Equivalences of classifying spaces completed at the prime two

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Abstract

We prove here the Martino-Priddy conjecture at the prime 2: the 2-completions of the classifying spaces of two finite groups G and G' are homotopy equivalent if and only if there is an isomorphism between their Sylow 2-subgroups which preserves fusion. This is a consequence of a technical algebraic result, which says that for a finite group G, the second higher derived functor of the inverse limit vanishes for a certain functor $\mathcal{Z}_G$ on the 2-subgroup orbit category of G. The proof of this result uses the classification theorem for finite simple groups.

Received by the editor 25 July 2003.

2000 *Mathematics Subject Classification.* Primary 55R37. Secondary 55P60, 20D05.

Key words and phrases. Classifying space, p-completion, finite simple groups.

Partially supported by UMR 7539 of the CNRS.

Introduction

In their paper [**MP**], John Martino and Stewart Priddy claimed (erroneously) to have shown that for any prime p and any pair G, G' of finite groups, the p-completed classifying spaces $BG^\wedge_p$ and $BG'^\wedge_p$ are homotopy equivalent if and only if the p-local structures of G and of G' are isomorphic in a sense to be made precise below. This was part of a program by those two authors and others to understand the close connection between homotopy properties of $BG^\wedge_p$ (when G is a finite group or a compact Lie group) and the p-local structure of G, using tools which came out of the Sullivan conjecture and its proofs.

In an earlier paper [**BLO**] in collaboration with Carles Broto and Ran Levi, we identified the obstruction groups to constructing a homotopy equivalence between $BG^\wedge_p$ and $BG'^\wedge_p$, given an isomorphism between the p-local structures of G and G'. For odd primes p, these groups have already been shown [**Ol**] to vanish in all cases. The main technical result of this paper, of which the Martino-Priddy conjecture is one consequence, is that these obstruction groups also vanish when $p = 2$. The proof of this result (like the proof of the conjecture for odd primes) depends on the classification theorem for finite simple groups.

Fix a prime p and a finite group G. The *p-subgroup orbit category* of G is the category $\mathcal{O}_p(G)$ whose objects are the p-subgroups of G, and where

$$\mathrm{Mor}_{\mathcal{O}_p(G)}(P, Q) = \mathrm{Map}_G(G/P, G/Q).$$

For the purposes of this paper, it will be more convenient to describe $\mathrm{Mor}_{\mathcal{O}_p(G)}(P, Q)$ in terms of the *transporter sets*

$$N_G(P, Q) = \{x \in G \mid xPx^{-1} \leq Q\}.$$

Then

$$\mathrm{Mor}_{\mathcal{O}_p(G)}(P, Q) = Q \backslash N_G(P, Q),$$

where a coset Qx for $x \in N_G(P, Q)$ corresponds to the map $(gP \mapsto gx^{-1}Q)$ between orbits.

A p-subgroup $P \leq G$ is called *p-centric* if $Z(P)$ is a Sylow p-subgroup of $C_G(P)$, or equivalently if $C_G(P) = Z(P) \times C'_G(P)$ for some subgroup $C'_G(P)$ of order prime to p. Define the functor

$$\mathcal{Z}_G \colon \mathcal{O}_p(G)^{\mathrm{op}} \longrightarrow \mathsf{Ab}$$

by setting $\mathcal{Z}_G(P) = Z(P)$ if P is p-centric in G and $\mathcal{Z}_G(P) = 0$ otherwise, and sending $Qx \in \mathrm{Mor}_{\mathcal{O}_p(G)}(P, Q)$ to the morphism $Z(Q) \xrightarrow{g \mapsto x^{-1}gx} Z(P)$ if P and Q are both p-centric. Our main algebraic result is the following.

Theorem A. *For any finite group G,*

$$\varprojlim_{\mathcal{O}_2(G)}{}^{i}(\mathcal{Z}_G) = 0 \qquad \text{for all } i \geq 2.$$

Proof. In Proposition 2.9, we show that $\varprojlim^i(\mathcal{Z}_G) = 0$ for all $i \geq 2$ if each nonabelian simple group L which appears in the decomposition series of G belongs to a certain class $\mathfrak{L}^{\geq 2}(2)$ (Definition 2.8). We then show that $\mathfrak{L}^{\geq 2}(2)$ contains all alternating groups (Theorem 5.1); all simple groups of Lie type in characteristic two including the Tits group (Theorem 6.2); all simple groups of Lie type in odd characteristic (Theorems 7.5 and 8.13); and all sporadic groups (Theorem 9.1). Theorem A then follows from the classification theorem for finite simple groups. □

Theorem A was motivated by studying equivalences between completed classifying spaces of finite groups. Let p be a prime, let G and G' be finite groups, and let $S \leq G$ and $S' \leq G'$ be Sylow p-subgroups. An isomorphism $\varphi\colon S \xrightarrow{\cong} S'$ is called *fusion preserving* if for all $P, Q \leq S$ and all $P \xrightarrow[\cong]{\alpha} Q$, α is conjugation by an element of G if and only if $\varphi(P) \xrightarrow[\cong]{\varphi\alpha\varphi^{-1}} \varphi(Q)$ is conjugation by an element of G'.

The Martino-Priddy conjecture states that for any prime p, and any pair G, G' of finite groups, $BG^\wedge_p \simeq BG'^\wedge_p$ if and only if there is a fusion preserving isomorphism between Sylow p-subgroups of G and G'. The "only if" part of the conjecture was proved by Martino and Priddy [**MP**]; and follows from the bijection

$$\operatorname{Rep}(P,G) \stackrel{\text{def}}{=} \operatorname{Hom}(P,G)/\operatorname{Inn}(G) \xrightarrow{\;\cong\;} [BP, BG^\wedge_p]$$

for any p-group P and any finite group G. This bijection was originally shown by Mislin [**Mi**], in the proof of his main theorem there. In fact, the main theorem in [**Mi**] states that a group homomorphism $G \longrightarrow G'$ induces an isomorphism in mod p cohomology (hence an equivalence $BG^\wedge_p \simeq BG'^\wedge_p$) if and only if there is a fusion preserving isomorphism between Sylow p-subgroups, so this can be regarded as a precursor to the theorem in [**MP**]. Martino and Priddy also claimed in [**MP**] to prove the "if" part of the conjecture, but an error in their proof was found on [**MP**, p. 129], where a certain functor fails to be well defined on morphisms.

By [**BLO**, Proposition 6.1], given a fusion preserving isomorphism between Sylow p-subgroups of G and G', the obstruction to extending it to a homotopy equivalence $BG^\wedge_p \simeq BG'^\wedge_p$ lies in $\varprojlim^2(\mathcal{Z}_G)$. This is a consequence of Dwyer's p-centric subgroup decomposition of a classifying space [**Dw**, §8], together with the obstruction theory for constructing maps on a homotopy colimit (cf. [**Wo**]). Hence Theorem A implies:

Theorem B (Martino-Priddy conjecture at the prime 2). *For any pair G and G' of finite groups with Sylow 2-subgroups $S \leq G$ and $S' \leq G'$, $BG^\wedge_2 \simeq BG'^\wedge_2$ if and only if there is a fusion preserving isomorphism $S \xrightarrow{\cong} S'$.*

In general, a map $f\colon X \longrightarrow Y$ induces an equivalence $X^\wedge_p \simeq Y^\wedge_p$ between the p-completions if and only if $H^*(f;\mathbb{F}_p)$ is an isomorphism. So Theorem B can also be regarded as a refinement of the theorem of Cartan and Eilenberg [**CE**, Theorem

XII.10.1], that $H^*(G;\mathbb{F}_p)$ is determined by any Sylow p-subgroup $S \le G$ and the G-fusion in S.

We next turn to the question of self equivalences of $BG^\wedge_p$. For any space X, let $\mathrm{Out}(X)$ denote the group of homotopy classes of self homotopy equivalences of X. For any finite group G, any prime p, and any Sylow p-subgroup $S \le G$, let $\mathrm{Aut}_{\mathrm{fus}}(S,G)$ be the group of fusion preserving automorphisms of S, let $\mathrm{Aut}_G(S)$ be the group of automorphisms induced by conjugation by elements of G (i.e., elements of $N_G(S)$), and set

$$\mathrm{Out}_{\mathrm{fus}}(S,G) = \mathrm{Aut}_{\mathrm{fus}}(S,G)/\,\mathrm{Aut}_G(S).$$

Theorem A, when combined with [**BLO**, Theorem 6.2], gives the following description, up to extension, of $\mathrm{Out}(BG^\wedge_p)$.

Theorem C. *For any finite group G with Sylow 2-subgroup $S \le G$, there is a short exact sequence*

$$1 \longrightarrow \varprojlim_{\mathcal{O}_2(G)}{}^1(\mathcal{Z}_G) \longrightarrow \mathrm{Out}(BG^\wedge_2) \longrightarrow \mathrm{Out}_{\mathrm{fus}}(S,G) \longrightarrow 1.$$

In [**Ol**], we showed that when p is odd, the groups $\varprojlim^i(\mathcal{Z}_G)$ vanish for all finite G and all $i \ge 1$. In contrast, when $p = 2$, the groups $\varprojlim^1(\mathcal{Z}_G)$ *can* be nonvanishing. Examples of this include the groups $G = PSL_2(q)$ when $q \equiv \pm 1 \pmod 8$, $G = A_n$ when $n \equiv 2,3 \pmod 4$, and $G = PSL_4(q)$ when $q \equiv 3 \pmod 4$. A simple proof of this when G has dihedral Sylow 2-subgroup is given in Proposition 1.6, and other cases are discussed in Chapter 10.

Theorem B says that the homotopy type of $BG^\wedge_p$ is determined by a Sylow p-subgroup S of G and the G-fusion in S, and so it is natural to ask whether and how that homotopy type can be recovered from the fusion in G. To make this more precise, consider the p-centric orbit category $\mathcal{O}^c_p(G)$: the full subcategory of $\mathcal{O}_p(G)$ whose objects are the p-centric subgroups of G. There is a functor

$$B\colon \mathcal{O}^c_p(G) \longrightarrow \mathsf{hoTop}$$

to the homotopy category which sends a p-centric subgroup $P \le G$ to BP, and which sends a morphism Qx (for $x \in N_G(P,Q)$) to the (homotopy class of) map $BP \longrightarrow BQ$ induced by $(g \mapsto xgx^{-1})$. This clearly depends only on the p-fusion in G. It is this functor which determines $BG^\wedge_p$ up to homotopy, in the following sense:

Theorem D. *For any finite group G and any prime p, the homotopy functor B has a unique lifting (unique up to homotopy) to a functor*

$$\widetilde{B}\colon \mathcal{O}^c_p(G) \longrightarrow \mathsf{Top}$$

and $\big[\underrightarrow{\mathrm{hocolim}}(\widetilde{B})\big]^\wedge_p \simeq BG^\wedge_p$.

Proof. One lifting of B is given by the functor $\widehat{B}(P) = EG \times_G (G/P)$, and $BG^\wedge_p \simeq \big[\underrightarrow{\mathrm{hocolim}}(\widehat{B})\big]^\wedge_p$ by the p-centric decomposition of Dwyer [**Dw**, §8]. So it remains only to show the uniqueness of this lifting. The functor B is a centric diagram in the sense of Dwyer and Kan by [**DK**, Theorem 5.1]. So by their obstruction theory [**DK**, Theorem 1.1], the obstructions to the uniqueness of a lifting

$\widetilde{B}$ lie in $\varprojlim^2(\mathcal{Z}_G)$. Since these groups vanish by Theorem A when $p = 2$, and by [**Ol**, Theorem A] when p is odd, the lifting is unique up to homotopy. □

We now turn to the proof of Theorem A. When G is p-constrained (in particular, when G is p-solvable), then it is relatively simple to prove that the functor $\mathcal{Z}_G$ is acyclic (i.e., $\varprojlim^i(\mathcal{Z}_G) = 0$ for all $i \geq 1$), and this is shown in Proposition 1.17. The general case is, however, much more complicated. When G is an arbitrary finite group, then $\mathcal{Z}_G$ can be filtered by subfunctors $\mathcal{Z}_G^K$, defined for all $K \lhd G$ by setting

$$\mathcal{Z}_G^K(P) = \begin{cases} Z(P) \cap K & \text{if } P \text{ is } p\text{-centric in } G \\ 0 & \text{otherwise.} \end{cases}$$

The idea of the proof of Theorem A is to filter G by a maximal sequence $1 = K_0 \lneqq K_1 \lneqq \cdots \lneqq K_{n-1} \lneqq K_n = G$ of normal subgroups, and then analyze higher limits of the quotient functors $\mathcal{Z}_G^{K_j}/\mathcal{Z}_G^{K_{j-1}}$. In particular, $\varprojlim^i(\mathcal{Z}_G) = 0$ for all $i \geq 2$ if the same holds for higher limits of $\mathcal{Z}_G^{K_j}/\mathcal{Z}_G^{K_{j-1}}$ for all j.

We first reduce the general computation of $\varprojlim^*\bigl(\mathcal{Z}_G^{K_j}/\mathcal{Z}_G^{K_{j-1}}\bigr)$ to the special case where $L = K_j$ is quasisimple (i.e., L is perfect and $L/Z(L)$ is simple) with p-group center $A = K_{j-1} = Z(L)$. This is done in Lemmas 2.1 and 2.4. We then observe that in this case, $\varprojlim^*\bigl(\mathcal{Z}_G^L/\mathcal{Z}_G^A\bigr)$ depends only on L and on $\mathrm{Aut}_G(L)$. This motivates the definition of new functors $\mathcal{Y}_L^\Gamma$ on $\mathcal{O}_p(\Gamma)$, defined for any quasisimple group L with p-group center and any $\Gamma \leq \mathrm{Aut}(L)$ which contains $\mathrm{Inn}(L)$, with the property that

$$\varprojlim^*(\mathcal{Z}_G^L/\mathcal{Z}_G^A) \cong \varprojlim^*(\mathcal{Y}_L^\Gamma)$$

for any $A \lhd L \lhd G$ as above with $\Gamma = \mathrm{Aut}_G(L)$. For example, if L is simple (and identified with $\mathrm{Inn}(L) \lhd \Gamma$), then $\mathcal{Y}_L^\Gamma = \mathcal{Z}_\Gamma^L$. The definition of $\mathcal{Y}_L^\Gamma$ in the general case is given in Definition 2.5.

To simplify notation in the rest of the paper, we then define $\mathfrak{L}^i(p)$ to be the class of simple groups L for which $\varprojlim^i(\mathcal{Y}_{\widetilde{L}}^\Gamma) = 0$ for all choices of central extensions $\widetilde{L}$ of L and $\Gamma \leq \mathrm{Aut}(\widetilde{L})$. We are thus reduced to proving that all simple groups lie in $\mathfrak{L}^{\geq 2}(2)$.

In Chapter 4, we define, for each nonabelian finite simple group L, certain sets $\mathfrak{R}^i(L\,;p)$ of p-subgroups which could "contribute" to $\varprojlim^i(-)$, in a way made precise in Definition 4.1. We then show (Proposition 4.2) that $L \in \mathfrak{L}^i(p)$ if there is a p-centric subgroup $Q \leq L$, which is weakly closed in a Sylow p-subgroup which contains it, and with the property that all subgroups in $\mathfrak{R}^i(L\,;p)$ contain Q up to conjugacy. This is the result which in almost all cases will be used to show $L \in \mathfrak{L}^{\geq 2}(2)$ (simple groups of Lie type in characteristic two are handled in a different way). The last half of Chapter 4 then consists of a series of propositions, each of which gives some conditions to be used when proving that certain subgroups do not lie in $\mathfrak{R}^{\geq 2}(L\,;2)$.

In this way, the paper splits into two halves. Chapters 1–4 involve homological algebra, and reduce the problem to a series of criteria stated in purely group theoretic terms. These criteria are then applied in Chapters 5–9 to the individual groups, to show that $L \in \mathfrak{L}^{\geq 2}(2)$ in all cases. Afterwards, in Chapter 10, some computations of $\varprojlim^1(\mathcal{Z}_G)$ are listed (mostly without proof).

Clearly, as a topologist writing a paper which depends very heavily on the structure of the individual finite simple groups, I had a lot of assistance. Michael Aschbacher, Ron Solomon, and Richard Lyons all gave extensive help in answering my questions about the structure of certain simple groups. I am grateful to Sergey Shpektorov and Ulrich Meierfrankenfeld for sending me their manuscript listing the maximal 2-local subgroups of the monster and the baby monster — which allowed me to fill in the last step in the proof of Theorem A. I had several helpful discussions with George Glauberman and Jesper Grodal during my short visits to the University of Chicago. Also, Jesper Grodal's theorems for computing higher limits using Steinberg complexes [**Gro**], while not used here directly, was used in many of the computations which led to this proof. I am especially indebted to my earlier collaborator, Yoav Segev, who (while not involved in this project) taught me much of what I know about the classification theorem, and especially about the finite simple groups of Lie type. I would also like to thank my colleagues at Northwestern University and the University of Wisconsin for their hospitality while working on many of the later stages of this project. Finally, I would like to thank both referees of this paper, and especially the referee of Chapters 5–9 whose detailed suggestions led to considerable improvements (and corrections) in those chapters.

General notation: We list here, for easy reference, the following notation which will be used throughout the paper.

- $\mathcal{S}_p(G)$ denotes the set of p-subgroups of G
- $\mathrm{Syl}_p(G)$ denotes the set of Sylow p-subgroups of G
- G_p denotes a Sylow p-subgroup of the group G, but *only* when it is abelian and a direct factor of G
- $G^\# = G \smallsetminus \{1\}$ for any group G
- $O_p(G)$ is the maximal normal p-subgroup of G
- $O_{p'}(G)$ is the maximal normal subgroup of G of order prime to p
- C_n, D_n, and Q_n denote cyclic, dihedral, and quaternion groups of order n
- A_n and Σ_n are the alternating and symmetric groups on n elements
- $\Omega_n(P)$ (for a p-group P) is the subgroup generated by all $g \in P$ such that $g^{p^n} = 1$
- $N_G(H,K) = \{x \in G \mid xHx^{-1} \le K\}$ (for $H, K \le G$)
- c_x denotes conjugation by x ($g \mapsto xgx^{-1}$)
- $\mathrm{Hom}_G(H,K) = \big\{c_x \in \mathrm{Hom}(H,K) \,\big|\, x \in N_G(H,K)\big\}$ (for $H, K \le G$)
- $\mathrm{Aut}_G(H) = \mathrm{Hom}_G(H,H) \cong N_G(H)/C_G(H)$, $\quad \mathrm{Out}_G(H) = \mathrm{Aut}_G(H)/\mathrm{Inn}(H)$
- A functor $F\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{Ab}$ is called *acyclic* if $\varprojlim^i(F) = 0$ for all $i > 0$.

CHAPTER 1

Higher limits over orbit categories

Throughout this chapter, p will be a fixed prime. We first fix our notation. For any finite group G, the ***p*-subgroup orbit category** of G is the category $\boldsymbol{\mathcal{O}_p(G)}$ whose objects are the p-subgroups of G, and where

$$\mathrm{Mor}_{\mathcal{O}_p(G)}(P,Q) = Q\backslash N_G(P,Q) \cong \mathrm{Map}_G(G/P, G/Q).$$

Recall that $N_G(P,Q) = \{x \in G \,|\, xPx^{-1} \le Q\}$ (the transporter). This can also be thought of as a category whose objects are orbits G/P of G and whose morphisms are G-maps, but for our purposes it is more convenient to let the objects be subgroups. For any homomorphism $G \xrightarrow{\ \rho\ } G'$ of groups, we let

$$\mathcal{O}_p(G) \xrightarrow{\ \rho_\#\ } \mathcal{O}_p(G')$$

denote the induced functor between orbit categories.

Theorem A is a statement about higher limits of the functor $\mathcal{Z}_G\colon \mathcal{O}_p(G)^{\mathrm{op}} \to \mathsf{Ab}$, defined for any finite group G by setting $\mathcal{Z}_G(P) = Z(P)$ if P is p-centric in G and $\mathcal{Z}_G(P) = 0$ otherwise. As described very briefly in the introduction, when $K \lhd G$ is a normal subgroup, we must also consider the subfunctor

$$\mathcal{Z}_G^K \subseteq \mathcal{Z}_G\colon \mathcal{O}_p(G)^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{-}\mathsf{mod},$$

defined by setting

$$\boldsymbol{\mathcal{Z}_G^K}(P) = \begin{cases} Z(P)\cap K & \text{if } P \text{ is } p\text{-centric in } G \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\mathcal{Z}_G = \mathcal{Z}_G^G$.

Mostly, this chapter is a collection of general results about higher limits of functors over orbit categories. However, some concrete computations of $\varprojlim^i(\mathcal{Z}_G)$ are given in Proposition 1.6 (including cases where $\varprojlim^1(\mathcal{Z}_G) \neq 0$); and the acyclicity of $\mathcal{Z}_G$ and $\mathcal{Z}_G^K$ is proved when G is p-constrained (in particular, when G is p-solvable) in Lemma 1.8 and Proposition 1.17.

1.1. The functor Λ^*

In [**JMO**], certain graded $\mathbb{Z}_{(p)}$-modules $\Lambda^*(G;M)$ are defined, for any prime p, any finite group G, and any $\mathbb{Z}_{(p)}[G]$-module M, by setting

$$\boldsymbol{\Lambda^*(G;M)} = \varprojlim_{\mathcal{O}_p(G)}{}^{*}(\Phi_M^G) \qquad \text{where} \qquad \Phi_M^G(P) = \begin{cases} M & \text{if } P = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that these depend on the prime p, even though that has been suppressed from the notation. We first list some of the basic properties of these groups.

PROPOSITION 1.1. *The following hold for any finite group G.*

(a) *Fix a p-subgroup $P \leq G$, and let $F\colon \mathcal{O}_p(G) \longrightarrow \mathbb{Z}_{(p)}$-mod be any functor which vanishes except on subgroups conjugate to P. Then*
$$\varprojlim_{\mathcal{O}_p(G)}{}^*(F) \cong \Lambda^*(N_G(P)/P; F(P)).$$

(b) *If $H \lhd G$ is a normal subgroup which acts trivially on the $\mathbb{Z}_{(p)}[G]$-module M, then*
$$\Lambda^*(G;M) \cong \begin{cases} \Lambda^*(G/H;M) & \text{if } (p,|H|)=1 \\ 0 & \text{otherwise.} \end{cases}$$

(c) *If $O_p(G) \neq 1$ (if G contains a nontrivial normal p-subgroup), then $\Lambda^*(G;M) = 0$ for all $\mathbb{Z}_{(p)}[G]$-modules M.*

(d) $\Lambda^i(\Sigma_3;(\mathbb{Z}/2)^2) \cong \begin{cases} \mathbb{Z}/2 & \text{if } i=1 \\ 0 & \text{if } i \neq 1 \end{cases}$*, where Σ_3 acts nontrivially on $(\mathbb{Z}/2)^2$.*

(e) *A short exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ of $\mathbb{Z}_{(p)}[G]$-modules induces a long exact sequence*
$$\longrightarrow \Lambda^i(G;M') \longrightarrow \Lambda^i(G;M) \longrightarrow \Lambda^i(G;M'') \longrightarrow \Lambda^{i+1}(G;M') \longrightarrow .$$

PROOF. See [**JMO**, Propositions 5.4, 5.5, 6.1, & 6.2]. Point (d) is a special case of [**JMO**, Proposition 6.2(i)], included here because that particular computation will be needed later, and because it is the simplest example where $\Lambda^i(G;M) \neq 0$ for $i > 0$. □

Proposition 1.1(b) motivates the following definition.

DEFINITION 1.2. *For any finite group G and any $\mathbb{Z}_{(p)}[G]$-module M, we say that G acts **p-faithfully** on M if the kernel of the action* $\mathrm{Ker}[G \longrightarrow \mathrm{Aut}(M)]$ *has order prime to p.*

The following lemma is an immediate consequence of Proposition 1.1(b,e).

LEMMA 1.3. *If the action of a finite group G on a finite $\mathbb{Z}_{(p)}[G]$-module M is not p-faithful, then $\Lambda^*(G;M) = 0$.* □

A ***radical p-subgroup*** of a finite group G is a p-subgroup $P \leq G$ such that $O_p(N_G(P)/P) = 1$; i.e., such that $N_G(P)/P$ has no nontrivial normal p-subgroups. If P is a p-subgroup of G which is not radical, then $\Lambda^*(N_G(P)/P;M) = 0$ for all $\mathbb{Z}_{(p)}[N_G(P)/P]$-modules M by Proposition 1.1(c).

Proposition 1.1 is usually applied by filtering an arbitrary functor on the orbit category $\mathcal{O}_p(G)$ in such a way that each quotient functor vanishes except on one conjugacy class. By Proposition 1.1(a), the higher limits of these quotient functors can then be described in terms of the graded groups Λ^*. The following lemma describes some ways this can be done, but covers only those cases which will be needed later.

LEMMA 1.4. *The following hold for any finite group G, and any functor*
$$F\colon \mathcal{O}_p(G)^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{-mod}.$$

(a) *Assume, for each radical p-subgroup $P \leq G$ such that $F(P) \neq 0$, that the $N_G(P)/P$-action on $F(P)$ is not p-faithful. Then* $\varprojlim^*(F) = 0$.

(b) *Let $\mathcal{H} \subseteq \mathcal{S}_p(G)$ be a subset such that $Q \in \mathcal{H}$ and $P \geq gQg^{-1}$ ($g \in G$) implies $P \in \mathcal{H}$. Then there is a quotient functor of F,*

$$F_{\mathcal{H}} \colon \mathcal{O}_p(G)^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{-}\mathtt{mod} \quad \textit{defined by} \quad F_{\mathcal{H}}(P) = \begin{cases} F(P) & \text{if } P \in \mathcal{H} \\ 0 & \text{if } P \notin \mathcal{H}. \end{cases}$$

If, for all radical p-subgroups $P \leq G$ such that $P \notin \mathcal{H}$, the action of $N_G(P)/P$ on $F(P)$ is not p-faithful, then

$$\varprojlim_{\mathcal{O}_p(G)}{}^*(F) \cong \varprojlim_{\mathcal{O}_p(G)}{}^*(F_{\mathcal{H}}) \cong \varprojlim_{\mathcal{O}_p^{\mathcal{H}}(G)}{}^*(F|_{\mathcal{O}_p^{\mathcal{H}}(G)}); \tag{1}$$

where $\mathcal{O}_p^{\mathcal{H}}(G) \subseteq \mathcal{O}_p(G)$ is the full subcategory with object set $\mathcal{H}$.

Proof. In the situation of (a), $\Lambda^*(N_G(P)/P; F(P)) = 0$ for all $P \in \mathcal{S}_p(G)$: either because $F(P) = 0$, or by Proposition 1.1(b,c). Hence $\varprojlim^*(F) = 0$ by Proposition 1.1(a), together with the obvious filtration of F and the exact sequences for higher limits of extensions of functors.

It remains to prove (b). For each $P \in \mathcal{S}_p(G)$, let $\Phi(P) \colon F(P) \longrightarrow F_{\mathcal{H}}(P)$ be the identity if $P \in \mathcal{H}$, and the trivial map otherwise ($F_{\mathcal{H}}(P) = 0$). By the assumption on $\mathcal{H}$, for any $\alpha \in \mathrm{Mor}_{\mathcal{O}_p(G)}(P, Q)$, with induced homomorphism $F(\alpha)$ from $F(Q)$ to $F(P)$, either $F_{\mathcal{H}}(Q) = F(Q)$ or $F_{\mathcal{H}}(P) = 0$ (or both). This proves that Φ is a natural epimorphism of functors, and thus that $F_{\mathcal{H}}$ is a quotient functor of F. If, for all radical p-subgroups $P \leq G$ such that $P \notin \mathcal{H}$, the action of $N_G(P)/P$ on $F(P)$ is not p-faithful, then $\varprojlim^*(\mathrm{Ker}(\Phi)) = 0$ by (a), and this proves the first isomorphism in (1).

Any injective resolution of $F|_{\mathcal{O}_p^{\mathcal{H}}(G)}$ can be extended to an injective resolution of $F_{\mathcal{H}}$ by assigning to all functors the value zero on objects not in $\mathcal{H}$, and this shows that $\varprojlim^*(F_{\mathcal{H}}) \cong \varprojlim^*(F|_{\mathcal{O}_p^{\mathcal{H}}(G)})$. □

Proposition 1.1(c) and Lemma 1.4 show the important role played by radical p-subgroups when working with higher limits of functors on orbit categories. The following lemma lists some conditions for a p-subgroup to be radical or not.

Lemma 1.5. *The following hold for any finite group G.*

(a) *If G splits as a product $G = \prod_{i \in I} G_i$, then each radical p-subgroup $P \leq G$ is of the form $P = \prod_{i \in I} P_i$, where $P_i \leq G_i$.*

(b) *Let $H \lhd G$ be any normal subgroup. Then for each radical p-subgroup P in G, $P \cap H$ is radical in H, and $P \geq H$ if H is a p-group. Conversely, if Q is radical in H, then $Q = P \cap H$ for some radical p-subgroup P in G.*

Proof. **(a)** If $P \leq G = \prod_{i \in I} G_i$ does not split as a product, then let $P_i \leq G_i$ be the image of P under the projection, and set $P' = \prod_{i \in I} P_i \leq G$. Then $P' \gneqq P$ by assumption, and $N_G(P') \geq N_G(P)$. Thus $N_{P'}(P)/P$ is a nontrivial normal p-subgroup of $N_G(P)/P$, and P is not p-radical.

(b) Fix $H \lhd G$. Assume first that $P \cap H$ is not radical in H, and set $Q = O_p(N_H(P\cap H)) \gneqq (P\cap H)$. Then $QP \gneqq P$ (since $Q \le H$), and so $N_{QP}(P)/P \ne 1$ by Lemma 1.10. Furthermore, $N_{QP}(P)$ is normalized by $N(P)$ since Q is, $N_{QP}(P)/P$ is thus a nontrivial normal p-subgroup of $N_G(P)/P$, and so P is not radical in G.

Conversely, fix a radical p-subgroup Q in H, and set $P = O_p(N_G(Q))$. Then $P \cap H$ is a normal p-subgroup of $N_H(Q)$ which contains Q, and hence $P \cap H = Q$. Clearly, $N_G(P) \ge N_G(Q)$; they are equal since any $x \in N_G(P)$ normalizes $P \cap H = Q$; and thus P is radical. □

To give a quick illustration of how these techniques are applied, we describe the computation of $\varprojlim^i(\mathcal{Z}_G)$ in certain very simple cases. In particular, the following proposition shows that these groups are nonvanishing when $i = 1$ and G is any of the simple groups A_6, A_7, or $PSL_2(q)$ for $q \equiv \pm 1 \pmod 8$.

PROPOSITION 1.6. *Fix a finite group G and a Sylow subgroup $S \in \mathrm{Syl}_2(G)$. Assume that S is a dihedral group of order ≥ 8, and let T_1, T_2 be S-conjugacy class representatives for the subgroups isomorphic to C_2^2. Then*

$$\varprojlim{}^1(\mathcal{Z}_G) \cong \mathbb{Z}/2$$

if $\mathrm{Aut}_G(T_i) \cong \Sigma_3$ *for* $i = 1, 2$, *and* $\varprojlim^1(\mathcal{Z}_G) = 0$ *otherwise. In all cases,* $\varprojlim^i(\mathcal{Z}_G) = 0$ *for all* $i \ge 2$.

PROOF. Assume $P \le S$ is such that $\Lambda^*(N_G(P)/P; \mathcal{Z}_G(P)) \ne 0$. Then P is 2-centric in G (since $\mathcal{Z}_G(P) = 0$ otherwise), so $PC_G(P)/P$ has odd order, and

$$\Lambda^*(\mathrm{Out}_G(P); \mathcal{Z}_G(P)) \cong \Lambda^*(N_G(P)/PC_G(P); Z(P)) \cong \Lambda^*(N_G(P)/P; Z(P)) \ne 0$$

by Proposition 1.1(b). Hence $O_2(\mathrm{Out}_G(P)) = 1$ (Proposition 1.1(c)). If P is cyclic of order ≥ 4 or dihedral of order ≥ 8, then $\mathrm{Out}(P)$ is a nontrivial 2-group, so these cases cannot occur. So we are left only with the cases

$$\Lambda^i(\mathrm{Out}_G(P); Z(P)) \cong \begin{cases} \mathbb{Z}/2 & \text{if } P = S,\ i = 0 \\ \mathbb{Z}/2 & \text{if } P \cong C_2^2,\ \mathrm{Aut}_G(P) \cong \Sigma_3,\ i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The computation of $\Lambda^*(\Sigma_3; (\mathbb{Z}/2)^2)$ follows from Proposition 1.1(d).

By Alperin's fusion theorem (cf. [**Gor**, Theorem 7.2.6]), and since $\langle T_1, T_2 \rangle = S$, T_1 and T_2 cannot be G-conjugate. So using Proposition 1.1(a) and the obvious filtration of $\mathcal{Z}_G$ by subgroups, we see that $\varprojlim^i(\mathcal{Z}_G) = 0$ for all $i \ge 2$; and obtain an exact sequence

$$0 \longrightarrow \varprojlim_{\mathcal{O}_2(G)}{}^0(\mathcal{Z}_G) \longrightarrow \mathbb{Z}/2 \longrightarrow (\mathbb{Z}/2)^k \longrightarrow \varprojlim_{\mathcal{O}_2(G)}{}^1(\mathcal{Z}_G) \longrightarrow 0,$$

where k is the number of $i = 1, 2$ such that $\mathrm{Aut}_G(T_i) \cong \Sigma_3$. If $k \ge 1$, then $Z(S)$ is G-conjugate to another subgroup of S, and this implies that $\varprojlim^0(\mathcal{Z}_G) = 0$. Thus $\varprojlim^1(\mathcal{Z}_G)$ has rank $k-1$ in this case, and is trivial otherwise. □

1.2. Fixed point and norm functors

Fix a finite group G. For any subgroup $H \le G$, set $\mathfrak{N}_H = \sum_{h\in H} h \in \mathbb{Z}[G]$. For any prime p and any $\mathbb{Z}_{(p)}[G]$-module M, we consider the functors

$$H^0M,\ \mathfrak{N}M\colon \mathcal{O}_p(G)^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{-}\mathtt{mod},$$

defined by setting

$$H^0M(P) = H^0(P;M) = M^P \qquad \text{and} \qquad \mathfrak{N}M(P) = \mathfrak{N}_P{\cdot}M.$$

The next proposition plays a central role — almost as important as that of the functors $\Lambda^*(G;M)$ — when computing higher limits over orbit categories.

PROPOSITION 1.7. *For any finite group G, any prime p, and any $\mathbb{Z}_{(p)}[G]$-module M,*

$$\varprojlim_{\mathcal{O}_p(G)}{}^i (H^0M) = \begin{cases} M^G & \text{if } i=0 \\ 0 & \text{if } i>0 \end{cases} \quad \textit{and} \quad \varprojlim_{\mathcal{O}_p(G)}{}^i (\mathfrak{N}M) = \begin{cases} \mathfrak{N}_G{\cdot}M & \text{if } i=0 \\ 0 & \text{if } i>0\,. \end{cases}$$

PROOF. Set $\widehat{H}^0M = H^0M/\mathfrak{N}M$; thus $\widehat{H}^0M(P) = \widehat{H}^0(P;M)$ for all P. These functors are all proto-Mackey functors in the sense of [**JM**], and hence are acyclic by [**JM**, Proposition 5.14]. The complete description of the higher limits of H^0M and $\widehat{H}^0M$ is shown in [**JMO**, Proposition 5.2], and that of $\mathfrak{N}M$ follows immediately. □

As a first application of Proposition 1.7, we show that $\mathcal{Z}_G$ is acyclic whenever G contains a normal p-centric subgroup.

LEMMA 1.8. *Assume G is a finite group with a normal p-subgroup $Q \lhd G$ which is p-centric in G. Then $\mathcal{Z}_G$ is acyclic. More generally, the functor $\mathcal{Z}_G^K$ is acyclic for any normal subgroup $K \lhd G$.*

PROOF. Let $\mathcal{O}_p^*(G) \subseteq \mathcal{O}_p(G)$ be the full subcategory whose objects are the p-subgroups which contain Q. By Lemma 1.5(b), all radical p-subgroups of G contain Q. So by Lemma 1.4(b), for all $K \lhd G$,

$$\varprojlim_{\mathcal{O}_p(G)}{}^*(\mathcal{Z}_G^K) \cong \varprojlim_{\mathcal{O}_p^*(G)}{}^*(\mathcal{Z}_G^K|_{\mathcal{O}_p^*(G)}).$$

Set $M = Z(Q)\cap K$. For any p-subgroup $P \le G$ containing Q, $Z(P) = Z(Q)^{P/Q}$ (since Q is p-centric in G), and hence

$$\mathcal{Z}_G^K(P) = Z(P) \cap K \cong M^{P/Q}.$$

Thus $\mathcal{Z}_G^K|_{\mathcal{O}_p^*(G)} \cong H^0M|_{\mathcal{O}_p^*(G)}$. So if we identify the categories $\mathcal{O}_p^*(G) \cong \mathcal{O}_p(G/Q)$ by sending $P \in \mathrm{Ob}(\mathcal{O}_p^*(G))$ to $P/Q \in \mathrm{Ob}(\mathcal{O}_p(G/Q))$, then

$$\varprojlim_{\mathcal{O}_p(G)}{}^*(\mathcal{Z}_G^K) \cong \varprojlim_{\mathcal{O}_p(G/Q)}{}^*\ (H^0M).$$

The functor H^0M is acyclic by Proposition 1.7, and so $\mathcal{Z}_G^K$ is also acyclic. Since $\mathcal{Z}_G = \mathcal{Z}_G^G$, this functor is also acyclic. □

1.3. Elementary group theory lemmas

We give here, for later reference, two standard lemmas in elementary group theory. We start with the "Frattini argument".

LEMMA 1.9. *For any finite group G, any normal subgroup $H \lhd G$, and any $P \in \mathrm{Syl}_p(H)$, $G = H{\cdot}N_G(P)$.*

PROOF. For any $g \in G$, $gPg^{-1} \in \mathrm{Syl}_p(H)$, and hence is H-conjugate to P. So there is $h \in H$ such that $hg \in N_G(P)$, and thus $g \in H{\cdot}N_G(P)$. □

We will also need the following lemma about normalizers of p-subgroups.

LEMMA 1.10. *Fix subgroups $P, H \le G$ such that P is a p-subgroup which normalizes H, and $P \cap H \notin \mathrm{Syl}_p(H)$. Then $P \cap H \notin \mathrm{Syl}_p(N_H(P))$, and*

$$p \big| |N_{HP}(P)/P| = |N_H(P)/(P \cap H)|.$$

PROOF. By assumption, $p\big|[H{:}P \cap H]$, and $[H{:}P \cap H] = [HP{:}P]$ since P normalizes H. Hence $N_{HP}(P)/P$ also has order a multiple of p (since $N_{HP}(P)/P$ is the fixed point set of a P-action on HP/P). Also,

$$N_{HP}(P)/P = \big(N_H(P){\cdot}P\big)/P \cong N_H(P)/\big(P \cap N_H(P)\big) = N_H(P)/(P \cap H),$$

and thus $P \cap H \notin \mathrm{Syl}_p(N_H(P))$. □

1.4. Reduction to smaller orbit categories

We next list some conditions which allow us, in certain situations, to reduce the computation of higher limits of a functor on $\mathcal{O}_p(G)$ to those of a functor on the orbit category of a smaller group.

LEMMA 1.11. *Fix a finite group G and a p-subgroup $Q \le G$. Then there is a well defined functor*

$$\Psi_Q^G \colon \mathcal{O}_p(N_G(Q)/Q) \longrightarrow \mathcal{O}_p(G)$$

such that $\Psi_Q^G(P/Q) = P$ for all $P/Q \le N_G(Q)/Q$. Define

$$\mathcal{T} = \big\{P \le G \,\big|\, Q \lhd P, \text{ and } x \in G,\ xQx^{-1} \lhd P \text{ implies } x \in N_G(Q)\big\}.$$

Then for any functor $F \colon \mathcal{O}_p(G)^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}$-mod which vanishes except on subgroups G-conjugate to subgroups in $\mathcal{T}$, the induced homomorphism

$$\varprojlim_{\mathcal{O}_p(G)}{}^{*}(F) \xrightarrow[\cong]{\Psi_Q^{G*}} \varprojlim_{\mathcal{O}_p(N_G(Q)/Q)}{}^{*} (F \circ \Psi_Q^G) \tag{1}$$

is an isomorphism.

PROOF. Set $\Psi = \Psi_Q^G$ for short. Clearly, Ψ is well defined on objects. To see that it is well defined on morphisms, recall first that

$$\mathrm{Mor}_{\mathcal{O}_p(G)}(P, P') = P' \backslash N_G(P, P'),$$

where $N_G(P,P')$ is the set of all $x\in G$ such that $xPx^{-1}\le P'$. Hence for any pair of objects P/Q and P'/Q in $\mathcal{O}_p(N_G(Q)/Q)$,

$$\begin{aligned}\mathrm{Mor}_{\mathcal{O}_p(N_G(Q)/Q)}(P/Q,P'/Q) &= (P'/Q)\backslash N_{N(Q)/Q}(P/Q,P'/Q)\\ &\cong P'\backslash N_{N(Q)}(P,P')\subseteq P'\backslash N_G(P,P') = \mathrm{Mor}_{\mathcal{O}_p(G)}(P,P');\end{aligned}$$

and Ψ is defined on morphism sets to be this inclusion.

Composition with Ψ is natural in F and preserves short exact sequences of functors. Hence if $F'\subseteq F$ is a pair of functors from $\mathcal{O}_p(G)$ to $\mathbb{Z}_{(p)}$-$\mathtt{mod}$, and the lemma holds for F' and for F/F', then it also holds for F by the 5-lemma. It thus suffices to prove that (1) is an isomorphism when F vanishes except on the G-conjugacy class of one subgroup $P\in\mathcal{T}$. When $P=Q$, then (1) is precisely the isomorphism $\varprojlim^*(F)\cong\Lambda^*(N(Q)/Q;F(Q))$ of Proposition 1.1(a).

Now let $P\in\mathcal{T}$ be arbitrary. By definition, $Q\lhd P$, and if $x\in G$ is such that $Q\lhd xPx^{-1}$, then $x^{-1}Qx\lhd P$ implies $x\in N_G(Q)$. Thus $N_G(P)\le N_G(Q)$, and $F\circ\Psi$ vanishes except on the $N_G(Q)/Q$-conjugacy class of P/Q. Let

$$\Psi' = \Psi_{P/Q}^{N(Q)/Q}\colon \mathcal{O}_p(N_G(P)/P)\longrightarrow \mathcal{O}_p(N_G(Q)/Q)$$

be the functor $\Psi'(R/P)=R/Q$ for p-subgroups $R\le N_G(P)\le N_G(Q)$ containing P. Then the following square commutes

$$\begin{array}{ccc}\varprojlim_{\mathcal{O}_p(G)}^*(F) & \xrightarrow{\Psi^*} & \varprojlim_{\mathcal{O}_p(N(Q)/Q)}^*(F\circ\Psi)\\ {\scriptstyle(\Psi\circ\Psi')^*}\Big\downarrow\cong & & {\scriptstyle\Psi'^*}\Big\downarrow\cong\\ \Lambda^*(N_G(P)/P;F(P)) & = & \Lambda^*(N_G(P)/P;F(P)),\end{array}$$

and the vertical maps are isomorphisms by Proposition 1.1(a) (see the proof of [**JMO**, Lemma 5.4] for the precise description of the isomorphisms). This shows that Ψ^* is an isomorphism. □

We will need three special cases of Lemma 1.11: the first and third will be applied in Chapter 3, and the second in Chapter 2.

Lemma 1.12. *Fix a finite group G and a normal subgroup $H\lhd G$. Fix a p-subgroup $Q\le H$, and let $F\colon\mathcal{O}_p(G)^{\mathrm{op}}\longrightarrow\mathbb{Z}_{(p)}$-$\mathtt{mod}$ be any functor such that $F(P)=0$ whenever $P\cap H$ is not G-conjugate to Q. Then the induced homomorphism*

$$\varprojlim_{\mathcal{O}_p(G)}^*(F)\ \xrightarrow[\cong]{\Psi_Q^{G*}}\ \varprojlim_{\mathcal{O}_p(N(Q)/Q)}^*(F\circ\Psi_Q^G),$$

is an isomorphism, where Ψ_Q^G is the functor of Lemma 1.11.

Proof. If $P\cap H=Q$, then $Q\lhd P$, and for any $x\in G$ such that $xQx^{-1}\le P$ we have $xQx^{-1}\le P\cap H=Q$ and hence $x\in N_G(Q)$. Thus $P\in\mathcal{T}$ in the notation of Lemma 1.11. □

The next lemma is really a special case of the last one, combined with other results shown earlier in the chapter.

LEMMA 1.13. *Fix a finite group G and a normal subgroup $H \lhd G$. Assume that $F\colon \mathcal{O}_p(G)^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{-}\mathsf{mod}$ is a functor such that $N_{HP}(P)/P$ acts trivially on $F(P)$ for all radical p-subgroups $P \le G$. Then for any $Q \in \mathrm{Syl}_p(H)$, the induced homomorphism*

$$\varprojlim_{\mathcal{O}_p(G)}{}^{*}(F) \xrightarrow[\cong]{\Psi_Q^{G*}} \varprojlim_{\mathcal{O}_p(N(Q)/Q)}{}^{*}(F \circ \Psi_Q^G),$$

is an isomorphism, where Ψ_Q^G is the functor of Lemma 1.11.

PROOF. Let F_0 be the functor $F_0(P) = F(P)$ if $P \cap H \in \mathrm{Syl}_p(H)$ and $F_0(P) = 0$ otherwise, regarded as a quotient functor of F. If $P \cap H \notin \mathrm{Syl}_p(H)$, then either P is not radical, or $p \big| |N_{HP}(P)/P|$ (Lemma 1.10) and thus the action of $N_G(P)/P$ is not p-faithful. Hence $\varprojlim^*(F) \cong \varprojlim^*(F_0)$ by Lemma 1.4(b); while $\varprojlim^*(F_0) \cong \varprojlim^*(F \circ \Psi)$ by Lemma 1.12. □

In the next lemma, recall that for any finite group G and any $S \in \mathrm{Syl}_p(G)$, a subgroup $P \le S$ is *weakly closed* in S with respect to G if P is not G-conjugate to any other subgroup of S.

