HIGHER LIMITS VIA STEINBERG REPRESENTATIONS

by Bob Oliver

Matematisk Institut
Ny Munkegade
8000 Aarhus C, Denmark

Recently, certain categories based on elementary abelian p-groups (i.e., finite p-groups of the form $E \cong (C_p)^k$) have played an important role as indexing categories for approximating spaces. The construction of approximations to classifying spaces in [JM], and the realization of a certain Dickson algebra as the cohomology algebra of a space in [DW2], both depended on the computation of higher derived functors of inverse limits over such categories. The purpose of this paper is to give a general procedure for doing this involving the Steinberg representation of $\mathrm{GL}_n(\mathbb{F}_p)$. One consequence is an upper bound for the degrees in which higher limits over such categories can be nonvanishing.

As one example, consider the category $\mathcal{A}_p(G)$, defined for any compact Lie group G as follows. An object in $\mathcal{A}_p(G)$ is a nontrivial elementary abelian p-subgroup $1 \neq E \subseteq G$. For any pair E_1, E_2 of such subgroups, $\mathrm{Mor}_{\mathcal{A}_p(G)}(E_1, E_2)$ is the set of monomorphisms from E_1 to E_2 which are composites of inclusions and conjugations in G. This is the category which (for finite G) was used by Quillen [Q1], for approximating $H^*(BG; \mathbb{F}_p)$ up to nilpotence. More recently, it was used by Jackowski & McClure [JM] as an indexing category for approximating the classifying space BG itself as a homotopy direct limit of (frequently) simpler spaces. The higher derived functors of inverse limits of certain covariant functors from $\mathcal{A}_p(G)$ to $\mathcal{A}b$ played an important role in [JM]. One consequence of the results here is that for any p-local covariant functor F on $\mathcal{A}_p(G)$, $\varprojlim^i(F) = 0$ for all $i \geq p\text{-rk}(G)$ (see Theorem 1).

Higher limits over certain orbit categories were handled in [JMO] by first filtering the functors in such a way that each of the quotient functors vanishes except on one single isomorphism class of objects, then analyzing the higher limits of those quotient functors, and finally using long exact sequences to recover the higher limits of the original functor. That process seems quite complicated, but it turned out to be very effective for computing those higher limits over orbit categories needed in [JMO], as well as in later papers by the same authors. The main idea of this paper is to use a similar filtering technique to get information about the higher limits over categories of elementary abelian groups such as $\mathcal{A}_p(G)$.

If $\mathcal{A}$ is the category of nontrivial subgroups of some fixed elementary abelian p-group A, and if $F : \mathcal{A} \to \mathcal{A}b$ is the functor which sends the full group A to $\mathbb{Z}$ and the proper subgroups to 0, then $\varprojlim^i_{\mathcal{A}}(F)$ is zero when $i \neq \mathrm{rk}(A) - 1$, and is isomorphic in a natural way to the dual of the Steinberg representation when $i = \mathrm{rk}(A) - 1$. This follows easily from Lemma 2 below, together with the classical description of Steinberg representations as homology groups of Tits buildings. What is surprising is that the higher limits can also be described in terms of Steinberg representations in more complicated cases. The following theorem is stated here only for $\mathcal{A}_p(G)$; but (as will be seen below) it works equally well for other categories of elementary abelian p-groups.

For any elementary abelian p-group E, we let St_E denote the Steinberg representation of $\mathrm{GL}(E)$.

Theorem 1. *Fix a prime p and a compact Lie group G. Write $\mathcal{A} = \mathcal{A}_p(G)$ for short.*

(i) Assume $F : \mathcal{A} \to \mathbb{Z}_{(p)}$-mod vanishes except on one (conjugacy class of) object E. Set $k = \mathrm{rk}(E)$, and $\Gamma = \mathrm{Aut}_{\mathcal{A}}(E) \subseteq \mathrm{GL}(E)$. Then

$$\varprojlim_{\mathcal{A}}{}^i(F) \cong \begin{cases} \mathrm{Hom}_\Gamma(\mathrm{St}_E, F(E)) & \text{if } i = k-1 \\ 0 & \text{if } i \neq k-1. \end{cases}$$

(ii) For each $k \geq 1$, set

$$\mathcal{E}_k = \mathcal{E}_k(G) = \big\{E \in \mathrm{Ob}(\mathcal{A}) : \mathrm{rk}(E) = k\big\}\big/(\text{isomorphisms}).$$

Then for any functor $F : \mathcal{A} \to \mathbb{Z}_{(p)}$-mod, $\varprojlim^*_{\mathcal{A}}(F)$ is isomorphic to the homology of a cochain complex $(C^*_{\mathrm{St}}(F), \delta)$, where

$$C^i_{\mathrm{St}}(F) \cong \prod_{E \in \mathcal{E}_{i+1}} \mathrm{Hom}_{\mathrm{Aut}_{\mathcal{A}}(E)}(\mathrm{St}_E, F(E)).$$

In particular, $\varprojlim^i(F) = 0$ for $i \geq \mathrm{rk}_p(G)$.

The two parts of Theorem 1 will be proven — also for certain other categories of elementary abelian p-groups — as Propositions 4 and 5 below.

Theorem 1 deals only with covariant functors on $\mathcal{A}_p(G)$. In other words, the limits are always taken in the direction of the smallest subgroups. This is the type of limit which arises in [JM]; but is the opposite of the limits used by Quillen [Q1] to approximate $H^*(BG; \mathbb{F}_p)$.

