$ be indices such that $a_k = j$, and such that u is obtained from w by replacing a_k by 0. If u is not equal to $w^{(s)}$ for any s, then there are indices $i < j < k < l$ such that $a_l = i$ (hence $a_i = a_j = 0$), and some $w^{(s)}$ obtained from w by a move of type $1'$ where a_l is replaced by 0. In other words,

$$w = \Big(i \;\overset{\frown}{j}\; \overset{\frown}{k} \;\overset{\frown}{l}\; \Big), \qquad u = \Big(i \;\overset{\frown}{j\;\;k}\; \overset{\frown}{l}\; \Big), \qquad \text{and} \qquad w^{(s)} = \Big(i \quad j \;\overset{\frown}{k}\; l \Big).$$

We now see that u is obtained from $w^{(s)}$ by a type 2 move followed by a type 1 move:



and thus $u \leq w^{(s)}$. □

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