LEMMA 1.14. *Fix a finite group G, and subgroups $Q \le S \in \mathrm{Syl}_p(G)$ such that Q is weakly closed in S with respect to G. Let $F\colon \mathcal{O}_p(G)^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{-}\mathsf{mod}$ be a functor such that $F(P) = 0$ whenever P does not contain a subgroup G-conjugate to Q. Then the induced homomorphism*

$$\varprojlim_{\mathcal{O}_p(G)}{}^{*}(F) \xrightarrow[\cong]{\Psi_Q^{G*}} \varprojlim_{\mathcal{O}_p(N(Q)/Q)}{}^{*}(F \circ \Psi_Q^G),$$

is an isomorphism, where Ψ_Q^G is the functor of Lemma 1.11.

PROOF. For each $P \ge Q$, there is $x \in G$ such that $xPx^{-1} \le S$; $xQx^{-1} = Q$ since Q is weakly closed in S, and thus $x \in N_G(Q)$. If $y \in G$ is such that $yQy^{-1} \lhd xPx^{-1}$, then $y \in N_G(Q)$ since Q is weakly closed in S (hence in xPx^{-1}). In particular, $Q \lhd xPx^{-1}$, and this shows that $xPx^{-1} \in \mathcal{T}$ in the notation of Lemma 1.11. Thus every P such that $F(P) \ne 0$ is G-conjugate to some $P' \in \mathcal{T}$, and the result follows from Lemma 1.11. □

The following lemma is very similar in nature to Lemma 1.11. Recall that $\varphi_\#$ denotes the functor between orbit categories induced by a group homomorphism φ.

LEMMA 1.15. *Fix a surjection $\varphi\colon G \twoheadrightarrow G'$ of finite groups. Then for any functor $\bar{F}\colon \mathcal{O}_p(G')^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{-}\mathsf{mod}$,*

$$\varprojlim_{\mathcal{O}_p(G)}{}^{*}(\bar{F} \circ \varphi_\#) \cong \varprojlim_{\mathcal{O}_p(G')}{}^{*}(\bar{F}).$$

More generally, set $H = \mathrm{Ker}(\varphi)$, and let $\mathcal{O}_p^*(G) \subseteq \mathcal{O}_p(G)$ be the full subcategory whose objects are the p-subgroups $P \le G$ such that $P \cap H \in \mathrm{Syl}_p(H)$. Then

$$\varprojlim_{\mathcal{O}_p(G)}{}^{*}(F) \cong \varprojlim_{\mathcal{O}_p(G')}{}^{*}(\bar{F}) \qquad (1)$$

for any $\bar{F}\colon \mathcal{O}_p(G')^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{-}\mathtt{mod}$ *and* $F\colon \mathcal{O}_p(G)^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{-}\mathtt{mod}$ *such that*

$$(\bar{F}\circ\varphi_{\#})|_{\mathcal{O}_p^*(G)} \cong F|_{\mathcal{O}_p^*(G)}, \tag{2}$$

and such that the action of $N_{HP}(P)/P$ *on* $F(P)$ *is trivial for all* $P\in\mathcal{S}_p(G)$.

Proof. It suffices to prove the last statement, since $N_{HP}(P)/P$ acts trivially on $\bar{F}(\varphi(P))$ for each P.

For all $P\in\mathcal{S}_p(G)$ such that $P\cap H\notin \mathrm{Syl}_p(H)$, $p\big||N_{HP}(P)/P|$ (see Lemma 1.10), and $N_{HP}(P)/P$ acts trivially on $F(P)$. Hence by Lemma 1.4(b),

$$\varprojlim_{\mathcal{O}_p(G)}{}^*(F) \cong \varprojlim_{\mathcal{O}_p^*(G)}{}^*\big(F|_{\mathcal{O}_p^*(G)}\big). \tag{3}$$

Let $\varphi_{\#}^*\colon \mathcal{O}_p^*(G) \longrightarrow \mathcal{O}_p(G')$ be the restriction of $\varphi_{\#}$. This is a bijection on isomorphism classes of objects, since $\varphi_{\#}^*(P)=P'$ (for p-subgroups $P\le G$ and $P'\le G'$) if and only if $P\in\mathrm{Syl}_p(\varphi^{-1}P')$. It is also surjective on all morphism sets, since for any pair of objects P,Q in $\mathcal{O}_p^*(G)$, and any $x\in N_G(PH,QH)$, $xPx^{-1}\le QH$ is H-conjugate to a subgroup of $Q\in\mathrm{Syl}_p(QH)$, and hence $x\in H{\cdot}N_G(P,Q)$.

Now assume $x,y\in N_G(P,Q)$ induce the same morphism in $\mathcal{O}_p(G')$. After replacing y by an appropriate element of Qy, we can assume that $y\in Hx=xH$. Set $y=xh$ (where $h\in H$). Then P and hPh^{-1} are both subgroups of $x^{-1}Qx$, so $[h,P]\le x^{-1}Qx\cap H = P\cap H$ (since $P\cap H\in\mathrm{Syl}_p(H)$), and thus $h\in N(P)$. In other words,

$$\mathrm{Mor}_{\mathcal{O}_p(G')}(\varphi(P),\varphi(Q)) \cong \mathrm{Mor}_{\mathcal{O}_p(G)}(P,Q)\big/(N_{HP}(P)/P)\,.$$

Also, $N_{HP}(P)/P$ has order prime to p since $P\in\mathrm{Syl}_p(HP)$. The category $\mathcal{O}_p(G')$ is thus equivalent to $\mathcal{O}_p^*(G)$ after dividing out by the action of certain subgroups of automorphisms of order prime to p. So by [**BLO**, Lemma 1.3],

$$\varprojlim_{\mathcal{O}_p(G')}{}^*(\bar{F}) \cong \varprojlim_{\mathcal{O}_p^*(G)}{}^*(\bar{F}\circ\varphi_{\#}^*),$$

and (1) follows from this together with (2) and (3). □

1.5. More higher limits of $\mathcal{Z}_G$

We have already given two examples (Proposition 1.6 and Lemma 1.8) of computations of higher limits of the functors $\mathcal{Z}_G$ or $\mathcal{Z}_G^K$, using the tools described at the beginning of this chapter. We now give two more applications, also using the last lemma.

We first show, for any finite group G, that Theorem A holds for G if and only if it holds for $G/O_{p'}(G)$. For use in the next chapter, we also state this in terms of the more general functors $\mathcal{Z}_G^K$.

Lemma 1.16. *Let G be a finite group, with normal subgroup $T\lhd G$ of order prime to p. Then for any p-subgroup $P\le G$, P is p-centric in G if and only if PT/T is p-centric in G/T. Hence*

$$\varprojlim_{\mathcal{O}_p(G)}{}^*(\mathcal{Z}_G) \cong \varprojlim_{\mathcal{O}_p(G/T)}{}^*(\mathcal{Z}_{G/T}),$$

and $\varprojlim{}^*(\mathcal{Z}_G^K)\cong\varprojlim{}^*(\mathcal{Z}_{G/T}^{KT/T})$ *for any* $K\lhd G$.

PROOF. Let $\varphi\colon G \longrightarrow G/T$ denote the projection. The isomorphisms between higher limits follows from Lemma 1.15, once we know that $\mathcal{Z}_G \cong \mathcal{Z}_{G/T} \circ \varphi_\#$ and $\mathcal{Z}_G^K \cong \mathcal{Z}_{G/T}^{KT/T} \circ \varphi_\#$. Since $\varphi_\#(P) = PT/T \cong P$, it remains only to show that P is p-centric in G if and only if PT/T is p-centric in G/T.

We will show that $C_{G/T}(PT/T) = C_G(P){\cdot}T/T$. Since $P \cong PT/T$, this will then imply that $Z(PT/T) \in \mathrm{Syl}_p(C_{G/T}(PT/T))$ (PT/T is p-centric) if and only if $Z(P) \in \mathrm{Syl}_p(C_G(P))$ (P is p-centric). One inclusion is clear: if $g \in C_G(P){\cdot}T$, then $[g, PT] \le T$, and so $gT \in C_{G/T}(PT/T)$.

Now assume $gT \in C_{G/T}(PT/T)$. Then $[g, P] \le T$, so P and gPg^{-1} are two Sylow p-subgroups of PT, and hence are T-conjugate. Choose $h \in T$ such that $hg \in N_G(P)$. Then $[hg, P] \le T \cap P = 1$, so $hg \in C_G(P)$, and thus $g \in C_G(P){\cdot}T$. □

If G is a finite group such that $O_{p'}(G) = 1$, then G is called *p-constrained* if there is a normal p-subgroup $Q \lhd G$ such that $C_G(Q) = Z(Q)$. More generally, if G is an arbitrary finite group, then G is called p-constrained if $G/O_{p'}(G)$ is p-constrained. Since any finite p-solvable group is p-constrained (cf. [**Gor**, Theorem 6.3.2]), the following proposition is an immediate corollary to Lemmas 1.16 and 1.8.

PROPOSITION 1.17. *For any finite group G which is p-constrained or p-solvable, $\mathcal{Z}_G$ is acyclic.*

1.6. Kan extensions and limits

In this section, for any (small) category $\mathcal{C}$, we let $\mathcal{C}$-mod denote the category of functors $calc^{\mathrm{op}} \longrightarrow \mathsf{Ab}$. If $H \le G$, and $F\colon \mathcal{O}_p(H)^{\mathrm{op}} \longrightarrow \mathsf{Ab}$ is any functor, then we define

$$F{\uparrow}_H^G\colon \mathcal{O}_p(G)^{\mathrm{op}} \longrightarrow \mathsf{Ab}$$

as follows. Fix an object P in $\mathcal{O}_p(G)$, let $\iota{\downarrow}P$ be the overcategory whose objects are the morphisms $Q \to P$ in $\mathcal{O}_p(G)$ for $Q \le H$, and let $\kappa_P\colon \iota{\downarrow}P \longrightarrow \mathcal{O}_p(H)$ be the forgetful functor which sends $(Q \to P)$ to Q. Set

$$(F{\uparrow}_H^G)(P) = \varprojlim_{\iota\downarrow P}(F \circ \kappa_P),$$

and let a morphism $P \longrightarrow P'$ in $\mathcal{O}_p(G)$ induce the obvious map between inverse limits. By [**McL**, §X.3, Theorem 1], $F{\uparrow}_H^G$ is a right Kan extension to F; i.e., $(-){\uparrow}_H^G$ is a right adjoint to the restriction functor from $\mathcal{O}_p(G)$-mod to $\mathcal{O}_p(H)$-mod.

LEMMA 1.18. *The following hold for any $H \le G$ and any $F\colon \mathcal{O}_p(H)^{\mathrm{op}} \longrightarrow \mathsf{Ab}$.*

(a) $\varprojlim_{\mathcal{O}_p(G)}{}^*(F{\uparrow}_H^G) \cong \varprojlim_{\mathcal{O}_p(H)}{}^*(F)$.

(b) *We can identify*

$$(F{\uparrow}_H^G)(P) = \Big(\bigoplus_{g\in G} F(H \cap gPg^{-1})\Big)^{P\times H} \cong \bigoplus_{HgP\in H\backslash G/P} F(H \cap gPg^{-1}); \qquad (1)$$

where $(x, y) \in P \times H$ acts on the first sum by sending the summand for g to the summand for xgy^{-1} via $F(y)$. When $aPa^{-1} \le Q$, the induced morphism

$$a^* = (F{\uparrow}_H^G)(a)\colon (F{\uparrow}_H^G)(Q) \longrightarrow (F{\uparrow}_H^G)(P)$$

satisfies $(a^*(\xi))_g = F(\mathrm{incl})(\xi_{ga^{-1}})$.

PROOF. The formulas in (b) follow directly from the definition of $(F{\uparrow}_H^G)(P)$ as an inverse limit. In particular, the term for $g \in G$ corresponds to the maximal object $H \cap gPg^{-1} \xrightarrow{g^{-1}} P$ in the overcategory $\iota{\downarrow}P$. Also, this formula shows that $(-){\uparrow}_H^G$ preserves exact sequences of functors.

From the fact that $(-){\uparrow}_H^G$ is a right adjoint to the restriction functor from $\mathcal{O}_p(G)$-mod to $\mathcal{O}_p(H)$-mod, it follows immediately that $(-){\uparrow}_H^G$ sends injectives in $\mathcal{O}_p(H)$-mod to injectives in $\mathcal{O}_p(G)$-mod. Also, if $\underline{\mathbb{Z}}$ denotes the constant functors with value $\mathbb{Z}$, then

$$\varprojlim_{\mathcal{O}_p(G)} (F{\uparrow}_H^G) \cong \mathrm{Hom}_{\mathcal{O}_p(G)\text{-}\mathsf{mod}}(\underline{\mathbb{Z}}, F{\uparrow}_H^G) \cong \mathrm{Hom}_{\mathcal{O}_p(H)\text{-}\mathsf{mod}}(\underline{\mathbb{Z}}, F) \cong \varprojlim_{\mathcal{O}_p(H)} (F).$$

Since $(-){\uparrow}_H^G$ sends an injective resolution of F to an injective resolution of $F{\uparrow}_H^G$,

$$\varprojlim_{\mathcal{O}_p(G)}{}^{*}(F{\uparrow}_H^G) \cong \varprojlim_{\mathcal{O}_p(H)}{}^{*}(F). \qquad \square$$

CHAPTER 2

Reduction to simple groups

Again, throughout this chapter, we fix a prime p. Our goal now is to study the higher limits of the quotient functors $\mathcal{Z}_G^{K_1}/\mathcal{Z}_G^{K_2}$, when G is a finite group, and $K_2 \lhd K_1 \lhd G$ are two normal subgroups such that K_1/K_2 is a minimal normal subgroup of G/K_2. Recall again that the subfunctors $\mathcal{Z}_G^{K_2} \subseteq \mathcal{Z}_G^{K_1} \subseteq \mathcal{Z}_G$ are defined by setting

$$\mathcal{Z}_G^{K_i}(P) = \begin{cases} Z(P) \cap K_i & \text{if } P \text{ is } p\text{-centric in } G \\ 0 & \text{otherwise.} \end{cases}$$

We will reduce the computation of higher limits of $\mathcal{Z}_G^{K_1}/\mathcal{Z}_G^{K_2}$ to the case where K_1/K_2 is a nonabelian simple group, $K_2 = Z(K_1)$ is an abelian p-group, and K_1 is perfect (thus a quasisimple group).

This involves a series of reductions. The general idea in each case is to first "simplify" the functors in question, by replacing them by quotient functors which vanish on certain smaller subgroups without changing the higher limits of the functors. In most cases, this means showing that $\Lambda^*(N(P)/P; -) = 0$ for certain subgroups P, and then applying Proposition 1.1(a) or Lemma 1.4. Afterwards, we compare the higher limits of these quotient functors, over orbit categories of different groups, using Lemma 1.13, 1.15, or 1.18.

Once we have reduced the problem to that of computing $\varprojlim^*(\mathcal{Z}_G^L/\mathcal{Z}_G^A)$, when L is quasisimple and $A = Z(L)$ is a p-group, we then observe that this computation depends only on L and on $\Gamma = \mathrm{Aut}_G(L)$. We can thus reformulate these results in terms of higher limits of a new functor $\mathcal{Y}_L^\Gamma$ (Proposition 2.7). Afterwards, we define $\mathfrak{L}^i(p)$ to be the set of all simple groups L for which $\varprojlim^i(\mathcal{Y}_{\widetilde{L}}^\Gamma) = 0$ whenever $\widetilde{L}/Z(\widetilde{L}) \cong L$ and $\mathrm{Inn}(\widetilde{L}) \leq \Gamma \leq \mathrm{Aut}(\widetilde{L})$. Thus by the results of this chapter, a nonabelian simple group L lies in $\mathfrak{L}^i(p)$ if and only if $\varprojlim^i(\mathcal{Z}_G^{K_1}/\mathcal{Z}_G^{K_2}) = 0$ whenever $K_1/K_2 \cong L^r$ for some $r \geq 1$. We are then left with the problem of showing, for each $i \geq 2$, that all simple groups lie in $\mathfrak{L}^i(2)$.

LEMMA 2.1. *Fix a finite group G, and subgroups $K_2 \subsetneqq K_1$ both normal in G, such that K_1/K_2 is a minimal normal subgroup of G/K_2. Then either $\mathcal{Z}_G^{K_1}/\mathcal{Z}_G^{K_2}$ is acyclic; or there is a finite group G_0 with normal subgroups $A \lhd K$ such that*

(a) *$K/A \cong K_1/K_2$ and is a minimal normal subgroup of G_0/A,*

(b) *A is a p-group and $A \leq Z(K)$, and*

(c) $\varprojlim_{\mathcal{O}_p(G)}^* \left(\mathcal{Z}_G^{K_1}/\mathcal{Z}_G^{K_2}\right) \cong \varprojlim_{\mathcal{O}_p(G_0)}^* \left(\mathcal{Z}_{G_0}^{K}/\mathcal{Z}_{G_0}^{A}\right).$

If, furthermore, K/A is an abelian p-group, then G_0 can be chosen to contain a normal p-subgroup $Q \lhd G_0$ which is p-centric in G_0.

PROOF. Set

$$H = \big\{g \in G \,\big|\, [g, K_1] \le K_2\big\} \lhd G$$

(i.e., $H/K_2 = C_{G/K_2}(K_1/K_2)$). Choose

$$Q \in \mathrm{Syl}_p(H),$$

and set

$$\big[G' = N_G(Q)\big] \quad \rhd \quad \big[K_1' = C_{K_1}(Q)\big] \quad \rhd \quad \big[K_2' = C_{K_2}(Q)\big].$$

Since $K_2 \lhd H$ and $Q \in \mathrm{Syl}_p(H)$,

$$K_2' = C_{K_2}(Q) = K_2 \cap C_H(Q) = K_2 \cap \big(Z(Q) \times \widehat{T}\big) = A' \times T,$$

where $A' = K_2 \cap Z(Q)$ is an abelian p-group, and where $\widehat{T}$ and $T = K_2 \cap \widehat{T}$ have order prime to p. Now set

$$\big[G_0 = G'/T\big] \quad \rhd \quad \big[K = K_1'/T\big] \quad \rhd \quad \big[A = K_2'/T \cong A'\big].$$

We will show that

$$\varprojlim_{\mathcal{O}_p(G)}{}^{*}\big(\mathcal{Z}_G^{K_1}/\mathcal{Z}_G^{K_2}\big) \cong \varprojlim_{\mathcal{O}_p(G')}{}^{*}\big(\mathcal{Z}_{G'}^{K_1'}/\mathcal{Z}_{G'}^{K_2'}\big) \cong \varprojlim_{\mathcal{O}_p(G_0)}{}^{*}\big(\mathcal{Z}_{G_0}^{K}/\mathcal{Z}_{G_0}^{A}\big). \tag{1}$$

The second isomorphism follows from Lemma 1.16.

Let

$$\mathcal{O}_p(G) \xleftarrow{\ \Psi\ } \mathcal{O}_p(G'/Q) \xrightarrow{\ \Psi'\ } \mathcal{O}_p(G')$$

be the functors of Lemma 1.13; thus $\Psi(P/Q) = P = \Psi'(P/Q)$. We claim that

$$\begin{aligned}\varprojlim_{\mathcal{O}_p(G)}{}^{*}(\mathcal{Z}_G^{K_1}/\mathcal{Z}_G^{K_2}) &\cong \varprojlim_{\mathcal{O}_p(G'/Q)}{}^{*}\ (\mathcal{Z}_G^{K_1}/\mathcal{Z}_G^{K_2} \circ \Psi) \\ &\cong \varprojlim_{\mathcal{O}_p(G'/Q)}{}^{*}\ (\mathcal{Z}_{G'}^{K_1'}/\mathcal{Z}_{G'}^{K_2'} \circ \Psi') \cong \varprojlim_{\mathcal{O}_p(G')}{}^{*}(\mathcal{Z}_{G'}^{K_1'}/\mathcal{Z}_{G'}^{K_2'}).\end{aligned} \tag{2}$$

For all $P \in \mathcal{S}_p(G)$, $N_{HP}(P)/P$ acts trivially on $(\mathcal{Z}_G^{K_1}/\mathcal{Z}_G^{K_2})(P)$ since H centralizes K_1/K_2 ($[H, K_1] \le K_2$ by definition of H), so the first isomorphism in (2) follows from Lemma 1.13. The proof of the third isomorphism is similar (just replace $H \lhd G$ by $Q \lhd G'$). For each $P/Q \in \mathcal{S}_p(G'/Q)$, $C_G(P) \le C_G(Q) \le G'$, and thus P is p-centric in G if and only if it is p-centric in G'. Hence

$$\big(\mathcal{Z}_G^{K_1}/\mathcal{Z}_G^{K_2} \circ \Psi\big)(P/Q) \cong \frac{Z(P) \cap K_1}{Z(P) \cap K_2} \cong \big(\mathcal{Z}_{G'}^{K_1'}/\mathcal{Z}_{G'}^{K_2'} \circ \Psi'\big)(P/Q)$$

if P is p-centric in G, and both groups are zero otherwise. So these two functors are isomorphic, this proves the second isomorphism in (2), and finishes the proof of the first isomorphism in (1).

If $K_1' = K_2'$, then $\mathcal{Z}_{G'}^{K_1'}/\mathcal{Z}_{G'}^{K_2'}$ is the zero functor, and the functors in (1) are all acyclic. Hence there is nothing more to prove in this case. So assume $K_1' \gneqq K_2'$ (and recall $K_1' \lhd G'$). We must show that $K/A \cong K_1'/K_2' \cong K_1/K_2$, and that this is a minimal normal subgroup of $G_0/A \cong G'/K_2'$.

Fix any $K' \gneqq K_2'$ such that $K' \le K_1'$ and $K' \lhd G'$. Since $K_2' = K_1' \cap K_2$, this implies $K' \not\le K_2$, so $K'K_2 \gneqq K_2$. The subgroup $K'K_2$ is normalized by

G', and is normalized by H since $[H, K'] \leq [H, K_1] \leq K_2$ by definition. Hence $K'K_2 \lhd HG' = G$, where the last equality follows from a Frattini argument again. Since K_1/K_2 is a minimal normal subgroup of G/K_2 and $1 \neq K'K_2/K_2 \lhd G/K_2$, we have $K'K_2 = K_1'K_2 = K_1$. Thus $K_1/K_2 \cong K'/K_2' = K_1'/K_2' \cong K/A$, and K/A is a minimal normal subgroup of $G_0/A \cong G'/K_2'$. This proves (a), and we already showed (c). As for (b), A is a p-group by construction, and $A \leq Z(K)$ since $A = K_2'/T \leq Z(Q){\cdot}T/T$ and $K = C_{K_1}(Q)/T$.

It remains to prove the last statement. Assume $K_1/K_2 \cong K/A$ is an abelian p-group. Then $H \geq K_1$ since $H/K_2 = C_{G/K_2}(K_1/K_2)$ by definition, and $QK_2 \geq K_1$ since $Q \in \mathrm{Syl}_p(H)$. For any $g \in C_G(Q)$, $[g, K_1] \leq [g, QK_2] \leq [g, K_2] \leq K_2$, so $g \in H$. This proves that Q is p-centric in G, and hence also in G'. So $QT/T \cong Q$ is a normal p-subgroup of $G_0 = G'/T$, which is p-centric by Lemma 1.16. □

As a first application of Lemma 2.1, we handle the case where the subquotient K_1/K_2 is abelian.

Proposition 2.2. *Let G be a finite group. If $K_2 \lhd K_1 \lhd G$ are subgroups, both normal in G, such that K_1/K_2 is abelian, then the functor $\mathcal{Z}_G^{K_1}/\mathcal{Z}_G^{K_2}$ is acyclic.*

Proof. It suffices to prove the acyclicity of $\mathcal{Z}_G^{K_1}/\mathcal{Z}_G^{K_2}$ when K_1/K_2 is a minimal normal subgroup of G/K_2. If K_1/K_2 has order prime to p, then $\mathcal{Z}_G^{K_1} = \mathcal{Z}_G^{K_2}$, and the result is clear. If K_1/K_2 is a p-group, then by Lemma 2.1, we can also assume that K_2 (hence K_1) is a p-group, and that there is a normal p-centric subgroup $Q \leq G$. Then $\mathcal{Z}_G^{K_1}$ and $\mathcal{Z}_G^{K_2}$ are both acyclic by Lemma 1.8, and the quotient functor is also acyclic. □

The following technical lemma will be needed twice in later reductions.

Lemma 2.3. *Let G be a finite group with normal subgroups $A \lhd K \lhd G$, such that $A \leq Z(K)$ is an abelian p-group. Let $P \leq G$ be a p-subgroup such that either $A \nleq P$, or $(P \cap K)/A$ is not p-centric in K/A, or $P \cap K$ is not p-centric in K. Then the action of $N_G(P)/P$ on $(PA \cap K)/A$ is not p-faithful.*

Proof. Set $H = \{x \in K \mid [x, P \cap K] \leq A\}$. Then P normalizes H. Also, $P \cap H \notin \mathrm{Syl}_p(H)$: this is clear if $A \nleq P$, and follows by definition (of p-centric) if $(P \cap K)/A$ is not p-centric in K/A or $(P \cap K)/A$ is not p-centric in K/A. Hence $p \big| |N_{HP}(P)/P|$ by Lemma 1.10. Since $N_{HP}(P)/P$ acts trivially on $(PA \cap K)/A$, this shows that the $N_G(P)/P$-action is not p-faithful. □

We next reduce the computation of $\varprojlim^*(\mathcal{Z}_G^{K_1}/\mathcal{Z}_G^{K_2})$ when K_1/K_2 is nonabelian to the case where K_1 is *quasisimple*; i.e., to the case where K_1/K_2 is nonabelian and simple, $K_2 = Z(K_1)$, and K_1 is perfect.

Lemma 2.4. *Fix a finite group G, and subgroups $A \leq K \leq G$, both normal in G, such that A is a p-group, $[A, K] = 1$, and K/A is nonabelian and a minimal normal subgroup of G/A. Then $A = Z(K)$, and we can write $K/A = \prod_{j \in J} L_j$, where each L_j is simple and a minimal normal subgroup of K/A. Let $K_j \leq K$ be such that $L_j = K_j/A$, and set $\widetilde{L}_j = [K_j, K_j]$, $A_j = \widetilde{L}_j \cap A = Z(\widetilde{L}_j)$, and*

$G_j = N_G(\widetilde{L}_j)$. *Then for any given* $j \in J$,

$$\varprojlim_{\mathcal{O}_p(G)}{}^{i}\,(\mathcal{Z}_G^K/\mathcal{Z}_G^A) \cong \varprojlim_{\mathcal{O}_p(G_j)}{}^{i}\,(\mathcal{Z}_{G_j}^{\widetilde{L}_j}/\mathcal{Z}_{G_j}^{A_j})$$

for all $i \geq 2$, *and there is a surjection*

$$\varprojlim_{\mathcal{O}_p(G_j)}{}^{1}\,(\mathcal{Z}_{G_j}^{\widetilde{L}_j}/\mathcal{Z}_{G_j}^{A_j}) \twoheadrightarrow \varprojlim_{\mathcal{O}_p(G)}{}^{1}(\mathcal{Z}_G^K/\mathcal{Z}_G^A).$$

PROOF. Since K/A is nonabelian and a minimal normal subgroup of G/A, it must be a product of nonabelian simple groups isomorphic to each other (cf. [**Gor**, Theorem 2.1.5]). Since we also assume $[A,K]=1$, this implies that $A = Z(K)$. So write $K/A = \prod_{j\in J} L_j$ where $L_j = K_j/A$ as above. For any $i \neq j$, $K_i \cap K_j = A$, and hence $[K_i, K_j] \leq A$. Since A is central, this means that the commutator map $[-,-]\colon K_i \times K_j \longrightarrow A$ is a homomorphism in each coordinate; e.g., $[gg',h] = [g,h][g',h]$ for $g, g' \in K_i$ and $h \in K_j$. Thus

$$[K_i, K_j] = 1 \quad \text{for all } i \neq j, \tag{1}$$

since K_i/A and K_j/A are simple and centralize A.

Set $\mathcal{Z}_j = \mathcal{Z}_{G_j}^{\widetilde{L}_j}/\mathcal{Z}_{G_j}^{A_j}$ for short. Define functors $F_1, F_2\colon \mathcal{O}_p(G)^{\mathrm{op}} \longrightarrow \mathsf{Ab}$ by setting

$$F_1(P) = \Big(\prod_{j\in J} \mathcal{Z}_j(P \cap G_j)\Big)^P \qquad \text{and} \qquad F_2 = \mathcal{Z}_G^K/\mathcal{Z}_G^A.$$

Each factor $\mathcal{Z}_j(P\cap G_j)$ in $F_1(P)$ can be identified with a subgroup of $L_j \cong \widetilde{L}_j/A_j$, and hence $F_1(P)$ can be identified with a subgroup of K/A. Under this identification, F_1 sends the morphism in $\mathcal{O}_p(G)$ represented by $g \in N_\Gamma(P,Q)$ to

$$F_1(Q) \xrightarrow{x\mapsto g^{-1}xg} F_1(P).$$

Let F_1' and F_2' be the functors

$$F_i'(P) = \begin{cases} F_i(P) & \text{if } P\cap K \text{ is } p\text{-centric in } K \\ 0 & \text{otherwise.} \end{cases}$$

In Step 1, we show that

$$\varprojlim_{\mathcal{O}_p(G)}{}^{*}(F_1) \cong \varprojlim_{\mathcal{O}_p(G_j)}{}^{*}\,(\mathcal{Z}_j) \quad \text{for all } j \in J; \tag{2}$$

and in Step 2 that

$$\varprojlim_{\mathcal{O}_p(G)}{}^{*}(F_1') \cong \varprojlim_{\mathcal{O}_p(G)}{}^{*}(F_1) \qquad \text{and} \qquad \varprojlim_{\mathcal{O}_p(G)}{}^{*}(F_2') \cong \varprojlim_{\mathcal{O}_p(G)}{}^{*}(F_2) \tag{3}$$

In Step 3, we identify F_1' as a subfunctor of F_2', and prove in Step 4 that F_2'/F_1' is acyclic. The lemma then follows from the relative exact sequence for higher limits of the pair $F_1' \subseteq F_2'$.

Step 1: For each $j \in J$, there is an obvious morphism of functors on $\mathcal{O}_p(G_j)$

$$F_1|_{\mathcal{O}_p(G_j)} \longrightarrow \mathcal{Z}_j,$$

defined by projection to the j-th factor, which is adjoint to a natural morphism

$$\omega\colon F_1 \longrightarrow \mathcal{Z}_j\!\uparrow_{G_j}^{G}.$$

Since G acts transitively on the factors $L_j \le K/A$ (otherwise K/A would not be a minimal normal subgroup), ω is an isomorphism of functors by the formula in Lemma 1.18(b) for Kan extensions. Hence (2) follows from Lemma 1.18(a).

Step 2: To prove (3), it suffices, using Lemma 1.4(b), to show that when $P \cap K$ is not p-centric in K, then neither of the actions of $N_G(P)/P$ on $F_1(P)$ or $F_2(P)$ is p-faithful. This follows from Lemma 2.3, since both of these groups $F_i(P)$ can be identified with subgroups of

$$(PA \cap K)/A \cong PA/A \cap \prod_{j\in J} \widetilde{L}_j/A_j \ge \prod_{j\in J} (PA \cap \widetilde{L}_j)/A_j.$$

Step 3: Set $\widetilde{K} = \prod_{j\in J} \widetilde{L}_j$ and $\widetilde{A} = \prod_{j\in J} A_j$. We will write R_p for the Sylow p-subgroup of a group R, but only in situations where it is a direct factor of R and abelian.

Fix a p-subgroup $P \le G$ such that $P \cap K$ is p-centric in K, and note that

$$C_{\widetilde{K}}(P) \cong \Big[\prod_{j\in J} C_{\widetilde{L}_j}(P \cap G_j)\Big]^P \qquad \text{and} \qquad C_{\widetilde{A}}(P) \cong \Big[\prod_{j\in J} C_{A_j}(P \cap G_j)\Big]^P.$$

Furthermore, the action of any $g \in P$ permutes the factors under each of these product decompositions, and is trivial whenever it sends a factor to itself. Hence

$$F_1(P) \cong \left[\prod_{j\in J} \frac{C_{\widetilde{L}_j}(P \cap G_j)_p}{C_{A_j}(P \cap G_j)}\right]^P \cong \frac{\big[\prod_{j\in J} C_{\widetilde{L}_j}(P \cap G_j)_p\big]^P}{\big[\prod_{j\in J} C_{A_j}(P \cap G_j)\big]^P} \cong C_{\widetilde{K}}(P)_p / C_{\widetilde{A}}(P).$$

We can thus write

$$F_1'(P) \cong C_{\widetilde{K}}(P)_p / C_{\widetilde{A}}(P) \qquad \text{and} \qquad F_2'(P) \cong C_K(P)_p / C_A(P) \tag{4}$$

for all $P \in \mathcal{S}_p(G)$ such that $P \cap K$ is p-centric in K. The natural map from $\widetilde{K}$ to K induces a natural morphism of functors $F_1' \longrightarrow F_2'$, and this is injective since $F_1'(P)$ and $F_2'(P)$ can both be identified as subgroups of $K/A \cong \widetilde{K}/\widetilde{A}$.

Step 4: Set $\Phi = F_1'/F_2'$ for short; it remains to show that Φ is acyclic. Fix $Q \in \mathrm{Syl}_p(K)$. Define

$$\bar{\Phi}\colon \mathcal{O}_p(N_G(Q)/Q)^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{-}\mathsf{mod} \qquad \text{by setting} \qquad \bar{\Phi}(P/Q) = \Phi(P).$$

We claim that

$$\varprojlim_{\mathcal{O}_p(G)}{}^{*}(\Phi) \cong \varprojlim_{\mathcal{O}_p(N_G(Q)/Q)}{}^{*}(\bar{\Phi}), \tag{5}$$

and that $\bar{\Phi}$ is acyclic. The isomorphism follows from Lemma 1.13, once we show that $N_{KP}(P)/P$ acts trivially on $\Phi(P)$ for each radical p-subgroup $P \le G$.

Recall that $K/A = \prod_{j\in J} K_j/A$, where the factors K_j/A are simple. Also, $\widetilde{L}_j = [K_j, K_j]$ is perfect (hence quasisimple), with center $Z(\widetilde{L}_j) = A_j = \widetilde{L}_j \cap A$.

Let $\mathcal{R}$ be the set of all radical p-subgroups of G. For any p-subgroup $P \le G$, we write

$$P_0 = P \cap K \qquad \text{and} \qquad P_j = P \cap K_j \quad (\text{all } j \in J).$$

For any $P \in \mathcal{R}$, $P \geq A$ and P_0 is radical in K by Lemma 1.5(b), and $P_0/A = \prod_{j\in J} P_j/A$ by Lemma 1.5(a). If P_0 is p-centric in K, then following sequence is exact by (4):

$$1 \longrightarrow C_{\widetilde{A}}(P) \longrightarrow C_{\widetilde{K}}(P)_p \times C_A(P) \longrightarrow C_K(P)_p \longrightarrow \Phi(P) \longrightarrow 1\,. \quad (6)$$

We claim that the sequence

$$1 \longrightarrow \widetilde{A} \longrightarrow C_{\widetilde{K}}(P_0) \times A \longrightarrow C_K(P_0) \longrightarrow 1 \quad (7)$$

is exact. This is obtained from the exact sequence $1 \to \widetilde{A} \to \widetilde{K} \times A \to K \to 1$ by taking fixed subgroups of the conjugation action of P_0. So the exactness of (7) will follow upon showing that $C_{\widetilde{K}}(P_0) \times A$ surjects onto $C_K(P_0)$. Any $g \in K$ can be written $g = a\cdot\prod_{j\in J} g_j$ for $a \in A$ and $g_j \in \widetilde{L}_j$, $[g, P_0] = 1$ implies $[g_j, P_j] = 1$ for each j (since $[P_j, \widetilde{L}_i] \leq [K_j, K_i] = 1$ for $i \neq j$ by (1)), and thus ga^{-1} is the image of $(g_j)_{j\in J} \in C_{\widetilde{K}}(P_0)$.

After taking fixed points of the P/P_0-action on (7), we get an exact sequence

$$1 \longrightarrow C_{\widetilde{A}}(P) \longrightarrow C_{\widetilde{K}}(P)_p \times C_A(P) \longrightarrow C_K(P)_p \longrightarrow H^1(P/P_0; \widetilde{A}).$$

A comparison with (6) shows that $\Phi(P)$ is contained in $H^1(P/P_0; \widetilde{A})$ as a module over $N(P)/P$. Each coset in $N_{KP}(P)/P$ contains an element of $N_K(P)$, so this group acts trivially on $\Phi(P)$ since it acts trivially on $P/P_0 \cong PK/K$ and on $\widetilde{A}$. This finishes the proof of the isomorphism in (5), using Lemma 1.13.

Finally, when $P_0 = P \cap K = Q$, the first three terms in the exact sequence (6) are acyclic as functors on $\mathcal{O}_p(N_G(Q)/Q)$ by Proposition 1.7. (For example, $C_{\widetilde{K}}(P) = C_{\widetilde{K}}(Q)^{P/Q}$ is the fixed subgroup of the P/Q-action.) So $\bar{\Phi}$ is also acyclic, and this finishes the proof that F_1'/F_2' is acylic. □

We have now reduced the computation of $\varprojlim^*(\mathcal{Z}_G^{K_1}/\mathcal{Z}_G^{K_2})$ to the case where K_1 is quasisimple and $K_2 = Z(K_1)$ is an abelian p-group. It turns out that this only depends on K_1 and $\mathrm{Aut}_G(K_1)$. To make this precise, we define certain functors $\mathcal{Y}_L^\Gamma$ as follows. As usual, c_x denotes conjugation by an element x; and we write K_p for the Sylow p-subgroup of K when it is a direct factor and abelian.

Definition 2.5. *Fix a finite group L and a group of automorphisms $\Gamma \leq \mathrm{Aut}(L)$ which contains $\mathrm{Inn}(L)$. Define $\mathcal{Y}_L^\Gamma\colon \mathcal{O}_p(\Gamma)^{\mathrm{op}} \longrightarrow \mathsf{Ab}$ by setting*

$$\mathcal{Y}_L^\Gamma(P) = \begin{cases} \{c_x \in \mathrm{Inn}(L) \,|\, x \in C_L(P)\}_p & \text{if } P\cap\mathrm{Inn}(L) \text{ is } p\text{-centric in } \mathrm{Inn}(L) \\ 0 & \text{otherwise.} \end{cases}$$

For any $P, Q \leq \Gamma$ and any $g \in N_\Gamma(P, Q)$,

$$\mathcal{Y}_L^\Gamma\Big(P \xrightarrow{\;Qg\;} Q\Big) = \Big(\mathrm{Aut}_{C_L(Q)}(L)_p \xrightarrow{c_x \mapsto c_{g^{-1}xg}} \mathrm{Aut}_{C_L(P)}(L)_p\Big)\,.$$

Thus whenever $P \cap \mathrm{Inn}(L)$ is p-centric in $\mathrm{Inn}(L)$,

$$\mathcal{Y}_L^\Gamma(P) \leq C_{\mathrm{Inn}(L)}(P \cap \mathrm{Inn}(L))_p = Z(P \cap \mathrm{Inn}(L)).$$

Lemma 2.6. *Fix a finite group G with quasisimple normal subgroup $L \lhd G$, and assume that $A = Z(L)$ is a p-group. Set $\Gamma = \mathrm{Aut}_G(L)$. Then*

$$\varprojlim_{\mathcal{O}_p(G)}{}^*(\mathcal{Z}_G^L/\mathcal{Z}_G^A) \cong \varprojlim_{\mathcal{O}_p(\Gamma)}{}^*(\mathcal{Y}_L^\Gamma).$$

PROOF. We consider the following three sets of p-subgroups of G:

$$\mathcal{H}_0 = \{P \in \mathcal{S}_p(G) \mid Z(P) \cap L \in \mathrm{Syl}_p(C_L(P))\}$$
$$\mathcal{H}_1 = \{P \in \mathcal{S}_p(G) \mid Z(P) \in \mathrm{Syl}_p(C_G(P))\} = \{P \in \mathcal{S}_p(G) \mid P \text{ } p\text{-centric in } G\}$$
$$\mathcal{H}_2 = \{P \in \mathcal{S}_p(G) \mid P \geq A \text{ and } (P \cap L)/A \text{ } p\text{-centric in } L/A\}.$$

We first claim that $\mathcal{H}_1 \subseteq \mathcal{H}_0 \supseteq \mathcal{H}_2$. The first inclusion is clear: if $Z(P) \in \mathrm{Syl}_p(C_G(P))$, then $Z(P) \cap L \in \mathrm{Syl}_p(C_L(P))$. To see the second, note that if $(P\cap L)/A$ is centric in L/A, then $P\cap L$ is centric in L, so $C_L(P\cap L) = Z(P\cap L)\times T$ for some T of order prime to p, and hence (after taking P-centralizers) $C_L(P) = (Z(P) \cap L) \times C_T(P)$. Clearly, all three of these sets are closed under G-conjugacy and overgroups.

For any $P \in \mathcal{H}_0$, $Z(P) \cap L$ is central in $C_L(P)$ and a Sylow subgroup; hence $C_L(P) = (Z(P) \cap L) \times T$ for some T of order prime to p. Hence it makes sense to define functors $F_0, F_1, F_2 \colon \mathcal{O}_p(G)^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{-}\mathsf{mod}$, by setting

$$F_i(P) = \begin{cases} (Z(P) \cap L)/(Z(P) \cap A) = C_L(P)_p/C_A(P) & \text{if } P \in \mathcal{H}_i \\ 0 & \text{otherwise.} \end{cases}$$

In all cases, when $P, Q \in \mathcal{H}_i$, a coset Qg for $g \in N_G(P,Q)$ induces the morphism $(x \mapsto g^{-1}xg)$ from $F_i(Q)$ to $F_i(P)$. Since $\mathcal{H}_0$ contains $\mathcal{H}_1$ and $\mathcal{H}_2$, F_1 and F_2 can be considered quotient functors of F_0. For each $P \in \mathcal{H}_0 \smallsetminus \mathcal{H}_1$, $C_G(P)\cdot P/P$ has order a multiple of p and acts trivially on $F_0(P)$, so the action of $N_G(P)/P$ is not p-faithful. For each $P \in \mathcal{H}_0 \smallsetminus \mathcal{H}_2$, the action of $N_G(P)/P$ on $F_0(P) \leq (PA \cap L)/A$ fails to be p-faithful by Lemma 2.3. Since $F_1 = \mathcal{Z}_G^K/\mathcal{Z}_G^A$, Lemma 1.4(b) now applies to show that

$$\varprojlim_{\mathcal{O}_p(G)}{}^{*}(\mathcal{Z}_G^K/\mathcal{Z}_G^A) \cong \varprojlim_{\mathcal{O}_p(G)}{}^{*}(F_0) \cong \varprojlim_{\mathcal{O}_p(G)}{}^{*}(F_2). \quad (1)$$

It remains to show that $\varprojlim^*(F_2) \cong \varprojlim^*(\mathcal{Y}_L^\Gamma)$. Let $c\colon G \twoheadrightarrow \Gamma$ be the surjection which sends $g \in G$ to $c_g \in \mathrm{Aut}(L)$. Set $H = \mathrm{Ker}(c) = C_G(L)$, and let $\mathcal{O}_p^*(G) \subseteq \mathcal{O}_p(G)$ be the full subcategory whose objects are the p-groups $P \leq G$ such that $P \cap H \in \mathrm{Syl}_p(H)$. By Lemma 1.15, it suffices to show that

(a) $(\mathcal{Y}_L^\Gamma \circ c_\#)|_{\mathcal{O}_p^*(G)} \cong F_2|_{\mathcal{O}_p^*(G)}$; and

(b) the action of $N_{HP}(P)/P$ on $F_2(P)$ is trivial for all p-subgroups $P \leq G$.

Point (b) is clear, since H centralizes L.

For any $P \leq G$ such that $P \cap H \in \mathrm{Syl}_p(H)$,

$$(\mathcal{Y}_L^\Gamma \circ c_\#)(P) = \mathcal{Y}_L^\Gamma(\mathrm{Aut}_P(L))$$
$$\cong \begin{cases} C_L(P)_p/C_A(P) & \text{if } \mathrm{Aut}_P(L) \cap \mathrm{Inn}(L) \text{ } p\text{-centric in } \mathrm{Inn}(L) \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

Also, $\mathrm{Inn}(L) \cong L/A$, and (since $H \cap L = A$):

$$\mathrm{Aut}_P(L) \cap \mathrm{Inn}(L) = \mathrm{Aut}_{PH}(L) \cap \mathrm{Inn}(L) = \mathrm{Aut}_{PH\cap L}(L) \cong (PH \cap L)/A.$$

In particular, $PH \cap L$ is a p-group since $\mathrm{Aut}_P(L)$ is. Since $P \cap H \in \mathrm{Syl}_p(H)$, $P \in \mathrm{Syl}_p(PH)$, so $P \cap H \in \mathrm{Syl}_p(PH \cap L)$; and since $PH \cap L$ is a p-group, this implies that $PH \cap L = P \cap L$. Thus $\mathrm{Aut}_P(L) \cap \mathrm{Inn}(L)$ is p-centric in $\mathrm{Inn}(L)$ if and only if $(P \cap L)/A$ is p-centric in L/A, so $\mathcal{Y}_L^\Gamma|_{\mathcal{O}_p^*(G)} \cong F_2|_{\mathcal{O}_p^*(G)}$ by (2), and this finishes the proof of (a). □

Lemma 2.6 finishes the process of reducing the general computation of $\varprojlim^*(\mathcal{Z}_G)$ to that of $\varprojlim^*(\mathcal{Y}_L^\Gamma)$ when L is quasisimple and $\mathrm{Inn}(L) \le \Gamma \le \mathrm{Aut}(L)$. These results are now summarized in the following:

PROPOSITION 2.7. *Fix a finite group G and normal subgroups $K_2 \le K_1$ in G, such that K_1/K_2 is a minimal subgroup of G/K_2. If K_1/K_2 is abelian, or if there is a p-subgroup $Q \le G$ such that $[Q, K_1] \le K_2$ and $C_{K_1}(Q) \le K_2$, then the functor $\mathcal{Z}_G^{K_1}/\mathcal{Z}_G^{K_2}$ is acyclic. Otherwise, there is a quasiperfect group L with p-group center such that K_1/K_2 is isomorphic to a product of copies of $L/Z(L)$, a subgroup $\Gamma \le \mathrm{Aut}(L)$ containing $\mathrm{Inn}(L)$, and (for each i) a homomorphism*

$$\varprojlim_{\mathcal{O}_p(\Gamma)}{}^i (\mathcal{Y}_L^\Gamma) \longrightarrow \varprojlim_{\mathcal{O}_p(G)}{}^i \left(\mathcal{Z}_G^{K_1}/\mathcal{Z}_G^{K_2}\right)$$

which is onto when $i = 1$ and an isomorphism when $i \ge 2$.