One of the main results in [JM] was a theorem which says that any *Mackey functor* F defined on $\mathcal{A}_p(G)$ is acyclic: i.e., $\varprojlim^i(F) = 0$ for all $i \geq 0$. Theorem 1 is intended in part to supplement that result, in that it provides a means to compute higher limits for functors which are not Mackey functors.

The abstract definition of $\varprojlim^*_{\mathcal{C}}(F)$ in terms of an injective resolution of F is not very useful when making specific calculations. The following lemma describes these higher limits as the homology of an explicit cochain complex. Alternatively, it can be thought of as saying that they are the cohomology of a certain sheaf over the nerve of $\mathcal{C}$.

Lemma 2. *Let $\mathcal{C}$ be any small category, and let $F : \mathcal{C} \to \mathcal{A}b$ be any covariant functor. Then $\varprojlim^*_{\mathcal{C}}(F) \cong H^*(C^*(\mathcal{C}; F), \delta)$, where*

$$C^n(\mathcal{C}; F) = \prod_{x_0 \to \cdots \to x_n} F(x_n) \tag{1}$$

for all $n \geq 0$; and where for $U \in C^n(\mathcal{C}; F)$,

$$\begin{aligned}\delta(U)(x_0 \to \ldots \to x_n \xrightarrow{\varphi} x_{n+1}) = \sum_{i=0}^{n} (-1)^i U(x_0 \to \ldots \widehat{x_i} \cdots \to x_{n+1}) \\ + (-1)^{n+1} F(\varphi)(U(x_0 \to \cdots \to x_n)).\end{aligned} \tag{2}$$

Proof. Let $\mathcal{C}$-*mod* denote the category of covariant functors from $\mathcal{C}$ to $\mathcal{A}b$. For any F in $\mathcal{C}$-*mod*,

$$\varprojlim_{\mathcal{C}}(F) \cong \mathrm{Mor}_{\mathcal{C}\text{-}mod}(\underline{\mathbb{Z}}, F),$$

where $\underline{\mathbb{Z}}$ denotes the constant functor with values $\mathbb{Z}$. So if (P_*, ∂) is any projective resolution of $\underline{\mathbb{Z}}$ in $\mathcal{C}$-*mod*, then $\varprojlim^*(F)$ is the cohomology of the cochain complex

$$\big(\mathrm{Mor}_{\mathcal{C}\text{-}mod}(P_*, F)\, , \, \mathrm{Mor}(\partial, F)\big).$$

For each $n \geq -1$, define the functor $P_n : \mathcal{C} \to \mathcal{A}b$ as follows. For each object x in $\mathcal{C}$, let $P_n(x)$ be the free abelian group with basis the set of all sequences $x_0 \to \ldots \to x_n \to x$ of morphisms in $\mathcal{C}$ ending in x. For any morphism f in $\mathcal{C}$, $P_n(f)$ is defined by composition in the obvious way. Note that $P_{-1} \cong \underline{\mathbb{Z}}$. Define boundary maps $\partial : P_n \to P_{n-1}$ by setting

$$\partial([x_0 \to \cdots \to x_n \to x]) = \sum_{i=0}^{n} (-1)^i [x_0 \to \ldots \widehat{x_i} \cdots \to x_n \to x].$$

For each x, the chain complex

$$\ldots \xrightarrow{\partial} P_2(x) \xrightarrow{\partial} P_1(x) \xrightarrow{\partial} P_0(x) \xrightarrow{\partial} P_{-1}(x) \to 0$$

is split by the maps $([\cdots \to x_n \to x] \mapsto [\cdots \to x_n \to x \xrightarrow{\mathrm{Id}} x])$; and hence is exact. Thus, (P_*, ∂) is a resolution of $\underline{\mathbb{Z}}$. Also, for any F,

$$\mathrm{Mor}_{\mathcal{C}\text{-}mod}(P_n, F) \cong \prod_{x_0 \to \cdots \to x_n} F(x_n).$$

This shows that P_n is projective, and that $\big(\mathrm{Mor}_{\mathcal{C}\text{-}mod}(P_*, F), \mathrm{Mor}(\partial, F)\big)$ is isomorphic to the complex $(C^*(\mathcal{C}; F), \delta)$ defined in (1) and (2) above. $\square$

The description of higher limits given in Lemma 2 is well known, but we have been unable to find it in the literature — aside from a rather obscurely formulated version in [BK, XI.6.2].

As was noted above, Theorem 1 holds for a wide range of categories based on elementary abelian p-groups; and similar results seem likely to hold for other, related categories. Hence, for the sake of other possible applications, we want to prove Theorem 1 — or at least its essence — in as much generality as possible.

The natural setting for filtering functors and reducing to "single object functors" seems to be that of what we here call *ordered categories*. We define an ordered category to be a category where all endomorphisms are automorphisms. This is the condition formulated by Lück in [Lü] (where he called them "EI-categories"). If $\mathcal{C}$ is such a category, then the set of isomorphism classes in $\mathcal{C}$ is partially ordered by the relation $[x] \le [y]$ if $\mathrm{Mor}(x, y) \neq \emptyset$. And if $\mathcal{C}$ has only finitely many isomorphism classes of objects, then it is easy to see that for any $F : \mathcal{C} \to \mathcal{A}b$, F can be filtered by a sequence $0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_k = F$, such that each F_i/F_{i-1} vanishes except on one isomorphism class of objects.

In order to formulate the results presented here, some more structure on the category is needed. We define a *category with subobjects* to be a pair $\mathcal{C} \supseteq \mathcal{I}$ of categories such that $\mathrm{Ob}(\mathcal{I}) = \mathrm{Ob}(\mathcal{C})$, and such that the following two conditions are satisfied:

(a) $|\mathrm{Mor}_{\mathcal{I}}(x, y)| \le 1$ for any pair of objects x, y; and

(b) each morphism $f \in \mathrm{Mor}_{\mathcal{C}}(x, y)$ can be written in a unique way as a composite $f = i \circ a$, where $a \in \mathrm{Iso}_{\mathcal{C}}(x, x')$ for some x', and $i \in \mathrm{Mor}_{\mathcal{I}}(x', y)$.

The idea for categories with subobjects comes, of course, from the categories of sets, groups, etc. with monomorphisms, where the subcategories consist of inclusions of subobjects. With this in mind, for any category with subobjects $(\mathcal{C}, \mathcal{I})$, and any morphism $f : x \underset{\cong}{\xrightarrow{a}} x' \overset{i}{\hookrightarrow} y$ as above, we write $f(x)$ or $\mathrm{Im}(f)$ for the object x'.

Similarly, we write $x \subseteq y$ if $\mathrm{Mor}_{\mathcal{I}}(x,y) \neq \emptyset$. Note that an inclusion $x \overset{i}{\hookrightarrow} y$ is an isomorphism in $\mathcal{C}$ only if $x = y$ and $i = \mathrm{Id}_x$.

The categories $\mathcal{A}_p(G)$ defined above, and other categories based on elementary abelian p-groups, are all ordered categories, and can all be made into categories with subobjects in an obvious way. In contrast, the orbit categories dealt with in [JMO] are also ordered, but cannot be given the structure of categories with subobjects.

For any category with subobjects $(\mathcal{C}, \mathcal{I})$, and any object x in $\mathcal{C}$, we let $\mathcal{C}_{<x} \subseteq \mathcal{C}_x \subseteq \mathcal{C}$ denote the full subcategories

$$\mathrm{Ob}(\mathcal{C}_x) = \{y \in \mathrm{Ob}(\mathcal{C}) : y \subseteq x\} \qquad \text{and} \qquad \mathrm{Ob}(\mathcal{C}_{<x}) = \{y \in \mathrm{Ob}(\mathcal{C}) : y \subsetneqq x\}.$$

These are equivalent to the full subcategories of objects y such that $[y] \leq [x]$, or $[y] \leq [x]$ and $[y] \neq [x]$, respectively. It is these last categories which appear when one studies functors on $\mathcal{C}$ which vanish except on objects isomorphic to x. The main idea is to compare them with the categories $\mathcal{I}_x = \mathcal{C}_x \cap \mathcal{I}$ and $\mathcal{I}_{<x} = \mathcal{C}_{<x} \cap \mathcal{I}$ of subobjects of x. Note that the automorphism group $\mathrm{Aut}_{\mathcal{C}}(x)$ acts in a natural way on $\mathcal{I}_x$ and $\mathcal{I}_{<x}$ — this follows from property (b) in the definition — but not on $\mathcal{C}_x$ or $\mathcal{C}_{<x}$.

Proposition 3. *Let $(\mathcal{C}, \mathcal{I})$ be any (small) ordered category with subobjects. Fix an object x in $\mathcal{C}$, and set $\Gamma = \mathrm{Aut}_{\mathcal{C}}(x)$. Let $F : \mathcal{C} \to \mathcal{A}b$ be any covariant functor such that $F(y) = 0$ for $y \not\cong x$, and regard $F(x)$ as a $\mathbb{Z}[\Gamma]$-module. Then the following hold.*

(1) $H_*(B\mathcal{C}_x, B\mathcal{C}_{<x}) \cong H_*(E\Gamma \times_\Gamma B\mathcal{I}_x, E\Gamma \times_\Gamma B\mathcal{I}_{<x})$

(2) There are isomorphisms

$$\underset{\overleftarrow{\mathcal{C}}}{\lim}{}^*(F) \cong \underset{\overleftarrow{\mathcal{C}_x}}{\lim}{}^*(F) \cong H^*_\Gamma(B\mathcal{I}_x, B\mathcal{I}_{<x}; F(x)),$$

where $H^*_\Gamma(-;-)$ denotes Borel cohomology with twisted coefficients. The first isomorphism is induced by the inclusion of categories. The second isomorphism is induced by the chain homomorphism Ψ_*, where

$$\Psi_n : \bigoplus_{p+q=n} \mathrm{Hom}_{\mathbb{Z}[\Gamma]}\big(C_q(E\Gamma) \otimes C_p(B\mathcal{I}_x, B\mathcal{I}_{<x}), F(x)\big) \longrightarrow C^n(\mathcal{C}_x; F) \cong \prod_{x_0 \to \cdots \to x_{n-1} \to x} F(x)$$

satisfies the formula

$$\Psi_n(U)\Big(\big[y_0 \xrightarrow{f_1} \dots \xrightarrow{f_{p-1}} y_{p-1} \xrightarrow{f_p} x \xrightarrow{\gamma_1} x \to \dots \xrightarrow{\gamma_q} x\big]\Big)$$
$$= \gamma_q \cdots \gamma_1 \cdot U\Big(\big(1, \gamma_1, \gamma_2\gamma_1, \dots, \gamma_q \cdots \gamma_1\big) \otimes \big[\mathrm{Im}(f_p \cdots f_1) \hookrightarrow \dots \hookrightarrow \mathrm{Im}(f_p) \hookrightarrow x\big]\Big).$$

for $y_i \subsetneqq x$ *and* $U \in \mathrm{Hom}(C_q(E\Gamma) \otimes C_p(B\mathcal{I}_x, B\mathcal{I}_{<x}), F(x))$.