PROOF. The abelian case was handled in Proposition 2.2. The cases where K_1/K_2 is nonabelian follow from Lemmas 2.1, 2.4, and 2.6. □

In order to avoid repeating these conditions about central extensions and groups of automorphisms throughout the remaining chapters, we define the following classes of finite simple groups. Recall that p is a fixed prime, and that the functors $\mathcal{Y}_L^\Gamma$ depend on p.

DEFINITION 2.8. *For each $i \ge 1$, let $\boldsymbol{\mathfrak{L}^i(p)}$ be the class of finite nonabelian simple groups L with the property that*

$$\varprojlim_{\mathcal{O}_p(\Gamma)}{}^i (\mathcal{Y}_{\widetilde{L}}^\Gamma) = 0$$

for each quasisimple group $\widetilde{L}$ such that $Z(\widetilde{L})$ is a p-group and $\widetilde{L}/Z(\widetilde{L}) \cong L$, and each subgroup $\Gamma \le \mathrm{Aut}(\widetilde{L})$ which contains $\mathrm{Inn}(\widetilde{L})$. Also, $\mathfrak{L}^{\ge i}(p)$ denotes the intersection of the classes $\mathfrak{L}^j(p)$ for all $j \ge i$.

The important consequence of Proposition 2.7 is:

PROPOSITION 2.9. *For any finite group G and any $i \ge 1$, $\varprojlim^i(\mathcal{Z}_G) = 0$ if each nonabelian simple group L which appears in the decomposition series for G lies in $\mathfrak{L}^i(p)$.*

Our goal now, throughout the rest of the paper, is to show that every finite nonabelian simple group lies in $\mathfrak{L}^{\ge 2}(2)$.

CHAPTER 3

A relative version of Λ-functors

We now have the problem of proving, for each nonabelian finite simple group L, that higher limits vanish for certain functors $\mathcal{Y}^{\Gamma}_{\widetilde{L}}$, defined for all central extensions $\widetilde{L}$ of L and all groups of outer automorphisms $\Gamma/\operatorname{Inn}(\widetilde{L})$ of $\widetilde{L}$. We want to find a way of doing all of these computations simultaneously (for a given L) as far as possible. The functors for a central extension $\widetilde{L}$ can be regarded as subfunctors of the ones for L itself (this will be made more explicit in the next chapter). So the main problem is to deal simultaneously with different groups of automorphisms of L, which involves handling different functors over different orbit categories. The main idea for doing this is to use the same filtration for all of these functors: a filtration indexed by p-centric subgroups of L.

This motivates the following definition of a relative version of the functors $\Lambda^*(G;M)$: it will be used to describe the higher limits of the subquotients which occur under this filtration of $\mathcal{Y}^{\Gamma}_{\widetilde{L}}$. Throughout this chapter, p denotes a fixed prime.

Definition 3.1. *Fix a pair $H \lhd G$ of finite groups and a $\mathbb{Z}_{(p)}[G]$-module M. Let $\Phi^{G,H}_M : \mathcal{O}_p(G)^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}$-$\mathtt{mod}$ be the functor defined by*

$$\boldsymbol{\Phi^{G,H}_M}(P) = \begin{cases} M^P & \text{if } P \cap H = 1 \\ 0 & \text{otherwise;} \end{cases}$$

and define

$$\boldsymbol{\Lambda^*(G,H;M)} = \varprojlim_{\mathcal{O}_p(G)}{}^{*}(\Phi^{G,H}_M).$$

Note in particular that $\Phi^G_M = \Phi^{G,G}_M$, and hence $\Lambda^*(G;M) = \Lambda^*(G,G;M)$. At the other extreme, when $H = 1$, $\Phi^{G,1}_M = H^0M$, and so $\Lambda^0(G,1;M) = M^G$ and $\Lambda^i(G,1;M) = 0$ for $i > 0$ by Proposition 1.7.

The importance of these groups arises from the following generalization of [**JMO**, Proposition 5.4]. For any $H \lhd G$, we say that a functor $F\colon \mathcal{O}_p(G)^{\mathrm{op}} \longrightarrow \mathsf{Ab}$ is *H-controlled* if for all P, the inclusion of $(P\cap H)$ in P induces an isomorphism $F(P) \cong F(P\cap H)^P$. Thus the functors $\Phi^{G,H}_M$ defined above are H-controlled.

Proposition 3.2. *Fix a finite group G, a prime p, a normal subgroup $H \lhd G$, and a p-subgroup $Q \le H$. Let $F\colon \mathcal{O}_p(G) \longrightarrow \mathbb{Z}_{(p)}$-$\mathtt{mod}$ be any H-controlled functor which vanishes except on subgroups $P \le G$ such that $P \cap H$ is G-conjugate to Q. (Thus $F(P) = F(Q)^{P/Q}$ whenever $P \cap H = Q$.) Then*

$$\varprojlim_{\mathcal{O}_p(G)}{}^{*}(F) \cong \Lambda^*\big(N_G(Q)/Q, N_H(Q)/Q; F(Q)\big).$$

Proof. This is a special case of Lemma 1.12. □

The idea now is, given an H-controlled functor $F\colon \mathcal{O}_p(G) \longrightarrow \mathbb{Z}_{(p)}\text{-}\mathtt{mod}$, to filter it by subfunctors F_i in such a way that for each quotient functor F_i/F_{i-1}, $F_i/F_{i-1}(P) = 0$ except for those $P \le G$ such that $P \cap H$ lies in one G-conjugacy class of p-subgroups of H. Then each $\varprojlim^*(F_i/F_{i-1})$ is described via Proposition 3.2 and the $\Lambda^*(G, H; -)$. When L is quasisimple and $\mathrm{Inn}(L) \le \Gamma \le \mathrm{Aut}(L)$, the functors $\mathcal{Y}_L^\Gamma$ need not be $\mathrm{Inn}(L)$-controlled, but we will see that the same techniques can be used to compute higher limits of these functors.

For any G, and any short exact sequence of $\mathbb{Z}_{(p)}[G]$-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

the induced sequence of functors $0 \longrightarrow \Phi^G_{M'} \longrightarrow \Phi^G_M \longrightarrow \Phi^G_{M''} \longrightarrow 0$ is also exact, and hence induces a long exact sequence of the groups $\Lambda^*(G; -)$. This is *not* in general the case for the relative groups $\Lambda^*(G, H; -)$ when $H \lhd G$ is a proper normal subgroup. For example, from the above remarks about $\Lambda^*(G, 1; -)$, and the fact that $M \mapsto M^G$ is not an exact functor, it is clear that a short exact sequence of $\mathbb{Z}_{(p)}[G]$-modules will not in general induce a long exact sequence of the groups $\Lambda^*(G, 1; -)$. This is why the statement of point (b) in the next proposition is so detailed.

Most of the properties of the relative groups $\Lambda^*(G, H; M)$ listed in the following proposition generalize properties of the groups $\Lambda^*(G; M)$ proven in [**JMO**].

Proposition 3.3. *Fix a prime p, a finite group G, a normal subgroup $H \lhd G$, and a $\mathbb{Z}_{(p)}[G]$-module M. Then the following hold.*

(a) *If $(p, |H|) = 1$, then $\Lambda^0(G, H; M) \cong M^G$, and $\Lambda^i(G, H; M) = 0$ for all $i > 0$. If $p\big||H|$, then $\Lambda^0(G, H; M) = 0$.*

(b) *$\Lambda^*(G, H; M) = 0$ if the kernel of the action of H on M has order a multiple of p. More generally, if $M_0 \subseteq M$ is a G-invariant submodule, then*
- *$\Lambda^*(G, H; M_0) \cong \Lambda^*(G, H; M)$ if the kernel of the H-action on M/M_0 has order a multiple of p; and*
- *$\Lambda^*(G, H; M) \cong \Lambda^*(G, H; M/M_0)$ if the kernel of the H-action on M_0 has order a multiple of p.*

(c) *$\Lambda^*(G, H; M) = 0$ if $O_p(H) \ne 1$.*

(d) *If $K \lhd G$ is a normal subgroup which acts trivially on the $\mathbb{Z}_{(p)}[G]$-module M, and $H \cap K$ has order prime to p, then*

$$\Lambda^*(G, H; M) \cong \Lambda^*(G/K, HK/K; M).$$

(e) *Assume the Sylow p-subgroups of H are cyclic, or (if $p = 2$) quaternion. Then $\Lambda^i(G, H; M) = 0$ for all $i \ge 2$.*

Proof. **(a)** If $(p, |H|) = 1$, then $P \cap H = 1$ for all p-subgroups $H \le G$, and hence $\Phi^{G,H}_M = H^0 M$ in the notation of Proposition 1.7. So by that proposition, $\varprojlim^0(\Phi^{G,H}_M) = M^G$ and its higher limits vanish. If $p\big||H|$, then $\Phi^{G,H}_M(S) = 0$ for $S \in \mathrm{Syl}_p(G)$, and hence $\varprojlim^0(\Phi^{G,H}_M) = 0$.

(b) The first statement is a special case of either of the other two.

Assume first that $K \overset{\text{def}}{=} \text{Ker}[H \longrightarrow \text{Aut}(M/M_0)] \lhd G$ has order a multiple of p. If $P \le G$ is such that $P \cap H = 1$, then $1 = P \cap K \notin \text{Syl}_p(K)$, so $N_{PK}(P)/P$ has order a multiple of p by Lemma 1.10, and acts trivially on M/M_0. Since this holds for all such P, $\varprojlim^*(\Phi_M^{G,H}/\Phi_{M_0}^{G,H}) = 0$ by Lemma 1.4(a), and hence $\Lambda^*(G,H;M) \cong \Lambda^*(G,H;M_0)$.

Now assume that $K \overset{\text{def}}{=} \text{Ker}[H \longrightarrow \text{Aut}(M_0)] \lhd G$ has order a multiple of p. There is an exact sequence of functors on $\mathcal{O}_p(G)$

$$0 \longrightarrow \Phi_{M_0}^{G,H} \longrightarrow \Phi_M^{G,H} \longrightarrow \Phi_{M/M_0}^{G,H} \longrightarrow \Psi \longrightarrow 0,$$

where $\Psi(P) \subseteq H^1(P;M_0)$ if $P \cap H = 1$ and $\Psi(P) = 0$ otherwise. We have just seen that $\varprojlim^*(\Phi_{M_0}^{G,H}) = 0$, and hence we will be done upon showing that $\varprojlim^*(\Psi) = 0$. For each $P \le G$ such that $P \cap H = 1$, $N_{PK}(P)/P$ has order a multiple of p (Lemma 1.10) and acts trivially on $\Psi(P) \subseteq H^1(P;M_0)$. The action of $N_G(P)/P$ on $\Psi(P)$ thus fails to be p-faithful for any such P, and so all higher limits of Ψ vanish by Lemma 1.4(a).

(c) If $O_p(H) \ne 1$, then for all p-subgroups $P \le G$ such that $P \cap H = 1$,

$$O_p(N_G(P)/P) \ge N_{O_p(H)\cdot P}(P)/P \ne 1.$$

Hence no such subgroup is radical in G, and so $\Lambda^*(G,H;M) = 0$ by Lemma 1.4(a).

(d) Let $\varphi\colon G \longrightarrow G/K$ be the projection, and let $\varphi_\#$ denote the induced functor between orbit categories. Let $\mathcal{O}_p^*(G) \subseteq \mathcal{O}_p(G)$ be the full subcategory whose objects are those $P \le G$ such that $P \cap K \in \text{Syl}_p(K)$. We claim that

$$\Phi_M^{G,H}|_{\mathcal{O}_p^*(G)} \cong \big(\Phi_M^{G/K,HK/K} \circ \varphi_\#\big)|_{\mathcal{O}_p^*(G)}. \tag{1}$$

This is clear, once we have checked that for any p-subgroup $P \le G$ such that $P\cap K \in \text{Syl}_p(K)$, $P \cap H = 1$ if and only if $PK/K \cap HK/K = 1$. If $PK/K \cap HK/K = 1$, then $P \cap H \le K$, and hence $P \cap H = 1$ since P is a p-group and $K \cap H$ has order prime to p.

Conversely, if $P\cap H = 1$, then $PK\cap H$ has order prime to p, since $P \in \text{Syl}_p(PK)$ and thus any element of $PK \cap H$ of p-power order would be G-conjugate to an element of $P \cap H$. Hence $PK \cap H \le K$, since any element of $PK\smallsetminus K$ has order a multiple of p. It follows that $PK \cap HK \le K$, and $PK/K \cap HK/K = 1$.

This finishes the proof of (1). Also, $N_{KP}(P)/P$ acts trivially on $\Phi_M^{G,H}(P)$ for all P, since K acts trivially on M. Lemma 1.15 now applies to show that

$$\Lambda^*(G,H;M) = \varprojlim_{\mathcal{O}_p(G)}{}^*(\Phi_M^{G,H}) \cong \varprojlim_{\mathcal{O}_p(G/K)}{}^*(\Phi_M^{G/K,HK/K}) = \Lambda^*(G/K,HK/K;M).$$

(e) Fix $T \in \text{Syl}_p(G)$, set $S = T \cap H \in \text{Syl}_p(H)$, and let $Z \le S$ be the unique subgroup of order p. In particular, Z is weakly closed in T with respect to G.

Regard $\Phi_M^{G,H}$ as a subfunctor of the functor H^0M which sends P to M^P for all P. Then $H^0M/\Phi_M^{G,H}$ sends a p-subgroup $P \le G$ to M^P if P contains a subgroup conjugate to Z and sends P to 0 otherwise. So by Lemma 1.14,

$$\varprojlim_{\mathcal{O}_p(G)}{}^*(H^0M/\Phi_M^{G,H}) \cong \varprojlim_{\mathcal{O}_p(N_G(Z)/Z)}{}^*\big(P/Z \mapsto M^P\big),$$

and this last functor is acyclic by Proposition 1.7. Since H^0M is also acyclic by Proposition 1.7, the exact sequence for the extension of functors now shows that $\Lambda^i(G,H;M) = \varprojlim^i(\Phi_M^{G,H}) = 0$ for all $i \geq 2$. □

The next proposition describes the role of these relative functors $\Lambda^*(G,H;M)$ when computing higher limits of the $\mathcal{Y}_{\widetilde{L}}^{\Gamma}$. For these purposes, it is useful to consider the following subquotient functors of $\mathcal{Y}_{\widetilde{L}}^{\Gamma}$, defined for any p-centric subgroup $Q \leq L$:

$$(\mathcal{Y}_{\widetilde{L}}^{\Gamma})_{\geq Q}(P) = \begin{cases} \mathcal{Y}_{\widetilde{L}}^{\Gamma}(P) & \text{if } P \geq Q', \text{ some } Q' \ \Gamma\text{-conjugate to } Q \\ 0 & \text{otherwise} \end{cases}$$

$$(\mathcal{Y}_{\widetilde{L}}^{\Gamma})_{Q}(P) = \begin{cases} \mathcal{Y}_{\widetilde{L}}^{\Gamma}(P) & \text{if } P \cap L \text{ is } \Gamma\text{-conjugate to } Q \\ 0 & \text{otherwise} \end{cases}$$

Thus $(\mathcal{Y}_{\widetilde{L}}^{\Gamma})_Q$ is a subfunctor of $(\mathcal{Y}_{\widetilde{L}}^{\Gamma})_{\geq Q}$, and this is a quotient functor of $\mathcal{Y}_{\widetilde{L}}^{\Gamma}$. When Γ and $\widetilde{L}$ are clear from context, we drop them, and just write $\mathcal{Y}_Q \subseteq \mathcal{Y}_{\geq Q}$, etc.

Lemma 3.4. *Fix a simple group L, a central extension $\widetilde{L} \xrightarrow{\pi} L$ such that $\widetilde{L}$ is quasisimple and $\mathrm{Ker}(\pi)$ is a p-group, and a subgroup $\Gamma \leq \mathrm{Aut}(\widetilde{L})$ which contains $\mathrm{Inn}(\widetilde{L}) \cong L$. Then the following hold for any p-centric subgroup $Q \leq L$.*

(a) *If $Q \notin \mathrm{Syl}_p(L)$, then* $\varprojlim_{\mathcal{O}_p(\Gamma)}^*((\mathcal{Y}_{\widetilde{L}}^{\Gamma})_Q) \cong \Lambda^*(N_\Gamma(Q)/Q, N_L(Q)/Q; \mathcal{Y}_{\widetilde{L}}^{\Gamma}(Q))$.

(b) *If $S \in \mathrm{Syl}_p(L)$, and Q is p-centric in L and weakly closed in S with respect to Γ, then for all $i \geq 0$,*

$$\varprojlim_{\mathcal{O}_p(\Gamma)}^i((\mathcal{Y}_{\widetilde{L}}^{\Gamma})_{\geq Q}) \cong \begin{cases} (\pi^{-1}Q)^{N_\Gamma(Q)}/Z(\widetilde{L})^{\Gamma} & \text{if } i = 0 \\ 0 & \text{if } i > 0. \end{cases}$$

Proof. Let $F^1 \supseteq F^0$ be the following functors on $\mathcal{O}_p(\Gamma)$:

$$F^1(P) = \begin{cases} (\pi^{-1}(P \cap L))^P & \text{if } P \cap L \text{ is } p\text{-centric in } L \\ Z(\widetilde{L})^P & \text{otherwise} \end{cases} \quad \text{and} \quad F^0(P) = Z(\widetilde{L})^P.$$

Here, we identify $L = \mathrm{Inn}(\widetilde{L}) \lhd \Gamma$. By definition, $\mathcal{Y} = \mathcal{Y}_{\widetilde{L}}^{\Gamma} \cong F^1/F^0$. Define quotient functors $F^i_{\geq Q}$ of F^i and subfunctors $F^i_Q \subseteq F^i_{\geq Q}$ in analogy with the definitions for $\mathcal{Y}$. Thus

$$\mathcal{Y}_{\geq Q} \cong F^1_{\geq Q}/F^0_{\geq Q} \qquad \text{and} \qquad \mathcal{Y}_Q \cong F^1_Q/F^0_Q$$

By Proposition 3.2,

$$\varprojlim_{\mathcal{O}_p(\Gamma)}^*(F^i_Q) \cong \Lambda^*\big(N_\Gamma(Q)/Q, N_L(Q)/Q; F^i_Q(Q)\big)$$

for $i = 0,1$. Since the action of $N_L(Q)/Q$ on $F^0(Q) = Z(\widetilde{L})$ is trivial, Proposition 3.3(b) implies that if $Q \notin \mathrm{Syl}_p(L)$, then

$$\varprojlim_{\mathcal{O}_p(\Gamma)}^*(\mathcal{Y}_Q) \cong \varprojlim_{\mathcal{O}_p(\Gamma)}^*(F^1_Q/F^0_Q) \cong \Lambda^*\big(N_\Gamma(Q)/Q, N_L(Q)/Q; F^1(Q)/F^0(Q)\big)$$
$$\cong \Lambda^*\big(N_\Gamma(Q)/Q, N_L(Q)/Q; \mathcal{Y}_{\widetilde{L}}^{\Gamma}(Q)\big).$$

Now assume that Q is p-centric in L, and weakly closed in some $S \in \mathrm{Syl}_p(L)$ with respect to Γ. By definition, for all $P \geq Q$ in Γ, $F^i(P) = F^i(Q)^P$ for $i = 0, 1$. Hence by Lemma 1.14 and Proposition 1.7,

$$\varprojlim{}^*(F^i_{\geq Q}) \cong \begin{cases} 0 & \text{if } * > 0 \\ F^i(Q)^{N_\Gamma(Q)} & \text{if } * = 0. \end{cases}$$

Thus $\mathcal{Y}_{\geq Q} \cong F^1_{\geq Q}/F^0_{\geq Q}$ is acyclic, and

$$\varprojlim{}^0(\mathcal{Y}_{\geq Q}) \cong (\pi^{-1}Q)^{N_\Gamma(Q)}/Z(\widetilde{L})^\Gamma. \qquad \square$$

We will need a much stronger vanishing theorem for the $\Lambda^*(G, H; M)$ than what was shown in Proposition 3.3. In the following lemma, for any set $H_1, \ldots, H_m \leq G$ of subgroups, we write $\mathbf{N}_G(H_1, \ldots, H_m)$ to denote the subgroup of elements of G which normalize all of them. A *radical p-chain of length n* in G is a sequence

$$O_p(G) = P_0 \subsetneqq P_1 \subsetneqq \cdots \subsetneqq P_n$$

of distinct p-subgroups of G such that $P_i = O_p(\mathbf{N}_G(P_0, \ldots, P_i))$ for all i, and such that $P_n \in \mathrm{Syl}_p(\mathbf{N}_G(P_0, \ldots, P_{n-1}))$. Note that the first condition implies that $P_i \lhd P_j$ for $i < j$, and that P_i/P_{i-1} is a radical p-subgroup of $\mathbf{N}_G(P_0, \ldots, P_{i-1})/P_{i-1}$ for all i.

Recall that for any finite group G and any $H \leq G$, we set $\mathfrak{N}_H = \sum_{h\in H} h \in \mathbb{Z}[G]$.

Proposition 3.5. *Fix a finite group G, a normal subgroup $H \lhd G$, and a pair of finite $\mathbb{Z}_{(p)}[G]$-modules $M' \subseteq M$. Assume, for some $n \geq 1$, that the induced map $\Lambda^i(G, H; M') \longrightarrow \Lambda^i(G, H; M)$ is not onto for $i = n$, or is not injective for $i = n + 1$. Then there is a radical p-chain*

$$1 = P_0 \subsetneqq P_1 \subsetneqq P_2 \subsetneqq \cdots \subsetneqq P_n$$

of length n in H such that $\mathfrak{N}_{\mathbf{N}_H(P_1,\ldots,P_n)} \cdot (M/M') \neq 0$. Also, if $p \cdot M \leq M'$, then M/M' (when regarded as an $\mathbb{F}_p[P_n]$-module) contains a copy of the free module $\mathbb{F}_p[P_n]$; and in particular

$$\mathrm{rk}_p(M) \geq \mathrm{rk}(M/M') \geq |P_n| \geq p^n.$$

Proof. Since the proof is fairly complicated, we first explain how it works when $H = G$. In this case, $\Lambda^n(G; M/M') = \varprojlim{}^n(\Phi^G_{M/M'}) \neq 0$ by assumption. We set $V = M/M'$ for short, and regard Φ^G_V as a subfunctor of the acyclic functor $\mathfrak{N}V$ of Proposition 1.7: the functor which sends P to $\mathfrak{N}_P \cdot V$. Thus $\varprojlim{}^n(\mathfrak{N}V) = 0$, and hence $\varprojlim{}^{n-1}(\mathfrak{N}V/\Phi^G_V) \neq 0$. This in turn implies, by Proposition 1.1(a) and an appropriate filtration of the functor $\mathfrak{N}V/\Phi^G_V$, that there is a p-subgroup $1 \neq P \leq G$ such that $\Lambda^{n-1}(N_G(P)/P; \mathfrak{N}_P \cdot V) \neq 0$. Also, P is radical by Proposition 1.1(c). If $n = 1$, then $N_G(P)/P$ has order prime to p by Proposition 3.3(a); so $P \in \mathrm{Syl}_p(G)$, $(1 \lhd P)$ is a radical p-chain of length one, and $\mathfrak{N}_{N_G(P)} \cdot V = \big(\mathfrak{N}_P \cdot V\big)^{N_G(P)/P} \neq 0$. Otherwise, if $n > 1$, then we repeat this procedure, and eventually get a radical p-chain of length n which satisfies the required conditions.

When dealing with the general case, the above argument gets into trouble at the first step. We do not know that $\Lambda^n(G, H; V) \neq 0$, since an extension of modules need not induce a long exact sequence of the $\Lambda^*(G, H; -)$. Hence, instead of proving a result about higher limits of the functor $\Phi_V^{G,H}$, we prove it for a larger class of functors on $\mathcal{O}_p(G)$, which includes the quotient functor $\Phi_M^{G,H}/\Phi_{M'}^{G,H}$ as well as other functors which are needed to carry through the induction step.

More precisely, for any $\mathbb{Z}_{(p)}[G]$-module V, let $\mathcal{M}_V^G$ be the set of all Mackey subfunctors $\Psi \subseteq H^0M$. In other words, $\Psi \in \mathcal{M}_V^G$ if and only if it satisfies the following conditions for all $P \in \mathcal{S}_p(G)$:

- $\Psi(P) \le V^P$,
- $\Psi(gPg^{-1}) = g(\Psi(P))$ for all $g \in G$, and
- $Q \le P$ implies $\mathfrak{N}_Q^P(\Psi(Q)) \le \Psi(P) \le \Psi(Q)$.

Here, $\mathfrak{N}_Q^P\colon V^Q \longrightarrow V^P$ is the relative norm map: $\mathfrak{N}_Q^P(x) = \sum_{gQ\in P/Q} gx$.

In particular, we need to consider the functor

$$\Delta_V^{G,H} \in \mathcal{M}_V^G \qquad \text{defined by} \qquad \Delta_V^{G,H}(P) = (\mathfrak{N}_{P\cap H}\cdot V)^P \quad \forall\, P.$$

Clearly, $\Delta_V^{G,H}(gPg^{-1}) = g(\Delta_V^{G,H}(P))$ for all $g \in G$. So to see that $\Delta_V^{G,H} \in \mathcal{M}_V^G$, it remains only to check that for all $Q \le P \in \mathcal{S}_p(G)$, $\mathfrak{N}_Q^P(\Delta_V^{G,H}(Q)) \le \Delta_V^{G,H}(P)$ and $\Delta_V^{G,H}(P) \le \Delta_V^{G,H}(Q)$; and these follow easily from the definition.

Next, for any functor $\Psi\colon \mathcal{O}_p(G)^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{-}\mathtt{mod}$, we define

$$\Psi_{[H]}\colon \mathcal{O}_p(G)^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{-}\mathtt{mod} \quad \text{by setting} \quad \Psi_{[H]}(P) = \begin{cases} \Psi(P) & \text{if } P\cap H = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{N}_V^{G,H}$ be the set of all functors on $\mathcal{O}_p(G)$ of the form $\Psi_{[H]}$ for $\Psi \in \mathcal{M}_V^G$.

Step 1: We first show:

for any $\Phi \in \mathcal{N}_V^{G,H}$ and any $n \ge 1$ with $\varprojlim^n(\Phi) \ne 0$, there exists a radical p-subgroup $P \le H$ such that $\mathfrak{N}_P\cdot V \ne 0$, and a functor $\Phi' \in \mathcal{N}_{\mathfrak{N}_P\cdot V}^{N_G(P)/P,N_H(P)/P}$ such that $\varprojlim^{n-1}(\Phi') \ne 0$. (1)

To see this, let $\Psi_0 \in \mathcal{M}_V^G$ be such that $\Phi = (\Psi_0)_{[H]}$. Set $\Psi = \Psi_0 \cap \Delta_V^{G,H}$; then $\Psi \in \mathcal{M}_V^G$ since an intersection of two Mackey subfunctors is again a Mackey subfunctor. Thus $\Psi(P) = \Psi_0(P) = \Phi(P)$ if $P\cap H = 1$, and so $\Phi = \Psi_{[H]} \subseteq \Psi$. Since Ψ is acyclic [**JM**, Proposition 5.14], $\varprojlim^{n-1}(\Psi/\Phi) \ne 0$.

For each $P \in \mathcal{S}_p(H)$, let $\Phi_P\colon \mathcal{O}_p(G) \longrightarrow \mathbb{Z}_{(p)}\text{-}\mathtt{mod}$ be the functor $\Phi_P(Q) = \Psi(Q)$ if $Q\cap H$ is G-conjugate to P and $\Phi_P(Q) = 0$ otherwise. Thus $\Phi = \Phi_1$; and the Φ_P are the subquotients of a certain filtration of Ψ. Hence there is some P such that $\varprojlim^{n-1}(\Psi_P) \ne 0$. By Proposition 1.1(a,c), for some p-subgroup $Q \le G$ such that $Q\cap H = P$, Q is radical in G, and hence P is radical in H by Lemma 1.5(b). Furthermore, if we define $\Phi' \subseteq \Psi'$ on $\mathcal{O}_p(N_G(P)/P)$ by setting $\Psi'(Q/P) = \Psi(Q)$ and $\Phi'(Q/P) = \Phi_P(Q)$, then $\Psi' \in \mathcal{M}_{\mathfrak{N}_P\cdot V}^{N_G(P)/P}$, and hence

$$\Phi' = \Psi'_{[N_H(P)/P]} \in \mathcal{N}_{\mathfrak{N}_P\cdot V}^{N_G(P)/P,N_H(P)/P}.$$

Finally, by Lemma 1.12, the functors Φ_P and Φ' have the same higher limits, and in particular $\varprojlim^{n-1}(\Phi') \ne 0$. This finishes the proof of (1).

Step 2: We next claim, by induction on n, that for any $\Phi \in \mathcal{N}_V^{G,H}$ with $\varprojlim^n(\Phi) \neq 0$, there exists a radical p-chain $1 = P_0 \subsetneqq P_1 \subsetneqq \cdots \subsetneqq P_n$ of length n such that $\mathfrak{N}_K{\cdot}V \neq 0$, where $K = \mathbf{N}_H(P_1, \ldots, P_n)$. (2)

To prove this, let P and Φ' be as in (1). If $n = 1$, then

$$0 \neq \varprojlim^0(\Phi') \subseteq \Lambda^0(N_G(P)/P, N_H(P)/P; \mathfrak{N}_P{\cdot}V)$$

implies that $(p, |N_H(P)/P|) = 1$, hence that $P \in \mathrm{Syl}_p(H)$, and

$$(\mathfrak{N}_P{\cdot}V)^{N_H(P)} = \mathfrak{N}_{N_H(P)}{\cdot}V \neq 0$$

by Proposition 3.3(a). Thus $(1 \subsetneqq P)$ is a radical p-chain, and (2) holds in this case.

If $n > 1$, then by the induction hypothesis (applied to Φ'), there is a radical p-chain

$$1 \neq P_2/P \subsetneqq \cdots \subsetneqq P_n/P \qquad \text{in} \qquad N_H(P)/P \tag{3}$$

such that $\mathfrak{N}_{K/P}{\cdot}(\mathfrak{N}_P{\cdot}V) = \mathfrak{N}_K{\cdot}V \neq 0$, where K is defined by setting $K/P = \mathbf{N}_{N_H(P)/P}(P_2/P, \ldots, P_n/P)$. Since P is radical in H, $O_p(N_H(P)/P) = 1$, and the sequence $1 \subsetneqq P \subsetneqq P_2 \subsetneqq \cdots \subsetneqq P_n$ is a radical p-chain of length n in H. Also, $K = \mathbf{N}_H(P, P_2, \ldots, P_n)$, and this finishes the proof of (2).

Step 3: By assumption, $M' \subseteq M$ are $\mathbb{Z}_{(p)}[G]$-modules such that

$$\varprojlim^n\left(\Phi_M^{G,H}/\Phi_{M'}^{G,H}\right) \neq 0.$$

Set $V = M/M'$ for short.

Consider the functor $\Psi = H^0M/H^0M'$ on $\mathcal{O}_p(G)$; thus $\Psi(P) = M^P/(M')^P$ for all P. This is a quotient of Mackey functors, hence itself a Mackey functor, and a Mackey subfunctor of H^0V. Thus $\Psi \in \mathcal{M}_V^G$, and so

$$\Phi_M^{G,H}/\Psi_{M'}^{G,H} = \Psi_{[H]} \in \mathcal{N}_V^{G,H}.$$

Hence by (2), there is a radical p-chain $1 = P_0 \subsetneqq P_1 \subsetneqq \cdots \subsetneqq P_n$ of length n such that $\mathfrak{N}_K{\cdot}V \neq 0$, where $K = \mathbf{N}_H(P_1, \ldots, P_n)$.

Now assume that $pM \leq M'$, and thus that $pV = 0$. Choose $x \in V$ such that $\mathfrak{N}_{P_n}{\cdot}x \neq 0$, and consider the P_n-linear homomorphism $\mathbb{F}_p[P_n] \xrightarrow{\alpha} V$ defined by setting $\alpha(g) = gx$. Thus $\mathrm{Ker}(\alpha)$ is an ideal in $\mathbb{F}_p[P_n]$ which does not contain $\mathfrak{N}_{P_n}$, and hence is the zero ideal (cf. [**Se**, §8.3, Prop. 26]). So α is injective, and $V = M/M'$ contains a copy of $\mathbb{F}_p[K]$. The lower bounds for $\mathrm{rk}_p(M) \geq \mathrm{rk}(M/M')$ are now immediate. □

CHAPTER 4

Subgroups which contribute to higher limits

As noted earlier, our remaining goal is to prove that every nonabelian finite simple group lies in the class $\mathfrak{L}^{\geq 2}(2)$ (see Definition 2.8). Once we have shown this, then Theorem A will follow as a consequence of Proposition 2.9. The aim of this chapter is to prove a series of propositions, whose hypotheses are stated in purely group theoretic terms (without reference to higher limits or Λ^*'s), which can then be used in later chapters to carry out a case-by-case check that $L \in \mathfrak{L}^{\geq 2}(2)$.

The following definition is mostly of interest when L is simple, but in a few cases (when carrying out inductive arguments) we need to also deal with almost simple groups. For simplicity in notation, for any group L with $Z(L) = 1$, we identify $L = \mathrm{Inn}(L)$ as a subgroup of $\mathrm{Aut}(L)$. Recall that for any p-group P and any $n \geq 1$,

$$\Omega_n(P) \stackrel{\text{def}}{=} \langle g \in P \,|\, g^{p^n} = 1 \rangle.$$

DEFINITION 4.1. *Fix a centerfree group L and a prime $p \big| |L|$.*

(a) *For each $i \geq 1$, let* $\boldsymbol{\mathfrak{R}^i(L\,;p)}$ *be the set of all p-subgroups $P \leq L$ with the property that for some $\Gamma \leq \mathrm{Aut}(L)$ which contains $\mathrm{Inn}(L)$, and some $N_\Gamma(P)$-invariant subgroup $Z' \subseteq Z(P)$,*

$$\Lambda^i(N_\Gamma(P)/P, N_L(P)/P; Z') \neq 0.$$

(b) *For each $i \geq 1$, set*

$$\boldsymbol{\mathfrak{E}^i(L\,;p)} = \big\{\Omega_1(Z(P)) \,\big|\, P \in \mathfrak{R}^i(L\,;p)\big\}.$$

(c) *An elementary abelian p-subgroup $E \leq L$ is called* **pivotal** *if $E = \Omega_1(Z(P))$ for some $P \in \mathrm{Syl}_p(C_L(E))$, and $O_p(\mathrm{Aut}_L(E)) = 1$.*

We also write $\mathfrak{R}^{\geq i}(L\,;p) = \bigcup_{j\geq i} \mathfrak{R}^j(L\,;p)$, and similarly for $\mathfrak{E}^{\geq i}(L\,;p)$. In addition, for any p-subgroup $Q \leq L$, we let $\mathfrak{R}^i(L\,;p)_{\ngeq Q}$ and $\mathfrak{R}^{\geq i}(L\,;p)_{\ngeq Q}$ denote the set of subgroups in $\mathfrak{R}^i(L\,;p)$ and $\mathfrak{R}^{\geq i}(L\,;p)$, respectively, which do not contain any subgroup conjugate to Q.

As will be seen in the next proposition (or in its proof), we can think of $\mathfrak{R}^i(L\,;p)$ as the set of p-subgroups of L which could "contribute" to $\varprojlim^i(\mathcal{Y}_{\widetilde{L}}^{\Gamma})$, for some quasicentric group $\widetilde{L}$ with $\widetilde{L}/Z(\widetilde{L}) \cong L$, and some $\Gamma \leq \mathrm{Aut}(\widetilde{L})$ which contains $\mathrm{Inn}(\widetilde{L}) \cong L$. Of course, the existence of an element of $\mathfrak{R}^i(L\,;p)$ does not mean that $\varprojlim^i(\mathcal{Y}_{\widetilde{L}}^{\Gamma})$ is nonvanishing for some Γ and $\widetilde{L}$: a nonzero "contribution" to $\varprojlim^i(-)$ by a subgroup in $\mathfrak{R}^i(L\,;p)$ could be cancelled by a contribution to $\varprojlim^{i\pm 1}(-)$.

PROPOSITION 4.2. *Fix a prime p, a finite simple group L, $S \in \mathrm{Syl}_p(L)$, and $i \geq 1$. Assume there is a subgroup $Q \leq S$ which is p-centric in L and weakly closed in S with respect to* $\mathrm{Aut}(L)$, *such that $\mathfrak{R}^i(L\,;p)_{\ngeq Q} = \varnothing$. Then $L \in \mathfrak{L}^i(p)$. In particular, $L \in \mathfrak{L}^i(p)$ if $\mathfrak{R}^i(L\,;p) = \varnothing$.*

PROOF. Fix Γ and $\widetilde{L}$, and let $\mathcal{Y}_{\ngeq Q}$ be the subfunctor of $\mathcal{Y}^{\Gamma}_{\widetilde{L}}$ defined by setting $\mathcal{Y}_{\ngeq Q}(P) = \mathcal{Y}^{\Gamma}_{\widetilde{L}}(P)$ if P does not contain a subgroup conjugate to Q and $\mathcal{Y}_{\ngeq Q}(P) = 0$ otherwise. For each $P \leq L \cong \mathrm{Inn}(\widetilde{L})$ which does not contain a subgroup conjugate to Q, we defined $\mathcal{Y}^{\Gamma}_{\widetilde{L}}(P)$ to be a certain $N_\Gamma(P)/P$-invariant subgroup of $Z(P)$ (Definition 2.5). Thus $P \notin \mathfrak{R}^i(L\,;p)$ implies that

$$\Lambda^i(N_\Gamma(P)/P, N_L(P)/P; \mathcal{Y}^{\Gamma}_{\widetilde{L}}(P)) = 0.$$

Hence $\varprojlim^i(\mathcal{Y}_{\ngeq Q}) = 0$ by Lemma 3.4(a) and an appropriate filtration of $\mathcal{Y}_{\ngeq Q}$. Furthermore, $\varprojlim^i(\mathcal{Y}^{\Gamma}_{\widetilde{L}}/\mathcal{Y}_{\ngeq Q}) = 0$ by Lemma 3.4(b), and this finishes the proof of the proposition.

The last statement is just the case $Q \in \mathrm{Syl}_p(L)$. □

In the course of the next five chapters, we will prove that all simple groups lie in $\mathfrak{L}^{\geq 2}(2)$, and most cases this will be done using Proposition 4.2. The following proposition will, however, be needed when proving that simple groups of Lie type in characteristic two lie in $\mathfrak{L}^{\geq 2}(2)$ (and in fact, with the exception of $L_3(2)$, lie in $\mathfrak{L}^{\geq 1}(2)$).

PROPOSITION 4.3. *Let L be a simple group. Fix $S \in \mathrm{Syl}_p(L)$, and let $Q \lhd S$ be a p-subgroup with the following properties:*

(a) *Q is p-centric in L and weakly closed in S with respect to* $\mathrm{Aut}(L)$.

(b) *for each $P \leq S$ which does not contain Q, and which is p-centric and radical in L, P is weakly closed in S with respect to L, and $Z(N_L(P))_p = Z(N_L(PQ))_p$.*

Then $L \in \mathfrak{L}^{\geq 1}(p)$.

PROOF. Let $\mathcal{P}$ be the set of p-subgroups of S which are radical and p-centric in L. For any $P \in \mathcal{P}$, let $\widetilde{P}$ denote its inverse image in $\widetilde{L}$. We first check that for each $P \in \mathcal{P}$,

$$Z(N_{\widetilde{L}}(\widetilde{P}))_p = Z(N_{\widetilde{L}}(\widetilde{PQ}))_p. \tag{1}$$

By assumption, $Z(N_L(P))_p = Z(N_L(PQ))_p$. Also,

$$Z(N_{\widetilde{L}}(\widetilde{P}))/Z(\widetilde{L}) = \mathrm{Ker}\big[Z(N_L(P)) \xrightarrow{\psi_P} \mathrm{Hom}(N_L(P), Z(\widetilde{L}))\big]\,,$$

and similarly $Z(N_{\widetilde{L}}(\widetilde{PQ}))/Z(\widetilde{L}) = \mathrm{Ker}(\psi_{PQ})$. Here, ψ_P and ψ_{PQ} are defined by lifting to $\widetilde{L}$ and taking commutators. Thus ψ_P and ψ_{PQ} have the same domain. Since $Z(\widetilde{L})$ is a p-group and $N_L(P) \geq N_L(PQ) \geq S$ (P and Q being weakly closed in S), their target groups both inject into $\mathrm{Hom}(S, Z(\widetilde{L}))$. Thus $\mathrm{Ker}(\psi_P) = \mathrm{Ker}(\psi_{PQ})$, and this proves (1).

For each $R\in\mathcal{P}$, define functors $\mathcal{Y}_R$ and $\mathcal{Y}_{\geq R}$ on $\mathcal{O}_p(\Gamma)$ by setting

$$\mathcal{Y}_{\geq R}(P)=\begin{cases}\mathcal{Y}^{\Gamma}_{\widetilde{L}}(P) & \text{if } P\cap L\geq R',\ \text{ some } R'\ \Gamma\text{-conjugate to } R\\ 0 & \text{otherwise}\end{cases}$$

and

$$\mathcal{Y}_R(P)=\begin{cases}\mathcal{Y}^{\Gamma}_{\widetilde{L}}(P) & \text{if } P\cap L \text{ is } \Gamma\text{-conjugate to } R\\ 0 & \text{otherwise.}\end{cases}$$

Thus $\mathcal{Y}_{\geq R}$ is a quotient functor of $\mathcal{Y}^{\Gamma}_{\widetilde{L}}$, and $\mathcal{Y}_R$ is a subfunctor of $\mathcal{Y}_{\geq R}$.

We claim that $\varprojlim^*(\mathcal{Y}_R)=0$ for each $R\in\mathcal{P}$ which does not contain Q. It suffices to prove this when $N_\Gamma(R){\cdot}L=\Gamma$ (otherwise replace Γ by $N_\Gamma(R){\cdot}L$ without changing the higher limits). In this case, R and RQ are both weakly closed in S with respect to Γ. Hence by Lemma 3.4(b), $\mathcal{Y}_{\geq R}$ and $\mathcal{Y}_{\geq RQ}$ are acyclic, and by (1),

$$\varprojlim{}^0(\mathcal{Y}_{\geq R})\cong Z(N_{\widetilde{L}}(\widetilde{R}))_p^{\Gamma/L}/Z(\widetilde{L})^\Gamma\cong Z(N_{\widetilde{L}}(\widetilde{RQ}))_p^{\Gamma/L}/Z(\widetilde{L})^\Gamma\cong\varprojlim{}^0(\mathcal{Y}_{\geq RQ}).\quad(2)$$

Thus if we set $\mathcal{Y}^\bullet_{\geq R}=\operatorname{Ker}[\mathcal{Y}_{\geq R}\longrightarrow\mathcal{Y}_{\geq RQ}]$, then $\varprojlim^*(\mathcal{Y}^\bullet_{\geq R})=0$. We can assume inductively that $\varprojlim^*(\mathcal{Y}_P)=0$ for all $P\in\mathcal{P}$ such that $P\gneqq R$ and $P\ngeq Q$. Via the obvious filtration of $\mathcal{Y}^\bullet_{\geq R}$, we now see that $\varprojlim^*(\mathcal{Y}_R)=0$.

Thus $\varprojlim^*(\mathcal{Y}^\Gamma_{\widetilde{L}})\cong\varprojlim^*(\mathcal{Y}_{\geq Q})$, and $\mathcal{Y}_{\geq Q}$ is acyclic by Lemma 3.4(b) again. Since this holds for all Γ and $\widetilde{L}$, $L\in\mathfrak{L}^{\geq 1}(2)$. □

It remains to develop some tools for determining which subgroups belong to the sets $\mathfrak{R}^i(L\,;p)$. The first step is to study their connection with the sets $\mathfrak{E}^i(L\,;p)$ and with the pivotal subgroups of L, also defined in 4.1.

Proposition 4.4. *Fix a prime p, a finite centerfree group L, and $i\geq 1$. Then the following hold for any p-subgroup $P\leq L$ and any elementary abelian p-subgroup $E\leq L$.*

(a) *If $P\in\mathfrak{R}^i(L\,;p)$ and $E=\Omega_1(Z(P))$, then $P\in\operatorname{Syl}_p(C_L(E))$. If $E\in\mathfrak{E}^i(L\,;p)$ and $P\in\operatorname{Syl}_p(C_L(E))$, then $P\in\mathfrak{R}^i(L\,;p)$ and $E=\Omega_1(Z(P))$. In other words, there are inverse bijections*

$$\mathfrak{R}^i(L\,;p)/(\mathrm{conj})\ \underset{\mathfrak{rad}}{\overset{\mathfrak{elem}}{\rightleftarrows}}\ \mathfrak{E}^i(L\,;p)/(\mathrm{conj}),$$

where $\mathfrak{elem}(P)=\Omega_1(Z(P))$ and $\mathfrak{rad}(E)\in\operatorname{Syl}_p(C_L(E))$.

(b) *Assume $E=\Omega_1(Z(P))$ and $P\in\operatorname{Syl}_p(C_L(E))$. Then the natural map*

$$N_L(P)/P\twoheadrightarrow\operatorname{Aut}_L(E)$$

induced by restriction is a surjection, its kernel has order prime to p, and the action of $N_L(P)/P$ on $Z(P)$ factors through $\operatorname{Aut}_L(E)$. Furthermore, for any $\Gamma\leq\operatorname{Aut}(L)$ which contains $\operatorname{Inn}(L)$, the natural map $N_\Gamma(P)/P\twoheadrightarrow\operatorname{Aut}_\Gamma(E)$ is a surjection.

(c) *If $E\in\mathfrak{E}^i(L\,;p)$, then E is a pivotal subgroup.*

(d) *If E is a pivotal subgroup and $P\in\operatorname{Syl}_p(C_L(E))$, then P is a radical p-subgroup of L.*

(e) *Fix* $S \in \mathrm{Syl}_p(L)$. *Each pivotal* p*-subgroup of* L *is* L*-conjugate to a subgroup* $E \le S$ *such that* $E = \Omega_1(Z(C_S(E)))$, *and hence such that* $E \ge \Omega_1(Z(S))$.

(f) *Let* $E \le L$ *be an elementary abelian* p*-subgroup, and let* $\mathfrak{X} \subseteq C_L(E) \smallsetminus E$ *be a* $C_L(E)$*-conjugacy class of elements of order* p *such that* $|\mathfrak{X}|$ *is prime to* p. *Then no elementary abelian subgroup* $E' \ge E$ *such that* $E' \cap \mathfrak{X} = \varnothing$ *is pivotal in* L.

PROOF. **(a)** Assume $P \in \mathfrak{R}^i(L\,;p)$, and let $\Gamma \le \mathrm{Aut}(L)$ and $Z' \subseteq Z(P)$ be such that

$$\Lambda^i(N_\Gamma(P)/P, N_L(P)/P; Z') \ne 0.$$

Set $E = \Omega_1(Z(P))$ and $H = C_L(E)$. Then $P \le H$ by definition of E. Since $N_H(P)/P$ acts trivially on E, it acts trivially on $\Omega_n(Z')/\Omega_{n-1}(Z')$ for each n, and hence must have order prime to p by Proposition 3.3(b). So by Lemma 1.10, $P \in \mathrm{Syl}_p(H)$.

Now assume $E \in \mathfrak{E}^i(L\,;p)$. By definition, $E = \Omega_1(Z(P'))$ for some $P' \in \mathfrak{R}^i(L\,;p)$, and we just saw that $P' \in \mathrm{Syl}_p(C_L(E))$. Hence if $P \in \mathrm{Syl}_p(C_L(E))$, then $P = xP'x^{-1}$ for some $x \in C_L(E)$, so $P \in \mathfrak{R}^i(L\,;p)$, and $\Omega_1(Z(P)) = xEx^{-1} = E$.

(b) Fix $\mathrm{Inn}(L) \le \Gamma \le \mathrm{Aut}(L)$ (and we identify L with $\mathrm{Inn}(L) \lhd \Gamma$). Since $E = \Omega_1(Z(P))$, we have $N_\Gamma(P) \le N_\Gamma(E)$. Since P is a Sylow p-subgroup of $C_L(E) \lhd N_\Gamma(E)$ (and $N_\Gamma(P) \le N_\Gamma(E)$), $N_\Gamma(E) = C_L(E){\cdot}N_\Gamma(P)$ by a Frattini argument (Lemma 1.9). Hence $N_\Gamma(P)/P$ surjects onto $\mathrm{Aut}_\Gamma(E) \cong N_\Gamma(E)/C_\Gamma(E)$. When $\Gamma = L$, then the kernel of this surjection is $N_{C_L(E)}(P)/P$, which has order prime to p since $P \in \mathrm{Syl}_p(C_L(E))$.

It remains to show that $N_{C_L(E)}(P)/P$ acts trivially on $Z(P)$. It acts trivially on $E = \Omega_1(Z(P))$ by definition, and hence on $\Omega_n(Z(P))/\Omega_{n-1}(Z(P))$ for each n. By [**Gor**, Corollary 5.3.3], any group of automorphisms of $Z(P)$ with this property must be a p-group; and since $N_{C_L(E)}(P)/P$ has order prime to p, it must act trivially.

(c) Assume $E \in \mathfrak{E}^i(L\,;p)$, and fix $P \in \mathrm{Syl}_p(C_L(E))$. Then $E = \Omega_1(Z(P))$ by (a). So to show E is pivotal, it remains to show that $O_p(\mathrm{Aut}_L(E)) = 1$.

Let $Z' \le Z(P)$ and $\Gamma \le \mathrm{Aut}(L)$ be such that

$$\Lambda^i(N_\Gamma(P)/P, N_L(P)/P; Z') \ne 0.$$

By (b), $N_L(P)/P$ surjects onto $\mathrm{Aut}_L(E)$ with kernel $K/P \overset{\text{def}}{=} N_{C_L(E)}(P)/P$ of order prime to p, and K acts trivially on $Z(P)$ (hence on Z'). So by Proposition 3.3(d),

$$\Lambda^i\big(N_\Gamma(P)/K, \mathrm{Aut}_L(E); Z'\big) \ne 0,$$

and hence $O_p(\mathrm{Aut}_L(E)) = 1$ by Proposition 3.3(c).

(d) Assume E is pivotal and $P \in \mathrm{Syl}_p(C_L(E))$. By (b), $N_L(P)/P$ surjects onto $\mathrm{Aut}_L(E)$ with kernel of order prime to p. Since $O_p(\mathrm{Aut}_L(E)) = 1$ by definition of a pivotal subgroup, $O_p(N(P)/P) = 1$, and so P is radical.

(e) Fix $S \in \mathrm{Syl}_p(L)$. Any pivotal subgroup of L is L-conjugate to a subgroup E such that some $P \in \mathrm{Syl}_p(C_L(E))$ is contained in S. Thus $P = C_S(E)$, and so $E = \Omega_1(Z(C_S(E)))$ by definition of pivotal. Since $E \le P \le S$, we have $Z(S) \le C_S(E) = P$, so $Z(S) \le Z(P)$, and hence $\Omega_1(Z(S)) \le \Omega_1(Z(P)) = E$.

(f) Fix an elementary abelian p-subgroup $E \le L$, and a $C_L(E)$-conjugacy class $\mathfrak{X} \subseteq C_L(E)$ of elements of order p such that $|\mathfrak{X}|$ is prime to p. Assume $E' \ge E$ is such that $E' \cap \mathfrak{X} = \varnothing$. Then $C_L(E') \cap \mathfrak{X}$ is the fixed point set of an E'/E-action on $\mathfrak{X}$, and hence also has order prime to p. It is a union of $C_L(E')$-conjugacy classes, and thus contains at least one $C_L(E')$-conjugacy class $\mathfrak{X}'$ of order prime to p. Fix $x \in \mathfrak{X}'$. Then $x \notin E'$ since $E' \cap \mathfrak{X} = \varnothing$, and $C_{C_L(E')}(x)$ has index $|\mathfrak{X}'|$ in $C_L(E')$. Hence there is $P \in \mathrm{Syl}_p(C_L(E'))$ such that $x \in P$ and $[x, P] = 1$. Thus $x \in Z(P) \smallsetminus E'$, so $\Omega_1(Z(P)) \gneqq E'$, and hence E' is not pivotal. □

It remains to list some more conditions which can be used later to prove that certain subgroups are not in $\mathfrak{E}^{\ge 2}(L\,;2)$ or $\mathfrak{R}^{\ge 2}(L\,;2)$. Almost all of these will be based on the following, very general, proposition. Recall (from Chapter 3) that a *radical p-chain* in a group G is a sequence

$$O_p(G) = P_0 \lhd P_1 \lhd \cdots \lhd P_n$$

of distinct p-subgroups of G such that $P_i = O_p\big(\mathbf{N}_G(P_0, \ldots, P_i)\big)$ for all i (in particular, $P_i \lhd P_n$ for all i), and such that $P_n \in \mathrm{Syl}_p\big(\mathbf{N}_G(P_0, \ldots, P_{n-1})\big)$.

Proposition 4.5. *Fix a finite centerfree group L and an elementary abelian p-subgroup $E \le L$. Assume, for some $m \ge 1$, that $E \in \mathfrak{E}^m(L\,;p)$. Then for any sequence $1 = E_0 \le E_1 \le \cdots \le E_k = E$ of $N_{\mathrm{Aut}(L)}(E)$-invariant subgroups, there is some $1 \le i \le k$ such that*

(a) *the $\mathrm{Aut}_L(E)$-action on E_i/E_{i-1} is p-faithful; and*

(b) *there is a radical p-chain $1 = P_0 \lneqq P_1 \lneqq \cdots \lneqq P_m$ of length m in $\mathrm{Aut}_L(E)$ such that E_i/E_{i-1} (when regarded as a $\mathbb{F}_p[P_m]$-module) contains a copy of the free module $\mathbb{F}_p[P_m]$.*

Proof. Fix $P \in \mathrm{Syl}_p(C_L(E))$; then $P \in \mathfrak{R}^m(L\,;2)$ by Proposition 4.4(a). Let $\Gamma \le \mathrm{Aut}(L)$ and $Z' \le Z(P)$ be such that $\Lambda^m(N_\Gamma(P)/P, N_L(P)/P; Z') \ne 0$. By Proposition 4.4(b), the natural map $N_L(P)/P \longrightarrow \mathrm{Aut}_L(E)$ is surjective with kernel of order prime to p, and the action of $N_L(P)/P$ on $Z(P)$ (hence on Z') factors through $\mathrm{Aut}_L(E)$.

Since each quotient $\Omega_n(Z')/\Omega_{n-1}(Z')$ is isomorphic as an $\mathbb{F}_p[N_\Gamma(P)/P]$-module to a submodule of E, and since the E_i are all $\mathrm{Aut}_\Gamma(E)$-invariant (hence $N_\Gamma(P)/P$-invariant) by assumption, there is a sequence $1 = Z_0 \le Z_1 \le \cdots \le Z_k = Z'$ of $N_\Gamma(P)/P$-invariant subgroups such that each Z_i/Z_{i-1} is isomorphic to a subgroup of some E_j/E_{j-1}. Choose $i > 0$ such that

$$\Lambda^m(N_\Gamma(P)/P, N_L(P)/P; Z_{i-1}) = 0 \quad \text{and} \quad \Lambda^m(N_\Gamma(P)/P, N_L(P)/P; Z_i) \ne 0.$$

Then (a) follows from Proposition 3.3(b), and (b) from Proposition 3.5. □

We now, for most of the rest of the chapter, specialize to the case $p = 2$. The next proposition gives some special cases of Proposition 4.5.

Proposition 4.6. *Fix a finite centerfree group L and an elementary abelian 2-subgroup $E \le L$. Let $1 = E_0 \le E_1 \le \cdots \le E_k = E$ be any sequence of $N_{\mathrm{Aut}(L)}(E)$-invariant subgroups such that at least one of the following conditions holds for each i:*

(a) *The* $\mathrm{Aut}_L(E)$*-action on* E_i/E_{i-1} *is not* 2*-faithful; i.e., its kernel has order a multiple of* 2*.*

(b) $\mathrm{rk}(E_i/E_{i-1}) \leq 3$.

(c) $\mathrm{rk}(E_i/E_{i-1}) \leq 7$ *and the Sylow 2-subgroups of* $\mathrm{Aut}_L(E)$ *are neither dihedral nor semidihedral.*

(d) $\mathrm{rk}(E_i/E_{i-1}) < 2{\cdot}|R|$ *for each radical 2-subgroup* $1 \neq R \leq \mathrm{Aut}_L(E)$*.*

(e) $\mathrm{rk}(E_i/E_{i-1}) \leq 7$ *and* $\mathrm{Aut}_L(E) \cong SL_2(4)$*,* $GL_2(4)$*,* A_6*,* A_7*, or* Σ_6*.*

Then $E \notin \mathfrak{E}^{\geq 2}(L\,;2)$*.*

PROOF. Assume otherwise: assume $E \in \mathfrak{E}^m(L\,;2)$ for some $m \geq 2$. By Proposition 4.5, there is some i such that the $\mathrm{Aut}_L(E)$-action on E_i/E_{i-1} is p-faithful (i.e., (a) does not hold), and a radical 2-chain $1 = P_0 \subsetneqq P_1 \subsetneqq \cdots \subsetneqq P_m$ in $\mathrm{Aut}_L(E)$ such that E_i/E_{i-1} contains $\mathbb{F}_p[P_m]$ as a summand. In particular, $\mathrm{rk}(E_i/E_{i-1}) \geq |P_m| \geq 2^m$, and this is impossible if either of the conditions (b) or (d) holds. Condition (e) is a special case of (d), since none of those groups has a radical 2-subgroup of order 2.

If $\mathrm{rk}(E_i/E_{i-1}) \leq 7$, then $|P_m| \leq 4$, so $m = 2$, $P_1 = \langle x \rangle$ for some involution x, and $P_2 = C_S(x)$ for some $S \in \mathrm{Syl}_2(\mathrm{Aut}_L(E))$. By [**Hp**, III.14.23], $|C_S(x)| = 4$ implies that S has maximal class (its central series has length $k-1$ if $|S| = 2^k$). By [**Gor**, Theorem 5.4.5], each 2-group of maximal class is dihedral, quaternion, or semidihedral. Together with Proposition 3.3(e), this shows that S must be dihedral or semidihedral.

Thus none of the conditions (a)–(e) hold for E_i/E_{i-1}. □

We next look at some cases where $\mathrm{Aut}_L(E)$ is a symmetric group or general linear group. Similar results involving other classical groups in characteristic two will be shown in Proposition 6.5.

PROPOSITION 4.7. *Fix a finite centerfree group* L*, a pivotal 2-subgroup* $E \leq L$*, and* $N_{\mathrm{Aut}(L)}(E)$*-invariant subgroups* $1 = E_0 \leq E_1 \leq \cdots \leq E_k = E$*. Let* $\mathrm{Aut}_L(E_i/E_{i-1})$ *denote the image of* $\mathrm{Aut}_L(E)$ *in* $\mathrm{Aut}(E_i/E_{i-1})$*. Assume, for each* $1 \leq i \leq k$*, that* E_i/E_{i-1} *either satisfies one of the conditions (a–e) in Proposition 4.6, or satisfies one of the following conditions: either*

(a) $\mathrm{Aut}_L(E_i/E_{i-1}) \cong GL_n(2)$ *for some* n *and* $\mathrm{rk}(E_i/E_{i-1}) < 2^n$*; or*

(b) $\mathrm{Aut}_L(E_i/E_{i-1}) \cong \Sigma_n$ *or* A_n*, and* E_i/E_{i-1} *is the permutation representation on* $(\mathbb{Z}/2)^n$ *or* $(\mathbb{Z}/2)^{n-1}$*; or*

(c) $\bigl(\mathrm{Aut}_L(E_i/E_{i-1}), E_i/E_{i-1}\bigr) \cong \bigl(GL_m(2) \wr \Sigma_n, (\mathbb{Z}/2)^{mn}\bigr)$ *for some* $m \geq 3$ *and* $n \geq 2$*.*

Then $E \notin \mathfrak{E}^{\geq 2}(L\,;2)$*.*

PROOF. Set $E' = E_i/E_{i-1}$ and $G = \mathrm{Aut}_L(E_i/E_{i-1})$ for short. By Propositions 4.5 and 4.6, it suffices to show that there is no radical 2-chain $1 \subsetneqq P_1 \subsetneqq \cdots \subsetneqq P_k$ of length $k \geq 2$ in G such that E' contains a copy of $\mathbb{F}_2[P_k]$ as a summand.

When $G \cong GL_n(2)$, then the smallest radical 2-subgroups of G have order 2^{n-1} (see Lemma 6.1 for a description of the radical 2-subgroups). Hence by Proposition 4.6(d), $E \in \mathfrak{E}^{\geq 2}(L\,;2)$ implies $\mathrm{rk}(E_i/E_{i-1}) \geq 2^n$.

Now assume $G \cong \Sigma_n$ or A_n, regarded as acting on a set X of n elements, such that E_i/E_{i-1} is isomorphic to a submodule or quotient module of $\mathbb{F}_2(X)$. Then for any 2-subgroup $P \leq G$, E_i/E_{i-1} contains a free submodule $\mathbb{F}_2[P]$ only if the action of P on X has at least one free orbit. Furthermore, no radical 2-subgroup $P \leq G$ can have $m > 1$ free orbits: this is clear if $|P| = 2$, $m = 2$, and $G = A_n$ (since $N_G(P) \cong A_n \cap (D_8 \times A_{n-4})$ in this case); and holds in the other cases since $O_2(N_{\Sigma_n}(P)/P)$ contains a copy of P^{m-1}. Thus there is no radical 2-chain of length ≥ 2 such that P_2 has a free orbit, and so (b) follows from Proposition 4.5.

The third case now follows since for any radical 2-chain $\{P_i\}$ in $GL_m(2) \wr \Sigma_n$ of length ≥ 2, either $P_2 \cap (GL_m(2))^n \neq 1$ and contains a nontrivial radical 2-subgroup of $(GL_m(2))^n$ (Lemma 1.5), in which case the module contains no summand $\mathbb{F}_2[P_2]$; or $P_2 \cap (GL_m(2))^n = 1$, and an argument similar to that used for $G \cong \Sigma_n$ shows that the module contains no summand $\mathbb{F}_2[P_2]$. □

It is important to note that there *is* an action of Σ_5 on $(\mathbb{Z}/2)^4$ with the property that $\Lambda^2(\Sigma_5; (\mathbb{Z}/2)^4) \cong \mathbb{Z}/2$. This is the action obtained by identifying $\Sigma_5 \cong \Sigma L_2(4)$ — the group $SL_2(4) \cong A_5$ extended by its field automorphism — and $(\mathbb{Z}/2)^4 \cong (\mathbb{F}_4)^2$. Note that this action is transitive on the nonzero elements in the module, unlike the permutation action. To see this computation, let $M = (\mathbb{F}_4)^2$ denote the module, and consider the functors $\Phi_M^{\Sigma_5} \subseteq \mathfrak{N}M$ over $\mathcal{O}_2(\Sigma_5)$. Then $\varprojlim^*(\mathfrak{N}M) = 0$ by Proposition 1.7, and so for all $i \geq 1$,

$$\Lambda^i(\Sigma_5; M) = \varprojlim{}^i(\Phi_M^{\Sigma_5}) \cong \varprojlim{}^{i-1}(\mathfrak{N}M/\Phi_M^{\Sigma_5}).$$

The only nontrivial radical 2-subgroups $P \leq \Sigma_5$ such that $\mathfrak{N}_P{\cdot}M \neq 0$ are the subgroups of order 2 generated by a transposition in Σ_5, or equivalently conjugate to the field automorphism $\theta \in \Sigma L_2(4)$. Set $P_0 = \langle\theta\rangle$; then by Proposition 1.1(a,d),

$$\begin{aligned}\Lambda^i(\Sigma_5; M) &\cong \varprojlim{}^{i-1}(\mathfrak{N}M/\Phi_M^{\Sigma_5}) \cong \Lambda^{i-1}(N(P_0)/P_0; \mathfrak{N}_{P_0}{\cdot}M) \\ &\cong \Lambda^{i-1}(GL_2(\mathbb{F}_2); (\mathbb{F}_2)^2) \cong \begin{cases}\mathbb{Z}/2 & \text{if } i = 2 \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

The following variant of Proposition 4.7 will be needed when handling classical groups in odd characteristic.

Proposition 4.8. *Fix a finite centerfree group L and a pivotal 2-subgroup $E \leq L$, and set $\Gamma = \mathrm{Aut}_L(E)$ for short. Assume there is a finite set X with Γ-action such that E is isomorphic to a Γ-submodule of the permutation representation $\mathbb{F}_2(X)$, or of $\mathbb{F}_2(X)/\langle\sum_{x\in X} x\rangle$. Let $X_1, \ldots, X_k \subseteq X$ be the Γ-orbits, and assume that the Γ-action contains all permutations whose restriction to each X_i is an even permutation. Then $E \notin \mathfrak{E}^{\geq 2}(L\,;2)$.*

Proof. Assume otherwise. By Proposition 4.5, there is a radical 2-chain $1 \subsetneqq P_1 \subsetneqq \cdots \subsetneqq P_k$ of length $k \geq 2$ in Γ such that X contains a free orbit of P_k.

Let $X_i \subseteq X$ be a Γ-orbit which contains a free P_k-orbit. Let m be the number of free orbits of P_1 in X_i (thus $m \geq [P_2{:}P_1] \geq 2$), and let $Y \subseteq X_i$ be the union of those orbits. Let $H \cong P_1 \wr \Sigma_m$ be the group of all permutations of X which centralize P_1 and are the identity on $X\smallsetminus Y$. The subsets $Y \subseteq X_i$ are both invariant under the $C_\Gamma(P_1)$-action on X, so $\Gamma\cap H \lhd C_\Gamma(P_1)$ is the kernel of the map from $C_\Gamma(P_1)$ to the symmetric group on $X\smallsetminus Y$.

Since Γ contains all even permutations of X_i (hence of Y) by assumption, $\Gamma\cap H$ has index at most 2 in H. Also, $O_2(H)\cong P_1^m$ if $m>2$ and $O_2(H)=H$ if $m=2$. In either case,

$$P_1 \ngeq O_2(\Gamma\cap H) \le O_2(C_\Gamma(P_1)) \le O_2(N_\Gamma(P_1))$$

since $\Gamma\cap H \lhd C_\Gamma(P_1) \lhd N_\Gamma(P_1)$, so $O_2(N_\Gamma(P_1)) \gneqq P_1$, and this contradicts the assumption that P_1 is a radical 2-subgroup of Γ. □

We end the chapter with the following proposition, which will be useful when working with the sporadic groups.

PROPOSITION 4.9. *Let L be any finite centerfree group, let $E\le L$ be a pivotal p-subgroup, and fix $P\in \mathrm{Syl}_p(C_L(E))$. In particular, $E=\Omega_1(Z(P))$. Let $H\le L$ be a subgroup which contains $N_L(E)\ge N_L(P)$.*

(a) *Assume $O_p(H)\ne 1$, and set $E_0=E\cap Z(O_p(H))$. Then $E_0\ne 1$, $P\ge O_p(H)$, $[E,O_p(H)]=1$, and E_0 is an $\mathrm{Aut}_L(E)$-invariant submodule of E. In particular, in this case, $E\le Z(O_p(H))$ if E is $\mathrm{Aut}_L(E)$-irreducible or if $O_p(H)$ is p-centric in H.*

(b) *Assume $P\in\mathfrak{R}^n(L;p)$ (some $n\ge 1$). Let $H'\lhd H$ be a characteristic subgroup which is quasisimple, and set $K=C_H(H')$. Assume that $N_{\mathrm{Aut}(L)}(E)\le N_{\mathrm{Aut}(L)}(H)$, and that $P\cap H'\notin \mathrm{Syl}_p(H')$. Then $PK/K\in\mathfrak{R}^n(H/K;p)$.*

PROOF. **(a)** Since E is pivotal, P is a radical p-subgroup of L (Proposition 4.4(d)), and hence a radical p-subgroup of H (since $N_L(P)=N_H(P)$). So $P\ge O_p(H)$ by Lemma 1.5(b), and $E\le Z(P)$ centralizes $O_p(H)$. In particular,

$$E_0 \overset{\text{def}}{=} E\cap Z(O_p(H)) = \Omega_1\bigl(Z(P)\cap Z(O_p(H))\bigr),$$

and so $E_0\ne 1$ since $Z(P)\cap Q\ne 1$ for any $1\ne Q\lhd P$. Also, E_0 is $\mathrm{Aut}_L(E)$-invariant, since $N_L(E)=N_H(E)$ normalizes both E and $O_p(H)$, and so $E_0\le Z(O_p(H))$ if E is $\mathrm{Aut}_L(E)$-irreducible.

If $O_p(H)$ is p-centric, then $Z(O_p(H))\lhd C_H(O_p(H))$ is the unique Sylow p-subgroup. Since $E\le C_H(O_p(H))$ is a p-subgroup, $E\le Z(O_p(H))$ in this case.

(b) We now assume that $P\in\mathfrak{R}^n(L;p)$ (hence $E\in\mathfrak{E}^n(L;p)$), that $H'\lhd H$ is a characteristic subgroup which is quasisimple, that $P\cap H'\notin\mathrm{Syl}_p(H')$, and that $N_{\mathrm{Aut}(L)}(E)\le N_{\mathrm{Aut}(L)}(H)$. Let $\Gamma\le \mathrm{Aut}(L)$ and $Z'\le Z(P)$ be such that

$$\Lambda^n(N_\Gamma(P)/P, N_L(P)/P; Z')\ne 0.$$

Consider the group

$$Z_0'=\mathrm{Aut}_{Z'}(H')\cong Z'/C_{Z'}(H').$$

By assumption, $N_\Gamma(P)\le N_\Gamma(E)\le N_\Gamma(H)$, and $N_\Gamma(H)\le N_\Gamma(H')$ since H' is characteristic in H. Hence $C_{Z'}(H')$ is an $N_\Gamma(P)$-invariant subgroup of Z'. Since

$P\cap H'\notin \mathrm{Syl}_p(H')$, $N_{PH'}(P)/P$ has order a multiple of p by Lemma 1.10. This group acts trivially on $C_{Z'}(H')$, and thus

$$\Lambda^n(N_\Gamma(P)/P, N_L(P)/P; Z_0')\neq 0$$

by Proposition 3.3(b).

Set

$$\overline{K}=C_L(H').$$

The action of $N_L(P)/P$ on Z_0' is p-faithful by Proposition 3.3(b). Since $P\le H$ normalizes H' and hence $\overline{K}$, $N_{P\overline{K}}(P)/P$ acts trivially on $Z_0'\le \mathrm{Aut}(H_0)$. Hence $N_{P\overline{K}}(P)/P$ has order prime to p, and $P\in \mathrm{Syl}_p(P\overline{K})$ by Lemma 1.10 again.

Now set

$$\begin{aligned} P_0 &= \mathrm{Aut}_P(H')\cong PK/K, \\ L_0 &= \mathrm{Aut}_H(H')\cong H/K, \qquad\qquad (K=C_H(H')) \\ \Gamma_0 &= \mathrm{Aut}_{N_\Gamma(H)}(H'). \end{aligned}$$

We must show that $P_0\in\mathfrak{R}^n(L_0\,;p)$. Note first that $\mathrm{Inn}(H')\le L_0\lhd\Gamma_0\le\mathrm{Aut}(H')$. Since H' is quasisimple, $\mathrm{Inn}(H')\cong H'/Z(H')$ is nonabelian and simple, and the natural map from $\mathrm{Aut}(H')$ to $\mathrm{Aut}(H'/Z(H'))$ is injective. Thus L_0 and Γ_0 can be considered as groups of automorphisms of $\mathrm{Inn}(H')$, and in particular, Γ_0 can be considered a group of automorphisms of the centerfree group L_0.

For any $x\in N_\Gamma(H)$ such that $c_x\in N_{\Gamma_0}(P_0)$,

$$xPx^{-1}\le \big(P{\cdot}C_\Gamma(H')\big)\cap L = P{\cdot}C_L(H')=P\overline{K}.$$

Since $P\in\mathrm{Syl}_p(P\overline{K})$ and P normalizes $\overline{K}$, this implies there is $y\in\overline{K}=C_L(H')$ such that $yx\in N_\Gamma(P)$, and $c_{yx}=c_x\in\mathrm{Aut}(H')$. This (together with the relation $N_\Gamma(P)\le N_\Gamma(H)$) shows that there is a surjection

$$N_\Gamma(P)\xrightarrow{(x\mapsto c_x)}\!\!\!\!\rightarrow N_{\Gamma_0}(P_0).$$

A similar argument, involving elements $x\in H$ such that $c_x\in N_{L_0}(P_0)$, shows that $N_L(P)=N_H(P)$ surjects onto $N_{L_0}(P_0)$.

We thus have surjections

$$N_\Gamma(P)/P\xrightarrow{\kappa_\Gamma}\!\!\!\!\rightarrow N_{\Gamma_0}(P_0)/P_0 \quad\text{and}\quad N_L(P)/P=N_H(P)/P\xrightarrow{\kappa_L}\!\!\!\!\rightarrow N_{L_0}(P_0)/P_0.$$

Furthermore, $\mathrm{Ker}(\kappa_L)\cong N_{PK}(P)/P\le N_{P\overline{K}}(P)/P$ has order prime to p. So by Proposition 3.3(d),

$$\Lambda^n(N_{\Gamma_0}(P_0)/P_0, N_{L_0}(P_0)/P_0; Z_0')\cong\Lambda^n(N_\Gamma(P)/P; N_L(P)/P; Z_0')\neq 0,$$

and this proves that $P_0\in\mathfrak{R}^n(L_0\,;p)$. □

CHAPTER 5

Alternating groups

We are now ready to go through the list of all finite nonabelian simple groups L, to show in each case that $L \in \mathfrak{L}^{\geq 2}(2)$. When L is of Lie type in characteristic 2, this will be done using Proposition 4.3. In all other cases, we prove $L \in \mathfrak{L}^{\geq 2}(2)$ with the help of Proposition 4.2. This means either showing that $\mathfrak{R}^{\geq 2}(L\,;2) = \varnothing$ (equivalently $\mathfrak{E}^{\geq 2}(L\,;2) = \varnothing$); or else choosing a 2-centric subgroup $Q \leq L$ which is weakly closed in a Sylow subgroup of L, and then showing that $\mathfrak{R}^{\geq 2}(L\,;2)_{\not\geq Q} = \varnothing$ (equivalently $\mathfrak{E}^{\geq 2}(L\,;2)_{\not\leq Z(Q)} = \varnothing$).

We will be constantly referring to Proposition 4.4 for some of the most basic properties of subgroups $P \in \mathfrak{R}^i(L\,;2)$ and $E = \Omega_1(Z(P)) \in \mathfrak{E}^i(L\,;2)$, as well as the correspondence between these two sets. Propositions 4.5, 4.6, 4.7, and 4.8, all of which give restrictions on the pairs $(E, \mathrm{Aut}_L(E))$ for $E \in \mathfrak{E}^i(L\,;2)$, will then be used to further eliminate subgroups from these sets; while Proposition 4.9 will be used in some cases to reduce to a problem involving a smaller group. Together with Lemma 1.5, these will be the only results from earlier chapters used in this part of the proof.

The easiest case to consider is that of the alternating groups.

THEOREM 5.1. *For any $n \geq 5$, $A_n \in \mathfrak{L}^{\geq 2}(2)$.*

PROOF. Set $L = A_n$. When $n \leq 7$, then $\mathrm{rk}_2(L) = 2$, so $\mathfrak{R}^{\geq 2}(L\,;2) = \varnothing$ by Proposition 4.6(b), and $L \in \mathfrak{L}^{\geq 2}(2)$ by Proposition 4.2. So we can assume that $n \geq 8$, and hence that $\mathrm{Aut}(L) = \Sigma_n$ (cf. [**Sz**, 3.2.17]).

We regard Σ_n as the group of permutations of the set $\mathbf{n} = \{1, \ldots, n\}$. For any $H \leq \Sigma_n$, let $\mathrm{supp}(H)$ denote the *support* of H: the set of elements in $\mathbf{n}$ which are moved by H. For each $i \geq 1$, let $\mathcal{E}_{2^i}$ be the set of subgroups $E \leq \Sigma_n$, such that $E \cong (C_2)^i$, and the action of E on $\mathbf{n}$ contains one free orbit and is otherwise fixed. These subgroups (for each i such that $2^i \leq n$) clearly make up one conjugacy class in Σ_n.

Fix some $S \in \mathrm{Syl}_2(L)$, and let $Q \leq S$ be the subgroup generated by all $E \leq S$ such that $E \in \mathcal{E}_4$. For any $E, E' \in \mathcal{E}_4$ with $\mathrm{supp}(E) \cap \mathrm{supp}(E') \neq \varnothing$, either $\mathrm{supp}(E) = \mathrm{supp}(E')$ and hence $E = E'$; or $\langle E, E' \rangle$ acts transitively on $\mathrm{supp}(E) \cup \mathrm{supp}(E')$, a set of order 5, 6, or 7, and thus $\langle E, E' \rangle$ is not a 2-group. So if $E, E' \in \mathcal{E}_4$ and $E, E' \leq S$, then either $E = E'$ or they have disjoint support. In other words, Q is the direct product of $[n/4]$ subgroups in $\mathcal{E}_4$ with disjoint support, and is weakly closed in S with respect to Σ_n by construction. Also, $C_{A_n}(Q) \cong Q \times A_k$ where $k = n - 4{\cdot}[n/4] \leq 3$, so Q is 2-centric in $L = A_k$. By Proposition 4.2, it remains only to show that $\mathfrak{R}^{\geq 2}(A_n\,;2)_{\not\geq Q} = \varnothing$.

Fix $P \in \mathfrak{R}^{\geq 2}(A_n\,;2)_{\ngeq Q}$, and set $E = \Omega_1(Z(P))$. Then $P \in \mathrm{Syl}_2(C_{A_n}(E))$ by Lemma 4.4(a). Each union of k orbits of E of order q (orbits under the action on $\mathbf{n}$) which are isomorphic as E-orbits contributes a factor $E_q \wr \Sigma_k$ (for $E_q \in \mathcal{E}_q$) to $C_{\Sigma_n}(E)$. Since each Sylow 2-subgroup of Σ_k is a product of iterated wreath products $C_2 \wr \cdots \wr C_2$, this shows that P is the intersection with A_n of a product of subgroups of the form $E_q \wr C_2 \wr \cdots \wr C_2$ for $E_q \in \mathcal{E}_q$.

Write

$$P = A_n \cap \Big(W(q_1, k_1) \times W(q_2, k_2) \times \cdots \times W(q_t, k_t) \Big),$$

where each $W(q,k)$ is a group of the form $E_q \wr C_2 \wr C_2 \cdots \wr C_2$ (wreath product k times) which acts on a subset of order $q{\cdot}2^k$ in $\mathbf{n}$. We always assume $q \geq 2$, except that there is one factor $(q,k) = (1,0)$ in the above factorization for each point of $\mathbf{n}$ fixed by all of E. Thus, we are assuming $n = \sum_{i=1}^{t} q_i{\cdot}2^{k_i}$: the set $\mathbf{n}$ is partitioned, and each factor of P acts on one summand of the partition.

Since $P \in \mathrm{Syl}_2(C_{A_n}(E))$, and the factors $W(1,0)$ represent the points fixed by E, they can occur with multiplicity at most three; and with multiplicity at most one if E contains orbits of order 2 (i.e., if there are factors $W(2,k)$).

We now collect together factors with the same parameters (q,k). Thus, after changing parameters and changing t, we write

$$P = A_n \cap \Big(W(q_1, k_1)^{\times r_1} \times W(q_2, k_2)^{\times r_2} \times \cdots \times W(q_t, k_t)^{\times r_k} \Big) \quad \big(n = \textstyle\sum_{i=1}^{t} r_i q_i 2^{k_i}\big)$$

where $(q_i, k_i) \neq (q_j, k_j)$ for $i \neq j$. Then $\mathrm{Aut}_{\Sigma_n}(E)$ is a product of terms, one for each factor with $q_i \neq 1$; and the factor for $W(q,k)^{\times r}$ has the form $\mathrm{Aut}(E_q) \wr \Sigma_r$, and acts on a quotient of E of the form $(E_q)^{\times r}$ (or its intersection with an alternating group if $(q,k) = (2,0)$).

Assume P is not a Sylow 2-subgroup of A_n. Then, after permuting the factors in the above decomposition, we can assume that $q_1 \geq 2$, and that $q_1 \geq 4$ or $r_1 \geq 2$. We also assume that $q_1 \geq 4$ if there are any terms with $q_i \geq 4$, and that $(q_1, k_1, r_1) \neq (4, 0, 1)$ unless this is the only term with $q_i \geq 4$. Since $n \geq 8$, this means that if $(q_1, k_1, r_1) = (4, 0, 1)$, then there is at least one factor $W(q_i, k_i)^{\times r_i}$ with $q_i = 2$.

Let $E_0 \leq E$ and $P_0 \leq P$ be the subgroups of elements whose projection to the first factor is the identity. Set $(q,k,r) = (q_1, k_1, r_1)$ and $m = n - rq{\cdot}2^k$, and consider E_0 and P_0 as subgroups of A_m.

If $q_1 \geq 4$, then define an involution $\sigma_0 \in \mathrm{Aut}_{\Sigma_n}(E)$ by letting it act on each factor E_{q_1} in $Z(W(q_1, k_1)^{\times r_1}) \cong (E_{q_1})^{\times r_1}$ via some given automorphism of E_{q_1} (under some identification of the different factors), and letting σ_0 act trivially on the other factors $Z(W(q_i, k_i)^{\times r_i})$. If $(q_1, k_1, r_1) \neq (4, 0, 1)$, then set $\sigma = \sigma_0 \in \mathrm{Aut}_{A_n}(E)$. If $(q_1, k_1, r_1) = (4, 0, 1)$, then by the above remark there is some orbit of E of order 2; let τ be the transposition which exchanges the elements in that orbit, and set $\sigma = \sigma_0 \times \tau$. In both cases, $\sigma \in \mathrm{Aut}_{A_n}(E)$. Finally, if $q_1 = 2$, then $r_1 \geq 2$, and we choose $\sigma \in \mathrm{Aut}_{A_n}(E)$ to exchange two of the factors $W(q_1, k_1)$.

Thus, in all cases, there is an element $\sigma \in \mathrm{Aut}_{A_n}(E)$ of order 2 which centralizes E_0. In particular, $\mathrm{Aut}_{A_n}(E)$ does not act 2-faithfully on E_0. So by Proposition 4.6(a), $\mathrm{Aut}_{A_n}(E)$ must act 2-faithfully on E/E_0, and so $N_{A_n}(P)/P$ also acts 2-faithfully on E by Proposition 4.4(b). Since $N_{A_m}(P_0)/P_0 \leq N_{A_n}(P)/P)$ acts trivially on E/E_0, $N_{A_m}(P_0)/P_0$ has odd order. Thus P_0 is a Sylow 2-subgroup of A_m, and hence is the intersection with A_m of a Sylow 2-subgroup $S' \leq \Sigma_m$. By the choice of the first term, either $q_1 \geq 4$ and hence $W(q_1, k_1)^{\times r_1} \leq A_n$, or $q_i \leq 2$ for all i and hence none of the factors is contained in A_n. In either case, S' can be chosen such that

$$P = A_n \cap \bigl(W(q,k)^{\times r} \times S'\bigr);$$

and $P_0 = P \cap S'$ and $E_0 = E \cap S'$. Since P does not contain any subgroup conjugate to Q, we must have either $q \geq 8$, or $q = 2$ and $k = 0$.

In the notation of Proposition 4.7, $(\mathrm{Aut}_L(E/E_0), E/E_0)$ is isomorphic to one of the pairs $(GL_s(2) \wr \Sigma_r, (Z/2)^{rs})$ (if $q = 2^s$, $s \geq 3$), or $(\Sigma_r, (\mathbb{Z}/2)^r)$ or $(\Sigma_r, (\mathbb{Z}/2)^{r-1})$ (if $q = 2$). In either case, since the $\mathrm{Aut}_L(E)$-action on E_0 is not 2-faithful, Proposition 4.7(b,c) applies to show that $E \notin \mathfrak{E}^{\geq 2}(A_n\,;2)$, and hence that $P \notin \mathfrak{R}^{\geq 2}(A_n\,;2)$. □

By extending the above arguments, one can show that $\mathfrak{R}^{\geq 3}(A_n\,;2) = \varnothing$ for all n, and

$$\mathfrak{R}^2(A_n\,;2) = \Bigl\{ \bigl(E_4 \underbrace{\wr C_2 \wr \cdots \wr C_2}_{k\text{ times}}\bigr)^{\times 2} \times S' \Bigm| E_4 \in \mathcal{E}_4,\ 2^{k+3} \leq n,\ S' \in \mathrm{Syl}_2(A_{n-2^{k+3}}) \Bigr\}.$$

CHAPTER 6

Groups of Lie type in characteristic two

We next consider simple groups of Lie type in characteristic two. The arguments in this chapter, and in particular the proof of Theorem 6.2, have been greatly simplified using suggestions of the referee.

We first summarize the notation we use for structures in these groups (in arbitrary characteristic p). We refer to [**GLS3**] and [**Ca1**] as general references. Any finite simple group L of Lie type in characteristic p is of the form $L = O^{p'}(C_G(\sigma))$, where $G = \mathbb{G}(\bar{\mathbb{F}}_p)$, and $\sigma \in \mathrm{Aut}(G)$ is the composite of a field automorphism and (possibly) a graph automorphism which permutes the root subgroups of G. We fix a σ-invariant Borel subgroup $\bar{B} \le G$, and let $\bar{U} \lhd \bar{B}$ be its maximal unipotent subgroup. Let $\Sigma \supseteq \Sigma^+ \supseteq I$ denote the sets of roots, of positive roots, and of primitive roots of $\mathbb{G}$; and let $\overline{X}_r \le G$ denote the root subgroup of the root $r \in \Sigma$. Thus $\bar{U}$ is generated by the root subgroups X_r for $r \in \Sigma^+$.

Let $\widehat{\Sigma} \supseteq \widehat{I}$ denote the sets of equivalence classes in $\Sigma \supseteq I$ as defined in [**GLS3**, Definition 2.3.1] and [**Ca1**, §13.2]. In particular, for each $r \in \Sigma$, r is equivalent to $\tau(r)$, as well as to all positive linear combinations of r and $\tau(r)$ which are roots. Also, $\widehat{I} = I/\tau$. For each $\hat{r} \in \widehat{\Sigma}$, set $\overline{X}_{\hat{r}} = \prod_{r\in\hat{r}} \overline{X}_r$ and $X_{\hat{r}} = C_{\overline{X}_{\hat{r}}}(\sigma)$. Then $U \overset{\text{def}}{=} \bar{U} \cap L$ is generated by the $X_{\hat{r}}$ for all equivalence classes $\hat{r} \in \widehat{\Sigma}$ of positive roots, and $B \overset{\text{def}}{=} \bar{B} \cap L = N_L(U)$. Finally, for each τ-invariant subset $J \subseteq I$, let $\langle J \rangle \subseteq \Sigma$ denote the set of $\mathbb{Z}$-linear combinations of elements of J which are roots, and set

$$\mathfrak{P}_J = \big\langle B, X_{-\hat{r}} \,|\, \hat{r} \subseteq \langle J \rangle \big\rangle \qquad \text{and} \qquad U_J = \big\langle X_{\hat{r}} \,|\, \hat{r} \subseteq \Sigma^+ \smallsetminus \langle J \rangle \big\rangle.$$

In particular, $\mathfrak{P}_\varnothing = B$, $U_\varnothing = U$, $\mathfrak{P}_I = L$, and $U_I = 1$.

We let ${}^2A_n(q)$, ${}^2D_n(q)$, ${}^3D_4(q)$, and ${}^2E_6(q)$ denote the Steinberg groups defined over the field $\mathbb{F}_{q^2}$ or $\mathbb{F}_{q^3}$. But in contrast to the notation used in [**GLS3**], we let ${}^2B_2(q) = Sz(q)$ and ${}^2F_4(q)$ denote the Suzuki and Ree groups defined as subgroups of $B_2(q)$ and $F_4(q)$ (for q an odd power of 2).

Lemma 6.1. *Let $L = {}^t\mathbb{G}(q)$ be a finite simple group of Lie type over the field $\mathbb{F}_q$ of characteristic p. Set $n = \mathrm{rk}(L) = |\widehat{I}|$. Then the following hold.*

(a) *For each τ-invariant subset $J \subsetneqq I$, $U_J = O_p(\mathfrak{P}_J)$, $\mathfrak{P}_J = N_L(U_J)$, and $C_L(U_J) = Z(U_J)$. The subgroups $\mathfrak{P}_J$ are the only subgroups of L which contain B. Also, for any pair of τ-invariant subsets $J, K \subseteq I$, $U_J U_K = U_{J\cap K}$.*

(b) *A subgroup $P \le U$ is radical in L if and only if $P = U_J$ for some τ-invariant $J \subseteq I$.*

(c) *Each subgroup U_J is weakly closed in U with respect to L.*

(d) *If $p = 2$ and $q \geq 4$, then $Z(B) = 1$.*

(e) *If $p = 2$, $n > 1$, and $L \neq L_3(q)$ or $Sp_4(q)$, then $C_L(Z(U)) \not\leq B$.*

PROOF. **(a)** The first statement is shown in [**GLS3**, Theorem 2.6.5], and the second in [**GLS3**, Theorem 2.6.5(b)]. The relation $U_J U_K = U_{J\cap K}$ is immediate from the definition of the U_J.

(b,c) Each subgroup U_J is radical in L by (a). By the Borel-Tits theorem (cf. [**GLS3**, Corollary 3.1.5]), if $P \leq L$ is a radical p-subgroup, then there is a parabolic subgroup $\mathfrak{P}$ of L — a subgroup L-conjugate to one of the $\mathfrak{P}_J$ — such that $P = O_p(\mathfrak{P})$ and $\mathfrak{P} = N_L(P)$. Thus each radical p-subgroup of L is L-conjugate to one of the U_J, and both points (b) and (c) will follow once we show that each U_J is weakly closed in U with respect to L.

The following argument is based on that in the proof of [**AS**, Lemma I.2.5]. Assume otherwise, and let U_J be maximal among those subgroups of this form which are not weakly closed in U with respect to L. By Alperin's fusion theorem, there is a radical subgroup $Q \leq U$ such that $Q \gneqq U_J$ — hence $Q = U_K$ for some $K \subsetneqq J$ — and an element $x \in N_L(Q) = \mathfrak{P}_K$ such that $xU_Jx^{-1} \neq U_J$. But this is impossible, since $\mathfrak{P}_K \leq \mathfrak{P}_J = N_L(U_J)$.

(d) By [**Ca1**, Theorem 6.3.1], for any $r \in \Sigma$ fixed by τ, $\langle X_r, X_{-r}\rangle$ is the image of a homomorphism ϕ_r defined on $SL_2(q)$, which sends (strict) upper and lower triangular matrices to the root subgroups X_r and X_{-r}, respectively. Let $D \leq SL_2(q)$ be the subgroup of diagonal matrices. The images $\phi_r(D)$ commute with each other (for all $r \in \Sigma$), $\phi_r(D)$ normalizes X_s for all r and s, the $\phi_r(D)$ generate the abelian subgroup H of diagonal elements of $\mathbb{G}(q)$, and $N_{\mathbb{G}(q)}(U) = UH$ (cf. [**Ca1**, §7.1]). In particular, when $q = 2^k$ for $k \geq 2$, $C_{X_r}(\phi_r(D)) = 1$.

Thus when $L = \mathbb{G}(2^k)$ for $k \geq 2$, then H acts on U with trivial fixed subgroup, and hence $Z(B) = C_B(Z(U)) = 1$. If L is one of the Steinberg groups ${}^2A_n(2^k)$, ${}^2D_n(2^k)$, ${}^3D_4(2^k)$, or ${}^2E_6(2^k)$ for $k \geq 2$, then $Z(U) = X_s$ where s is the highest positive root [**GLS3**, Theorem 3.3.1], $\phi_s(SL_2(2^k))$ is contained in L and contains X_s, and hence $Z(B) = Z(N_L(U)) = 1$. Similarly, if L is a Suzuki group ${}^2B_2(2^{2k+1})$ or a Ree group ${}^2F_4(2^{2k+1})$ for some $k \geq 1$, then the center of the Borel subgroup is trivial: this follows from the description (cf. [**Ca1**, Theorem 13.7.4]) of the diagonal elements in these groups; or (more explicitly) from [**HB3**, §XI.3] for the Suzuki groups and from [**Sh**, §II] for the Ree groups.

(e) The center of U is described as follows.

(i) If L is a Chevalley group $\mathbb{G}(q)$, or one of the Steinberg groups ${}^2A_n(q)$, ${}^2D_n(q)$, ${}^3D_4(q)$, or ${}^2E_6(q)$, and $L \not\cong Sp_{2n}(q)$ or $F_4(q)$, then $Z(U) = X_\alpha$ where α is the highest root in the root system.

(ii) If $L \cong F_4(q)$ or $Sp_{2n}(q)$, then $Z(U) = X_\alpha X_{\bar\alpha}$, where α is the highest root and $\bar\alpha$ is the highest short root.