(3) There is a spectral sequence

$$E_2^{pq} \cong H^p(\Gamma; H^q(B\mathcal{I}_x, B\mathcal{I}_{<x}; F(x))) \Longrightarrow \varprojlim_{\mathcal{C}}{}^{p+q}(F).$$

(4) Assume, for some ring $R \subseteq \mathbb{Q}$, *that* $F(x)$ *is an* R*-module, and that* $H_*(B\mathcal{I}_x, B\mathcal{I}_{<x}; R)$ *is* $R[\Gamma]$*-projective. Then for each* $i \geq 0$,

$$\varprojlim_{\mathcal{C}}{}^i(F) \cong \mathrm{Hom}_\Gamma\big(H_i(B\mathcal{I}_x, B\mathcal{I}_{<x}), F(x)\big).$$

Proof. Let $\bar{\mathcal{C}}_x \subseteq \mathcal{C}$ be the full subcategory whose objects are those y such that $[y] \leq [x]$, i.e., such that $\mathrm{Mor}_{\mathcal{C}}(y,x) \neq \emptyset$. From the formula in Lemma 2, it is clear that $\varprojlim_{\mathcal{C}}{}^*(F) \cong \varprojlim_{\bar{\mathcal{C}}_x}{}^*(F)$. And $\varprojlim_{\bar{\mathcal{C}}_x}{}^*(F) \cong \varprojlim_{\mathcal{C}_x}{}^*(F)$, since the categories are equivalent (every object in $\bar{\mathcal{C}}_x$ is isomorphic to an object of $\mathcal{C}_x$). So from now on, we can work within the category $\mathcal{C}_x$.

Consider the subcategory $\mathcal{C}_x^1 \subseteq \mathcal{C}_x$ defined by setting $\mathrm{Ob}(\mathcal{C}_x^1) = \mathrm{Ob}(\mathcal{C}_x)$, $\mathrm{Mor}_{\mathcal{C}_x^1}(y,y') = \mathrm{Mor}_{\mathcal{C}}(y,y')$ if $y \subsetneqq x$, and $\mathrm{Mor}_{\mathcal{C}_x^1}(x,x) = \{\mathrm{Id}_x\}$. Let Γ act on $\mathcal{C}_x^1$ via the identity on objects, via the identity on $\mathrm{Mor}(\mathcal{C}_{<x})$, and via composition on $\mathrm{Mor}_{\mathcal{C}}(y,x)$ for $y \subsetneqq x$.

Step 1 Let

$$C(i): C_*(B\mathcal{I}_x, B\mathcal{I}_{<x}) \longrightarrow C_*(B\mathcal{C}_x^1, B\mathcal{C}_{<x})$$

be the inclusion. Define a retraction

$$r: C_*(B\mathcal{C}_x^1, B\mathcal{C}_{<x}) \to C_*(B\mathcal{I}_x, B\mathcal{I}_{<x})$$

by setting

$$\begin{aligned} r\big(\big[y_0 \xrightarrow{f_0} y_1 \xrightarrow{f_1} \cdots \to y_k \xrightarrow{f_k} x\big]\big) \\ = \big[f_k \cdots f_0(y_0) \hookrightarrow f_k \cdots f_1(y_1) \hookrightarrow \cdots \hookrightarrow f_k(y_k) \hookrightarrow x\big]. \end{aligned}$$

Both of these are homomorphisms of chain complexes, $r \circ C(i)$ is the identity on $C_*(B\mathcal{I}_x, B\mathcal{I}_{<x})$, and r is $\mathbb{Z}[\Gamma]$-linear. We first show that these homomorphisms induce a $\mathbb{Z}[\Gamma]$-linear isomorphism

$$H_*(r): H_*(B\mathcal{C}_x^1, B\mathcal{C}_{<x}) \xrightarrow{\cong} H_*(B\mathcal{I}_x, B\mathcal{I}_{<x}). \tag{5}$$

Set $X_* = \mathrm{Coker}(C(i)) = C_*(B\mathcal{C}^1_x, B\mathcal{C}_{<x} \cup B\mathcal{I}_x)$. We must show that X_* is exact. To do this, define $D : X_n \to X_{n+1}$ by setting

$$
\begin{aligned}
&D\big([y_0 \xrightarrow{f_0} y_1 \xrightarrow{f_1} \cdots \to y_k \xrightarrow{f_k} x]\big) \\
&\qquad = \sum_{i=0}^{k} (-1)^i \big[y_0 \to \cdots \to y_i \xrightarrow{f_k \cdots f_i} f_k \cdots f_i(y_i) \hookrightarrow \cdots \hookrightarrow f_k(y_k) \hookrightarrow x\big].
\end{aligned}
$$

Fix an element $[y_0 \xrightarrow{f_0} \cdots \to y_k \xrightarrow{f_k} x]$ as above, and set $y'_i = f_k \cdots f_i(y_i)$. Then

$$
\begin{aligned}
&D\partial\big([y_0 \to \cdots \to y_k \to x]\big) \\
&\qquad = D\left(\sum_{i=0}^{k} (-1)^i \big[y_0 \to \cdots \widehat{y_i} \cdots \to y_k \to x\big]\right) \\
&\qquad = \sum_{j<i} (-1)^{i+j} \big[y_0 \to \cdots \to y_j \to y'_j \hookrightarrow \cdots \widehat{y'_i} \cdots \hookrightarrow y'_k \hookrightarrow x\big] \\
&\qquad\quad + \sum_{j>i} (-1)^{i+j-1} \big[y_0 \to \cdots \widehat{y_i} \cdots \to y_j \to y'_j \hookrightarrow \cdots \hookrightarrow y'_k \hookrightarrow x\big]
\end{aligned}
$$

and

$$
\begin{aligned}
&\partial D\big([y_0 \to \cdots \to y_k \to x]\big) \\
&\qquad = \partial\left(\sum_{j=0}^{k} (-1)^i \big[y_0 \to \cdots \to y_j \to y'_j \hookrightarrow \cdots \hookrightarrow y'_k \hookrightarrow x\big]\right) \\
&\qquad = \sum_{i<j} (-1)^{i+j} \big[y_0 \to \cdots \widehat{y_i} \cdots \to y_j \to y'_j \hookrightarrow \cdots \hookrightarrow y'_k \hookrightarrow x\big] \\
&\qquad\quad + \sum_{i>j} (-1)^{i+j-1} \big[y_0 \to \cdots \to y_j \to y'_j \hookrightarrow \cdots \widehat{y'_i} \cdots \hookrightarrow y'_k \hookrightarrow x\big] \\
&\qquad\quad + \sum_{i=0}^{k} \Big(\big[y_0 \to \cdots \to y_{j-1} \to y'_j \hookrightarrow \cdots \hookrightarrow y'_k \hookrightarrow x\big] \\
&\qquad\qquad\qquad - \big[y_0 \to \cdots \to y_j \to y'_{j+1} \hookrightarrow \cdots \hookrightarrow y'_k \hookrightarrow x\big]\Big).
\end{aligned}
$$

And since $[y'_0 \hookrightarrow \cdots \hookrightarrow y'_k \hookrightarrow x]$ vanishes in $C_*(B\mathcal{C}^1_x, B\mathcal{C}_{<x} \cup B\mathcal{I}_x)$, this gives

$$
(D\partial + \partial D)\big([y_0 \to \cdots \to y_k \to x]\big) = -\big[y_0 \to \cdots \to y_k \to x\big].
$$

Thus, $D\partial + \partial D = -\mathrm{Id}$, and so X_* is exact.

Step 2 Let $C_*(E\Gamma)$ denote the usual chain complex for $E\Gamma$: $C_n(E\Gamma)$ is the free abelian group with basis consisting of $(n+1)$-tuples $(\gamma_0,\ldots,\gamma_n)$, and Γ acts by right multiplication. Define

$$\Phi : C_*(E\Gamma)\otimes C_*(B\mathcal{C}^1_x, B\mathcal{C}_{<x}) \xrightarrow{\;\cong\;} C_*(B\mathcal{C}_x, B\mathcal{C}_{<x})$$

by setting

$$\Phi\Big(\big(\gamma_0,\gamma_1,\ldots,\gamma_m\big)\otimes\big[y_0\to\ldots\to y_{k-1}\xrightarrow{f} x\big]\Big) = \\ \big[y_0\to\ldots\to y_{k-1}\xrightarrow{(\gamma_0\circ f)} x \xrightarrow{\gamma_1\gamma_0^{-1}} x\to\ldots\xrightarrow{\gamma_m\gamma_{m-1}^{-1}} x\big].$$

Here, $y_i\in \mathrm{Ob}(\mathcal{C}_{<x})$ for each i. Note that we have dropped the degenerate simplices; at least those in $B\mathcal{C}^1_x$ which involve Id_x. Clearly, Φ factors through an isomorphism on $C_*(E\Gamma)\otimes_{\mathbb{Z}\Gamma} C_*(B\mathcal{C}^1_x, B\mathcal{C}_{<x})$ (it sends a basis to a basis), and commutes with boundary maps. It thus induces an isomorphism

$$\Phi_* : H_*(E\Gamma\times_\Gamma B\mathcal{C}^1_x, E\Gamma\times_\Gamma B\mathcal{C}_{<x}) \xrightarrow{\;\cong\;} H_*(B\mathcal{C}_x, B\mathcal{C}_{<x}). \tag{6}$$

Together with Step 1, this proves point (1).

Consider the cochain complex $(C^*(\mathcal{C};F),\delta)$ of Lemma 2, whose cohomology is $\varprojlim^*(F)$. Define isomorphisms

$$\widehat{\Psi}_n : \sum_{p+q=n} \mathrm{Hom}_{\mathbb{Z}[\Gamma]}\Big(C_q(E\Gamma)\otimes C_p(B\mathcal{C}^1_x, B\mathcal{C}_{<x})\,,\,F(x)\Big) \\ \xrightarrow{\;\cong\;} C^n(\mathcal{C}_x;F) \cong \prod_{x_0\to\cdots\to x_{n-1}\to x} F(x)$$

by setting, for $y_i \subsetneqq x$ and $U\in \mathrm{Hom}(C_q\otimes C_p, F(x))$,

$$\widehat{\Psi}_n(U)\Big(\big[y_0\to\ldots\to y_{p-1}\xrightarrow{f} x\xrightarrow{\gamma_1} x\to\ldots\xrightarrow{\gamma_q} x\big]\Big) \\ = (\gamma_q\cdots\gamma_1)\cdot U\Big(\big(1,\gamma_1,\gamma_2\gamma_1,\ldots,\gamma_q\cdots\gamma_1\big)\otimes\big[y_0\to\ldots\to y_{p-1}\xrightarrow{f} x\big]\Big).$$