(iii) If $L \cong {}^2F_4(q)$ (where $q \geq 8$ is an odd power of 2), and $L = C_{\bar{L}}(\sigma)$ where $\bar{L} = F_4(q)$, then $Z(U) = C_{\overline{X}_\alpha \overline{X}_{\bar{\alpha}}}(\sigma)$, where $\overline{X}_r$ denotes the root group of r in $\bar{L}$, and α and $\bar{\alpha}$ are as in (ii).

Points (i) and (ii) are shown in [**GLS3**, Theorem 3.3.1], and point (iii) is shown in [**Ree**, Theorem 4.14].

To prove that $C_L(Z(U)) \not\leq B$, it suffices to find an equivalence class $\hat{\beta} \subseteq \Sigma^+$ such that $[X_{-\hat{\beta}}, Z(U)] = 1$. By [**GLS3**, Theorem 1.12.1], two root groups $\overline{X}_r$ and $\overline{X}_s$ in $\mathbb{G}(\bar{\mathbb{F}}_2)$ commute (in characteristic two) if and only if either $r+s$ is not a root; or $\mathbb{G} = B_n$ or F_4, r and s are short roots, and $r+s$ is a long root (in particular, $r \perp s$). Using this, together with the tables of roots in [**Bb**, pp. 250–275], one checks directly that in all cases except where $L = L_2(q)$, $U_3(q)$, $Sz(q)$, $L_3(q)$, or $Sp_4(q)$ (where only the last two have rank ≥ 2), there is a class $\hat{\beta}$ such that $[\overline{X}_{-\beta}, \overline{X}_\alpha] = 1$ and $[\overline{X}_{-\beta}, \overline{X}_{\bar{\alpha}}] = 1$ for $\beta \in \hat{\beta}$ and $\alpha, \bar{\alpha}$ as above. □

We now have the tools needed to prove the main result of this chapter.

THEOREM 6.2. *Let L be a finite simple group of Lie type in characteristic two, or the Tits group ${}^2F_4(2)'$. Then $L \in \mathfrak{L}^{\geq 2}(2)$, and $L \in \mathfrak{L}^{\geq 1}(2)$ if $L \not\cong PSL_3(2)$.*

PROOF. **Case 1:** Assume first that $L \not\cong {}^2F_4(2)'$. If $L = PSL_3(2)$, then $\mathrm{rk}_2(L) = 2$, and the theorem follows from Proposition 4.6(b). Since none of the groups $L_2(2)$, $Sz(2)$, $U_3(2)$, or $Sp_4(2)$ are simple, one of the two conditions (d) or (e) in Lemma 6.1 applies in all other cases. If (d) applies, then the theorem follows from Proposition 4.3, applied with $Q = S$.

Now assume that condition (e) applies; thus $C_L(Z(U)) \not\leq B$. Consider the subgroup $B{\cdot}C_L(Z(U)) \gneqq B$. By Lemma 6.1(a), there is some $\varnothing \neq K \subseteq I$ such that $B{\cdot}C_L(Z(U)) = \mathfrak{P}_K$. Set $D = I \smallsetminus K \subsetneqq I$, and set $Q = U_D$. We claim that Proposition 4.3 applies with this choice of Q. By Lemma 6.1(a), Q is 2-centric in L.

Now, $\mathrm{Aut}(L) = L{\cdot}N_{\mathrm{Aut}(L)}(U)$ by a Frattini argument (Lemma 1.9). By Lemma 6.1(c), Q is weakly closed in U with respect to L. Hence if $\alpha \in \mathrm{Aut}(L)$ is such that $\alpha(Q) \leq U$, then α can be chosen to normalize U and thus permute the positive roots of L. Hence $\alpha(Q) = U_{D'}$ for some $D' \subseteq I$, $\alpha(\mathfrak{P}_K) = \mathfrak{P}_{K'}$ where $K' = I \smallsetminus D'$, but $\alpha(\mathfrak{P}_K) = \mathfrak{P}_K$ since $\alpha(U) = U$ and $\mathfrak{P}_K = C_L(Z(U)){\cdot}N_L(U)$. Thus $K' = K$, $D' = D$, $\alpha(Q) = Q$, and so Q is weakly closed in U with respect to $\mathrm{Aut}(L)$.

Fix a radical 2-subgroup $P \leq L$ such that $Q \not\leq P \leq U$. By Lemma 6.1(b), $P = U_J$ for some $J \subseteq I$, and $N_L(P) = \mathfrak{P}_J$. Also, $PQ = U_{J \cap D} = U_{J \smallsetminus K}$ (Lemma 6.1(a)), and $N_L(PQ) = \mathfrak{P}_{J \smallsetminus K}$. Since

$$N_L(P) = \mathfrak{P}_J = \mathfrak{P}_{J \smallsetminus K} \mathfrak{P}_{J \cap K} = N_L(PQ){\cdot}\mathfrak{P}_{J \cap K},$$

and since P and PQ both contain their centralizers (Lemma 6.1(a)),

$$Z(N_L(P)) = C_L(\mathfrak{P}_J) = Z(N_L(PQ)) \cap C_L(\mathfrak{P}_{J \cap K}).$$

Since $\mathfrak{P}_K = B{\cdot}C_L(Z(U))$, $C_L(\mathfrak{P}_{J \cap K}) \geq C_L(\mathfrak{P}_K) \geq Z(B) \geq Z(N_L(PQ))$, and thus $Z(N_L(P)) = Z(N_L(PQ))$. We have now shown that the hypotheses of Proposition 4.3 all hold, and this finishes the proof of the theorem in this case.

Case 2: Now assume L is the Tits group ${}^2F_4(2)'$. By Lemma 1.5(b), the radical 2-subgroups of L are precisely the subgroups $L\cap P$ for 2-radical subgroups $P \le {}^2F_4(2)$. Hence by Lemma 6.1, the subgroups of $L\cap U \in \mathrm{Syl}_2(L)$ which are radical in L are exactly the subgroups $L\cap U_J$ for τ-invariant subsets $J\subseteq I$. Since I has just two τ-orbits, this means that there are exactly two such proper subgroups.

By [**P1**] (and in the notation used there), there are 2-subgroups $J, K \le L$, with normalizers $H = N_L(J)$ and $N = N_L(K)$, such that H and N both contain $T \in \mathrm{Syl}_2(L)$, and

$$O_2(H) = J, \quad H/J \cong C_5 \rtimes C_4, \qquad O_2(N) = K, \quad N/K \cong \Sigma_3.$$

Thus J and K are the two radical subgroups of L which are proper subgroups of T. Since $|J| \ne |K|$, this implies that J and K are both weakly closed in T with respect to $\mathrm{Aut}(L)$. Also, by [**P1**] again, $Z(H) = Z(T)$ has order 2 (H is chosen to be the centralizer of an involution), while $Z(K) \cong C_2^2$. Hence $L \in \mathfrak{L}^{\ge 1}(2)$ by Proposition 4.3, applied with $Q = K$. □

Throughout the rest of the chapter, we prove some related results about groups of Lie type in characteristic two; results which will be needed in later chapters. First, since it will frequently be referred to later on, we note Witt's lemma (over any field).

Theorem 6.3. *Let V be a finite dimensional vector field over a field K. Let $\mathfrak{b}$ be a nondegenerate symplectic, unitary, or quadratic form on V. (If a unitary form, then with respect to some $\theta \in \mathrm{Aut}(K)$ of order 2.) Then for any pair of subspaces $W_1, W_2 \subseteq V$, and any isomorphism $\alpha\colon W_1 \longrightarrow W_2$ which preserves $\mathfrak{b}$, α extends to an automorphism of V which preserves $\mathfrak{b}$.*

Proof. See, e.g., [**A2**, §20]. A stronger version of this (which allows $\mathfrak{b}$ to be degenerate) is proven in [**Ta**, Theorem 7.4]. □

We have described the radical 2-subgroups of a group of Lie type in characteristic 2 in terms of the root system. But in the case of the classical groups, it is often more useful to translate this into a description in terms of the action on the natural (defining) module of the group.

Proposition 6.4. *Let G be one of the groups $Sp_{2n}(2)$, $GU_n(2)$, or $SO^{\pm}_{2n}(2)$ (for $n \ge 2$), or $\Omega^{\pm}_{2n}(2)$ (for $n \ge 3$). Let V be the natural G-module (a vector space over $\mathbb{F}_2$ or $\mathbb{F}_4$), with bilinear form $\mathfrak{b}$, and (in the orthogonal case) with orthogonal form $\mathfrak{q}$. Let $1 \ne P \le G$ be a nontrivial radical 2-subgroup, and set $W = C_V(P)$. Then either*

(a) *W is an isotropic subspace with respect to $\mathfrak{b}$ and $\mathfrak{q}$, P contains all elements of G which induce the identity on W and on $W^\perp/W$, and $\mathrm{Aut}_{N_G(P)}(W) = \mathrm{Aut}(W)$; or*

(b) *$G = SO^{\pm}_{2n}(2)$, $|P| = 2$, P is generated by an orthogonal transvection, and W is a codimension 1 subspace on which $\mathfrak{q}$ is nondegenerate.*

Proof. Let P and $W = C_V(P)$ be as above. Let $G_W \le G$ be the subgroup of all elements $\alpha \in G$ such that $\alpha(W) = W$. Then $N_G(P) \le G_W$, so P is a radical 2-subgroup of G_W; and in particular, $P \ge O_2(G_W)$.

Set $W_0 = W \cap W^\perp$ if G is symplectic or unitary, and $W_0 = \mathrm{Ker}(\mathfrak{q}|_{W\cap W^\perp})$ if G is orthogonal. Let $U \le G$ be the subgroup of elements which induce the identity on W_0 and on ${W_0}^\perp/W_0$ (and hence also on $V/{W_0}^\perp$). All elements of G_W leave W_0 and ${W_0}^\perp$ invariant, and hence $U \lhd G_W$. Also, U is a 2-group (cf. [**Gor**, Corollary 5.3.3]), and thus $U \le O_2(G_W) \le P$. If $W_0 \ne 0$ and $W_0 \subsetneqq W$, then $W_0 \subsetneqq {W_0}^\perp$, and each automorphism of ${W_0}^\perp$ which induces the identity on W_0 and on the quotient extends to an element of U by Witt's lemma. This proves that $W_0 = C_V(P)$, which contradicts the definition of W. Thus either $W = W_0$ is totally isotropic, or $W_0 = 0$.

Assume $W_0 = 0$. Then either $V = W \oplus W^\perp$; or G is orthogonal and $W \cap W^\perp = \langle x\rangle$ for some x with $\mathfrak{q}(x) = 1$. Also, P leaves $W^\perp$ invariant, and (since it is a 2-group) fixes some element $y \in W^\perp$. If $W \cap W^\perp = 0$, then $C_V(P) \ge W + \langle y\rangle$, which contradicts the definition of W. If $W \cap W^\perp = \langle x\rangle$ and $W^\perp \supsetneqq \langle x\rangle$, then the action of P on $W^\perp/\langle x\rangle$ fixes some coset $y+\langle x\rangle$ for $y \notin \langle x\rangle$, $P(y) = y$ since $\mathfrak{q}(y) \ne \mathfrak{q}(y+x)$, and again this contradicts the assumption on W. We are thus left with the case where G is orthogonal, $\dim(V/W) = 1$, $W^\perp = \langle x\rangle \le W$ where $\mathfrak{q}(x) = 1$; and thus P is generated by the unique orthogonal transvection which fixes W. Also, $G = SO^\pm_{2n}(2)$ since the orthogonal transvections are not in $\Omega^\pm_{2n}(2)$ [**Di**, §20].

Now assume $W = W_0$ is isotropic. It remains to prove that every $\alpha \in \mathrm{Aut}(W)$ extends to an element of $N_G(P)$; it suffices to do this when α is a transvection on W. Assume first that G is not $\Omega^\pm_{2n}(2)$. We can identify $W^\perp = W \oplus W'$, where $\mathfrak{b}$ is nondegenerate on W'. For any $\alpha \in \mathrm{Aut}(W)$, $\alpha \oplus \mathrm{Id}_{W'}$ extends by Witt's lemma to some $\widehat{\alpha} \in G$. For any $\beta \in P$, $[\widehat{\alpha}, \beta]$ induces the identity on W and on $W^\perp/W$ (since $\widehat{\alpha}$ induces the identity on $W^\perp/W$ and β induces the identity on W), and hence $[\widehat{\alpha}, \beta] \in P$. This shows that $\widehat{\alpha} \in N_G(P)$, and hence that $\mathrm{Aut}_{N_G(P)}(W) = \mathrm{Aut}(W)$.

It remains only to prove the last statement when $G \cong \Omega^\pm_{2n}(2)$ and $n \ge 3$; i.e., to show that the automorphism $\widehat{\alpha} \in SO^\pm_{2n}(2)$ constructed above actually lies in G. To show this, we use Dickson's characterisation of G: an element $\beta \in SO^\pm_{2n}(2)$ lies in G if and only if $\mathrm{Im}(\beta - \mathrm{Id})$ has even rank (cf. [**Ta**, Theorems 11.41 & 11.44]. To make the above construction of $\widehat{\alpha}$ more precise, set $V = W \oplus W' \oplus W^*$, where $W^* = \mathrm{Hom}(W, \mathbb{F}_2)$, and where $\mathfrak{q}(w, w', \varphi) = \varphi(w) + \mathfrak{q}'(w')$ for some quadratic form $\mathfrak{q}'$ on W'. For $\alpha \in \mathrm{Aut}(W)$, set $\widehat{\alpha} = \alpha \oplus \mathrm{Id}_{W'} \oplus (\alpha^*)^{-1}$, where $\alpha^* \in \mathrm{Aut}(W^*)$ is the dual of α. This is clearly orthogonal, and $\mathrm{rk}(\widehat{\alpha} - \mathrm{Id}_V)$ is even since

$$\mathrm{rk}((\alpha^*)^{-1} - \mathrm{Id}) = \mathrm{rk}(\mathrm{Id} - \alpha^*) = \mathrm{rk}(\alpha - \mathrm{Id}). \qquad \square$$

We already established, in Chapter 4, several criteria for proving that certain pivotal subgroups of L are not in $\mathfrak{E}^{\ge 2}(L\,;2)$. The following additional conditions will be needed in later chapters.

Proposition 6.5. *Fix a finite centerfree group L, a pivotal 2-subgroup $E \le L$, and $N_{\mathrm{Aut}(L)}(E)$-invariant subgroups $1 = E_0 \le E_1 \le \cdots \le E_k = E$. Let $\mathrm{Aut}_L(E_i/E_{i-1})$ denote the image of $\mathrm{Aut}_L(E)$ in $\mathrm{Aut}(E_i/E_{i-1})$. Assume, for each $1 \le i \le k$, that E_i/E_{i-1} either satisfies one of the conditions (a–f) in Proposition 4.6, or satisfies one of the following conditions: either*

(a) $(\mathrm{Aut}_L(E_i/E_{i-1}), E_i/E_{i-1}) \cong (Sp_{2n}(2); (\mathbb{Z}/2)^{2n})$ *for $n \ge 2$; or*

(b) $(\mathrm{Aut}_L(E_i/E_{i-1}), E_i/E_{i-1}) \cong (\Omega^\pm_n(2); (\mathbb{Z}/2)^n)$ *for $n \ge 5$; or*

(c) $(\mathrm{Aut}_L(E_i/E_{i-1}), E_i/E_{i-1}) \cong (SO^{\pm}_{2n}(2); (\mathbb{Z}/2)^{2n})$ *for* $n \geq 3$*; or*

(d) $(\mathrm{Aut}_L(E_i/E_{i-1}), E_i/E_{i-1}) \cong (G_2(2), (\mathbb{Z}/2)^6)$.

Then $E \notin \mathfrak{E}^{\geq 2}(L\,; 2)$.

PROOF. This is closely related to a theorem of Grodal [**Gro**, Theorem 4.1], but does not seem to follow from that result (at least not easily) in the generality we need it here.

Set $E' = E_i/E_{i-1}$ and $G = \mathrm{Aut}_L(E_i/E_{i-1})$ for short. By Proposition 4.5, it suffices to show that there is no radical 2-chain $1 \subsetneqq P_1 \subsetneqq \cdots \subsetneqq P_k$ of length $k \geq 2$ in G such that E' contains a copy of $\mathbb{F}_2[P_k]$ as a summand. In particular, it suffices to show that $|P_k| > \dim(E')$.

When $G \cong G_2(2)$, the smallest nontrivial radical 2-subgroups of G are those U_J for $|J| = 1$, and have order 2^5. Thus $|P_2| \geq 2^6 > 6 = \dim(E')$, so the proposition holds in this case.

Now assume G is one of the groups $Sp_{2n}(2) \cong \Omega_{2n+1}(2)$ (for $n \geq 2$) or $SO^{\pm}_{2n}(2)$ (for $n \geq 3$). Set $V = \mathbb{F}_2^{2n}$, and regard G as the group of isometries of a symplectic form $\mathfrak{b}$ on V, or of a quadratic form $\mathfrak{q}$ on V with associated symplectic form $\mathfrak{b}$. Fix a radical 2-subgroup $1 \neq P \leq G$. Set $W = C_V(P)$, the fixed subspace of P, and assume first that W is an isotropic subspace of V. Then by Proposition 6.4, P contains all elements of G which are the identity on W and $W^{\perp}/W$. In particular, each automorphism of $W^{\perp}$ which induces the identity on W and on $W^{\perp}/W$ extends to an element of P by Witt's lemma. So if $k = \mathrm{rk}(W)$, then $|P| \geq 2^{k(2n-2k)}$, and $|P| \geq 2^{2n-2}$ if $k < n$. If $k = n$, then choose a basis $\{v_i\}$ such that $W = \langle v_1, \ldots, v_n \rangle$, and $\mathfrak{b}(v_i, v_j) = 1$ exactly when $|i-j| = n$. Also, in the orthogonal case, we can assume the basis is such that $\mathfrak{q}(\sum_{i=1}^{2n} \lambda_i v_i) = \sum_{i=1}^{n} \lambda_i \lambda_{n+i}$. Thus $\mathfrak{b}$ has matrix $\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$, and if G is symplectic, then P contains all automorphisms with matrix $\left(\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right)$ with $X = X^t$, and thus $|P| \geq 2^{n(n+1)/2}$. If G is orthogonal, then P contains all automorphisms with matrix $\left(\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right)$, where $X = X^t$ has zeroes along the diagonal, and thus $|P| \geq 2^{n(n-1)/2}$. Hence the second term of any radical 2-chain in G has order at least $2\cdot|P| \geq 2^{2n-1} > 2n+1 \geq \dim(E')$ in the symplectic case ($n \geq 2$), or $2\cdot|P| \geq 2^{2n-2} > 2n \geq \dim(E')$ in the even dimensional orthogonal case ($n \geq 3$).

By Proposition 6.4 again, it remains to consider the case where $G \cong SO^{\pm}_{2n}(2)$, and $P \leq G$ is a subgroup of order 2 generated by an orthogonal transvection τ. Set $\mathrm{Im}(\tau - \mathrm{Id}) = \langle x \rangle$. Then $\mathrm{Ker}(\tau - \mathrm{Id}) = x^{\perp}$, and $N(P)$ is the group of all elements of G which send x to itself. Also, $\mathfrak{q}(x) = 1$ (see [**Di**, §19] for details), and hence the restriction of $\mathfrak{q}$ to $W \stackrel{\text{def}}{=} x^{\perp}$ is nondegenerate. Each element of $\mathrm{Aut}(W, \mathfrak{q}) \cong SO_{2n-1}(2)$ extends to an element of $G = \mathrm{Aut}(V, \mathfrak{q})$, by Witt's lemma, and thus $N_G(P)/P \cong SO_{2n-1}(2)$. We have already seen that every nontrivial radical subgroup of $SO_{2n-1}(2)$ has order at least 2^{2n-3}, so the second term in any radical 2-chain in G starting with P has order at least $2^{2n-2} > 2n = \dim(E')$.

When $G = \Omega^{\pm}_{2n}(2)$ for $n \geq 3$, then the above arguments still apply, except that possibly only half of the elements constructed in $SO^{\pm}_{2n}(2)$ lie in G. (In fact, they do all lie in G, but we don't need that for the estimates here.) Thus each nontrivial radical 2-subgroup $P \le G$ has order at least 2^{2n-4}, so $2{\cdot}|P| \ge 2^{2n-3} > 2n = \dim(E')$. □

The following is a stronger version of one special case of Theorem 6.2. It will be needed when handling some of the sporadic groups.

LEMMA 6.6. *Set $L = PSU_6(2)$. Then for any $\Gamma \le \mathrm{Aut}(L)$ containing $\mathrm{Inn}(L)$, $\mathfrak{R}^{\ge 2}(\Gamma\,;2) = \varnothing$.*

PROOF. Fix $V \cong \mathbb{F}_4^6$, and let $\mathfrak{b}$ be a hermitian form on V. We set $G = SU(V,\mathfrak{b}) \cong SU_6(2)$, $Z = Z(G)$, and identify $L = G/Z$. Since $|Z| = 3$, each 2-subgroup $P \le L$ lifts to a unique 2-subgroup $P' \le G$, and $N_L(P) = N_G(P')/Z$. Hence $O_2(N_L(P)) \cong O_2(N_G(P')) = O_2(N_{GU_6(2)}(P'))$; and so the three groups L, G, and $GU_6(2)$ have the same radical 2-subgroups under this identification.

A general description of the outer automorphism group of a finite simple group of Lie type is given in [**GLS3**, Theorem 2.5.12]. In the notation of that theorem, when $L = PSU_6(2) = {}^2A_5(2)$, then $\mathrm{Outdiag}(L) \cong C_3$ (generated by conjugation by matrices of determinant $\ne 1$), $\Phi_L \cong C_2$ (generated by the field automorphism $\varphi(A) = \bar{A}$) and acts on $\mathrm{Outdiag}(L)$ via $x \mapsto x^2$, and $\Gamma_L = 1$. Thus $\mathrm{Out}(L) \cong \Sigma_3$. So we can identify

$$\widehat{\Gamma} \overset{\text{def}}{=} \mathrm{Aut}(L) \cong PGU_6(2){\cdot}\langle\varphi\rangle.$$

Fix $P \in \mathfrak{R}^{\ge 2}(\Gamma\,;2)$, and set $E = \Omega_1(Z(P)) \in \mathfrak{E}^{\ge 2}(\Gamma\,;2)$. In particular, P is a radical 2-subgroup of Γ (Proposition 4.4(c,d)), and so $P_0 \overset{\text{def}}{=} P \cap L$ is a radical 2-subgroup of L by Lemma 1.5(b). Also, $P_0 \ne 1$, since $P \in \mathrm{Syl}_p(C_\Gamma(E))$ is 2-centric. We identify P_0 with its lifting to a radical 2-subgroup of G, or of $GU_6(2)$, as described above.

Set $W = C_V(P_0)$. By Proposition 6.4, W is totally isotropic (with respect to $\mathfrak{b}$), and P_0 contains the subgroup U_W of all unitary automorphisms which induce the identity on W and on $W^\perp/W$. Let $Z_W \le U_W$ be the subgroup of those elements which induce the identity on $W^\perp$ (and hence also on V/W). We claim that $Z_W = Z(P_0)$.

To see this, decompose $V = W \oplus W' \oplus W''$, where $W \oplus W' = W^\perp$ (possibly $W' = 0$), and where $W' \perp (W \oplus W'')$. Fix an $\mathbb{F}_4$-basis for V such that the matrix of $\mathfrak{b}$ with respect to the matrix (when written in 3×3 blocks based on the above decomposition) is $\left(\begin{smallmatrix} 0&0&I\\ 0&I&0\\ I&0&0 \end{smallmatrix}\right)$. Then, with respect to this basis, every element of P_0 has the form $\left(\begin{smallmatrix} I&A&B\\ 0&R&C\\ 0&0&I \end{smallmatrix}\right)$. Let $\theta \in \mathrm{Aut}(\mathbb{F}_4)$ be the field automorphism, and write $A^* = \theta(A^t)$ for any matrix A over $\mathbb{F}_4$. Then

$$U_W = \Bigl\{\beta(X,Y) = \left(\begin{smallmatrix} I&X&Y\\ 0&I&X^*\\ 0&0&I \end{smallmatrix}\right) \Bigm| Y + Y^* = XX^*\Bigr\}$$

and $Z_W = \bigl\{\beta(0,Y) \bigm| Y + Y^* = 0\bigr\}$. Also, for each matrix X over $\mathbb{F}_4$ of the appropriate size, there is Y such that $\beta(X,Y) \in U_W$: this follows from Witt's lemma or by a direct check.

It is clear from this description that $[Z_W, P_0] = 1$, and thus that $Z_W \le Z(P_0)$. Fix $\alpha \in P_0 \smallsetminus Z_W$ with matrix $\left(\begin{smallmatrix} I&A&B\\ 0&R&C\\ 0&0&I \end{smallmatrix}\right)$; thus $A \ne 0$ or $R \ne I$. If $R \ne I$, choose $\beta(X,Y) \in U_W \le P$ such that $XR \ne X$, and then $[\alpha, \beta(X,Y)] \ne 1$. So $\alpha \notin Z(P_0)$

in this case. If $A \neq 0$, choose $\beta(X,Y) \in U_W$ such that $AX \neq 0$. If $[\alpha, \beta(X,Y)] = 1$, then $AX^* = XC$ (compare the $(1,3)$-components of the two products). But if this holds, then $A(\epsilon X) \neq (\epsilon X)^* C$ (where $\epsilon \in \mathbb{F}_4^*$ has order 3), and thus $[\alpha, \beta(\epsilon X, Y)] \neq 1$. So $\alpha \notin Z(P_0)$, and this finishes the proof that $Z(P_0) = Z_W$.

Set $r = \mathrm{rk}(W)$. By the above description of $Z(P_0)$, we can identify it with the additive group of matrices $Y \in M_r(\mathbb{F}_4)$ such that $Y^* = Y$. In particular, $Z(P_0) \cong {C_2}^{r^2}$. Furthermore, $\mathrm{Aut}_L(Z(P_0)) \cong \mathrm{Aut}_G(Z(P_0))$ is generated by matrices of the form $\begin{pmatrix} A & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & (A^*)^{-1} \end{pmatrix}$ with determinant one, and such a matrix acts on $Z(P_0)$ by $Y \mapsto AYA^*$. Thus $\mathrm{Aut}_L(Z(P_0)) \cong PSL_r(4)$ or $PGL_r(4)$.

If $P = P_0 \leq L$, then $E = Z(P_0)$ has rank $r^2 \geq 4$, and so $r = 2$ or 3. Also, depending on the choice of $\Gamma \leq \mathrm{Aut}(L)$, $\mathrm{Aut}_\Gamma(E) \cong PSL_r(4)$, $PGL_r(4)$, $P\Sigma L_r(4)$, or $P\Gamma L_r(4)$ (the last two are the extensions of the first two by the field automorphism φ of $PSL_r(4)$). All elements of order 2 in the coset $PSL_r(4)\cdot\varphi$ are conjugate to elements in $UT_r(4)\cdot\langle\varphi\rangle$, where $UT_r(4)$ is the Sylow subgroup of upper triangular matrices. Also, all involutions in $UT_r(4)\cdot\varphi$ are $UT_r(4)$-conjugate to φ. For example, when $r = 3$, every involution has the form $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}\cdot\varphi$ for $a, c \in \mathbb{F}_2$ and $b + \theta(b) = ac$, and a direct check shows that this is equal to $X\varphi X^{-1}$ for some $X \in UT_3(4)$. We thus conclude that all involutions in $PSL_r(4)\cdot\varphi$ are $PSL_r(4)$-conjugate to φ.

By Proposition 4.5, there is a radical 2-chain $1 \lneqq P_1 \lneqq \cdots \lneqq P_m$ in $\mathrm{Aut}_L(E)$, for some $m \geq 2$, such that E contains a copy of the free module $\mathbb{F}_2[P_m]$. In particular, $|P_m| \leq \mathrm{rk}(E) = r^2$. Since the smallest radical 2-subgroups of $PSL_r(4)$ have order $4^{r-1} > r^2$, this shows that $P_1 \cap PSL_r(4) = 1$, and thus that $P\Sigma L_r(4) \leq \mathrm{Aut}_L(E)$ and (up to conjugacy) $P_1 = \langle\varphi\rangle$. Then $P_2/\langle\varphi\rangle$ must be a radical 2-subgroup of $N_{\mathrm{Aut}_L(E)}(\langle\varphi\rangle)/\langle\varphi\rangle \cong GL_r(2)$, and thus $|P_2/\langle\varphi\rangle| \geq 2^{r-1}$. But if E contains a copy of $\mathbb{F}_2[P_2]$, then $\mathrm{rk}(C_E(\varphi)) \leq \mathrm{rk}(E) - |P_2/\langle\varphi\rangle| \leq r^2 - 2^{r-1}$; i.e., $\mathrm{rk}(C_E(\varphi)) \leq 2$ (if $r = 2$) or 5 (if $r = 3$). But $C_E(\varphi)$ is the group of symmetric matrices over $\mathbb{F}_2$, thus has rank 3 or 6, respectively, and so this situation is impossible. In conclusion, P cannot be a subgroup of L.

If $P \nleq L$, then since all involutions in $\mathrm{Aut}_{\widehat{\Gamma}}(Z(P_0)) \cong P\Gamma L_r(4)$ not in $PSL_r(4)$ are conjugate to φ, E is isomorphic to the group of symmetric matrices in $M_r(\mathbb{F}_2)$, and $\mathrm{Aut}_\Gamma(E)) \cong GL_r(2)$ with the action $(A, X) \mapsto AXA^t$. Since $\mathrm{rk}(E) \geq 4$, this means $r = 3$, $\mathrm{rk}(E) = 6$, and is impossible by Proposition 4.6(d) since $GL_3(2)$ contains no radical 2-subgroup of order 2. □

CHAPTER 7

Classical groups of Lie type in odd characteristic

We next show that when q is an odd prime power, the simple classical groups $PSL_n(q)$, $PSU_n(q)$, $PSp_{2n}(q)$, and $P\Omega_n^{\pm}(q)$ are all in $\mathfrak{L}^{\geq 2}(2)$. We refer to [**Di**] or [**Ca1**, §1] for definitions and descriptions of these groups.

It will be convenient to write the general linear, unitary, symplectic, and orthogonal groups in the form $G = GL(V, \mathfrak{b})$, where V is a vector space over a finite field K of odd characteristic, $\mathfrak{b}$ is a trivial, hermitian, symplectic, or quadratic form, and $GL(V, \mathfrak{b})$ denotes the group of all automorphisms of G which preserve this form. Also, we write $PGL(V, \mathfrak{b}) = GL(V, \mathfrak{b})/\{\lambda\,\mathrm{Id} \in G \,|\, \lambda \in K\}$, and let $\pi\colon GL(V, \mathfrak{b}) \to PGL(V, \mathfrak{b})$ denote the projection. The following technical lemma will be needed when dealing with decompositions of representations supporting such forms.

Lemma 7.1. *Fix a finite dimensional vector space V over a finite field K of odd characteristic p, and let $\mathfrak{b}$ be a nondegenerate symmetric, symplectic, or hermitian form on V. Let H be a finite group of order prime to p which acts linearly on V and preserves the form $\mathfrak{b}$. Assume, for each irreducible KH-submodule $W \subseteq V$, that W supports some nondegenerate form $\mathfrak{b}_W$ of the same type as $\mathfrak{b}$. Then there are irreducible KH-submodules $W_1, \ldots, W_k \subseteq V$ such that $\mathfrak{b}|_{W_i}$ is nondegenerate for each i, and such that $V = W_1 \oplus \cdots \oplus W_k$ is an orthogonal direct sum.*

Proof. Let $\theta \in \mathrm{Aut}(K)$ be the automorphism of order 2 if $\mathfrak{b}$ is hermitian, and set $\theta = \mathrm{Id}$ if $\mathfrak{b}$ is symplectic or symmetric. For any K-vector space U, let $U^{*\theta}$ denote the dual of U where the K-linear structure is twisted by θ.

For any irreducible KH-submodule $W \subseteq V$, $\mathfrak{b}|_W$ is either nondegenerate or zero. If $\mathfrak{b}|_W$ is nondegenerate, then $V = W \oplus W^\perp$, an orthogonal direct sum, and $\mathfrak{b}|_{W^\perp}$ is also nondegenerate. So it suffices to show, whenever $V \neq 0$, that there is some irreducible submodule in V on which the form is nonzero.

Assume otherwise: assume $\mathfrak{b}|_W = 0$ for all irreducible KH-submodules $W \subseteq V$. Fix one such submodule $U \subseteq V$, and let $U' \subseteq V$ be any irreducible submodule such that $U' \oplus U^\perp = V$. Then $\mathfrak{b}|_U = 0$, $\mathfrak{b}|_{U'} = 0$, and there is an isomorphism

$$\varphi\colon U \xrightarrow{\;\cong\;} (U')^{*\theta}$$

defined by setting $\varphi(u)(u') = \mathfrak{b}(u, u')$. By assumption, there is a nondegenerate form $\mathfrak{b}'$ on U' of the same type as $\mathfrak{b}$, and we define

$$\psi\colon U' \xrightarrow{\;\cong\;} (U')^{*\theta}$$

by setting $\psi(u_1')(u_2') = \mathfrak{b}'(u_1', u_2')$. Set

$$W = \{u + \psi^{-1}\varphi(u) \,|\, u \in U\}:$$

an irreducible KH-submodule isomorphic to U and to U'. For all $u_1, u_2 \in U$,

$$\begin{aligned}\mathfrak{b}(u_1 + \psi^{-1}\varphi(u_1), u_2 + \psi^{-1}\varphi(u_2)) &= \varphi(u_1)(\psi^{-1}\varphi(u_2)) + \epsilon\cdot\theta\bigl(\varphi(u_2)(\psi^{-1}\varphi(u_1))\bigr) \\ &= 2\cdot\mathfrak{b}'(\psi^{-1}\varphi(u_1), \psi^{-1}\varphi(u_2)),\end{aligned}$$

where $\epsilon = -1$ if $\mathfrak{b}$ is symplectic and $\epsilon = 1$ otherwise. This shows that $\mathfrak{b}|_W$ is nondegenerate (since $\mathfrak{b}'$ is nondegenerate on U'), and contradicts our assumption about V. □

Recall that the modular character χ_V of an $\mathbb{F}_q[G]$-module V is defined by identifying $\mathbb{F}_q^\times$ with a subgroup of $\mathbb{C}^\times$, and then letting $\chi_V(g) \in \mathbb{C}$ (when $(|g|, q) = 1$) be the sum of the eigenvalues of $V \xrightarrow{g} V$ lifted to $\mathbb{C}$. We always consider this in the case where G has order prime to q, and hence when two representations with the same character are isomorphic. See [**Se**, §18] for more details.

Lemma 7.2. *Assume $G = GL(V, \mathfrak{b})$, where V is a finite dimensional vector space over a finite field K of odd characteristic p, and $\mathfrak{b}$ is a nondegenerate symplectic, quadratic, or hermitian form, or the trivial form. Fix $H \le G$ of order prime to p, let $\chi\colon H \longrightarrow \mathbb{C}$ be the character of V as an H-representation, and let $\mathrm{Aut}_\chi(H)$ be the group of automorphisms $\alpha \in \mathrm{Aut}(H)$ such that $\chi \circ \alpha = \chi$. Then $\mathrm{Aut}_\chi(H) = \mathrm{Aut}_G(H)$ if any of the following conditions hold:*

(a) *G is a linear or unitary group; or*

(b) *G is an orthogonal or symplectic group, and there is $z \in Z(H)$ such that $z^2 v = -v$ for all $v \in V$; or*

(c) *there is no irreducible KH-submodule $W \subseteq V$ such that $\mathfrak{b}|_W \ne 0$.*

Furthermore,

(d) *$[\mathrm{Aut}_\chi(H){:}\,\mathrm{Aut}_G(H)] \le 2$ if G is an orthogonal or symplectic group and $V \cong W^k$ (the direct sum of k copies of W) for some irreducible H-representation W.*

Proof. Let $\theta \in \mathrm{Aut}(K)$, and $U^{*\theta}$ (for a K-vector space U) be as in the proof of Lemma 7.1. When $\mathfrak{b} \ne 0$, we let

$$\widehat{\mathfrak{b}}\colon V \xrightarrow{\ \cong\ } V^{*\theta}$$

be the isomorphism $\widehat{\mathfrak{b}}(v)(w) = \mathfrak{b}(v, w)$ for $v, w \in V$ (and similarly for other forms which occur in the proof below). This map is KH-linear when we let H act on $V^{*\theta}$ by setting $(h\varphi)(v) = \varphi(h^{-1}v)$ for $h \in H$ and $\varphi \in V^{*\theta}$.

For $\alpha \in \mathrm{Aut}(H)$, let $\mathrm{Aut}_{K\alpha}(V)$ denote the set of *α-linear automorphisms* of V; i.e., the set of automorphisms φ of V such that $\varphi(gv) = \alpha(g)\varphi(v)$ for all $g \in H$ and $v \in V$. Equivalently, $\alpha \in \mathrm{Aut}(H)$ is conjugation by φ, and so $\alpha \in \mathrm{Aut}_G(H)$ if and only if there is some $\varphi \in \mathrm{Aut}_{K\alpha}(V)$ which preserves the form $\mathfrak{b}$. Since $H \le GL(V)$ has order prime to p, the two representations of H on V induced by the inclusion $H \le GL(V)$ and by its composite with α are isomorphic if and only if their characters χ and $\chi \circ \alpha$ are equal, and thus

$$\mathrm{Aut}_\chi(H) = \mathrm{Aut}_{GL(V)}(H). \tag{1}$$

This proves (a) in the linear case.

We now study how this can be done while preserving a hermitian, symplectic, or symmetric form $\mathfrak{b}$. Write $V = V_0 \oplus V'$, where V_0 is generated by all irreducible KH-submodules which support forms of the same type as $\mathfrak{b}$, and V' is generated by all others. This is an orthogonal direct sum (since $\mathrm{Hom}_{KH}(V_0, (V')^{*\theta}) = 0$), and hence $\mathfrak{b}|_{V_0}$ and $\mathfrak{b}|_{V'}$ are nondegenerate. By Lemma 7.1, V_0 splits as an orthogonal direct sum of irreducible representations. We can thus decompose V as an orthogonal direct sum

$$V = \left(\bigoplus_{i=1}^{k} W_i \right) \oplus V', \tag{2}$$

where each W_i is an irreducible KH-module, and where V' is generated by all submodules of V of isomorphism types which do not support forms of the same type as $\mathfrak{b}$. In particular, the restriction of $\mathfrak{b}$ to each irreducible submodule of V' is zero.

The above conditions determine V' uniquely, and determine the W_i up to isomorphism (and permutation of indices). Also, none of the W_i is isomorphic to any irreducible submodule of V'. Hence by (1), for any $\alpha \in \mathrm{Aut}_\chi(H)$, there is $\varphi \in \mathrm{Aut}_{K\alpha}(V)$ which permutes these summands (and which sends V' to itself).

We are now ready to prove points (b), (c), and (d), and the remaining case of point (a).

(c) Assume $V = V'$. For each irreducible submodule $U \subseteq V'$, we can choose U' such that $V' = U' \oplus U^\perp$ (orthogonal complement in V'). Then $\mathfrak{b}$ vanishes on U and on U' by definition of V', and $\mathfrak{b}$ induces an isomorphism $U' \cong U^{*\theta}$. So $\mathfrak{b}|_{U \oplus U'}$ is nondegenerate. We can thus decompose V' as an orthogonal direct sum

$$V' = \bigoplus_{j=1}^{\ell} (U_j \oplus U'_j);$$

where for each j, U_j and U'_j are irreducible KH-modules, $\mathfrak{b}|_{U_j} = 0$ and $\mathfrak{b}|_{U'_j} = 0$, and $\mathfrak{b}$ defines an isomorphism $U_j \cong (U'_j)^{*\theta}$.

Fix $\alpha \in \mathrm{Aut}_\chi(H)$. By (1), there is $\varphi \in \mathrm{Aut}_{K\alpha}(V)$, and this can be chosen such that for some permutation σ of $\{1, \ldots, \ell\}$, $\varphi(U_j \oplus U'_j) = U_{\sigma(j)} \oplus U'_{\sigma(j)}$ for each j. Since $\mathfrak{b}$ vanishes on $\varphi(U_j)$ and $\varphi(U'_j)$ by assumption, it induces KH-linear isomorphisms $U_j \cong (U'_j)^{*\theta}$ and $\varphi(U_j) \cong (\varphi(U'_j))^{*\theta}$; and $\varphi|_{U_j \oplus U'_j}$ preserves $\mathfrak{b}$ if and only if $\varphi|_{U_j}$ and $(\varphi|_{U'_j})^*$ commute with these isomorphisms in the obvious way. Thus upon replacing φ by its composite with appropriate elements of $\mathrm{Aut}_{KH}(U_j)$ for all j (and the identity on each U'_j), we can arrange that $\varphi \in GL(V, \mathfrak{b}) = G$. This shows that $\alpha \in \mathrm{Aut}_G(H)$ in this case, and proves (c).

(a,b) Assume that $\mathfrak{b}$ is a hermitian form, or that $\mathfrak{b}$ is symmetric or symplectic and there is some $z \in Z(H)$ such that z^2 acts on V via $-\mathrm{Id}$. We must show that $\mathrm{Aut}_G(H) = \mathrm{Aut}_\chi(H)$. By (2) and (c), we are reduced to the case $V' = 0$; i.e., the case where $V = \bigoplus_{i=1}^k W_i$ is an orthogonal direct sum of irreducible submodules. By (1), for any $\alpha \in \mathrm{Aut}_\chi(H)$, there is an α-linear automorphism φ of V which permutes the summands W_i, and we will be done upon showing that φ can be chosen to preserve the form $\mathfrak{b}$. It thus suffices to show, for any irreducible KH-module W, and any two nondegenerate H-invariant forms $\mathfrak{b}$, $\mathfrak{b}'$ of the given type (symmetric, symplectic, or hermitian) on W, that there is some $\psi \in \mathrm{Aut}_{KH}(W)$ such that $\mathfrak{b}'(v,w) = \mathfrak{b}(\psi(v),\psi(w))$ for all $v,w \in W$.

Set $\widehat{K} = \mathrm{End}_{KH}(W)$, a field. We regard W as a $\widehat{K}H$-module. For each $\sigma \in \widehat{K}$, set $\beta(\sigma) = \widehat{\mathfrak{b}}^{-1}\sigma^*\widehat{\mathfrak{b}} \in \mathrm{Aut}_{KH}(W) = \widehat{K}$. Thus $\sigma^* \circ \widehat{\mathfrak{b}} = \widehat{\mathfrak{b}} \circ \beta(\sigma)$, and this translates to the relation

$$\mathfrak{b}(v,\sigma(w)) = \mathfrak{b}(\beta(\sigma)(v),w)$$

for all $v,w \in W$. Also, for all $v,w \in W$ and all $\sigma \in \widehat{K}$,

$$\mathfrak{b}(\beta^2(\sigma)(v),w) = \mathfrak{b}(v,\beta(\sigma)(w)) = \epsilon{\cdot}\theta(\mathfrak{b}(\beta(\sigma)(w),v)) = \epsilon{\cdot}\theta(\mathfrak{b}(w,\sigma(v))) = \mathfrak{b}(\sigma(v),w)$$

(where $\epsilon = -1$ if $\mathfrak{b}$ is symplectic and $\epsilon = 1$ if $\mathfrak{b}$ is symmetric or hermitian), and so $\beta^2 = \mathrm{Id}$. Let $\widehat{K}_0 \subseteq \widehat{K}$ be the fixed subfield of β.

By definition, $\beta|_K = \theta$, and thus $\beta \neq \mathrm{Id}$ if $\mathfrak{b}$ is hermitian. If $\mathfrak{b}$ is symmetric or symplectic, and $z \in Z(H)$ is such that z^2 acts via $-\mathrm{Id}$, let $\zeta \in \mathrm{Aut}_{KH}(W) = \widehat{K}$ denote the action of z. Then $\mathfrak{b}(v,zw) = \mathfrak{b}(z^{-1}v,w)$ for all $v,w \in W$, and so $\beta(\zeta) = \zeta^{-1} = -\zeta$. Thus in all cases, $\beta \neq \mathrm{Id}$, and so $[\widehat{K}{:}\widehat{K}_0] = 2$. Set $\widehat{q} = |\widehat{K}_0|$, so $|\widehat{K}| = \widehat{q}^2$, and $\beta(\tau) = \tau^{\widehat{q}}$ for all $\tau \in \widehat{K}$.

Set $\sigma = \widehat{\mathfrak{b}}^{-1} \circ \widehat{\mathfrak{b}}' \in \mathrm{Aut}_{KH}(W)$. Then $\mathfrak{b}'(v,w) = \mathfrak{b}(\sigma(v),w)$ for all $v,w \in W$. Since the forms have the same type of symmetry, $\mathfrak{b}(v,\sigma(w)) = \mathfrak{b}(\sigma(v),w)$ for all $v,w \in W$, and thus $\sigma \in \widehat{K}_0$. Also, $[(\widehat{K})^\times{:}(\widehat{K}_0)^\times] = \widehat{q}+1$; so we can choose $\psi \in \widehat{K}$ such that $\psi{\cdot}\beta(\psi) = \psi^{\widehat{q}+1} = \sigma$. Then $\mathfrak{b}'(v,w) = \mathfrak{b}(\psi(v),\psi(w))$ for all $v,w \in W$, and this finishes the proof.

(d) Assume now that $\mathfrak{b}$ is a symplectic or symmetric form, and that $V \cong W^k$ for some irreducible KH-module W. If W does not support a form of the same type as $\mathfrak{b}$, then we are in the situtation of (c), and $\mathrm{Aut}_G(H) = \mathrm{Aut}_\chi(H)$. So assume that there is a nondegenerate form $\mathfrak{b}_W$ on W of the same type. Let $V = \bigoplus_{i=1}^k W_i$ be the decomposition of (2), and fix KH-linear isomorphisms $\varphi_i\colon W \longrightarrow W_i$ for each i. Let $\mathfrak{b}_i$ be the form on W which makes φ_i into an isometry; i.e., $\mathfrak{b}_i(v,w) = \mathfrak{b}(\varphi_i(v),\varphi_i(w))$ for all $v,w \in W$.

Set $\widehat{K} = \mathrm{End}_{KH}(W)$, a field. As in the proof of (b) above, there is an automorphism $\beta \in \mathrm{Aut}(\widehat{K})$ such that $\mathfrak{b}_W(v,\sigma(w)) = \mathfrak{b}_W(\beta(\sigma)(v),w)$ for all $v,w \in W$ and all $\sigma \in \widehat{K}$, and $\mathrm{Aut}_G(H) = \mathrm{Aut}_\chi(H)$ if $\beta \neq \mathrm{Id}$. So we can assume that $\beta = \mathrm{Id}$; i.e., that

$$\mathfrak{b}_W(v,\sigma(w)) = \mathfrak{b}_W(\sigma(v),w) \tag{3}$$

for all $v,w \in W$ and $\sigma \in \widehat{K}$.