Equivalently, for $V\in C^n(\mathcal{C}_x;F)$,

$$\widehat{\Psi}_n^{-1}(V)\Big(\big(\gamma_0,\ldots,\gamma_q\big)\otimes\big[y_0\to\ldots\to y_{p-1}\xrightarrow{f} x\big]\Big) \\ = \gamma_q^{-1}\cdot V\Big(\big[y_0\to\ldots\to y_{p-1}\xrightarrow{\gamma_0 f} x\xrightarrow{\gamma_1\gamma_0^{-1}} x\to\ldots\xrightarrow{\gamma_q\gamma_{q-1}^{-1}} x\big]\Big).$$

These commute with the coboundary homomorphisms, and hence induce an isomorphism

$$\lim_{\substack{\longleftarrow \\ \mathcal{C}_x}}{}^*(F) \cong H^*_\Gamma\big(E\Gamma{\times_\Gamma}B\mathcal{C}^1_x, E\Gamma{\times_\Gamma}B\mathcal{C}_{<x}; F(x)\big)$$

where the homology groups are taken with twisted coefficients. And point (2) now follows upon composing this with the isomorphism of Step 1.

The last two points follow from the usual spectral sequence for the cohomology of the Borel construction. □

Before proving Theorem 1, we first look at two other types of categories based on elementary abelian p-groups.

For any space X such that $H^*(X;\mathbb{F}_p)$ is Noetherian, define the category $\mathcal{A}_p(X)$ as follows. An object in $\mathcal{A}_p(X)$ is a pair (E,ψ), where E is an elementary abelian p-group and $\psi : BE \to X$ is a homotopy class of maps such that $H^*(\psi;\mathbb{F}_p)$ is a finite morphism (i.e., it makes $H^*(BE;\mathbb{F}_p)$ into a finitely generated module over $H^*(X;\mathbb{F}_p)$). A morphism in $\mathcal{A}_p(X)$ from (E_1,ψ_1) to (E_2,ψ_2) is a monomorphism $\varphi : E_1 \rightarrowtail E_2$ such that $\psi_2 \circ B\varphi \simeq \psi_1$. By a theorem of Dwyer & Zabrodsky [DZ], for any compact Lie group G, $\mathcal{A}_p(BG)$ is equivalent to the category $\mathcal{A}_p(G)$ defined earlier.

For any unstable noetherian algebra K over the Steenrod algebra A_p, let $\mathcal{A}_p(K)$ denote the category whose objects are pairs (E,ψ), where $E \neq 1$ is an elementary abelian p-group and $\psi : K \to H^*(BE;\mathbb{F}_p)$ is a finite A_p-algebra homomorphism. A morphism from (E_1,ψ_1) to (E_2,ψ_2) is a monomorphism $\varphi : E_1 \rightarrowtail E_2$ such that $H^*(B\varphi;\mathbb{F}_p) \circ \psi_2 = \psi_1$. These categories were first defined and used by Rector [Re]. By a theorem of Lannes ([La1, Théorème 0.4] or [La2, Théorème 0.4]), $\mathcal{A}_p(X) \cong \mathcal{A}_p(H^*(X;\mathbb{F}_p))$ if X is simply connected and $H^*(X;\mathbb{F}_p)$ is noetherian. Dwyer & Wilkerson, in [DW1] and [DW2], have shown the usefulness of $\mathcal{A}_p(K)$, and the importance of higher limits of functors over $\mathcal{A}_p(K)$, when trying to determine whether K can be realized as the cohomology algebra of a space.

For convenience, an object of any of the categories $\mathcal{A}_p(G)$, $\mathcal{A}_p(X)$, or $\mathcal{A}_p(K)$ will be denoted (E,ψ), where ψ is an inclusion $E \hookrightarrow G$, a map $BE \to X$, or a homomorphism $K \to H^*(BE;\mathbb{F}_p)$, respectively. Recall that for any elementary abelian p-group $E \cong (\mathbb{Z}/p)^k$, we write St_E to denote the Steinberg representation of $\mathrm{GL}(E)$.

Proposition 4. *Assume $\mathcal{A}$ is one of the categories $\mathcal{A}_p(G)$ for a compact Lie group G, $\mathcal{A}_p(X)$ for a space X such that $H^*(X;\mathbb{F}_p)$ is noetherian, or $\mathcal{A}_p(K)$ for an unstable noetherian algebra K over the Steenrod algebra A_p. Let*

$$F : \mathcal{A} \longrightarrow \mathbb{Z}_{(p)}\text{-}mod$$

be a covariant functor which vanishes except on the isomorphism class of the object (E,ψ)*. Set* $k=\mathrm{rk}(E)$*, and let* $\Gamma=\mathrm{Aut}_{\mathcal{A}}(E,\psi)\subseteq\mathrm{GL}(E)$*. Then*

$$\lim_{\overleftarrow{\mathcal{A}}}{}^{i}(F)=\begin{cases}\mathrm{Hom}_{\Gamma}(\mathrm{St}_E,F(E,\psi)) & \text{if } i=k-1\\ 0 & \text{if } i\neq k-1.\end{cases}$$