For each $\varphi \in N_{GL(W)}(H)$, there is a unique automorphism $\omega(\varphi)$ in $\widehat{K}^\times = \mathrm{Aut}_{KH}(W)$ such that

$$\mathfrak{b}_W(\varphi(v),\varphi(w)) = \mathfrak{b}_W(\omega(\varphi)(v),w) \tag{4}$$

for all $v,w \in V$ (defined by $\omega(\varphi) = \widehat{\mathfrak{b}}_W^{-1}\varphi^*\widehat{\mathfrak{b}}_W\varphi$). Since $C_{GL(W)}(H) = \mathrm{Aut}_{KH}(W) = \widehat{K}^\times$, and since $\omega(\sigma) = \sigma^2$ for all $\sigma \in \widehat{K}^\times$ by (3), ω factors through a homomorphism

$$\bar{\omega}\colon \mathrm{Aut}_{GL(W)}(H) \cong N_{GL(W)}(H)/C_{GL(W)}(H) \longrightarrow \widehat{K}^\times/\{\sigma^2 \,|\, \sigma \in \widehat{K}^\times\} \cong \mathbb{Z}/2.$$

Now, $\chi = k{\cdot}\chi_W$, where χ_W is the character of W, and hence $\mathrm{Aut}_\chi(H) = \mathrm{Aut}_{\chi_W}(H) = \mathrm{Aut}_{GL(W)}(H)$ by (1). Since the index of $\mathrm{Ker}(\bar\omega)$ in $\mathrm{Aut}_{GL(W)}(H)$ is at most two, it remains only to show that $\mathrm{Aut}_G(H) \geq \mathrm{Ker}(\bar\omega)$. Fix some $\alpha \in \mathrm{Ker}(\bar\omega)$, and let $\psi_0 \in N_{GL(W)}(H)$ be any α-linear automorphism. Then $\omega(\psi_0) = \tau^2 = \omega(\tau{\cdot}\mathrm{Id})$ for some $\tau \in \widehat{K}^\times$; and so $\psi \overset{\text{def}}{=} \tau^{-1}\psi_0 \in \mathrm{Ker}(\omega)$ and is α-linear. By (4), $\psi \in \mathrm{Aut}(W, \mathfrak{b}_W)$.

For each $i = 1, \dots, k$, let $\sigma_i \in \widehat{K}^\times$ be the unique element such that

$$\mathfrak{b}_i(v, w) = \mathfrak{b}_W(\sigma_i(v), w)$$

for each $v, w \in W$. Conjugation by ψ induces an automorphism of the cyclic group $\widehat{K}^\times = \mathrm{Aut}_{KH}(W)$. So for each $i = 1, \dots, k$, there is $\tau_i \in \widehat{K}^\times$ such that $\tau_i^2 = \sigma_i^{-1}(\psi\sigma_i\psi^{-1})$. Set $\psi_i = \tau_i\psi \in \mathrm{Aut}(W)$, an α-linear automorphism. For all i and all $v, w \in W$,

$$\begin{aligned}\mathfrak{b}_i(\psi_i(v), \psi_i(w)) &= \mathfrak{b}_W(\sigma_i\tau_i\psi(v), \tau_i\psi(w)) = \mathfrak{b}_W(\sigma_i\tau_i^2\psi(v), \psi(w)) \\ &= \mathfrak{b}_W((\psi\sigma_i\psi^{-1})\psi(v), \psi(w)) = \mathfrak{b}_W(\sigma_i(v), w) = \mathfrak{b}_i(v, w).\end{aligned}$$

Thus $\psi_i \in \mathrm{Aut}(W, \mathfrak{b}_i)$ and is α-linear for each i, and so

$$\bigoplus_{i=1}^{k} \varphi_i\psi_i\varphi_i^{-1} \in \mathrm{Aut}(V)$$

is an α-linear automorphism of V which preserves $\mathfrak{b}$. Hence $\alpha \in \mathrm{Aut}_G(H)$, and this finishes the proof of (d). □

In general, for $g \in GL(V, \mathfrak{b})$, we write $-g = (-\mathrm{Id}) \circ g$. For any $H \leq PGL(V, \mathfrak{b})$ and any $\epsilon \in \mathrm{Hom}(H, \{\pm1\})$, we let $\mathcal{I}_\epsilon \in \mathrm{Aut}(\pi^{-1}H)$ denote the automorphism

$$\mathcal{I}_\epsilon(g) = \begin{cases} g & \text{if } \epsilon(\pi(g)) = 1 \\ -g & \text{if } \epsilon(\pi(g)) = -1. \end{cases}$$

The resulting map

$$\mathrm{Hom}(H, \{\pm1\}) \xrightarrow{\epsilon \mapsto \mathcal{I}_\epsilon} \mathrm{Aut}(\pi^{-1}H)$$

is a homomorphism of groups.

The next lemma will be needed later as an explicit way of constructing automorphisms.

Lemma 7.3. *Let V be a finite dimensional vector space over a finite field K of odd characteristic p, and let $\mathfrak{b}$ be a nondegenerate symplectic, quadratic, or hermitian form, or the trivial form. Set $G = GL(V, \mathfrak{b})$ and $\overline{G} = PGL(V, \mathfrak{b})$ for short, and let $\pi\colon G \to \overline{G}$ be the projection. Fix $H \leq \overline{G}$ of order prime to p, set $\widetilde{H} = \pi^{-1}(H) \leq G$, and let $T \leq Z(\widetilde{H})$ be an elementary abelian 2-subgroup. Set*

$$\Delta_T = \bigl\{\epsilon \in \mathrm{Hom}(H, \{\pm1\}) \bigm| \pi(T) \leq \mathrm{Ker}(\epsilon),\ \mathcal{I}_\epsilon \in \mathrm{Aut}_G(\widetilde{H})\bigr\}.$$

For $\epsilon \in \Delta_T$ and $x \in T$, define $\alpha_{\epsilon,x} \in \mathrm{Aut}(\widetilde{H})$ by setting

$$\alpha_{\epsilon,x}(g) = \begin{cases} g & \text{if } \epsilon(\pi(g)) = 1 \\ xg & \text{if } \epsilon(\pi(g)) = -1. \end{cases}$$

Then $\alpha_{\epsilon,x} \in \mathrm{Aut}_G(\widetilde{H})$ for all $\epsilon \in \Delta_T$ and $x \in T$.

Proof. Let $T \le Z(\widetilde{H})$ and $\Delta_T \le \mathrm{Hom}(H, \{\pm 1\})$ be as described. Fix $x \in T$, and let $V_\pm$ be its ± 1-eigenspaces. Then $V = V_+ \oplus V_-$ is an orthogonal direct sum of $\widetilde{H}$-invariant subspaces.

Fix some $\epsilon \in \Delta_T$, and set $\alpha = \mathcal{I}_\epsilon \in \mathrm{Aut}_G(\widetilde{H})$. Let $\varphi \in G = GL(V, \mathfrak{b})$ be such that $c_\varphi = \alpha$; i.e., $\varphi \in N_G(\widetilde{H})$ and $\varphi g \varphi^{-1} = \alpha(g)$ for $g \in \widetilde{H}$. In particular, $\varphi x \varphi^{-1} = \alpha(x) = x$, and so φ preserves the $V_\pm$. Define $\psi \in GL(V, \mathfrak{b})$ by setting $\psi|_{V_+} = \mathrm{Id}$ and $\psi|_{V_-} = \varphi|_{V_-}$. Then $\psi g \psi^{-1} = \alpha_{\epsilon,x}(g)$ for $g \in \widetilde{H}$, and thus $\alpha_{\epsilon,x} \in \mathrm{Aut}_G(\widetilde{H})$. □

The last lemma will now be applied to get information about elementary abelian 2-subgroups of the projective classical groups.

Lemma 7.4. *Let V be a finite dimensional vector space over a finite field K of odd characteristic, and let $\mathfrak{b}$ be a nondegenerate symplectic, quadratic, or hermitian form, or the trivial form. Set $G = GL(V, \mathfrak{b})$ and $\bar{G} = PGL(V, \mathfrak{b})$, let $\pi\colon G \to \bar{G}$ be the projection, and set $Z = \mathrm{Ker}(\pi) = \{u{\cdot}\mathrm{Id} \in G \,|\, u \in K^\times\}$. Let $E \le \bar{G}$ be an elementary abelian subgroup, set $\widetilde{E} = \pi^{-1}(E) \le G$, and let $\chi\colon \widetilde{E} \longrightarrow \mathbb{C}$ be the character of V as an $\widetilde{E}$-representation. Set $\widetilde{E}_0 = Z{\cdot}\Omega_1(Z(\widetilde{E}))$, $E_0 = \pi(\widetilde{E}_0) \le \bar{G}$, and*

$$\Delta = \bigl\{\epsilon \in \mathrm{Hom}(E, \{\pm 1\}), \bigm| \mathcal{I}_\epsilon \in \mathrm{Aut}_G(\widetilde{E})\bigr\}.$$

Then the following hold.

(a) *There is an elementary abelian subgroup $T \le Z(\widetilde{E})$, a cyclic subgroup $Z' \le Z(\widetilde{E})$ such that $Z \le Z'$ and $[Z'{:}Z] \le 2$, and a subgroup $X \le \widetilde{E}$ such that $Z = Z(X)$ and $[X, X] \le \{\pm\,\mathrm{Id}\}$, such that*

$$\widetilde{E} = T \times \bigl(Z' \times_Z X\bigr), \qquad Z(\widetilde{E}) = T \times Z', \qquad \text{and} \qquad \widetilde{E}_0 = T \times Z.$$

Here, $Z' \times_Z X$ denotes the central product, where $Z \le Z'$ is identified with $Z = Z(X)$.

(b) $\chi(g) = 0$ *for all* $g \in \widetilde{E} \smallsetminus \widetilde{E}_0$.

(c) Δ *is a subgroup of* $\mathrm{Hom}(E, \{\pm 1\})$*, and* $\Delta \ge \mathrm{Hom}(E/E_0, \{\pm 1\})$.

Proof. Since $Z(\widetilde{E})/Z$ is elementary abelian, we can write $Z(\widetilde{E}) = Z' \times T$ for some elementary abelian subgroup T and some cyclic subgroup Z' such that $Z \le Z'$ and $[Z'{:}Z] \le 2$. Choose elements $g_1, \dots, g_r \in \widetilde{E}$ whose images form a basis for the elementary abelian 2-group $\widetilde{E}/Z(\widetilde{E})$, and set $X = \langle Z, g_1, \dots, g_r\rangle$. Since $E \cong \widetilde{E}/Z$ is elementary abelian, $g_i^2 \in Z$ and $[g_i, g_j] \le Z$ for all i, j, so $X \cap Z(\widetilde{E}) = Z$, and thus $Z(X) = Z$. Also, the commutator map

$$\delta\colon \widetilde{E}/Z(\widetilde{E}) \times \widetilde{E}/Z(\widetilde{E}) \longrightarrow Z$$

is bilinear and nondegenerate, and hence its image lies in $\{\pm\,\mathrm{Id}\}$. By construction,

$$\widetilde{E} = T \times \bigl(Z' \times_Z X\bigr), \qquad Z(\widetilde{E}) = T \times Z', \qquad \text{and} \qquad \widetilde{E}_0 = T \times Z,$$

and this proves (a).

The set Δ is a subgroup of $\mathrm{Hom}(E, \{\pm 1\})$, since it is the inverse image of $\mathrm{Aut}_G(\widetilde{E})$ under the homomorphism from $\mathrm{Hom}(E, \{\pm 1\})$ to $\mathrm{Aut}(\widetilde{E})$ which sends ϵ to $\mathcal{I}_\epsilon$. So it remains only to prove (b), and the second statement in (c). Point (b) follows from (c) since $\chi(g)=0$ whenever there is $\alpha\in\mathrm{Aut}_G(\widetilde{E})$ such that $\alpha(g)=-g$ — and this is the case whenever there is $\epsilon\in\Delta$ such that $\epsilon(\pi(g))=-1$.

Since the commutator map δ is bilinear and nondegenerate, it defines an isomorphism $\mathrm{Hom}(\widetilde{E}/Z(\widetilde{E}),\{\pm1\})\cong\widetilde{E}/Z(\widetilde{E})$. Hence for any $\epsilon\in\Delta$ such that $\mathrm{Ker}(\epsilon)\ge\pi(Z(\widetilde{E}))$, there is $g\in\widetilde{E}$ such that $[g,h]=\epsilon(\pi(h))\cdot\mathrm{Id}$ (so $ghg^{-1}=\mathcal{I}_\epsilon(h)$) for all $h\in\widetilde{E}$. In other words, $\mathcal{I}_\epsilon=c_g\in\mathrm{Aut}_G(\widetilde{E})$ is an inner automorphism whenever $\epsilon(\pi(Z(\widetilde{E})))=1$. This proves (b) and (c) when $Z(\widetilde{E})=\widetilde{E}_0$ (equivalently, when $Z'=Z$); and also proves in the general case that $\chi(g)=0$ for $g\in\widetilde{E}\smallsetminus Z(\widetilde{E})$.

Now assume $[Z':Z]=2$. If $\mathfrak{b}$ is symmetric or symplectic, then $|Z|=2$, so $|Z'|=4$, and $Z'=\langle z\rangle$ where z^2 acts on V via $-\mathrm{Id}$. Hence in all cases, $\mathrm{Aut}_G(\widetilde{E})=\mathrm{Aut}_\chi(\widetilde{E})$ by Lemma 7.2(a,b). Thus (c) follows from (b) in this case; and to prove (b), it remains to show that $\chi(g)=0$ for all $g\in Z(\widetilde{E})\smallsetminus\widetilde{E}_0$.

Fix such a g, and recall that $Z(\widetilde{E})=Z'\times T$ and $\widetilde{E}_0=Z\times T$, where T is elementary abelian. Thus $g^2\in Z$, a generator, and so g^2 acts on V via $u\cdot\mathrm{Id}$ for some $u\in K^\times$. Let $\overline{K}$ be the algebraic closure of K, and let $\zeta\in\overline{K}^\times$ be such that $\zeta^2=u$. If $\zeta\notin K$, then the $Z(\widetilde{E})$-irreducible summands of V are all 2-dimensional, all induced from 1-dimensional $\widetilde{E}_0$-representations, and so their characters all vanish on $Z(\widetilde{E})\smallsetminus\widetilde{E}_0$. Thus (b) holds in this case.

So assume now that $\zeta\in K$. In particular, G is not linear ($\mathfrak{b}\ne0$), since otherwise $\zeta\cdot\mathrm{Id}\in Z$. Thus $\mathfrak{b}$ defines an isomorphism $V\cong V^{*\theta}$ of $K\widetilde{E}$-modules, where $\theta\in\mathrm{Aut}(K)$ and $V^{*\theta}$ are as in the proof of Lemma 7.1. The only eigenvalues of the action of g on V are $\pm\zeta$. Let m be the multiplicity of ζ and m' the multiplicity of $-\zeta$ (so $m+m'=\dim(V)$). Then the action of g on $V^{*\theta}$ has eigenvalues $\theta(\zeta)^{-1}$ with multiplicity m and $-\theta(\zeta)^{-1}$ with multiplicity m'. Hence either $\zeta=\theta(\zeta)^{-1}$ or $m=m'$. But if $\zeta\cdot\theta(\zeta)=1$, then $\zeta\cdot\mathrm{Id}\in GL(V,\mathfrak{b})$, hence lies in Z, and this contradicts the assumption that $\zeta^2\cdot\mathrm{Id}$ generates Z. We thus conclude that $m=m'$, and hence that $\chi(g)=0$. □

We are now ready to show that all classical groups of Lie type in odd characteristic lie in $\mathfrak{L}^{\ge2}(2)$.

THEOREM 7.5. *Let q be an odd prime power, and let L be one of the simple groups $PSL_n(q)$ ($n\ge2$), $PSU_n(q)$ ($n\ge3$), $PSp_{2n}(q)$ ($n\ge2$), or $P\Omega_n^\pm(q)$ ($n\ge5$). Then $\mathfrak{E}^{\ge2}(L;2)=\varnothing$, and hence $L\in\mathfrak{L}^{\ge2}(2)$.*

PROOF. Write $L=[\overline{G},\overline{G}]$, where $\overline{G}=PGL(V,\mathfrak{b})$, V is a vector space of dimension n or $2n$ over the field $K=\mathbb{F}_q$ or $\mathbb{F}_{q^2}$, and $\mathfrak{b}$ is a nondegenerate symplectic, quadratic, or hermitian form, or the trivial form. Since $PSL_4(q)\cong P\Omega_6^+(q)$, $PSU_4(q)\cong P\Omega_6^-(q)$, and $PSp_4(q)\cong P\Omega_5(q)$, we can assume that $\dim_K(V)\ne4$.

Set $G=GL(V,\mathfrak{b})$, let $\pi\colon G\to\overline{G}$ be the projection, and set $Z=\mathrm{Ker}(\pi)=Z(G)$. Set $\widetilde{L}=[G,G]$. Thus G is one of the groups $GL_n(q)$, $GU_n(q)$, $Sp_{2n}(q)$, or $GO_n^\pm(q)$, $\overline{G}=G/Z$, and $L=\widetilde{L}/(Z\cap\widetilde{L})$.

Fix $E\in\mathfrak{E}^{\ge2}(L;2)$, and set $\widetilde{E}=\pi^{-1}(E)$. Define

$$\Delta=\bigl\{\epsilon\in\mathrm{Hom}(E,\{\pm1\})\bigm|\mathcal{I}_\epsilon\in\mathrm{Aut}_G(\widetilde{E})\bigr\}.$$

Set
$$\widetilde{E}_0 = Z{\cdot}\Omega_1(Z(\widetilde{E})), \qquad \widetilde{E}_1 = \bigcap_{\epsilon\in\Delta} \operatorname{Ker}(\epsilon\circ\pi),$$
and $E_i = \pi(\widetilde{E}_i)$. By Lemma 7.4(c),
$$\widetilde{E}_1 \le \widetilde{E}_0 \qquad \text{and hence} \qquad E_1 \le E_0.$$

Since Δ is a subgroup of $\operatorname{Hom}(E,\{\pm1\})$ (Lemma 7.4(c)), and since E_1 is defined to be the intersection of the kernels of all $\epsilon\in\Delta$,
$$\Delta = \bigl\{\epsilon\in\operatorname{Hom}(E,\{\pm1\}) \bigm| \operatorname{Ker}(\epsilon)\ge E_1\bigr\} \cong \operatorname{Hom}(E/E_1,\{\pm1\}). \tag{1}$$
Also, for any $g\in\widetilde{E}\smallsetminus\widetilde{E}_1$, there is some $\epsilon\in\Delta$ such that $\epsilon(\pi(g))=-1$, and hence some $\alpha\in\operatorname{Aut}_G(\widetilde{E})$ ($\alpha=\mathcal{I}_\epsilon$) such that $\alpha(g)=-g$. So if $\chi\colon\widetilde{E}\longrightarrow\mathbb{C}$ denotes the character of V as an $\widetilde{E}$-representation, then $\chi(g)=\chi(-g)=-\chi(g)$, and hence $\chi(g)=0$. We have now shown that
$$\chi(g)=0 \quad \text{for all } g\in\widetilde{E}\smallsetminus\widetilde{E}_1. \tag{2}$$

Set $r=\operatorname{rk}(E_1)$, $s=\operatorname{rk}(E_0/E_1)$, $t=\operatorname{rk}(E/E_0)$, and $m=r+s+t=\operatorname{rk}(E)$. Let $\{e_1,\dots,e_m\}$ be an $\mathbb{F}_2$-basis for E, chosen and ordered such that $\{e_1,\dots,e_r\}\subseteq E_1$ and $\{e_1,\dots,e_{r+s}\}\subseteq E_0$ are bases for these subgroups. Let $\widetilde{e}_i\in\widetilde{E}$ be a lifting of $e_i\in E$, chosen such that if $i\le r+s$ (if $e_i\in E_0$), then $\widetilde{e}_i\in\Omega_1(Z(\widetilde{E}))$. Thus $\widetilde{E}_0=Z\times T_0$, where $T_0\overset{\text{def}}{=}\langle\widetilde{e}_1,\dots,\widetilde{e}_{r+s}\rangle\cong C_2^{r+s}$.

For each $1\le i<j\le m$ such that $i\le r+s$ and $j\ge r+1$, let $\alpha_{ij}\in\operatorname{Aut}(\widetilde{E})$ be the automorphism $\alpha_{ij}|_Z=\operatorname{Id}_Z$, $\alpha_{ij}(\widetilde{e}_k)=\widetilde{e}_k$ if $k\ne j$, and $\alpha_{ij}(\widetilde{e}_j)=\widetilde{e}_i\widetilde{e}_j$. (This is an automorphism since $\widetilde{e}_i$ is central of order 2.) Let $\epsilon_j\in\operatorname{Hom}(\widetilde{E},\{\pm1\})$ be the homomorphism $\epsilon_j(\widetilde{e}_k)=\widetilde{e}_k$ if $k\ne j$ and $\epsilon_j(\widetilde{e}_j)=-1$ (and $\epsilon_j(Z)=1$). Since $j>r$, $\epsilon_j\in\Delta$ by (1). Since $i\le r+s$, $\widetilde{e}_i\in T_0$, and hence $\alpha_{ij}\in\operatorname{Aut}_G(\widetilde{E})$ by Lemma 7.3, applied with $T=\langle\widetilde{e}_1,\dots,\widetilde{e}_i\rangle\le T_0$.

Let $A_0\le A\le\operatorname{Aut}(E)$ be the subgroups
$$A=C_{\operatorname{Aut}(E)}(E_1)\cap C_{\operatorname{Aut}(E)}(E/E_0) \qquad\text{and}\qquad A_0=A\cap C_{\operatorname{Aut}(E)}(E_0/E_1).$$
Thus A is the group of automorphisms which send E_0 and E_1 to themselves and induce the identity on E_1 and E/E_0, while A_0 is the subgroup of those which also induce the identity on E_0/E_1. With respect to the basis $\{e_1,\dots,e_m\}$, A is the group of automorphisms whose matrices have the form $\begin{pmatrix} I_r & Y & X\\ 0 & U & Z\\ 0 & 0 & I_t\end{pmatrix}$ for $U\in GL_s(\mathbb{F}_2)$ and arbitrary matrices X,Y,Z of the appropriate dimensions, while A_0 is the subgroup of automorphisms whose matrices have this form with $U=I_s$. We have just shown that $\operatorname{Aut}_{\overline{G}}(E)$ contains all automorphisms with elementary matrix e_{ij}, for $i<j$, $i\le r+s$, and $j>r$. In particular, since A_0 is generated by automorphisms with matrices e_{ij} for $i\le r<j$ or $i\le r+s<j$, this shows that $A_0\le\operatorname{Aut}_{\overline{G}}(E)$. Furthermore, since the basis chosen for E_0/E_1 was arbitrary, $\operatorname{Aut}_{\overline{G}}(E)$ also contains all automorphisms whose matrix has the above form when X,Y,Z all vanish and $U\in GL_s(\mathbb{F}_2)$ is a transvection. Since $GL_s(\mathbb{F}_2)$ is generated by transvections, we have now shown that
$$A_0\le A\le\operatorname{Aut}_{\overline{G}}(E). \tag{3}$$

Now, A_0 is a 2-subgroup of $\mathrm{Aut}_{\overline{G}}(E)$ (it has order $2^{rs+rt+st}$ by the above description in terms of matrices), and is a normal subgroup since E_0 and E_1 are both $\mathrm{Aut}_{\overline{G}}(E)$-invariant subgroups of E. Thus $A_0 \le O_2(\mathrm{Aut}_{\overline{G}}(E))$. Also, $A_0 \cap L = 1$ since $O_2(\mathrm{Aut}_L(E)) = 1$ (E is pivotal by Lemma 4.4(c)). Either $|\overline{G}/L| \le 4$ (if G is orthogonal), or $\overline{G}/L$ is cyclic. Hence A_0 is cyclic or of order 4, and in particular $\mathrm{rk}(A_0) \le 2$. Also, $m = r+s+t = \mathrm{rk}(E) \ge 4$ by Proposition 4.6(b). By the matrix description, $\mathrm{rk}(A_0) \ge r(s+t)$ and $\mathrm{rk}(A_0) \ge t(r+s)$. Thus if any two of the ranks r, s, t are nonzero, then $\mathrm{rk}(A_0) \ge 3$, which we have just shown is impossible. This shows that two of the three groups E_1, E_0/E_1, and E/E_0 must vanish, and we are reduced to considering the three cases $(E_0, E_1) = (1, 1)$, $(E, 1)$, or (E, E).

Case 1: Assume $E_0 = 1$. Then $\Omega_1(Z(\widetilde{E})) \le Z$, so $\widetilde{E} = \widetilde{P} \cong Z' \times_Z X$ by Lemma 7.4(a), where Z' is cyclic, $[Z'{:}Z] \le 2$, and X is such that $Z(X) = Z$ and $[X, X] \le \{\pm \mathrm{Id}\}$. If G is linear or unitary, or if G is orthogonal or symplectic and $[Z'{:}Z] = 2$, then $\mathrm{Aut}_G(\widetilde{E}) = \mathrm{Aut}_\chi(\widetilde{E})$ by Lemma 7.2(a,b). If G is orthogonal or symplectic and $Z' = Z$ $(= \{\pm \mathrm{Id}\})$, then for any $K\widetilde{E}$-submodule $V' \subseteq V$ with character χ', $\chi'(-\mathrm{Id}) = -\chi'(\mathrm{Id})$, and $\chi'(g) = 0$ for all $g \in \widetilde{E} \smallsetminus Z$ by Lemma 7.4(b). So by the independence of irreducible characters, $V \cong W^m$ for some irreducible $K\widetilde{E}$-module W, and $\mathrm{Aut}_G(\widetilde{E})$ has index at most two in $\mathrm{Aut}_\chi(\widetilde{E})$ in this case by Lemma 7.2(d).

By (2), the character χ of $\widetilde{E}$ on V vanishes on $\widetilde{E} \smallsetminus Z$, and hence

$$\mathrm{Aut}_\chi(\widetilde{E}) = \bigl\{\alpha \in \mathrm{Aut}(\widetilde{E}) \bigm| \alpha|_Z = \mathrm{Id}\bigr\}.$$

Set

$$\Gamma = \bigl\{\alpha/Z \bigm| \alpha \in \mathrm{Aut}_\chi(\widetilde{E})\bigr\} \le \mathrm{Aut}(E):$$

the image of $\mathrm{Aut}_\chi(\widetilde{E})$ under the projection to $\mathrm{Aut}(E)$. Since $\mathrm{Aut}_G(\widetilde{E})$ has index at most two in $\mathrm{Aut}_\chi(\widetilde{E})$, $\mathrm{Aut}_{\overline{G}}(E)$ has index at most two in Γ.

Set $Z'^2 = \{g^2 \mid g \in Z'\}$. Let

$$\delta \colon E \times E \longrightarrow \{\pm \mathrm{Id}\} \qquad \text{and} \qquad \sigma \colon E \longrightarrow Z/Z'^2$$

denote the commutator and squaring maps: $\delta(\pi(g), \pi(h)) = [g, h]$ and $\sigma(\pi(g)) = g^2 \cdot Z'^2$. Then each automorphism of E which preserves these maps lifts to an automorphism of $\widetilde{E}$; i.e.,

$$\Gamma = \{\alpha \in \mathrm{Aut}(E) \mid \delta \circ (\alpha \times \alpha) = \delta \text{ and } \sigma \circ \alpha = \sigma\}.$$

Also, δ is a nondegenerate symplectic form on $X/Z \le E$ (since $Z = Z(X)$), and hence $\mathrm{rk}(X/Z)$ is even. Set $2m = \mathrm{rk}(X/Z)$. Since $\mathrm{rk}(E) \ge 4$ and $\mathrm{rk}(E) \le \mathrm{rk}(X/Z) + 1$, we must have $2m \ge 4$.

Assume first that $|Z'| \ge 4$ and $\sigma \ne 1$. In particular, $\mathrm{Aut}_G(\widetilde{E}) = \mathrm{Aut}_\chi(\widetilde{E})$, and hence $\mathrm{Aut}_{\overline{G}}(E) = \Gamma$, in this case. Also, $Z' = Z$ (so δ is nondegenerate), $[\widetilde{E}, \widetilde{E}] \le Z'^2$, and hence σ is linear. Set $E_2 = \mathrm{Ker}(\sigma)$ and $E_3 = (E_2)^\perp$ (the orthogonal complement with respect to δ); thus $\mathrm{rk}(E_3) = 1$ and $\mathrm{rk}(E_2) = 2m - 1$. Each automorphism $\alpha \in \Gamma$ leaves E_2 and E_3 invariant, and hence each $\alpha \in \Gamma$ which induces the identity on E_3, E_2/E_3, and E/E_2 lies in $O_2(\Gamma)$. The transvection $\alpha \in \mathrm{Aut}(E)$ such that $\alpha|_{E_2} = \mathrm{Id}$ and $\mathrm{Im}(\alpha - \mathrm{Id}) = E_3$ preserves σ and δ and hence lies in Γ. Also, by Witt's lemma (see Theorem 6.3), each automorphism of E_2 which induces the identity on E_3 and on E_2/E_3 extends to a unique automorphism of E

which preserves δ, and hence which lies in $O_2(\Gamma)$. From this, one sees that $O_2(\Gamma) = O_2(\mathrm{Aut}_{\widetilde{G}}(E))$ is noncyclic and $|O_2(\Gamma)| \geq 2^{2m-1} \geq 8$. Since $\mathrm{Aut}_{\widetilde{G}}(E)/\mathrm{Aut}_L(E)$ is cyclic or has order 4, this proves that $O_2(\mathrm{Aut}_L(E)) \neq 1$, which contradicts the fact that E is pivotal (Lemma 4.4(c)). Thus, this case cannot occur.

If $|Z'| \geq 4$ and $\sigma = 1$, then $\Gamma \cong Sp_{2m}(2)$: the group of automorphisms which preserve δ. If $|Z'| = |Z| = 2$, then $\widetilde{E}$ is an extraspecial 2-group, σ is a quadratic form associated to the bilinear form δ, and hence $\Gamma \cong SO^{\pm}_{2m}(2)$. Hence if $m \geq 3$, then $\mathrm{Aut}_L(E)$ is isomorphic to $Sp_{2m}(2)$, $SO^{\pm}_{2m}(2)$, or $\Omega^{\pm}_{2m}(2)$, with the usual action on $\widetilde{E}/Z' \cong (\mathbb{Z}/2)^{2m}$; and by Proposition 6.5, E cannot lie in $\mathfrak{E}^{\geq 2}(L;2)$. If $m = 2$ and $\Gamma = Sp_4(2) \cong \Sigma_6$ [**Hp**, II.9.22], then $\mathrm{Aut}_L(E)$ has index at most two in Γ, and $E \notin \mathfrak{E}^{\geq 2}(L;2)$ by Proposition 4.7(b). If $m = 2$ and $\Gamma = SO^{-}_4(2)$, then $\Gamma \cong \Sigma_5$ and acts on $E \cong (\mathbb{Z}/2)^4$ by the permutation action modulo its fixed component (since E contains exactly five involutions which lift to involutions in $\widetilde{E}$ and their only relation in E is that their product is trivial); and so again, $[\Gamma\colon \mathrm{Aut}_L(E)] \leq 2$ and $E \notin \mathfrak{E}^{\geq 2}(L;2)$ by Proposition 4.7(b).

It remains to consider the case where $\Gamma \cong SO^{+}_4(2)$, and thus where $\widetilde{E} \cong D_8 \times_{C_2} D_8 \cong Q_8 \times_{C_2} Q_8$. As was noted above, this group has a unique irreducible K-representation W on which its central involution acts via $(-\mathrm{Id})$. More precisely, let U be an irreducible 2-dimensional KD_8-module; then $W = U \otimes_K U$ is 4-dimensional, and the unique irreducible $K\widetilde{E}$-module on which the central involution acts via $-\mathrm{Id}$. Also,

$$\mathrm{End}_{K\widetilde{E}}(W) \cong \mathrm{End}_{KD_8}(U) \otimes_K \mathrm{End}_{KD_8}(U) \cong K.$$

Furthermore, U is self-dual (since its character vanishes on $D_8 \smallsetminus Z(D_8)$), so it supports a D_8-invariant symmetric or symplectic form $\mathfrak{b}_U$, and $\mathfrak{b}_U \otimes \mathfrak{b}_U$ is an $\widetilde{E}$-invariant symmetric form on W (cf. [**A1**, 9.1]).

Thus $V \cong W^m$ for some $m \geq 2$, since we are assuming $\dim(V) \neq 4$. Any $g \in C_L(E)$ lifts to $\widetilde{g} \in \widetilde{L}$ whose conjugation action on $\widetilde{E}$ is the identity modulo Z, hence an inner automorphism of $\widetilde{E}$; and thus $C_L(E) = (C_{\widetilde{L}}(\widetilde{E})/Z) \times E$. Hence any $P \in \mathrm{Syl}_2(C_L(E))$ has the form $P = P' \times E$ for some $P' \in \mathrm{Syl}_2(C_{\widetilde{L}}(\widetilde{E})/Z)$. Also, $E = \Omega_1(Z(P))$ by Proposition 4.4(a), so $P' = 1$, and $C_{\widetilde{L}}(\widetilde{E})/Z$ must have odd order.

Assume the irreducible $K\widetilde{E}$-module W supports a form of the same type as $\mathfrak{b}$. Then by Lemma 7.1, V splits nontrivially as an orthogonal direct sum irreducible submodules, and P' contains all automorphisms which are the identity on certain summands and $-\mathrm{Id}$ on others. So $P \neq 1$, and this case cannot occur. We can thus eliminate the linear case ($\mathfrak{b} = 0$), and also the orthogonal case since W was shown above to support a symmetric form. If $\mathfrak{b}$ is a hermitian form, and K_0 is the fixed subfield of the involution $\theta \in \mathrm{Aut}(K)$, then the symmetric form on the irreducible $K_0\widetilde{E}$-module W_0 extends to a hermitian form on $W \cong K \otimes_{K_0} W_0$. So the only case which remains to eliminate is the symplectic case.

Assume $\mathfrak{b}$ is symplectic, and thus that $G \cong Sp_{4m}(K)$. Identify $V = V_0 \otimes_K W$, where $V_0 \cong K^m$ (with no action). Since $\mathrm{End}_{K\widetilde{E}}(W) \cong K$ as shown above (and since W is self-dual), the space of all $K\widetilde{E}$-bilinear forms on $V \cong W^m$, and the space $\mathrm{End}_{K\widetilde{E}}(W)$, are both m^2-dimensional K-vector spaces. Thus each bilinear form on V is of the form $\mathfrak{b}_0 \otimes \mathfrak{b}_W$ for some unique bilinear form $\mathfrak{b}_0$ on V_0, and each $K\widetilde{E}$-linear endomorphism of W is of the form $f\otimes \mathrm{Id}_W$ for some unique $f\in \mathrm{End}_K(V_0)$. Let $\mathfrak{b}_0$ be the unique form on V_0 such that $\mathfrak{b}=\mathfrak{b}_0\otimes\mathfrak{b}_W$. Then $\mathfrak{b}_0$ is skew-symmetric and nondegenerate since $\mathfrak{b}$ is (and by the uniqueness of $\mathfrak{b}_0$). Thus $\mathfrak{b}_0$ is a symplectic form on V_0 (hence $m=\dim_K(V_0)$ is even); and $C_G(\widetilde{E})$ is the group of all $f\otimes\mathrm{Id}_W\in\mathrm{Aut}_{K\widetilde{E}}(V)$ such that $f\in\mathrm{Aut}_K(V_0,\mathfrak{b}_0)$. Hence $C_G(\widetilde{E})\cong Sp_m(K)$. Since $Sp_m(K)/\{\pm\mathrm{Id}\}$ has even order for all even $m\geq 2$, this contradicts the above assertion that $C_G(\widetilde{E})/Z$ must have odd order.

This finishes the proof of Case 1: no subgroup $E\leq L$ of this type can be in $\mathfrak{E}^{\geq 2}(L\,;2)$.

Case 2: Assume $E_1=1$ and $E_0=E$. By (3), $\mathrm{Aut}_{\overline{G}}(E)=\mathrm{Aut}(E)$. Since $\mathrm{Aut}(E)\cong GL_m(2)$ is a perfect group (recall $m=\mathrm{rk}(E)\geq 4$), this shows that $\mathrm{Aut}_L(E)=\mathrm{Aut}(E)$. Hence $E\notin\mathfrak{E}^{\geq 2}(L\,;2)$ by Proposition 4.7(a).

Case 3: Now assume $E_1=E$. In this case, $C_L(E)$ is the image in L of $C_{\widetilde{L}}(\widetilde{E})$, and hence $\widetilde{E}$ must contain the center of $C_{\widetilde{L}}(\widetilde{E})$. Write $\widetilde{E}=Z\times T$ where T is elementary abelian, and let $V=\bigoplus_{i=1}^k V_i$ be the decomposition into eigenspaces for the characters of T. This is an orthogonal decomposition, and $C_G(\widetilde{E})$ is the product of the groups $GL(V_i,\mathfrak{b}|_{V_i})$. We claim that

$$\widetilde{E}=Z\cdot\bigl\{\varphi\in\widetilde{L}\,\big|\,\varphi|_{V_i}=\pm\mathrm{Id}\ \text{ for all } i\bigr\}. \tag{4}$$

Since $E=\Omega_1(Z(P))$ for some $P\in\mathrm{Syl}_2(C_L(E))$ (Proposition 4.4(a)), $\widetilde{E}$ contains each involution in the center of $C_{\widetilde{L}}(\widetilde{E})$, and thus $\widetilde{E}$ contains all involutions in $\widetilde{L}$ which act on each V_i by $\pm\mathrm{Id}$. The opposite inclusion follows since by construction, $\widetilde{E}=Z\times T$ for some elementary abelian group T which acts on each V_i via $\pm\mathrm{Id}$.

Now, each element $\varphi\in N_G(\widetilde{E})$ must preserve the decomposition of V as a sum of V_i's, and any $\varphi\in G$ which preserves the decomposition normalizes $\widetilde{E}$ by (4). We have already seen that $C_G(\widetilde{E})$ is the group of elements of G which send each V_i to itself. Thus $\mathrm{Aut}_G(\widetilde{E})\cong\mathrm{Aut}_{\overline{G}}(E)$ can be regarded as a group of permutations of the set $\{1,\ldots,k\}$ — a product of one symmetric group for each isometry class of pairs $(V_i,\mathfrak{b}|_{V_i})$ — and $\mathrm{Aut}_L(E)$ contains the corresponding product of alternating groups. By (4) again, we can identify E as an $\mathrm{Aut}_{\overline{G}}(E)$-invariant subgroup of $\mathbb{F}_2^k/\langle(1,\ldots,1)\rangle$, where $\mathrm{Aut}_{\overline{G}}(E)$ permutes the canonical basis of $\mathbb{F}_2^k$. But this situation is impossible for $E\in\mathfrak{E}^{\geq 2}(L\,;2)$ by Proposition 4.8. □

CHAPTER 8

Exceptional groups of Lie type in odd characteristic

In addition to the five families of Chevalley groups $G_2(q)$, $F_4(q)$, and $E_n(q)$, it remains to consider the twisted groups ${}^2G_2(3^{2k+1})$, ${}^3D_4(q)$, and ${}^2E_6(q)$ for odd q. The following cases are easy.

PROPOSITION 8.1. *Assume L is one of the groups $G_2(q)$ or ${}^3D_4(q)$ for any odd prime power q, or ${}^2G_2(3^{2k+1})$ for some $k \geq 1$. Then $L \in \mathfrak{L}^{\geq 2}(2)$.*

PROOF. By [**Gr2**, Theorem 6.1], $\mathrm{rk}_2(G_2(\bar{\mathbb{F}}_q)) = 3$. Hence $G_2(q)$ and ${}^2G_2(q)$ have 2-rank at most 3 (in fact, equal to three in all cases). Furthermore, the tables of orders of the groups of Lie type in [**Ca1**] or [**GLS3**] show that for odd q,

$$[{}^3D_4(q){:}G_2(q)] = q^6(q^8 + q^4 + 1)$$

is odd, and thus $\mathrm{rk}_2({}^3D_4(q)) = \mathrm{rk}_2(G_2(q))$. So $\mathfrak{E}^{\geq 2}(L\,;2) = \varnothing$ by Proposition 4.6(b), and hence $L \in \mathfrak{L}^{\geq 2}(2)$ by Proposition 4.2. □

The groups $F_4(q)$ are almost as easy to handle.

PROPOSITION 8.2. *Assume, for some odd prime power q, that $L = F_4(q)$. Then $\mathfrak{R}^{\geq 2}(L\,;2) = \varnothing$, and $L \in \mathfrak{L}^{\geq 2}(2)$.*

PROOF. We regard L as a subgroup of $G = F_4(\bar{\mathbb{F}}_q)$. There are two conjugacy classes of involutions in G, denoted **2A** or **2B** in [**Gr2**, Table VI]. By [**Gr2**, Theorem 7.3], G contains a unique conjugacy class of maximal elementary abelian 2-subgroups, represented by $E_5 = T_{(2)} \times \langle\theta\rangle$, where T is a maximal torus, $T_{(2)}$ is its 2-torsion subgroup, and $\theta \in N_G(T)$ is an element which inverts T. Furthermore, the elements of type **2B** in E_5 form (together with the identity) a subgroup $E_2 \leq E_5$ of rank 2.

Thus for any elementary abelian 2-subgroup $E \leq L$, there is a subgroup $E_0 \leq E$ such that $E \cap \mathbf{2B} = E_0{}^{\#}$ and $E \cap \mathbf{2A} = E \smallsetminus E_0$, and such that $\mathrm{rk}(E_0) \leq 2$ and $\mathrm{rk}(E/E_0) \leq 3$. In particular, E_0 is $N_{\mathrm{Aut}(G)}(E)$-invariant, and thus $N_{\mathrm{Aut}(L)}(E)$-invariant. Hence $\mathfrak{E}^{\geq 2}(L\,;2) = \varnothing$ by Proposition 4.6(b), and so $L \in \mathfrak{L}^{\geq 2}(2)$ by Proposition 4.2. □

In order to deal with the remaining cases, we need to look more closely at the algebraic groups G over $\bar{\mathbb{F}}_q$, and the endomorphisms σ of G, for which $L = C_G(\sigma)$ is a finite group of Lie type. Here, $C_G(\sigma)$ denotes the subgroup of elements of G fixed by σ. Our general references for the properties of algebraic groups are [**Hum**] and [**Ca2**]. Note in particular that connected algebraic groups always have maximal tori (products of copies of $F^\times$) which are unique up to conjugacy [**Hum**, §21.3]. For an arbitrary algebraic group G, we let G^0 denote the connected component of the identity, a normal subgroup of finite index [**Hum**, §7.3]; and let $\pi_0(G) = G/G^0$ denote the finite group of connected components of G.

A connected algebraic group over an algebraically closed field F is *reductive* if it has no nontrivial normal unipotent subgroup. Equivalently, G is reductive if and only if it is the central product over a finite group of a semisimple group and a torus [**Hum**, §§19.5 & 27.5]. We first note the following well known results about centralizers and normalizers of subgroups of a maximal torus in a reductive group.

Lemma 8.3. *Let G be an algebraic group over an algebraically closed field F whose identity component G^0 is reductive. Fix a maximal torus $T \le G$, and set $W = N_G(T)/T$ (regarded as a group acting on T). Let Φ be the set of roots of G, regarded as elements of $\mathrm{Hom}(T, F^\times)$. For each $\alpha \in \Phi$, let $X_\alpha \le G$ denote the corresponding root subgroup ($X_\alpha \cong F$). Then for any subgroup $H \le T$,*

$$C_G(H)^0 = \langle T, X_\alpha \,|\, \alpha(s) = 1, \text{ all } s \in H\rangle$$

is a reductive group, with root system $\{\alpha \in \Phi \,|\, \alpha(s) = 1, \text{ all } s \in H\}$. Also,

$$C_G(H) = C_G(H)^0{\cdot}\{wT \in W \,|\, wsw^{-1} = s \text{ all } s \in H\};$$

$\mathrm{Aut}_G(H) = \mathrm{Aut}_W(H)$; and two elements $x, y \in T$ are $C_G(H)$-conjugate if and only if they are $C_W(H)$-conjugate.

Proof. The description of $C_G(H)^0$ in (a) is shown in [**Ca2**, Theorem 3.5.3] when G is connected and $H = \langle s\rangle$ is cyclic, and the proof given there also applies in the more general case.

Now, T is a maximal torus in $C_G(H)^0$, and all maximal tori of $C_G(H)^0$ are conjugate in $C_G(H)^0$. Each element of $N_G(H)$ conjugates T to another maximal torus of $C_G(H)^0$, and hence by a Frattini argument, $C_G(H) = C_G(H)^0{\cdot}C_{N_G(T)}(H)$ and $N_G(H) = C_G(H)^0{\cdot}N_{N_G(T)}(H)$. This proves the descriptions of $C_G(H)$ and $\mathrm{Aut}_G(H)$. Similarly, for any $x, y \in T$ and $g \in C_G(H)$ such that $y = gxg^{-1}$, T and gTg^{-1} are two maximal tori of the reductive group $C_G(H, y)^0$, hence conjugate in $C_G(H, y)^0$, so there is $a \in C_G(H, y)$ such that $a(gTg^{-1})a^{-1} = T$, and $ag \in C_{N_G(T)}(H)$ conjugates x into y. □

We adopt the terminology used in [**GLS3**], and write *Steinberg endomorphism* to mean a surjective algebraic endomorphism of an algebraic group G over $\bar{\mathbb{F}}_q$ whose fixed subgroup is finite. (When G is semisimple, all Steinberg endomorphisms are group automorphisms, but their inverses are not in general algebraic.) All finite simple groups of Lie type can be constructed as (commutator subgroups of) fixed subgroups of Steinberg endomorphisms. The following result is one of the key properties of these endomorphisms.

Proposition 8.4. (Lang-Steinberg theorem) *Fix a prime p and a connected algebraic group G over $\bar{\mathbb{F}}_p$. Let σ be any Steinberg endomorphism of G. Then every element of G is of the form $x^{-1}\sigma(x)$ for some $x \in G$.*

Proof. See [**St**, Theorem 10.1]. The proof is also sketched in [**Ca2**, §1.17]. □

The following proposition is a special case of [**GLS3**, Theorem 2.1.5]. It describes, in many cases, the relationship between conjugacy classes and normalizers in a connected algebraic group with those in the subgroup fixed by a Steinberg endomorphism.

PROPOSITION 8.5. *Let G be any connected algebraic group over $\bar{\mathbb{F}}_q$. Fix a Steinberg endomorphism σ of G. Let $H \leq C_G(\sigma)$ be any subgroup, and let $\mathcal{H}$ be the set of $C_G(\sigma)$-conjugacy classes of subgroups $H' \leq C_G(\sigma)$ which are G-conjugate to H. Set*

$$N = N_G(H), \quad C = C_G(H), \quad \text{and} \quad \bar{C} = \pi_0(C) = C/C^0$$

for short. Let $\bar{g}$ denote the class in $\bar{C}$ of $g \in C$, and let $[H']$ denote the class in $\mathcal{H}$ of $H' \leq C_G(\sigma)$. Let N act on $\bar{C}$ by sending $(x, \bar{g})$ (for $x \in N$ and $g \in C$) to $\overline{xg\sigma(x)^{-1}}$. Let $\bar{C}/N$ and $\bar{C}/C$ denote sets of orbits of these actions of N and C, respectively, and let $N{\cdot}\bar{g} \in \bar{C}/N$ and $C{\cdot}\bar{g} \in \bar{C}/C$ denote the orbits of $\bar{g} \in \bar{C}$. Then the following hold:

(a) *For all $x \in G$, $xHx^{-1} \leq C_G(\sigma)$ if and only if $x^{-1}\sigma(x) \in C$.*

(b) *There is a bijection*

$$\omega\colon \mathcal{H} \xrightarrow{\cong} \bar{C}/N,$$

where for any $x \in G$ such that $xHx^{-1} \leq C_G(\sigma)$, $\omega([xHx^{-1}]) = N{\cdot}\overline{x^{-1}\sigma(x)} \in \bar{C}/N$.

(c) *For any $x \in G$ such that $xHx^{-1} \leq C_G(\sigma)$, the isomorphism from $\mathrm{Aut}(xHx^{-1})$ to $\mathrm{Aut}(H)$ induced by x (i.e., $\alpha \mapsto c_x^{-1}\alpha c_x$) sends $\mathrm{Aut}_{C_G(\sigma)}(H')$ onto the stabilizer of $C{\cdot}\overline{x^{-1}\sigma(x)} \in \bar{C}/C$: the stabilizer under the action of $\mathrm{Aut}_G(H) \cong N/C$ on $\bar{C}/C$ induced by the action (as defined above) of $N = N_G(H)$ on $\bar{C}$.*

PROOF. We first check that $N_G(H)$ does act on $C = C_G(H)$, and hence on $\bar{C} = C/C^0$. If $g \in N_G(H)$ and $x \in C_G(H)$, then for all $h \in H$,

$$\big(gx\sigma(g)^{-1}\big){\cdot}h{\cdot}\big(gx\sigma(g)^{-1}\big)^{-1} = gx\sigma(g^{-1}hg)x^{-1}g^{-1} = g(g^{-1}hg)g^{-1} = h$$

(since $g^{-1}hg \in H \leq C_G(\sigma)$), and hence $gx\sigma(g)^{-1} \in C_G(H)$.

If $x \in G$ is such that $xHx^{-1} \leq C_G(\sigma)$, then $xhx^{-1} = \sigma(xhx^{-1}) = \sigma(x)h\sigma(x)^{-1}$ for all $h \in H$, and thus $x^{-1}\sigma(x) \in C_G(H)$. Conversely, if $x^{-1}\sigma(x) \in C_G(H)$, then $xhx^{-1} = \sigma(x)h\sigma(x)^{-1} = \sigma(xhx^{-1})$ for all $h \in H$, and hence $xHx^{-1} \leq C_G(\sigma)$. This proves (a).

Let $x, y \in G$ be such that $xHx^{-1}, yHy^{-1} \leq C_G(\sigma)$, and are $C_G(\sigma)$-conjugate. Let $a \in C_G(\sigma)$ be such that $xHx^{-1} = ayHy^{-1}a^{-1}$, and set $g = (ay)^{-1}x \in N = N_G(H)$. Thus $x = ayg$,

$$x^{-1}\sigma(x) = g^{-1}\big(y^{-1}a^{-1}\sigma(a)\sigma(y)\big)\sigma(g) = g^{-1}\big(y^{-1}\sigma(y)\big)\sigma(g),$$

and so $N{\cdot}\overline{x^{-1}\sigma(x)} = N{\cdot}\overline{y^{-1}\sigma(y)} \in \bar{C}/N$. Thus $\omega([xHx^{-1}])$ depends only on the $C_G(\sigma)$-conjugacy class of the subgroup xHx^{-1}, and not on x.

This shows that ω is well defined. To see that ω is onto, fix some $z \in C_G(H)$. By the Lang-Steinberg theorem, there is $x \in G$ such that $x^{-1}\sigma(x) = z$. Then $xHx^{-1} \leq C_G(\sigma)$ by (a), and $N{\cdot}\bar{z} = \omega([xHx^{-1}]) \in \mathrm{Im}(\omega)$.

To see that ω is injective, it now suffices to consider the case where $x, y \in G$ are such that $x^{-1}\sigma(x) \equiv y^{-1}\sigma(y) \pmod{C_G(H)^0}$. Let $z \in C_G(H)^0$ be such that $x^{-1}\sigma(x) = y^{-1}\sigma(y){\cdot}z$. Then

$$x^{-1}\sigma(x) = y^{-1}\bigl(\sigma(y)z\sigma(y)^{-1}\bigr)\sigma(y);$$

and since $yHy^{-1} = \sigma(y)H\sigma(y)^{-1}$,

$$(xy^{-1})^{-1}\sigma(xy^{-1}) = \sigma(y)z\sigma(y)^{-1} \in C_G(yHy^{-1})^0.$$

By the Lang-Steinberg theorem, $(xy^{-1})^{-1}\sigma(xy^{-1}) = g^{-1}\sigma(g)$ for some element $g \in C_G(yHy^{-1})^0$. Then $gyx^{-1} \in C_G(\sigma)$, and so xHx^{-1} and yHy^{-1} are $C_G(\sigma)$-conjugate.

It remains to prove (c): to describe $\mathrm{Aut}_{C_G(\sigma)}(H')$ for $H' = xHx^{-1} \le C_G(\sigma)$. Consider the isomorphism

$$\gamma\colon \mathrm{Aut}(H') \longrightarrow \mathrm{Aut}(H) \tag{1}$$

defined by $\gamma(\alpha) = c_x^{-1}\alpha c_x$. In particular, $\gamma(c_g) = c_{x^{-1}gx}$ for $g \in N_G(H')$. We must show that $\gamma(\mathrm{Aut}_{C_G(\sigma)}(H'))$ is the stabilizer of $C{\cdot}\overline{x^{-1}\sigma(x)} \in \bar{C}/C$ under the action of $\mathrm{Aut}_G(H)$.

Assume first that $c_g \in \mathrm{Aut}_G(H)$ stabilizes $C{\cdot}\overline{x^{-1}\sigma(x)}$. Upon replacing g by ag for some appropriate $a \in C_G(H)$, we can assume that g fixes the class of $x^{-1}\sigma(x)$ in $\bar{C} = \pi_0(C_G(H))$ itself. Thus

$$g\bigl(x^{-1}\sigma(x)\bigr)\sigma(g)^{-1} = (xg^{-1})^{-1}\sigma(xg^{-1}) \equiv x^{-1}\sigma(x) \pmod{C_G(H)^0},$$

and hence (as already shown above) there is $z \in C_G(H')$ such that $x(xg^{-1})^{-1}z^{-1} \in C_G(\sigma)$. Thus, $xgx^{-1}z^{-1} \in N_{C_G(\sigma)}(H')$, and γ sends conjugation by this element to c_g.

Conversely, for any $g \in N_{C_G(\sigma)}(H')$,

$$(x^{-1}gx){\cdot}\bigl(x^{-1}\sigma(x)\bigr){\cdot}\sigma(x^{-1}gx)^{-1} = x^{-1}g\sigma(g)^{-1}\sigma(x) = x^{-1}\sigma(x);$$

and hence $x^{-1}gx$ stabilizes $C{\cdot}\overline{x^{-1}\sigma(x)}$. □

For example, if $H = T_{(2)}$ is the 2-torsion in a σ-invariant maximal torus $T \le G$ (and $H \le C_G(\sigma)$), then $C_G(H)$ is generated by T, together with the element of $W = N(T)/T$ which inverts T if there is such an element. Hence $\pi_0(C_G(H))$ has at most two elements. So either there are two $C_G(\sigma)$-conjugacy classes of subgroups G-conjugate to H (if some element of W inverts T), or there is just one such class (if no element of W inverts T).

The following easy corollary of Proposition 8.5 will be useful.

COROLLARY 8.6. *Fix a connected algebraic group G over $\bar{\mathbb{F}}_q$ and a Steinberg endomorphism σ of G, and assume $L \lhd C_G(\sigma)$ is a simple subgroup of index d. Assume $E \in \mathfrak{E}^{\ge 2}(L;2)$, and set $k = |\pi_0(C_G(E))|$ and $2^n = |O_2(\mathrm{Aut}_G(E))|$. Then $2^n \le kd$; and $\mathrm{Aut}_L(E)$ is isomorphic to a subgroup of $\mathrm{Aut}_G(E)/O_2(\mathrm{Aut}_G(E))$ of index $\le kd/2^n$.*

PROOF. By Proposition 8.5, $\mathrm{Aut}_{C_G(\sigma)}(E)$ is isomorphic to some point stabilizer of an action of $\mathrm{Aut}_G(E)$ on a quotient set of $\pi_0(C_G(E))$. Thus $\mathrm{Aut}_{C_G(\sigma)}(E)$

has index at most k in $\mathrm{Aut}_G(E)$; and so $\mathrm{Aut}_L(E)$ has index $\le dk$ in $\mathrm{Aut}_G(E)$. Furthermore, $O_2(\mathrm{Aut}_L(E)) = 1$, since E is pivotal by Proposition 4.4(c), so $\mathrm{Aut}_L(E)$ is isomorphic to a subgroup of $\mathrm{Aut}_G(E)/O_2(\mathrm{Aut}_G(E))$ of index $\le kd/2^n$. In particular, $2^n \le kd$. □

It will also be useful at times to know that the group of components of the centralizer of a 2-group is again a 2-group, and to get upper bounds on its order. The following two propositions will be proven simultaneously.

PROPOSITION 8.7. *Let p be any prime, and let G be a connected reductive algebraic group over an algebraically closed field $\mathbb{F}$ of characteristic $\neq p$. Then for any finite p-subgroup $P \le G$, $\pi_0(C_G(P))$ is a p-group, and the identity connected component $C_G(P)^0$ of the centralizer is also a reductive algebraic group over $\mathbb{F}$.*

PROPOSITION 8.8. *Let p be any prime, and let G be a connected reductive algebraic group over an algebraically closed field $\mathbb{F}$ of characteristic $\neq p$. Set $T = Z(G)^0$, the largest normal toral subgroup, and let $H \lhd G$ be the largest normal semisimple subgroup. Let R be the fundamental group of H; i.e., $H \cong \widetilde{H}/R$ where $\widetilde{H}$ is of universal type. Then $\pi_0(C_G(\alpha))$ is a p-group, and*

$$|\pi_0(C_G(\alpha))| = p^r \qquad \text{where} \qquad r \le \mathrm{rk}(T) + \mathrm{rk}_p(R) \le \mathrm{rk}(G).$$

PROOF. First let G and P be as in Proposition 8.7. Let $P' \lhd P$ be a subgroup of index p; we can assume inductively that the proposition holds for P'. Thus $C_G(P')^0$ is reductive and has p-power index in $C_G(P')$. Fix any $g \in P{\smallsetminus}P'$. Then g acts via conjugation on $C_G(P')$ as an algebraic automorphism $\alpha \in \mathrm{Aut}(C_G(P'))$ of order p. Thus upon replacing G by $C_G(P')^0$ and P by $\langle\alpha\rangle$, we are reduced to proving Proposition 8.8, and also proving that $C_G(\alpha)^0$ is reductive in that situation.

Set $P = \langle\alpha\rangle \le \mathrm{Aut}(G)$ for short. Since $p = |\alpha|$ is prime to $\mathrm{char}(\mathbb{F})$, α is a semisimple automorphism. Assume first that

$$1 \longrightarrow K \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1$$

is a central extension of algebraic groups, where K is finite, and that α lifts to an automorphism $\widetilde{\alpha} \in \mathrm{Aut}(\widetilde{G})$ of order p. Let N_α denote the norm map for the α-actions:

$$N_\alpha(x) = x \cdot \alpha(x) \cdot \alpha^2(x) \cdots \alpha^{p-1}(x).$$

The induced exact sequence in cohomology takes the form

$$1 \longrightarrow \underset{=C_K(\widetilde{\alpha})}{H^0(P;K)} \longrightarrow \underset{=C_{\widetilde{G}}(\alpha)}{H^0(P;G)} \xrightarrow{\ \eta\ } \underset{=C_G(\alpha)}{H^0(P;G)} \longrightarrow H^1(P;K);$$

where $H^1(P;K) \cong \{x \in K \mid N_\alpha(x) = 1\}\big/\{x^{-1}\alpha(x) \mid x \in K\}$ is a subquotient of K, and is finite of exponent p since K is finite and $|P| = p$. Thus, in this situation,

(1) $C_{\widetilde{G}}(\widetilde{\alpha})^0$ is a finite covering group of $C_G(\alpha)^0$, and hence $C_G(\alpha)^0$ is reductive if $C_{\widetilde{G}}(\widetilde{\alpha})^0$ is; and

(2) $|\pi_0(C_G(\alpha))|$ divides $|\pi_0(C_{\widetilde{G}}(\widetilde{\alpha}))| \cdot p^{\mathrm{rk}_p(K)}$.

Recall that $H \lhd G$ is the maximal normal semisimple subgroup, and that $T = Z(G)^0$. Let $\widetilde{H}$ denote the universal central extension of H, so $H \cong \widetilde{H}/R$. By [**St**, 9.16], α lifts to a unique automorphism $\widetilde{\alpha}$ of $\widetilde{H}$, which also has order p (since

$\alpha^p = \mathrm{Id}_H$ has a unique lifting to $\widetilde{H}$). Also, $C_{\widetilde{H}}(\widetilde{\alpha})$ is connected and reductive by [**St**, Theorem 8.1] (and this is the deep result which lies behind these propositions). Hence by (1) and (2) (applied with $G = H$), $C_H(\alpha)^0$ is reductive, and $|\pi_0(C_H(\alpha))|$ divides $p^{\mathrm{rk}_p(R)}$. Upon applying (1) again to the surjection of $H \times T$ onto G, we see that $C_G(\alpha)^0$ is also reductive.

Set $T' = G/H$. The extension of H by T' induces an exact sequence

$$1 \longrightarrow C_H(\alpha) \longrightarrow C_G(\alpha) \longrightarrow C_{T'}(\alpha).$$

Furthermore, since $G = TH$, T is a finite covering group of T', so $C_T(\alpha)^0 \le C_G(\alpha)^0$ surjects onto $C_{T'}(\alpha)^0$ by (1). From this, we see that the sequence

$$\pi_0(C_H(\alpha)) \longrightarrow \pi_0(C_G(\alpha)) \longrightarrow \pi_0(C_{T'}(\alpha))$$

is exact, and hence that $|\pi_0(C_G(\alpha))|$ divides $|\pi_0(C_H(\alpha))| \cdot |\pi_0(C_{T'}(\alpha))|$. Since the first factor has order dividing $p^{\mathrm{rk}_p(R)}$, it remains only to show that $|\pi_0(C_{T'}(\alpha))|$ divides $p^{\mathrm{rk}(T)} = p^{\mathrm{rk}(T')}$. In particular, it suffices to show that $\pi_0(C_{T'}(\alpha))$ has exponent p. This follows upon observing that for any $x \in C_{T'}(\alpha)$, $x^p = N_\alpha(x)$ lies in the subgroup $N_\alpha(T')$, which is connected since T' is connected, and hence contained in $C_{T'}(\alpha)^0$. □

The next proposition will be used to show that certain elementary abelian subgroups are not pivotal. Note that the condition on σ^2 in the statement holds in all situations which occur for finite groups of Lie type, except those involving the triality automorphism and ${}^3D_4(q)$. In general, when T is a maximal torus in an algebraic group G, we let $T_{(2)}$ denote the subgroup of elements of order 2.

Proposition 8.9. *Let G be a connected reductive algebraic group over $\bar{\mathbb{F}}_q$, where q is odd, and fix a maximal torus T of G. Let σ be a Steinberg endomorphism of G such that $\sigma(T) = T$, and such that σ^2 is the identity on $W = N_G(T)/T$ and on $T_{(2)}$. Let $E \le C_T(\sigma)$ be an elementary abelian 2-subgroup. Assume there is an involution $x \in T\smallsetminus E$ such that the orbit of x under the $C_W(E)$-action on T has odd order. Then no subgroup of $C_G(\sigma)$ which is G-conjugate to E is pivotal in $C_G(\sigma)$. More generally, if $E \le \bar{E} \le G$ are such that $\bar{E}$ is also elementary abelian and x is not $C_G(E)$-conjugate to any element of $\bar{E}$, then for any $L \lhd C_G(\sigma)$ which contains $\{gxg^{-1} \mid g \in G\} \cap C_G(\sigma)$, no subgroup of L which is G-conjugate to $\bar{E}$ is pivotal in L.*

Proof. We will show, for any $E' \le L \lhd C_G(\sigma)$ which is G-conjugate to E, that there are $g \in G$ and $k \ge 1$ which satisfy the following conditions:

(a) $E' = gEg^{-1}$ and $g^{-1}\sigma^{2^k}(g) \in T$; and

(b) σ^{2^k} leaves $T' \stackrel{\mathrm{def}}{=} gTg^{-1}$ invariant and acts via the identity on $W' \stackrel{\mathrm{def}}{=} N_G(T')/T'$ and on $T'_{(2)}$.

We will also show, for any such g and k (and still with $T' = gTg^{-1}$), that

(c) $\mathrm{Aut}_{C_G(\sigma^{2^k})}(T') = \mathrm{Aut}_G(T')$; and

(d) there is a $C_L(E')$-conjugacy class $\mathfrak{X}_0$ of odd order whose elements are all $C_G(E')$-conjugate to $x' \stackrel{\mathrm{def}}{=} gxg^{-1}$.

Assume these have been shown, and let $\overline{E}' \le L$ be a subgroup G-conjugate to $\overline{E}$. Choose $h \in G$ such that $\overline{E}' = h\overline{E}h^{-1}$, and set $E' = hEh^{-1}$. Let g be as in (a) and (b) (and set $T' = gTg^{-1}$); thus $hg^{-1} \in N_G(E')$. By Lemma 8.3, $\mathrm{Aut}_G(E') = \mathrm{Aut}_{N_G(T')}(E')$; and by (c), $N_G(T') = C_G(T')\cdot N_{C_G(\sigma^{2^k})}(T')$. Since $C_G(E') \ge C_G(T')$, there are elements $x \in C_G(E')$ and $a \in N_{C_G(\sigma^{2^k})}(T')$ such that $hg^{-1} = xa$. So upon replacing g by $ag = x^{-1}h$, conditions (a) and (b) still hold, and $h \in C_G(E')\cdot g$.

Thus $\overline{E}'$ and $g\overline{E}g^{-1}$ are $C_G(E')$-conjugate, and $\overline{E}'$ contains no elements $C_G(E')$-conjugate to $x' = gxg^{-1}$ since $\overline{E}$ contains no elements $C_G(E)$-conjugate to x. Hence $\overline{E}' \cap \mathfrak{X}_0 = \varnothing$ by (d), and $\overline{E}'$ cannot be pivotal in L by Proposition 4.4(f).

It remains to prove points (a–d). Fix $g_0 \in G$ such that $E' = gEg^{-1}$; then $g_0^{-1}\sigma(g_0) \in C_G(E)$ by Proposition 8.5. Consider the homomorphism

$$C_W(E) \xrightarrow{\ \rho\ } \pi_0(C_G(E)),$$

which is surjective by Lemma 8.3. Since σ acts on W with order 2, and since $C_W(E)$ has an odd number of Sylow 2-subgroups, we can choose $S_0 \in \mathrm{Syl}_2(C_W(E))$ which is σ-invariant. Since $\pi_0(C_G(E))$ is a 2-group (Proposition 8.7), $\rho|_{S_0}$ is also surjective. Thus there is $a \in N_G(T)$ such that $aT \in S_0$ and $a \in g_0^{-1}\sigma(g_0)\cdot C_G(E)^0$; and by the Lang-Steinberg theorem, there is $g_1 \in G$ such that $g_1^{-1}\sigma(g_1) = a$. So by Proposition 8.5, $g_1Eg_1^{-1}$ is $C_G(\sigma)$-conjugate to $E' = g_0Eg_0^{-1}$. We can thus choose $g \in C_G(\sigma)\cdot g_1$ such that $E' = gEg^{-1}$, and $g^{-1}\sigma(g) = g_1^{-1}\sigma(g_1) = a$.

For each $m \ge 1$,

$$\begin{aligned} g^{-1}\sigma^{2m}(g) &= \big(g^{-1}\sigma(g)\big)\cdot\sigma\big(g^{-1}\sigma(g)\big)\cdots\sigma^{2m-1}\big(g^{-1}\sigma(g)\big) \\ &= a\cdot\sigma(a)\cdot\sigma^2(a)\cdots\sigma^{2m-1}(a) \in \big(a\cdot\sigma(a)\big)^m\cdot T, \end{aligned}$$

the last step since σ^2 acts via the identity on $W = N_G(T)/T$. Since $aT \in S_0$, it has 2-power order in W. So for k sufficiently large,

$$g^{-1}\sigma^{2^k}(g) \in \big(a\cdot\sigma(a)\big)^{2^{k-1}}T = T.$$

This proves (a). Write $\sigma^{2^k}(g) = gt$ for $t \in T$. For all $x \in N_G(T)$, $\sigma^{2^k}(gxg^{-1}) = gt\sigma^{2^k}(x)t^{-1}g^{-1}$; and since σ^2 sends T to itself and acts induces the identity on $T_{(2)}$ and $W = N_G(T)/T$, σ^{2^k} sends $T' = gTg^{-1}$ to itself and acts via the identity on $T'_{(2)}$ and on $W' = N_G(T')/T'$. Thus (b) also holds.

Now fix any g and k satisfying points (a) and (b), and set $T' = gTg^{-1}$. For any $x \in N_G(T')$, $x^{-1}\sigma^{2^k}(x) \in T'$ since σ^{2^k} acts via the identity on W'. By the Lang-Steinberg theorem, there is $t \in T'$ such that $t^{-1}\sigma^{2^k}(t) = x^{-1}\sigma^{2^k}(x)$, and hence $xt^{-1} \in C_G(\sigma^{2^k})$. Thus each coset of T' in $N_G(T')$ contains elements of $C_G(\sigma^{2^k})$, and this proves (c).

Let $T'_{(4)} \le T'$ be the subgroup generated by elements of order 4 in T'. If $x \in T'_{(4)}$, then $\sigma^{2^k}(x^2) = x^2$ since $x^2 \in T'_{(2)}$, and so $x^{-1}\sigma^{2^k}(x) \in T'_{(2)}$ is fixed by σ^{2^k}. Hence $\sigma^{2^{k+1}}(x) = x$ for all $x \in T'_{(4)}$, and $T'_{(4)} \le C_G(\sigma^{2^{k+1}})$.

Since any algebraic endomorphism of $\mathbb{F}^\times$ has the form $u \mapsto u^n$ for some $n \in \mathbb{Z}$, the group of algebraic automorphisms of T' is isomorphic to $GL_r(\mathbb{Z})$, where $r =$

$\mathrm{rk}(T')$. No nonidentity element of $1+4M_r(\mathbb{Z})$ can have finite order in $GL_r(\mathbb{Z})$ (this is seen by examining the binomial expansion of $(1+4X)^n$ for $n>1$), and hence no nonidentity element of $W'=N_G(T')/T'$ can centralize $T'_{(4)}$. Thus $C_G(T'_{(4)})=T'$ by Lemma 8.3.

Set $G_{k+1}=C_G(\sigma^{2^{k+1}})$ for short. Let $\mathfrak{X}'$ be the $C_{G_{k+1}}(E')$-orbit of $x'\stackrel{\text{def}}{=}gxg^{-1}$. By Lemma 8.3, any two elements of $\mathfrak{X}'\cap T'$ are $C_{W'}(E')$-conjugate. Since $\mathrm{Aut}_G(T')=\mathrm{Aut}_{C_G(\sigma^{2^k})}(T')$ by (c), each coset $gT'\in W$ contains elements of $G_{k+1}\geq C_G(\sigma^{2^k})$, so any element $C_{W'}(E')$-conjugate to x' lies in $\mathfrak{X}'$, and $\mathfrak{X}'\cap T'$ is the full $C_{W'}(E')$-orbit of x'. Since the $C_W(E)$-orbit of x has odd order by assumption, so does the $C_{W'}(E')$-orbit of x', and thus $\mathfrak{X}'\cap T'$ has odd order. Since $T'=C_G(T'_{(4)})$ and $T'_{(4)}\leq C_{G_{k+1}}(E')$, $\mathfrak{X}'\cap T'=C_{\mathfrak{X}'}(T'_{(4)})$ is the fixed set of an action of the 2-group $T'_{(4)}$ on $\mathfrak{X}'$. So $\mathfrak{X}'$ must also have odd order.

Since σ acts on $\mathfrak{X}'\subseteq G_{k+1}$ with order 2^{k+1}, the fixed point set $C_{\mathfrak{X}'}(\sigma)=\mathfrak{X}'\cap C_G(\sigma)$ of this action must also have odd order. Also, by assumption, $\mathfrak{X}'\cap C_G(\sigma)\subseteq L\lhd C_G(\sigma)$, and is a disjoint union of $C_L(E')$-conjugacy classes. Since the union $C_{\mathfrak{X}'}(\sigma)$ has odd order, at least one of those $C_L(E')$-conjugacy classes $\mathfrak{X}_0\subseteq\mathfrak{X}'\cap L$ must also have odd order, and this proves (d). □

We are now ready to consider the remaining exceptional groups $E_n(q)$, and ${}^2E_6(q)$. We adopt some of the notation used by Griess in [**Gr2**]. In particular, **2A** and **2B** will be used to denote conjugacy classes of elements of order 2.

In all cases, we let $E_6(\bar{\mathbb{F}}_q)$, $E_7(\bar{\mathbb{F}}_q)$, or $E_8(\bar{\mathbb{F}}_q)$ denote the adjoint (centerfree) forms of these groups, and let $\widetilde{E}_n(\bar{\mathbb{F}}_q)$ denote their universal covers. Thus $\widetilde{E}_6(\bar{\mathbb{F}}_q)$ is a 3-fold cover of $E_6(\bar{\mathbb{F}}_q)$ (when q is not a power of 3), and $\widetilde{E}_7(\bar{\mathbb{F}}_q)$ is a 2-fold cover of $E_7(\bar{\mathbb{F}}_q)$. In all cases, by [**Gr2**, Lemma 2.16], if $T\leq\widetilde{E}_n(\bar{\mathbb{F}}_q)$ is a maximal torus, there is a quadratic form $\mathfrak{q}\colon T_{(2)}\longrightarrow\mathbb{F}_2$ such that the Weyl group acts on $T_{(2)}$ as the full orthogonal group for the form $\mathfrak{q}$.

PROPOSITION 8.10. *Assume, for some odd prime power q, that L is one of the simple groups $E_6(q)$ or ${}^2E_6(q)$. Then $\mathfrak{R}^{\geq2}(E_6(q);2)=\varnothing$, and hence $L\in\mathfrak{L}^{\geq2}(2)$.*

PROOF. Set $G=E_6(\bar{\mathbb{F}}_q)$ (in adjoint form, with trivial center). Let $\psi^q\in\mathrm{Aut}(G)$ be the field automorphism induced by $(x\mapsto x^q)$, and let $\tau\in\mathrm{Aut}(G)$ be the graph automorphism with fixed subgroup $F_4(\bar{\mathbb{F}}_q)$. Set $\sigma=\psi^q$ if $L\cong E_6(q)$ or $\sigma=\psi^q\circ\tau$ if $L\cong{}^2E_6(q)$. In either case, we let L be the commutator subgroup of $C_G(\sigma)$. Note that $C_G(\sigma)=\mathrm{Inndiag}(L)$ contains L with index $(3,q-1)$ if $L=E_6(q)$, or with index $(3,q+1)$ if $L={}^2E_6(q)$.

Fix a maximal torus $T\leq G$ upon which σ acts via $(t\mapsto t^q)$ if $L=E_6(q)$, or via $(t\mapsto t^{-q})$ if $L={}^2E_6(q)$. To see that there is such a torus in the second case, note first that by construction of G and σ, there is a "standard" maximal torus in T_0 in G, such that $\psi^q(t)=t^q$ for all $t\in T_0$, and such that τ acts on T_0 by permuting its positive roots. Let $\widehat{W}_0\leq\mathrm{Aut}(T_0)$ be the group of all automorphisms which preserve the root system, and regard the Weyl group W_0 as a subgroup of $\widehat{W}_0$. Thus $\tau|_{T_0}\in\widehat{W}_0\smallsetminus W_0$. By [**Ca2**, §3.3], for any $w\in W_0$, there is another σ-invariant maximal torus $T_w=gT_0g^{-1}$ (some $g\in G$) such that $\sigma(gtg^{-1})=g(w\tau(t^q))g^{-1}$ for all $t\in T_0$. Also, $\widehat{W}_0/W_0$ has order 2 and $(t\mapsto t^{-1})\in\widehat{W}_0\smallsetminus W_0$ (cf. [**Bb**, p.261]). So there is $w \in W_0$ such that $w\tau(t) = t^{-1}$ for all t, and $\sigma(t) = t^{-q}$ for $t \in T_w$. Thus in either case, $T_{(2)} \le L$.

By [**Gr2**, Table VI], G has two conjugacy classes of involutions, denoted **2A** and **2B**. For any elementary abelian 2-subgroup $E \le G$, we define $\mathfrak{q}\colon E \longrightarrow \mathbb{F}_2$ by setting $\mathfrak{q}(x) = 1$ if $x \in \mathbf{2A}$ and $\mathfrak{q}(x) = 0$ otherwise. By [**Gr2**, Lemma 2.16], $\mathfrak{q}$ is a quadratic form on $T_{(2)}$, and the action of the Weyl group W on $T_{(2)}$ is that of the orthogonal group $SO(T_{(2)}, \mathfrak{q}) \cong SO_6^-(2)$.

By [**Gr2**, Theorem 8.2], there is a unique maximal elementary abelian 2-subgroup $W_5 \le E_6(\bar{\mathbb{F}}_q)$ which is not contained in a maximal torus, and W_5 is contained in a subgroup $F_4(\bar{\mathbb{F}}_q) \le E_6(\bar{\mathbb{F}}_q)$. By [**Gr2**, Lemma 2.16(i)], involutions in $F_4(\bar{\mathbb{F}}_q)$ of types **2A** and **2B** are sent under this embedding to involutions of types **2A** and **2B** in $E_6(\bar{\mathbb{F}}_q)$. So by [**Gr2**, Theorem 7.3(ii)], there is a subgroup $W_2 \le W_5$ of rank 2 such that $W_5 \cap \mathbf{2B} = W_2{}^{\#}$ and $W_5 \cap \mathbf{2A} = W_5 \smallsetminus W_2$. Thus for any elementary abelian 2-subgroup $E \le L$ which is not contained in a maximal torus of G, there is an $N_{\mathrm{Aut}(L)}(E)$-invariant subgroup $E_0 \le E$ such that $\mathrm{rk}(E_0) \le 2$ and $\mathrm{rk}(E/E_0) \le 3$, and $E \notin \mathfrak{E}^{\ge 2}(L\,;2)$ by Proposition 4.6(b).

Now fix some $E \in \mathfrak{E}^{\ge 2}(L\,;2)$. Since E cannot be G-conjugate to a subgroup of W_5, it must be G-conjugate to a subgroup $E' \le T_{(2)}$. Also, E is pivotal (Proposition 4.4(c)), and $\mathrm{rk}(E) \ge 4$ by Proposition 4.6(b). By Lemma 8.3(b), $\mathrm{Aut}_G(E') \cong SO(E', \mathfrak{q})$, since every element of $\mathrm{Aut}_G(E')$ is the restriction of the action of an element of the Weyl group $W \cong SO(T_{(2)}, \mathfrak{q})$.

Assume $\mathrm{rk}(E) = 4$. If $\mathfrak{q}|_E$ is singular, then $E \cap E^\perp$ is a proper subgroup of E which is $N_{\mathrm{Aut}(L)}(E)$-invariant, so $E \notin \mathfrak{E}^{\ge 2}(L\,;2)$ by Proposition 4.6(b). So we assume $\mathfrak{q}$ is nondegenerate on E, and hence on $E' \le T_{(2)}$. Then $T_{(2)} = E' \times E'^{\perp}$, $E'^{\perp}$ is $C_W(E')$-invariant, and hence there is $1 \ne x \in E'^{\perp}$ whose $C_W(E')$-orbit has odd order. By Proposition 8.9, no subgroup of L which is G-conjugate to E' is pivotal in L. In particular, $E \notin \mathfrak{E}^{\ge 2}(L\,;2)$ in this case.

Finally, if $\mathrm{rk}(E) \ge 5$, then we are in one of the following situations: either

(i) $\mathrm{rk}(E) = 6$ and $\mathrm{Aut}_G(E) \cong SO_6^-(2)$; or

(ii) $\mathrm{rk}(E) = 5$, $E = t^\perp$ for some $t \in T_{(2)}$ with $\mathfrak{q}(t) = 0$, and $\mathrm{Aut}_G(E) \cong 2^4{:}SO_4^-(2)$; or

(iii) $\mathrm{rk}(E) = 5$, $E = t^\perp$ for some $t \in T_{(2)}$ with $\mathfrak{q}(t) = 1$, and $\mathrm{Aut}_G(E) \cong \Omega_5(2) \cong \Sigma_6$.

If any nonidentity element $w \in W$ centralizes E, then w must be an orthogonal transvection, and hence E must be of type (iii) above (see [**Di**, §19]). Furthermore, the kernel of any root α (considered as an element of $\mathrm{Hom}(T, \bar{\mathbb{F}}_q^\times)$) must be of this form; and since there is a unique $SO(T_{(2)}, \mathfrak{q})$-orbit of subgroups of type (iii) ($SO(T_{(2)}, \mathfrak{q})$ acts transitively on the set of nonisotropic elements), every subgroup of type (iii) is the kernel of some root. So by Lemma 8.3, $C_G(E') = T$ if E' (or E) has type (i) or (ii), and $C_G(E')$ is connected if E' has type (iii). Since $C_G(E')$ is connected in all cases, $\mathrm{Aut}_{C_G(\sigma)}(E) = \mathrm{Aut}_G(E) \cong \mathrm{Aut}_G(E')$ by Proposition 8.5. Hence $\mathrm{Aut}_L(E) = \mathrm{Aut}_G(E)$ since $\mathrm{Aut}_G(E)$ has no normal subgroup of index 3. So

$E \notin \mathfrak{E}^{\geq 2}(L\,;2)$ by Proposition 4.6(c), since in all cases, the Sylow 2-subgroups of $\mathrm{Aut}_L(E)$ are neither dihedral nor semidihedral. □

Let $\widetilde{E}_7(q) \leq \widetilde{E}_7(\bar{\mathbb{F}}_q)$ denote the universal groups, with center Z of order 2, and set $E_7(\bar{\mathbb{F}}_q) = \widetilde{E}_7(\bar{\mathbb{F}}_q)/Z$ and $E_7(q) = \widetilde{E}_7(q)/Z$.

Proposition 8.11. *Assume, for some odd prime power q, that $L = E_7(q)$. Then $\mathfrak{R}^{\geq 2}(L\,;2) = \varnothing$, and hence $L \in \mathfrak{L}^{\geq 2}(2)$.*

Proof. Set $\widetilde{G} = \widetilde{E}_7(\bar{\mathbb{F}}_q)$, let $z \in Z(\widetilde{G})$ be the central involution, and set $G = \widetilde{G}/\langle z\rangle = E_7(\bar{\mathbb{F}}_q)$. Fix a Steinberg endomorphism σ of $\widetilde{G}$ such that $C_{\widetilde{G}}(\sigma) = \widetilde{L} \overset{\text{def}}{=} \widetilde{E}_7(q)$. We also let σ denote the induced endomorphism of G. Note, however, that L has index 2 in $C_G(\sigma) = \mathrm{Inndiag}(L)$: the extension of $L \cong E_7(q)$ by its diagonal automorphisms. Let $\widetilde{T} \leq \widetilde{G}$ be a σ-invariant maximal torus such that $\sigma(t) = t^q$ for all $t \in \widetilde{T}$, and set $T = \widetilde{T}/\langle z\rangle$. Set $T_{(2),0} = \widetilde{T}_{(2)}/\langle z\rangle$: the subgroup of elements of $T_{(2)}$ which lift to involutions in $\widetilde{T}$.

By [**Gr2**, Table VI], the group $\widetilde{G}$ has two conjugacy classes of noncentral involutions, denoted **2B** and **2C**, which are exchanged under multiplication by z. Define $\mathfrak{q}\colon \widetilde{T}_{(2)} \longrightarrow \mathbb{F}_2$ by setting $\mathfrak{q}(x) = 1$ if $x = z$ or $x \in \mathbf{2B}$, and $\mathfrak{q}(x) = 0$ if $x = 0$ or $x \in \mathbf{2C}$. Then $\mathfrak{q}$ is a quadratic form by [**Gr2**, Lemma 2.16], and

$$W(E_7) = SO(\widetilde{T}_{(2)}, \mathfrak{q}) \times C_2 \cong SO_7(2) \times C_2.$$

Let $\widetilde{\mathfrak{b}}$ denote the symplectic form on $\widetilde{T}_{(2)}$ associated to $\mathfrak{q}$; i.e., $\widetilde{\mathfrak{b}}(x,y) = \mathfrak{q}(xy) + \mathfrak{q}(x) + \mathfrak{q}(y)$. Since $\mathfrak{q}(xz) = \mathfrak{q}(x) + 1$ for all $x \in \widetilde{T}_{(2)}$, $\widetilde{\mathfrak{b}}(z,x) = 0$ for all x, and hence $\widetilde{\mathfrak{b}}$ factors through a symplectic form $\mathfrak{b}$ on $T_{(2),0} = \widetilde{T}_{(2)}/\langle z\rangle$. As described in [**Gr2**, Definition 2.15]), $\mathfrak{b}$ and $\widetilde{\mathfrak{b}}$ can be defined directly using the natural bilinear form on the Cartan subalgebra of the Lie algebra and the exponential map. Hence $\mathfrak{b}$ extends to a W-invariant form $\mathfrak{b}'\colon T_{(2)} \times \widetilde{T}_{(2)} \longrightarrow \mathbb{F}_2$, which in turn induces a W-equivariant isomorphism

$$\Psi\colon T_{(2)} \xrightarrow{\;\cong\;} \mathrm{Hom}(\widetilde{T}_{(2)}, \mathbb{F}_2).$$

This shows, for example, that there are two W-orbits in $T_{(2)} \smallsetminus T_{(2),0}$. These correspond to the elements $\lambda \in \mathrm{Hom}(\widetilde{T}_{(2)}, \mathbb{F}_2)$ with $\lambda(z) = 1$, and such that the quadratic form $\mathfrak{q}|_{\mathrm{Ker}(\lambda)}$ is nondegenerate of positive or negative type. (By Witt's lemma, two subgroups $F, F' \leq \widetilde{T}_{(2)}$ of index 2 lie in the same W-orbit if $\mathfrak{q}|_F$ and $\mathfrak{q}|_{F'}$ have the same type.) Since G contains two conjugacy classes of elements of order 4 whose square is z (denoted **4A** and **4H** in [**Gr2**, Table VI]), there are two G-conjugacy classes of elements in $T_{(2)} \smallsetminus T_{(2),0}$ which correspond to these two W-orbits.

For any $E = \widetilde{E}/Z \leq G$ such that $\widetilde{E}$ is an elementary abelian 2-group, we define $\mathfrak{q}$ on $\widetilde{E}$ and $\mathfrak{b}$ on $E \times E$ in the same way as described above when $\widetilde{E} = \widetilde{T}_{(2)}$. We claim that any $\alpha \in N_{\mathrm{Aut}(E_7(q))}(E)$ preserves $\mathfrak{b}$. By construction, this means showing that α lifts to some $\widetilde{\alpha} \in \mathrm{Aut}(\widetilde{E}_7(q))$ which sends involutions of each type **2B** and **2C** to themselves. This clearly holds for diagonal automorphisms, since they are conjugation by elements of $\widetilde{G}$. Any field automorphism fixes the involutions in some maximal torus of $\widetilde{G}$, and hence leaves each of **2B** and **2C** invariant. Since $\widetilde{E}_7(q)$ has no graph automorphisms, the claim now follows from Steinberg's theorem (see [**Ca1**, Theorem 12.5.1]).

We next note that no element of $C_G(\sigma)\smallsetminus L$ lifts to an involution in $\widetilde{G}$. If there were such an element, and it lifted to $x \in \widetilde{G}$, then x and xz would be exchanged by σ, and hence σ would exchange the classes **2B** and **2C**. That would imply that the only involution in $\widetilde{L} \cong \widetilde{E}_7(q)$ is the central element, which is clearly not true.

By [**Gr2**, Theorem 9.8], G contains two conjugacy classes of maximal elementary abelian 2-subgroups, both nontoral: M_8 of rank 8 and M_7 of rank 7. These lift to centric subgroups $\widetilde{M_8} \cong Q_8 \times C_2^6$ and $\widetilde{M_7} \cong Q_8 \times C_2^5$ in $\widetilde{G}$. Also, $M_8 = T_{(2)}{\cdot}\langle\theta\rangle$, the extension of 2-torsion in a maximal torus by an involution in $N_G(T)$ which inverts the torus; while $\widetilde{M_7} \le Q_8 \times F_4(\bar{\mathbb{F}}_q) \le \widetilde{G}$.

Fix $E \in \mathfrak{E}^{\ge 2}(L\,;2)$, and let $E_0 \le E$ be the subgroup of elements of E which lift to involutions in $\widetilde{L}$. Then E_0 is a subgroup by the above remarks, and is clearly $N_{\mathrm{Aut}(L)}(E)$-invariant. We now consider the different possibilities.

Case 1: Assume E is G-conjugate to a subgroup $E' \le M_7$. As noted above, $\widetilde{M_7} = Q \times V$, where $Q \cong Q_8$ (and $Z(Q) = \langle z\rangle$), and $V \cong C_2^5$ is a subgroup of $F_4(\bar{\mathbb{F}}_q) \le \widetilde{E}_7(\bar{\mathbb{F}}_q)$. We also identify V with its image in $M_7 \le L$. There is a subgroup $V_2 \le V$ such that the involutions in V_2 lie in one class in $F_4(\bar{\mathbb{F}}_q)$ and those in $V\smallsetminus V_2$ in the other ([**Gr2**, Theorem 7.3]). Also, any index two subgroup of V containing V_2 is toral in $F_4(\bar{\mathbb{F}}_q)$, hence in G, so the function $\mathfrak{q}$ as defined above is quadratic on this subgroup. Since $\widetilde{T}_{(2)}$ contains no rank four isotropic subspace, this is possible only if $V_2{}^{\#} \subseteq$ **2C** and $V\smallsetminus V_2 \subseteq$ **2B** (this also follows from [**Gr2**, Theorem 2.16]). Thus $\mathfrak{b}\colon V \times V \longrightarrow \mathbb{F}_2$ takes the form $\mathfrak{b}(x,y) = 1$ if $\mathrm{rk}(\langle V_2, x, y\rangle) = 4$, and $\mathfrak{b}(x,y) = 0$ otherwise.

Now set $E_1 = \{x \in E_0 \,|\, \mathfrak{b}(x, E_0) = 0\}$; this is again a $N_{\mathrm{Aut}(L)}(E)$-invariant subgroup of E. Let $E_1' \le E_0' \le E'$ be the corresponding subgroups of E' (thus $E_0' = E' \cap V$). The above description of $\mathfrak{b}|_V$ shows that $E_1' = E_0'$ ($\mathfrak{b}|_{E_0'} = 0$) if $\mathrm{rk}(\langle V_2, E_0'\rangle) < 4$), and $E_1' = E_0' \cap V_2$ otherwise. Thus either $E_1 = E_0$ and $\mathrm{rk}(E_0) \le 3$, or $\mathrm{rk}(E_1) \le 2$ and $\mathrm{rk}(E_0/E_1) \le 3$. Since $\mathrm{rk}(E/E_0) \le 2$, $E \notin \mathfrak{E}^{\ge 2}(L\,;2)$ in both cases by Proposition 4.6(b).

Case 2: Assume E is G-conjugate to a subgroup $E' \le M_8 = T_{(2)}{\cdot}\langle\theta\rangle$ as above. In particular, E_0 is a toral subgroup, conjugate to $E_0' = E' \cap T_{(2),0}$; and hence $\mathfrak{b}$ is a symplectic form on E_0 which is invariant under the action of $N_{\mathrm{Aut}(L)}(E)$. By Proposition 4.6(b) again, $\mathrm{rk}(E_0) \ge 4$.

Case 2a: Assume first that $\mathrm{rk}(E_0) = 6$. Thus $E' \cap T \ge T_{(2),0}$. When $E' \nleq T$, then since all elements of the coset θT are T-conjugate to θ, we can assume that $\theta \in E'$.

This leaves four possibilities for E', as described in the next table. To see the information about centralizers and automorphisms, note first that since $W \cong SO_7(2) \times C_2 \cong Sp_6(2) \times C_2$, θT is the only element of $W = N_G(T)/T$ which centralizes $T_{(2),0}$ or $T_{(2)}$. Since this is not a reflection, these subgroups have centralizer $T{\cdot}\langle\theta\rangle$ by Proposition 8.3, and the other two have centralizer $C_{T\cdot\langle\theta\rangle}(\theta) = M_8$. Also,

E'	$C_G(E')$	$\lvert\pi_0(C_G(E'))\rvert$	$\mathrm{Aut}_G(E') \cong$
$T_{(2),0}$	$T{\cdot}\langle\theta\rangle$	2	$Sp_6(2)$
$T_{(2)}$	$T{\cdot}\langle\theta\rangle$	2	$Sp_6(2)$
$T_{(2),0}{\cdot}\langle\theta\rangle$	M_8	2^8	$C_2^6 \rtimes Sp_6(2)$
$M_8 = T_{(2)}{\cdot}\langle\theta\rangle$	M_8	2^8	$C_2^7 \rtimes Sp_6(2)$

$\mathrm{Aut}_G(E') \cong Sp_6(2)$ in the first two cases by the description of the Weyl group again.