Proof. Let $\mathcal{I}_E\supseteq\mathcal{I}_{<E}$ denote the poset categories of nontrivial subgroups, and proper subgroups, of E, with the induced actions of $\mathrm{GL}(E)$. Then $B\mathcal{I}_E$ is contractible: it is the nerve of a category with final object. So by definition, for any elementary abelian p-group $E\cong(\mathbb{Z}/p)^k$ with $k\geq 1$,

$$H_i(B\mathcal{I}_E,B\mathcal{I}_{<E})\cong\left\{\begin{matrix}H_i(\text{point}) & \text{if } k=1\\ \widetilde{H}_{i-1}(B\mathcal{I}_{<E}) & \text{if } k>1\end{matrix}\right\}\cong\begin{cases}\mathrm{St}_E & \text{if } i=k-1\\ 0 & \text{if } i\neq k-1\end{cases}$$

as modules over $\mathbb{Z}[\mathrm{GL}(E)]\cong\mathbb{Z}[GL_n(\mathbb{F}_p)]$ (cf. [Lu, §1.13], where $B\mathcal{I}_{<E}$ is denoted $S_{II}(E)$). Also, $\mathcal{I}_{<E}$ and $\mathcal{I}_E$ can be identified with the subcategories $\mathcal{I}_{<(E,\psi)}\subseteq\mathcal{I}_{(E,\psi)}\subseteq\mathcal{A}_p(X)$ used in Proposition 3. Finally, $(\mathrm{St}_E)_{(p)}$ is projective as a $\mathbb{Z}_{(p)}[\mathrm{GL}(E)]$-module (cf. [Ro, Theorem 7.4]); and so Proposition 4 follows from Proposition 3. □

For any $k\geq 1$ and any elementary abelian p-group E of rank $k{+}1$, let $\mathcal{I}^2_{<E}\subseteq\mathcal{I}_{<E}$ be the subcategory of objects of codimension at least 2 in E; i.e., the category of proper subgroups of E of rank at most $k-1$. Define

$$R_E:\mathrm{St}_E\longrightarrow\bigoplus_{[E:A]=p}\mathrm{St}_A$$

to be the composite

$$\mathrm{St}_E\cong H_k(B\mathcal{I}_E,B\mathcal{I}_{<E})\xrightarrow{\ \partial\ }H_{k-1}(B\mathcal{I}_{<E},B\mathcal{I}^2_{<E})\cong\bigoplus_{[E:A]=p}\mathrm{St}_A\,.$$

Alternatively, R_E can be thought of as (up to sign) the homomorphism induced by truncating chains of subgroups.

Proposition 4 now implies the following:

Proposition 5. *Fix a prime* p*, and let* $\mathcal{A}$ *be one of the rings* $\mathcal{A}_p(G)$*,* $\mathcal{A}_p(X)$*, or* $\mathcal{A}_p(K)$ *as in Proposition 4. For each* $k\geq 1$*, set*

$$\mathcal{E}_k=\mathcal{E}_k(X)=\{(E,\psi)\in\mathrm{Ob}(\mathcal{A}):\mathrm{rk}(E)=k\}/(\text{isomorphisms})$$

Then for any functor

$$F:\mathcal{A}\to\mathbb{Z}_{(p)}\text{-}mod,$$

$\varprojlim^*_{\mathcal{A}}(F)$ *is isomorphic to the homology of a cochain complex* $(C^*_{\mathrm{St}}(F),\delta)$*, where*

$$C^i_{\mathrm{St}}(F) \cong \prod_{(E,\psi)\in\mathcal{E}_{i+1}} \mathrm{Hom}_{\mathrm{Aut}_{\mathcal{A}}(E)}(\mathrm{St}_E, F(E,\psi)).$$

In particular, if r *denotes the* p*-rank of* G*, or the Krull dimension of* $H^*(X;\mathbb{F}_p)$ *or* K*, then* $\varprojlim^i_{\mathcal{A}}(F) = 0$ *for* $i \geq r$*.*

The coboundary maps δ *are defined as follows. Fix an element* $c \in C^{i-1}_{\mathrm{St}}(F)$*, and choose some* $(E,\psi) \in \mathcal{E}_{i+1}$*. Then the projection of* $\delta(c)$ *onto the factor* $\mathrm{Hom}_{\mathrm{Aut}_{\mathcal{A}}(E)}(\mathrm{St}_E, F(E))$ *is the composite*

$$\mathrm{St}_E \xrightarrow{R_E} \bigoplus_{[E:A]=p} \mathrm{St}_A \xrightarrow{\oplus c(A)} \bigoplus_{[E:A]=p} F(A,\psi|A) \xrightarrow{\oplus F(\mathrm{incl})} F(E,\psi). \qquad (1)$$

Proof. By assumption (or by definition of Krull dimension), $\mathrm{rk}(E) \leq r$ for any (E,ψ) in $\mathcal{A}$. Define subfunctors $F = F_0 \supseteq F_1 \supseteq F_2 \supseteq \cdots \supseteq F_r = 0$ by setting

$$F_i(E,\psi) = \begin{cases} F(E,\psi) & \text{if } \mathrm{rk}(E) > i \\ 0 & \text{if } \mathrm{rk}(E) \leq i. \end{cases}$$

By Proposition 4, for each i,

$$\varprojlim^j(F_i/F_{i+1}) \cong \begin{cases} C^i_{\mathrm{St}}(F) & \text{if } j = i \\ 0 & \text{if } j \neq i. \end{cases}$$

The long exact sequences for extensions of functors now show that $\varprojlim^*_{\mathcal{A}}(F)$ is the cohomology of some cochain complex

$$0 \to C^0_{\mathrm{St}}(F) \xrightarrow{\delta} C^1_{\mathrm{St}}(F) \xrightarrow{\delta} \ldots \xrightarrow{\delta} C^{r-1}_{\mathrm{St}} \to 0.$$

It remains to check the formula for the boundary homomorphisms. Fix an object (E,ψ) in $\mathcal{A}$ of rank $i{+}1$, and write $\mathcal{I} = \mathcal{I}_E$ for short. The inclusion $\iota : \mathcal{I} \hookrightarrow \mathcal{A}$ induces homomorphisms $\iota^* : \varprojlim^*_{\mathcal{A}}(F_j) \to \varprojlim^*_{\mathcal{I}}(F_j|\mathcal{I})$ for each j, and similarly for the quotient functors. In particular, this yields the following commutative diagram

$$\begin{array}{ccc} C^{i-1}_{\mathrm{St}}(F) & \xrightarrow{\delta} & C^i_{\mathrm{St}}(F) \\ \iota^* \downarrow & & \iota^* \downarrow \\ \varprojlim^{i-1}_{\mathcal{I}}(F_{i-1}/F_i) & \xrightarrow{\delta_2} & \varprojlim^i_{\mathcal{I}}(F_i/F_{i+1}) \\ {\scriptstyle \cong \prod_{[E:A]=p} \mathrm{Hom}(\mathrm{St}_A, F(A))} & & {\scriptstyle \cong \mathrm{Hom}(\mathrm{St}_E, F(E))} \end{array}$$

And δ_2 is in turn induced by the coboundary homomorphism

$$R_E : \mathrm{St}_E \cong H_i(B\mathcal{I}_E, B\mathcal{I}_{<E}) \xrightarrow{\partial} H_{i-1}(B\mathcal{I}_{<E}, B\mathcal{I}^2_{<E}) \cong \bigoplus_{[E:A]=p} \mathrm{St}(A);$$

which shows that δ has the form given in (1). $\square$

For functors on $\mathcal{A}_p(X)$ or $\mathcal{A}_p(G)$ which are not p-local, the chain complex must be replaced by a spectral sequence. The same arguments as used above show:

Proposition 6. *Fix a prime p, let $\mathcal{A}$ be one of the rings $\mathcal{A}_p(G)$, $\mathcal{A}_p(X)$, or $\mathcal{A}_p(K)$ as before, and set*

$$\mathcal{E}_k = \mathcal{E}_k(X) = \{(E,\psi) \in \mathrm{Ob}(\mathcal{A}) : \mathrm{rk}(E) = k\}/(\text{isomorphisms})$$

for each $k \geq 1$. Then for any covariant functor

$$F : \mathcal{A} \to \mathcal{A}b,$$

there is a spectral sequence

$$E_1^{ij} \cong \prod_{(E,\psi)\in\mathcal{E}_{i+1}} H^j\big(\mathrm{Aut}_{\mathcal{A}}(E); \mathrm{Hom}(\mathrm{St}_E, F(E))\big) \Longrightarrow \varprojlim_{\mathcal{A}}{}^{i+j}(F);$$

where d_1 has the form described in Proposition 5 above.

I would like to thank Hans-Werner Henn for first suggesting that I look more closely at higher limits over these categories, and for correcting some mistakes in earlier versions of this paper. I would also like to thank all of my colleagues in Heidelberg for their hospitality during a very pleasant visit, during which this work was begun.

References

[BK] P. Bousfield & D. Kan, Homotopy limits, completions, and localizations, Lecture Notes in Math. 304, Springer-Verlag (1972)

[DW1] W. Dwyer & C. Wilkerson, A cohomology decomposition theorem (preprint)

[DW2] W. Dwyer & C. Wilkerson, A new finite loop space at the prime 2 (preprint)

[DZ] W. Dwyer and A. Zabrodsky, Maps between classifying spaces, Algebraic topology, Barcelona, 1986, Lecture Notes in Math. 1298, Springer-Verlag (1987), 106–119

[JM] S. Jackowski and J. McClure, Homotopy approximations for classifying spaces of compact Lie groups, Proceedings of Arcata conference in algebraic topology, Lecture Notes in Math. 1370, Springer-Verlag (1989), 221–234

[JMO] S. Jackowski, J. McClure, and B. Oliver, Homotopy classification of self-maps of BG via G-actions, Annals of Math. 135 (1992), 183–

[La1] J. Lannes, Sur la cohomologie modulo p des p-groupes abéliens élémentaires, Homotopy theory, Proceedings of the Durham symposium 1985, Cambridge Univ. Press (1987), 97–116

[La2] J. Lannes, Sur les espaces fonctionnels dont la source est la classifiant d'un p-groupe abélien élémentaire (preprint)

[Lü] W. Lück, Transformation groups and Algebraic K-theory, Lecture Notes in Math. 1408 (1989)

[Lu] G. Lusztig, The discrete series of GL_n over a finite field, Princeton Univ. Press (1974)

[Q1] D. Quillen, The spectrum of an equivariant cohomology ring I, Annals of Math. 94 (1971), 549–572

[Q2] D. Quillen, Homotopy properties of the poset of nontrivial p-subgroups of a group, Adv. in Math. 28 (1978), 101–128

[Re] D. Rector, Noetherian cohomology rings and finite loop spaces with torsion, Jour. Pure Appl. Algebra 32 (1984), 191–217

[Ro] M. Ronan, Buildings: Main ideas and applications: I, Bull. London Math. Soc. 24 (1992), 1-51