When $E' = M_8$, $\mathrm{Aut}_G(E')$ is described in [**Gr2**, Theorem 9.8]; and by the proof of that theorem, it is the group of all automorphisms of M_8 which send $T_{(2)}$ to itself by a W-automorphism. Hence by comparison, when $E' = T_{(2),0}{\cdot}\langle\theta\rangle$, $\mathrm{Aut}_G(E')$ is the group of all automorphisms of E' which send $T_{(2),0}$ to itself while preserving the form $\mathfrak{b}$, and thus $\mathrm{Aut}_G(E') \cong C_2^6 \rtimes Sp_6(2)$.

Thus in all cases, $|\pi_0(C_G(E'))| \leq 4{\cdot}|O_2(\mathrm{Aut}_G(E'))|$. Since L has index 2 in $C_G(\sigma)$, Corollary 8.6 shows that $\mathrm{Aut}_L(E)$ is isomorphic to a subgroup of $Sp_6(2)$ of index ≤ 8. Since $Sp_6(2)$ is simple and $|Sp_6(2)| > 8!$, it contains no proper subgroup of index ≤ 8. So in all cases, we must have $\mathrm{Aut}_L(E) \cong Sp_6(2)$ (with the standard action on E_0). Thus $E \notin \mathfrak{E}^{\geq 2}(L\,;2)$ by Proposition 6.5(a).

Case 2b: Assume $\mathrm{rk}(E_0) = 5$. Set $E_1' = E' \cap T_{(2)}$. If $E_1' \gneqq E_0'$ (so $\mathrm{rk}(E_1') = 6$), then $E_1' \smallsetminus E_0'$ must contain elements of both classes **4A** and **4H**: this follows from the above description of the types in terms of the isomorphism $T_{(2)} \cong \mathrm{Hom}(\widetilde{T}_{(2)}, \mathbb{F}_2)$. On the other hand, since θ inverts the torus (and every element of T is a square), each element of the coset θT (hence of $E' \smallsetminus E_1'$) is conjugate to θ by an element of T. This shows that in all cases, E_1' is an $\mathrm{Aut}_G(E')$-invariant subgroup of E': the subgroup generated by E_0' and those elements of E' not G-conjugate to elements in θT. Let E_1 be the corresponding subgroup of E.

The next table lists the four possibilities for $E_0 \leq E_1 \leq E$. Assuming this

$\mathrm{rk}(E_1)$	$\mathrm{rk}(E)$	$\lvert\pi_0(C_G(E'))\rvert$	$\lvert O_2(\mathrm{Aut}_G(E'))\rvert$	$\mathrm{Aut}_G(E')/O_2(\mathrm{Aut}_G(E'))$
5	5	$\leq 2^2$	$\geq 2^4$	$Sp_4(2)$
6	6	$\leq 2^2$	$\geq 2^4$	$Sp_4(2)$
5	6	$\leq 2^9$	$\geq 2^9$	$Sp_4(2)$
6	7	$\leq 2^9$	$\geq 2^{10}$	$Sp_4(2)$

data is correct, then by Corollary 8.6, E is not pivotal in the first, second, and fourth cases, and $\mathrm{Aut}_L(E) \cong Sp_4(2) \cong \Sigma_6$ in the third case. So $E \notin \mathfrak{E}^{\geq 2}(L\,;2)$ by Proposition 4.6(e).

It remains to check the data in the table. By Proposition 8.3, $\mathrm{Aut}_G(E_i') = \mathrm{Aut}_W(E_i')$ $(i = 0,1)$; and $\mathrm{Aut}_W(E_0') = \mathrm{Aut}(E_0', \mathfrak{b})$ is the group of all symplectic automorphisms. If $\mathrm{rk}(E_1') = 6$, then there is exactly one other subgroup $E_1'' \le T_{(2)}$ of rank 6 containing E_0' and not equal to $T_{(2),0}$. Under the isomorphism Ψ between $T_{(2)}$ and $\mathrm{Hom}(\widetilde{T}_{(2)}, \mathbb{F}_2)$, E_1' and E_1'' are sent to the groups of maps whose kernels contain x or xz, respectively, for some fixed element $x \in \widetilde{T}_{(2)} \smallsetminus \langle z \rangle$. Since x and xz are not G-conjugate, no element of W sends E_1' to E_1''; and thus $N_W(E_0') = N_W(E_1')$.

Thus $\mathrm{Aut}_G(E_0) = \mathrm{Aut}(E_0, \mathfrak{b}) \cong C_2^4 \rtimes Sp_4(2)$, and each such automorphism extends to at least one automorphism in $\mathrm{Aut}_G(E_1)$. Furthermore, since $\mathrm{Aut}_G(M_8)$ is the group of all $\alpha \in \mathrm{Aut}(M_8)$ such that $\alpha|_{T_{(2)}} \in \mathrm{Aut}_G(T_{(2)})$ [**Gr2**, Theorem 9.8], $\mathrm{Aut}_G(E)$ is the group of all automorphisms of E whose restriction to E_1 lies in $\mathrm{Aut}_G(E_1)$. This finishes the description of $\mathrm{Aut}_G(E)$ in the above table.

In all cases, $C_W(E_1')$ has order ≤ 4: it is generated by θ, and possibly the unique symplectic transvection of $T_{(2),0}$ which fixes E_0' (if it also fixes E_1'). Thus by Proposition 8.3, $|\pi_0(C_G(E_1))| \le 4$. When $E \gneqq E_1$, the upper bound for $|\pi_0(C_G(E))|$ then follows from Proposition 8.8.

Case 2c: Assume $\mathrm{rk}(E_0) = 4$. If $\mathfrak{b}|_{E_0}$ is degenerate, then $E_0 \cap E_0^\perp$ is a proper subgroup of E_0 which is $N_{\mathrm{Aut}(L)}(E)$-invariant, so $E \notin \mathfrak{E}^{\ge 2}(L;2)$ by Proposition 4.6(b). So we assume $\mathfrak{b}$ is nondegenerate on E_0, and hence on $E_0' \le T_{(2),0}$. Then $T_{(2),0} = E_0' \times E_0'^\perp$, $E_0'^\perp$ is $C_W(E_0')$-invariant, and hence there is $1 \ne x \in E_0'^\perp$ whose $C_W(E_0')$-orbit has odd order. By construction, x is not G-conjugate to any element of $E \smallsetminus E_0$. Hence by Proposition 8.9 (applied with $E = E_0'$ and $\overline{E} = E'$), no subgroup of L which is G-conjugate to E' is pivotal in L. In particular, $E \notin \mathfrak{E}^{\ge 2}(L;2)$ in this case. □

It remains only to consider the groups $E_8(q)$.

PROPOSITION 8.12. *Assume, for some odd prime power q, that $L = E_8(q)$. Then $\mathfrak{R}^{\ge 2}(L;2) = \varnothing$, and hence $L \in \mathfrak{L}^{\ge 2}(2)$.*

PROOF. Set $G = E_8(\bar{\mathbb{F}}_q)$. Fix a Steinberg endomorphism σ of G such that $C_G(\sigma) = L$. Let $T \le G$ be a σ-invariant maximal torus such that $\sigma(t) = t^q$ for all $t \in T$.

By [**Gr2**, Table VI], G has two conjugacy classes of involutions, denoted **2A** and **2B**. For any elementary abelian 2-subgroup $E \le G$, we define $\mathfrak{q}\colon E \longrightarrow \mathbb{F}_2$ by setting $\mathfrak{q}(x) = 1$ if $x \in$ **2A** and $\mathfrak{q}(x) = 0$ otherwise. By [**Gr2**, Lemma 2.16], $\mathfrak{q}$ is a quadratic form on $T_{(2)}$. Also, if $\theta \in N_G(T)$ is an involution which inverts T, then

$$W/\langle \theta T \rangle = \mathrm{Aut}_G(T)/\{\pm \mathrm{Id}\} \cong O(T_{(2)}, \mathfrak{q}) \cong SO_8^+(2).$$

Now let E be any elementary abelian subgroup of G. By [**Gr2**, Theorem 2.16], E is toral if and only if $\mathfrak{q}$ is quadratic on E and E is not **2B**-pure of rank 5. Let $\mathfrak{b}\colon E \times E \longrightarrow \mathbb{F}_2$ be the form $\mathfrak{b}(x,y) = \mathfrak{q}(x) + \mathfrak{q}(y) + \mathfrak{q}(xy)$. Using these functions, we define the following subgroups:

$$E_0 = \langle E \cap \mathbf{2A} \rangle = \langle x \in E \,|\, \mathfrak{q}(x) = 1 \rangle$$
$$E_1 = \{x \in E_0 \,|\, \mathfrak{b}(x, E_0) = 0\} = \{x \in E_0 \,|\, \mathfrak{q}(y) + \mathfrak{q}(yx) = \mathfrak{q}(x), \text{ all } y \in E_0\}$$
$$E_2 = E_1 \cap \mathfrak{q}^{-1}(0) = \{x \in E_0 \cap \mathbf{2B} \,|\, \mathfrak{q}(y) = \mathfrak{q}(xy), \text{ all } y \in E_0\} \cup \{1\}.$$

Note that E_1 and E_2 are subgroups: this is a formal consequence of the definition, and does not use any properties of $\mathfrak{q}$. For example, if $x, y \in E_1$, then for all $z \in E_0$,

$$\mathfrak{q}(xy) + \mathfrak{q}(z) + \mathfrak{q}(xyz) = \mathfrak{q}(xy) + (\mathfrak{q}(y) + \mathfrak{q}(yz)) + (\mathfrak{q}(x) + \mathfrak{q}(yz)) = 0;$$

so $xy \in E_1$. Also, $\mathfrak{q}|_{E_1}$ is linear, with kernel E_2.

By [**Gr2**, Theorem 2.17], G contains two conjugacy classes of maximal elementary abelian 2-subgroups, represented by $M_9 = T_{(2)} \times \langle \theta \rangle$ of rank 9 (where θ again inverts T), and M_8 of rank 8. All elements in $M_9 \smallsetminus T_{(2)}$ are of type **2B**.

Assume $E \in \mathfrak{E}^{\geq 2}(L\,; 2)$, with subgroups $E_2 \leq E_1 \leq E_0 \leq E$ defined as above. By Proposition 4.6(b),

$$\text{at least one } N_{\mathrm{Aut}(L)}(E)\text{-irreducible subquotient of } E \text{ has rank} \geq 4. \qquad (1)$$

We first check the following properties of E, which mostly follow from (1) and the properties of M_8 and M_9 stated in [**Gr2**].

(2) Assume $E_0 \neq 1$, and E is G-conjugate to a subgroup of M_8. Then $\mathrm{rk}(E_1) = \mathrm{rk}(E_2) \leq 2$, and there are subgroups $V', W' \leq E$ such that $E_2 = V' \cap W'$, $E \cap \mathbf{2A} = (V' \cup W') \smallsetminus E_2$, $E_0 = V'W'$, $\mathrm{rk}(V'/E_2) = \mathrm{rk}(W'/E_2) \in \{2, 3\}$, and $\mathrm{rk}(E/E_0) \leq 3 - \mathrm{rk}(V'/E_2)$.

To see this, fix some $M \geq E$ which is G-conjugate to M_8. By [**Gr2**, Theorem 2.17], there are subgroups $V, W \leq M$ of rank five, with intersection $V \cap W = X$ of rank two, such that $M \cap \mathbf{2A} = (V \cup W) \smallsetminus X$. Set $V' = V \cap E$, $W' = W \cap E$, and $X' = X \cap E = V' \cap W'$.

Clearly, $X' \leq E_2$; we claim they are equal. Assume otherwise, and fix some $g \in E_2 \smallsetminus X'$. Thus $g \in \mathbf{2B}$, so $g \in E \smallsetminus (V' \cup W')$, and for all $a \in E \smallsetminus \langle g \rangle$, a and ga have the same type. Then for all $a \in E \cap \mathbf{2A}$, $ga \in W' \smallsetminus X'$ if $a \in V' \smallsetminus X'$ and vice versa. In particular, if $a, a' \in V' \smallsetminus X'$, and if $g' \in E_2 \smallsetminus X'$ is another element, then $a^{-1}a' = (ag)^{-1}(a'g) \in V' \cap W' = X'$, $g^{-1}g' = (ag)^{-1}(ag') \in W'$, and $g^{-1}g' \in X'$ since it has type **2B** by assumption ($g \in E_2$). So $\mathrm{rk}(V'/X') = 1$ (recall E has at least one involution of type **2A**); and $E_2 = \langle X', g \rangle$. Then $\mathrm{rk}(E_2) = 1 + \mathrm{rk}(X') \leq 3$, and

$$\begin{aligned} \mathrm{rk}(E/E_2) = \mathrm{rk}(E/X') - 1 &= \mathrm{rk}(E/V') + \mathrm{rk}(V'/X') - 1 \\ &= \mathrm{rk}(E/V') \leq \mathrm{rk}(M/V) = 3. \end{aligned}$$

But this contradicts (1).

Thus $E_2 = X'$, and so $E_0 = V'W'$: the subgroup generated by elements of type **2A**. Since $X' = E_2$ is $N_{\mathrm{Aut}(L)}(E)$-invariant, so is the subset $V' \cup W' = X' \cup (E \cap \mathbf{2A})$. Furthermore, either V' and W' are each themselves $N_{\mathrm{Aut}(L)}(E)$-invariant, or they have the same rank and there is some element of $N_{\mathrm{Aut}(L)}(E)$ which exchanges them. The former case is impossible by (1), since the successive quotients in the series $1 \leq E_2 \leq V' \leq E$ all have rank ≤ 3. Thus $\mathrm{rk}(V') = \mathrm{rk}(W')$, so $\mathrm{rk}(V'W'/X')$ is even. Since $\mathrm{rk}(X') \leq 2$ and

$\mathrm{rk}(E/V'W') \le 3$, we must have $\mathrm{rk}(V'W'/X') = 4$ or 6. Also, this shows that $\mathrm{rk}(E/E_0) = \mathrm{rk}(E/V'W') = \mathrm{rk}(E/W') - \mathrm{rk}(V'/X') \le 3 - \mathrm{rk}(V'/E_2)$.

Finally, since V'/E_2 and W'/E_2 both have rank ≥ 2, for any $x \in E \cap \mathbf{2A} = (V' \cup W') \smallsetminus E_2$, there is some y such that $y, xy \in E \cap \mathbf{2A}$. This proves that $E_1 = E_2$, and finishes the proof of (2).

(3) Assume $E_0 \ne 1$ and E is G-conjugate to a subgroup of M_8, and let $V', W' \le E$ be as in (2). Then $\mathrm{Aut}_G(E)$ is the group of all $\alpha \in \mathrm{Aut}(E)$ such that $\alpha(E_2) = E_2$, and α either sends V' and W' to themselves or switches them.

By [**Gr2**, Theorem 2.17], $\mathrm{Aut}_G(M)$ is the group of all automorphisms of M which send X to itself, and which either send V and W to themselves or switch them. By restriction, we see that $\mathrm{Aut}_G(E_0)$ contains all automorphisms which induce the identity on X' and on E_0/X'. If $E = E_0$, then we are done.

If $\mathrm{rk}(E/E_0) = 1$, then $\mathrm{rk}(V'/X') = 2$, so $\mathrm{rk}(E_0/X') = 4$, and $\mathrm{rk}(E/E_0) = 1$. Choose elements $v_3 \in V$ and $w_3 \in W$ such that $v_3w_3 \in E \smallsetminus E_0$. Then $M = \langle E, v_3\rangle = \langle E, w_3\rangle$. For any $\alpha \in \mathrm{Aut}(E)$, we can write $\alpha(v_3w_3) = v_3w_3v'w'$ for some $v' \in V'$ and $w' \in W'$. Extend α to $\bar\alpha \in \mathrm{Aut}(M)$ by setting $\alpha(w_3) = w_3w'$ and $\alpha(v_3) = v_3v'$ if $\alpha(V') = V'$, or $\alpha(w_3) = w_3v'$ and $\alpha(v_3) = v_3w'$ if $\alpha(V') = W'$. Then $\bar\alpha \in \mathrm{Aut}_G(M)$ by [**Gr2**], and so $\alpha \in \mathrm{Aut}_G(E)$.

(4) Assume $E_0 \ne 1$; and that there is $g \in G$ such that $E' \overset{\text{def}}{=} gEg^{-1} \le M_9 = T_{(2)}\cdot\langle\theta\rangle$. Set $E_i' = gE_ig^{-1}$. Then $E_0' = E' \cap T_{(2)}$. Also, $\mathrm{Aut}_G(E)$ is the group of all $\alpha \in \mathrm{Aut}(E)$ such that $\alpha(E_0) = E_0$, and $\alpha|_{E_0}$ preserves the form $\mathfrak{q}$.

By [**Gr2**, Theorem 2.17], $M_9 \smallsetminus T_{(2)} \subseteq \mathbf{2B}$. Hence $E_0' = E' \cap T_{(2)}$. By [**Gr2**, Table I], $\mathrm{Aut}_G(M_9)$ is the group of all $\alpha \in \mathrm{Aut}(M_9)$ such that $\alpha|_{T_{(2)}}$ preserves the form $\mathfrak{q}$. If $\beta \in \mathrm{Aut}(E')$ is such that $\beta(E_0') = E_0'$ and $\beta|_{E_0'}$ preserves $\mathfrak{q}$, then $\beta|_{E_0'}$ extends to an isometry of $T_{(2)}$ by Witt's lemma, so β extends to an element of $\mathrm{Aut}_G(M_9)$, and thus $\beta \in \mathrm{Aut}_G(E')$.

(5) In all cases, if $E_0 \ne 1$, then $\mathrm{rk}(E_1) \le 2$ and $\mathrm{rk}(E/E_0) \le 1$; so $\mathrm{rk}(E_0/E_1) \in \{4, 6, 8\}$.

If E is G-conjugate to a subgroup of M_9, then $\mathfrak{q}|_{E_0}$ is quadratic, and hence $\mathrm{rk}(E_0/E_1) + 2\cdot\mathrm{rk}(E_1) \le 8$. Since one of $\mathrm{rk}(E_2)$, $\mathrm{rk}(E_1/E_2)$, or $\mathrm{rk}(E_0/E_1)$ must be ≥ 4 by (1), we either have $\mathrm{rk}(E_0/E_1) \ge 4$ (and is even) and $\mathrm{rk}(E_1) \le 2$, or $E_0 = E_2$ has rank 4. But in the last case, E_0 is **2B**-pure, which contradicts the definition of E_0 as being generated by elements of type **2A**. This proves (5) in this case, and it follows from (2) when E is conjugate to a subgroup of M_8.

(6) In all cases, if $E_0 \ne 1$, then

$$|O_2(\mathrm{Aut}_G(E))| = 2^N \quad \text{where} \quad N \ge \mathrm{rk}(E_2)\cdot\mathrm{rk}(E_0/E_2) + \mathrm{rk}(E_0)\cdot\mathrm{rk}(E/E_0).$$

By (3) or (4), $\mathrm{Aut}_G(E)$ contains all automorphisms of E which induce the identity on E_2, E_0/E_2, and E/E_0. Since these subgroups are all $\mathrm{Aut}_G(E)$-invariant, all such automorphisms lie in $O_2(\mathrm{Aut}_G(E))$; and the inequalities follow.

We are now ready to consider the individual cases. In Cases 1 and 2, we assume $E_0 \neq 1$: assuming E_0 is toral in Case 1 and nontoral in Case 2. The **2B**-pure subgroups are then handled in Cases 3 and 4: $\mathrm{rk}(E) = 4$ in Case 3 and $\mathrm{rk}(E) \geq 5$ in Case 4.

Case 1: Assume $E_0 = \langle E \cap \mathbf{2A} \rangle \neq 1$, and that E_0 is G-conjugate to a subgroup of $T_{(2)}$. Thus $\mathfrak{q}|_{E_0}$ is quadratic, $E_1 = E_0 \cap E_0^{\perp}$, and $E_2 = \mathrm{Ker}(\mathfrak{q}|_{E_1})$. By (5), $\mathrm{rk}(E_1) \leq 2$, and $\mathrm{rk}(E_0/E_1) \geq 4$.

Fix $g \in G$ such that $gE_0g^{-1} \leq T_{(2)}$, and set $E' = gEg^{-1}$ and $E'_i = gE_ig^{-1}$.

We claim that E_0 is a maximal toral subgroup of E. Assume otherwise: let $E_0 \lneqq \widehat{E} \leq E$ be such that $\mathfrak{q}|_{\widehat{E}}$ is quadratic. Let $\mathfrak{b}$ be the bilinear form associated to $\mathfrak{q}$. For any $x \in \widehat{E} \smallsetminus E_0$, $\mathfrak{b}(x, y) = \mathfrak{q}(y)$ for all $y \in E_0$, and so $\mathfrak{q}|_{E_0}$ is linear since $\mathfrak{b}$ is bilinear. But this would imply $E_1 = E_0$, which contradicts (5).

Case 1a: Assume $\mathfrak{q}$ is nondegenerate and $\mathrm{rk}(E_0) \leq 6$. If the symplectic form $\mathfrak{b}$ is nondegenerate, then $C_W(E'_0)$ leaves $E_0'^{\perp}$ invariant, and we can choose $1 \neq x \in E_0'^{\perp}$ whose $C_W(E'_0)$-orbit has odd order. Since E_0 is a maximal toral subgroup of E, x is not $C_G(E'_0)$-conjugate to any element of E'. Hence by Proposition 8.9, no subgroup of L which is G-conjugate to E' can be pivotal; and in particular, $E \notin \mathfrak{E}^{\geq 2}(L\,; 2)$.

It remains to consider the case $\mathrm{rk}(E_0) = 5$ and $\mathrm{rk}(E_1) = 1$. Write $E'_0 = E'_1 \times F$, where $\mathfrak{b}$ is nondegenerate on F, $\mathrm{rk}(F) = 4$, and $E'_1 = \langle a \rangle$ with $\mathfrak{q}(a) = 1$. Any nondegenerate quadratic form on $\mathbb{F}_2^4$ has an even number of nonisotropic elements (see Case 1b for more details on the two types of forms). Thus there are an odd number of nonisotropic elements in $F^{\perp} \smallsetminus \{a\}$, none of which lie in E'_0, and they are permuted by $C_W(E'_0)$. So we can choose $x \in F^{\perp} \smallsetminus \{a\}$ whose $C_W(E'_0)$-orbit has odd order; and Proposition 8.9 again applies to show that $E \notin \mathfrak{E}^{\geq 2}(L\,; 2)$.

Case 1b: Assume $\mathrm{rk}(E_0) = 8$; i.e., E is G-conjugate to $T_{(2)}$ or M_9. Then $C_G(T_{(2)}) = T{\cdot}\langle \theta \rangle$ and $C_G(M_9) = M_9$; while $\mathrm{Aut}_G(T_{(2)}) \cong SO_8^+(2)$ and $\mathrm{Aut}_G(M_9) \cong C_2^8 \rtimes SO_8^+(2)$. Thus in both cases, Corollary 8.6 applies to show that $\mathrm{Aut}_L(E)$ is isomorphic to a subgroup of index at most two in $SO_8^+(2)$; i.e., $\Omega_8^+(2)$ or $SO_8^+(2)$. In either case, $E \notin \mathfrak{E}^{\geq 2}(L\,; 2)$ by Proposition 6.5(b,c).

Case 1c: The remaining cases are all included in the next table.

Here, the last two rows represent the cases where $E \gneqq E_0$. Also, $C_G(E_0)^0_s$ means the maximal normal semisimple subgroup of $C_G(E_0)^0$; when the group is put in brackets we just give its universal cover without trying to identify the group itself. Assuming the table is correct, $|O_2(\mathrm{Aut}_G(E))| > |\pi_0(C_G(E))|$ in all cases except the last. So by Corollary 8.6, none of those subgroups can be pivotal. In the last case, $\mathrm{Aut}_G(E_0) \cong GO_7(2) \cong Sp_6(2)$, so Corollary 8.6 implies that $\mathrm{Aut}_L(E)$ has index at most 4 in $Sp_6(2)$, hence is equal to $Sp_6(2)$ (with the canonical action on E_0/E_1), and this is impossible by Proposition 6.5.

The lower bounds on $|O_2(\mathrm{Aut}_G(E))|$ follow from (6) in all cases. The upper bound on $|\pi_0(C_G(E))|$ when $\mathrm{rk}(E/E_0) = 1$ follows from the information given on $C_G(E_0)$, together with Proposition 8.8. To check the claims about $C_G(E_0)$, note first that there is one orthogonal transvection τ_x in $T_{(2)}$ for each $x \in T_{(2)}$ of type **2A**: $\tau_x(y) = y$ for $y \in x^{\perp}$ and $\tau_x(y) = xy$ otherwise (see [**Di**, §19]). Hence these are the reflections in W, and the subgroups $x^{\perp}$ are the kernels of the

$\mathrm{rk}(E_2)$	1	1	2	1	1	0
$\mathrm{rk}(E_1)$	1	1	2	2	1	1
$\mathrm{rk}(E_0/E_1)$	4	4	4	4	6	6
$\mathrm{type}(E_0/E_1)$	$+$	$-$	$+$	$\pm$	$+$	$\pm$
$C_W(E_0)$	$D_8 \times \langle\theta\rangle$	$\Sigma_4 \times \langle\theta\rangle$	$C_2 \times \langle\theta\rangle$	$C_2^2 \times \langle\theta\rangle$	$\langle\theta\rangle$	$C_2 \times \langle\theta\rangle$
$C_G(E_0)_s^0$	$SL_2 \times SL_2$	$[SL_4]$	1	$[SL_2]$	1	$[SL_2]$
$\lvert\pi_0(C_G(E_0))\rvert$	2^2	2	2^2	2^2	2	2
$\lvert O_2(\mathrm{Aut}_G(E_0))\rvert$	$\geq 2^4$	$\geq 2^4$	$\geq 2^8$	$\geq 2^5$	$\geq 2^6$	1
$\lvert\pi_0(C_G(E))\rvert$	$\leq 2^8$	$\leq 2^7$	$\leq 2^{10}$	$\leq 2^{10}$	$\leq 2^9$	$\leq 2^9$
$\lvert O_2(\mathrm{Aut}_G(E))\rvert$	$\geq 2^9$	$\geq 2^9$	$\geq 2^{14}$	$\geq 2^{11}$	$\geq 2^{13}$	2^7

roots. So by Proposition 8.3, the roots in $C_G(E_0)^0$ correspond precisely to the nonisotropic elements in the orthogonal complement of $E_0' \leq T_{(2)}$. In particular, when $\mathrm{rk}(E_0) = 7$ and $E_1 = \langle x\rangle$, this shows that $C_G(E_0) = T{\cdot}\langle\theta\rangle$ if x is isotropic, and $C_G(E_0) = H{\cdot}\langle\theta\rangle$ where $H_s \cong SL_2(\bar{\mathbb{F}}_q)$ if $\mathfrak{q}(x) = 1$.

Assume $E_0' = E_1' \times F$, where $F \leq T_{(2)}$ is orthogonal to E_1' and is nondegenerate of rank 4. Then $C_W(F) \cong \mathrm{Aut}_G(F^\perp)$, and $F^\perp$ has the same type of form (positive or negative) as F. If this form is positive, then there is a splitting $F^\perp = F_1 \times F_2$ such that $F^\perp \cap \mathbf{2A} = F_1^\# \cup F_2^\#$; and $C_{W/\langle\theta\rangle}(F) \cong GO_4^+(2) \cong \Sigma_3 \wr C_2$. This shows that $C_G(F)_s^0$ has type $SL_3(\bar{\mathbb{F}}_q) \times SL_3(\bar{\mathbb{F}}_q)$ up to covering. Also, $\pi_0(C_G(F)) \cong C_2^2$, generated by θ and an element which switches the two factors $SL_3(\bar{\mathbb{F}}_q)$. The information about $C_G(E_0)$ in the first, third, and fourth cases can now be computed directly, by checking roots and Weyl group elements left invariant by one or two involutions in this group. Note in particular the first case: since $C_G(F) \cong ((\bar{\mathbb{F}}_q^\times)^4 \times (SL_3(\bar{\mathbb{F}}_q))^2)/Z$ for some 3-subgroup Z, $C_G(E_0)^0$ is the centralizer of an involution which embeds diagonally in $SL_3(\bar{\mathbb{F}}_q)^2$, and hence has semisimple part $SL_2(\bar{\mathbb{F}}_q)^2$ (independently of Z).

If $\mathfrak{q}|_F$ and $\mathfrak{q}|_{F^\perp}$ have negative type, then $F^\perp$ contains 5 isotropic involutions which generate $F^\perp$ and whose product is trivial. Thus $C_{W/\langle\theta\rangle}(F) \cong \Sigma_5$ in this case, $C_G(F)_s^0 \cong SL_5(\bar{\mathbb{F}}_q)/Z$ for some Z, and $C_G(F)$ is connected. The information in the second case now follows upon taking the centralizer in $C_G(F)$ of one of the five isotropic involutions in $F^\perp$.

This finishes the proof of the information in the above table.

Case 2: Now assume that $E_0 \neq 1$, and that E_0 is not toral. By (4), E is not G-conjugate to a subgroup of M_9, and hence is G-conjugate to a subgroup of M_8.

Let $V', W' \le E_0$ be as in (2) and (3). If $\mathrm{rk}(V'/E_2) = \mathrm{rk}(W'/E_2) = 2$, then $\mathfrak{q}$ is a quadratic form on E_0: it sends all elements in $(V' \smallsetminus E_2) \cup (W' \smallsetminus E_2)$ to 1 and all others to 0. Hence E_0 is toral in this case by [**Gr2**, Theorem 9.2], in contradiction to our assumption. So without loss of generality, we can assume that $\mathrm{rk}(V'/E_2) = \mathrm{rk}(W'/E_2) = 3$. Then $E = E_0$ by (2).

Fix subgroups $\overline{V} \le V'$ and $\overline{W} \le W'$ of index 2 containing E_2, and set $\overline{E} = \overline{V}\,\overline{W} \le E$. Thus $[E{:}\overline{E}] = 4$. Also, $\mathfrak{q}|_{\overline{E}}$ is quadratic, and hence a toral subgroup by [**Gr2**, Theorem 9.2]. For any $\overline{E} \lneqq E' \lneqq E$, there are subgroups $F_0 \le F \le E'$ with $\mathrm{rk}(F) = 3$ and $\mathrm{rk}(F_0) \ge 2$, such that $F \cap \mathbf{2A} = (F_0)^{\#}$. Then $\mathfrak{q}|_F$ is not quadratic, and thus $E' \ge F$ are not toral. This shows that $\overline{E}$ is a maximal toral subgroup of E.

Case 2a: If $\mathrm{rk}(E_2) = 2$, then $E = M$; and so $C_G(E) = E$ and $|O_2(\mathrm{Aut}_G(E))| \ge 2^{12}$ by (6). Thus $|O_2(\mathrm{Aut}_G(E))| > |\pi_0(C_G(E))|$, so $E \notin \mathfrak{E}^{\ge 2}(L\,;2)$ by Corollary 8.6.

Case 2b: If $\mathrm{rk}(E_2) = 0$, then $E = V' \times W'$ where the elements of type $\mathbf{2A}$ are precisely those in $(V' \cup W') \smallsetminus 1$. Let $\overline{E} \le E$ be a maximal toral subgroup as constructed above. Thus $\mathrm{rk}(\overline{E}) = 4$, and $\overline{E} \cap V'$ and $\overline{E} \cap W'$ each has rank 2. Fix $\overline{E}' = g\overline{E}g^{-1} \le T_{(2)}$ for some $g \in G$, and set $E' = gEg^{-1}$. Then $C_W(\overline{E}')$ leaves $\overline{E}'^{\perp}$ invariant, and we can choose $1 \ne x \in \overline{E}'^{\perp}$ whose $C_W(\overline{E}')$-orbit has odd order. Since $\overline{E}'$ is a maximal toral subgroup of E', x is not $C_G(\overline{E}')$-conjugate to any element of E'. Hence by Proposition 8.9, no subgroup of L which is G-conjugate to E' can be pivotal, and in particular $E \notin \mathfrak{E}^{\ge 2}(L\,;2)$.

Case 2c: Now assume $\mathrm{rk}(E_2) = 1$. Then $|O_2(\mathrm{Aut}_G(E))| \ge 2^6$ by (6).

Fix $\overline{E} = \overline{V}\,\overline{W} \le E$ of index 4, as constructed above. Thus $\overline{E}$ is a maximal toral subgroup of E. Fix $g \in G$ such that $\overline{E}' \overset{\mathrm{def}}{=} g\overline{E}g^{-1} \le T_{(2)}$, and set $E' = gEg^{-1}$ and $E_i' = gE_ig^{-1}$.

Set $\Gamma = C_G(\overline{E})^0$ for short, and let $H = \Gamma_s$ be its maximal normal semisimple subgroup. In the proof of Case 1c, we saw that $H \cong SL_2(\bar{\mathbb{F}}_q)^2$, and that $C_G(\overline{E}) = \Gamma{\cdot}\langle a, b\rangle$, where a exchanges the two simple factors in H and b inverts a maximal torus in Γ. Since $\overline{E}$ is a maximal toral subgroup of E, $E \cap \Gamma = \overline{E}$, and so we can choose $a, b \in E$. Set $U = \langle a, b\rangle$. Thus $C_G(E) = C_\Gamma(U) \times U$; and there is an exact sequence

$$1 \longrightarrow C_H(U) \longrightarrow C_\Gamma(U) \longrightarrow C_{\Gamma/H}(U).$$

Then $C_H(U) = C_{SL_2(\bar{\mathbb{F}}_q)}(b)$ is connected by Steinberg's theorem (see Proposition 8.8), and the last term has order $\le 2^6$ since $\Gamma/H \cong (\bar{\mathbb{F}}_q^\times)^6$ is inverted by b. It follows that $|\pi_0(C_G(E))| \le 2^8$.

By Corollary 8.6, $\mathrm{Aut}_L(E)$ is isomorphic to a subgroup of index at most 4 in $\mathrm{Aut}_G(E)/O_2(\mathrm{Aut}_G(E))$. By (3) again, $\mathrm{Aut}_G(E)$ is the group of all automorphisms which either send V' and W' to themselves or switch them. Hence

$$\mathrm{Aut}_G(E)/O_2(\mathrm{Aut}_G(E)) \cong GL_3(2) \wr C_2.$$

Since $L_3(2)$ has no subgroups of index ≤ 4, this means that $\mathrm{Aut}_L(E)$ is isomorphic to $L_3(2) \wr C_2$ or $L_3(2) \times L_3(2)$. Since neither of these has dihedral or semidihedral Sylow 2-subgroup, $E \notin \mathfrak{E}^{\geq 2}(L\,; 2)$ by Proposition 4.6(c).

Case 3: Assume E is type **2B** pure of rank 4. Any such subgroup is toral [**Gr2**, Theorem 9.2], and thus G-conjugate to a maximal isotropic subgroup $E' \leq T_{(2)}$.

As noted in the proof of Case 1c, the kernels (in $T_{(2)}$) of the roots in E_8 are the orthogonal complements of involutions of type **2A**. None of these can contain E', and hence $C_G(E')^0 = T$ by Proposition 8.3. Also, $C_G(E')/T = C_W(T)$, and so $C_G(E')/\langle T, \theta\rangle$ is the group of all $\alpha \in \mathrm{Aut}(T_{(2)}, \mathfrak{q})$ such that $\alpha|_{E'} = \mathrm{Id}$.

To make this more explicit, we fix a basis $\{v_1, \ldots, v_8\}$ of $T_{(2)}$ such that $E' = \langle v_1, \ldots, v_4\rangle$, and such that $\mathfrak{q}(\sum_{i=1}^{8} r_i v_i) = \sum_{i=1}^{4} r_i r_{4+i}$. Let $\mathcal{S}$ be the additive group of symplectic 4×4 matrices; i.e., symmetric matrices with zeroes on the diagonal. By direct computation, the group of orthogonal automorphisms which leave E' invariant is the semidirect product

$$\bigl\{\left(\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right) \bigm| X \in \mathcal{S}\bigr\} \rtimes \bigl\{\left(\begin{smallmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{smallmatrix}\right) \bigm| A \in GL_4(2)\bigr\}.$$

Thus $C_G(E')/\langle T, \theta\rangle \cong \mathcal{S} \cong C_2^6$, and $\mathrm{Aut}_G(E') \cong GL_4(2)$. Furthermore, this shows that the conjugation action of $\mathrm{Aut}_G(E')$ on $C_G(E')/\langle T, \theta\rangle$ is that given by $(A, X) \mapsto AXA^t$, for $A \in GL_4(2)$ and $X \in \mathcal{S}$.

The stabilizer of $\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right) \in \mathcal{S}$ under this action of $GL_4(2)$ is just $Sp_4(2) \cong \Sigma_6$. If we set $Y = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in M_2(2)$, then the stabilizer of $\left(\begin{smallmatrix} Y & 0 \\ 0 & 0 \end{smallmatrix}\right) \in \mathcal{S}$ is the group of invertible matrices of the form $\left(\begin{smallmatrix} A & R \\ 0 & B \end{smallmatrix}\right)$, thus isomorphic to $C_2^4 \rtimes (\Sigma_3)^2$. Since these subgroups have index 28 and 35, respectively, in $GL_4(2)$, their $\mathrm{Aut}_G(E')$-orbits in $\mathcal{S}$ contain all 63 nonidentity elements.

Since each coset of T in $N_G(T)$ contains elements of L, the action of $\mathrm{Aut}_G(E')$ on the conjugacy classes of $\pi_0(C_G(E'))$ (as defined in Proposition 8.5) is just the conjugation action on the set of its conjugacy classes. Since we have identified $\pi_0(C_G(E'))/\langle\theta T\rangle$ with $\mathcal{S}$, this shows that the point stabilizers of the $\mathrm{Aut}_G(E')$-action on the set of conjugacy classes in $\pi_0(C_G(E'))$ all are isomorphic to subgroups of index at most two in one of the groups $GL_4(2)$, Σ_6, or $C_2^4 \rtimes (\Sigma_3)^2$; and thus that $\mathrm{Aut}_L(E)$ must be isomorphic to some such group. By Proposition 4.6(c), $\mathrm{Aut}_L(E)$ must have dihedral or semidihedral Sylow 2-subgroups, and this leaves only the possibility $\mathrm{Aut}_L(E) \cong A_6$ — which would contradict Proposition 4.6(e).

Case 4: Now assume that E is a **2B**-pure subgroup of rank ≥ 5. Cohen and Griess show in [**CG**] that any such subgroup of $E_8(\mathbb{C})$ has rank equal to 5 and has finite centralizer, and their argument also holds in $G = E_8(\bar{\mathbb{F}}_q)$. Namely, if $\chi_{\mathfrak{g}}$ denotes the character of the adjoint representation of G, then $\chi_{\mathfrak{g}}(x) = -8$ for all $x \in$ **2B** (see [**CG**, Table 4]) and $\chi_{\mathfrak{g}}(1) = \dim(\mathfrak{g}) = 248$. Since $\dim(C_G(E)) = \frac{1}{|E|} \cdot \sum_{x \in E} \chi_{\mathfrak{g}}(x) \geq 0$, we get $\mathrm{rk}(E) = 5$ and $\dim(C_G(E)) = 0$.

Fix any $E_4 \leq E$ of index 2. Since E_4 is toral, as noted above, we can choose $g \in G$, and set $E' = gEg^{-1}$ and $E_4' = gE_4g^{-1}$, such that $E_4' \leq T_{(2)}$. Then $C_G(E_4')^0 = T$ (see Case 3), so any $x \in E' \smallsetminus E_4'$ lies in $N_G(T)$, and $x \in \theta T$ since $C_T(x) \leq C_G(E')$ is finite. Since all **2B**-pure subgroups of rank 4 in $T_{(2)}$ are $N_G(T)$-conjugate (and since all elements of the coset θT are T-conjugate), this shows that all **2B**-pure subgroups of rank 5 in G are G-conjugate. In particular, we can assume

$E' = E'_4{\cdot}\langle\theta\rangle$. From the description of $C_G(E'_4)$ given in Case 3, this also shows that $|C_G(E')| = 2^{15}$.

By [**CLSS**, Theorem 1 & Lemma 2.17], there is a unique L-conjugacy class of elementary abelian 2-subgroups $F \le L$ of rank 5 such that $C_L(F)/F = C_G(F)/F \cong C_2^{10}$ and $\mathrm{Aut}_L(F) = \mathrm{Aut}_G(F) = \mathrm{Aut}(F)$. Since all involutions of F are conjugate, they must be of type **2B**, and hence F is G-conjugate to E. Since $N_G(F) \le L$, its action on $C_G(F)$ (as defined in Proposition 8.5) is the conjugation action. So $\mathrm{Aut}_L(E)$ is a stabilizer subgroup for the action of $\mathrm{Aut}_G(F)$ by conjugation on the set of conjugacy classes in $C_G(F)$; or equivalently, the conjugation action of $\mathrm{Aut}_G(E') \cong GL_5(2)$ on $C_G(E')/(\mathrm{conj})$.

We first determine the stabilizer subgroups of the conjugation action of $\mathrm{Aut}_G(E)$ on $C_G(E)/E \cong C_2^{10}$; i.e., the subgroups $\mathrm{Stab}(\mathrm{Aut}(E'), xE')$ for $x \in C_G(E') \smallsetminus E'$. To do this, fix $E' = E'_4{\cdot}\langle\theta\rangle \le T_{(2)}{\cdot}\langle\theta\rangle$ as above which is G-conjugate to E. Fix a basis $\{v_1, \dots, v_8\}$ for $T_{(2)}$ as in Case 3, such that $E'_4 = \langle v_1, \dots, v_4\rangle$.

Set $x = v_8$, $F = \langle E', x\rangle$, and $E'_3 = \langle v_1, v_2, v_3\rangle$. Then $F \cap \mathbf{2A} = v_4v_8E'_3$, so $\langle F \cap \mathbf{2A}\rangle \cap E' = E'_3$ is invariant under the action of any element of $\mathrm{Stab}(\mathrm{Aut}(E'), xE')$. The subgroup $C_2^6 \rtimes (GL_3(2) \times \Sigma_3)$ of all automorphisms which leave E'_3 invariant has index $5{\cdot}31$ in $GL_5(2)$, and thus the $\mathrm{Aut}(E')$-orbit of xE' in $C_G(E')/E'$ has order $\ge 5{\cdot}31$.

Next, since $C_G(E')$ is not elementary abelian, we can choose $y \in C_G(E')$ such that $y^2 \ne 1$; and hence (since $\mathrm{Aut}_G(E')$ acts transitively on involutions) such that $y^2 = \theta$. Then any element of $\mathrm{Stab}(\mathrm{Aut}(E'), yE')$ fixes θ. Also, y (and any other element which normalizes E'_4) normalizes $C_G(E'_4)^0 = T$. So by the computations in Case 3 (and with respect to the basis $\{v_i\}$), y has matrix $\left(\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right)$ for some symplectic matrix X. Fix $X' \in M_4(\mathbb{Z}/4)$ whose reduction mod 2 is X. By looking at the conjugation action of y on $T_{(4)}$, we see that the action of y has matrix $\left(\begin{smallmatrix} I & X' \\ 0 & I \end{smallmatrix}\right) + 2Y$ for some $Y \in M_8(\mathbb{Z}/2)$, and the relation $\bigl(\left(\begin{smallmatrix} I & X' \\ 0 & I \end{smallmatrix}\right) + 2Y\bigr)^2 = -I$ implies

$$\left(\begin{smallmatrix} 0 & X \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & X \\ 0 & 0 \end{smallmatrix}\right)Y + Y\left(\begin{smallmatrix} 0 & X \\ 0 & 0 \end{smallmatrix}\right) = I \qquad \text{in } M_8(\mathbb{Z}/2).$$

Thus X has rank 4. Hence all elements in yE'_4 are $T_{(2)}$-conjugate; and since E'_4 could be replaced by any other subgroup of E' complementary to θ, this shows that all elements of yE' are $C_G(E')$-conjugate. Also, X defines a symplectic form $\widehat{\mathfrak{b}}$ on E'_4 (with respect to the basis $\{v_1, \dots, v_4\}$), and this has the property that $\widehat{\mathfrak{b}}(a,b) = \mathfrak{b}(a,b')$ for any $a, b \in E'_4$ and any $b' \in T_{(2)}$ such that $[y, b'] = b$. Thus $\widehat{\mathfrak{b}}$ depends only on E'_4 and the class yE', and hence is preserved by any $\alpha \in \mathrm{Stab}(\mathrm{Aut}(E'), yE')$ which leaves E'_4 invariant. This shows that the index of $\mathrm{Stab}(\mathrm{Aut}(E'), yE')$ in $\mathrm{Aut}(E', \theta)$ (the group of $\alpha \in \mathrm{Aut}(E')$ such that $\alpha(\theta) = \theta$) is at least $[GL_4(2){:}Sp_4(2)] = 28$, and hence that $[\mathrm{Aut}(E'){:}\mathrm{Stab}(\mathrm{Aut}(E'), yE')] \ge 31{\cdot}28$. Also, if $\mathrm{Stab}(\mathrm{Aut}(E'), yE')$ has index exactly 28 in $\mathrm{Aut}(E', \theta) \cong C_2^4 \rtimes GL_4(2)$, then it must contain the normal subgroup C_2^4 (since $Sp_4(2)$ acts transitively on its involutions), and hence is isomorphic to $C_2^4 \rtimes Sp_4(2)$.

We have thus found two distinct $\mathrm{Aut}(E')$-orbits in $C_G(E')/E'$ (one containing elements of order 2 in $C_G(E')$ and the other elements of order 4), and these orbits have orders $\ge 5{\cdot}31$ and $\ge 28{\cdot}31$. So this accounts for all except the fixed orbit, the inequalities are equalities, and the point stabilizers of these orbits must be the subgroups described above. The point stabilizers of elements in $C_G(E')/(\mathrm{conj}) \smallsetminus E'$

thus either have index ≤ 32 in $C_2^6 \rtimes (GL_3(2) \times \Sigma_3)$, or are isomorphic to $C_2^4 \rtimes Sp_4(2)$ (since all elements of yE' were shown to be $C_G(E')$-conjugate). Also, the point stabilizers of elements in E' are $C_2^4 \rtimes GL_4(2)$ or $GL_5(2)$ itself. Since $O_2(\mathrm{Aut}_L(E)) = 1$ (E is pivotal), this implies $\mathrm{Aut}_L(E) \cong GL_5(2)$, which contradicts Proposition 4.7(a). □

The results of this chapter are now summarized in the following theorem.

Theorem 8.13. *Fix an odd prime power q. Assume L is a simple group, isomorphic to one of the groups $G_2(q)$, ${}^2G_2(q)$, $F_4(q)$, ${}^3D_4(q)$, $E_6(q)$, ${}^2E_6(q)$, $E_7(q)$, or $E_8(q)$. Then $L \in \mathfrak{L}^{\geq 2}(2)$.*

CHAPTER 9

Sporadic groups

It remains to consider the sporadic simple groups. The following standard shorthand notation for referring to certain groups will be frequently used throughout the proof:

- $2^n = C_2^n$ is an elementary abelian 2-group;
- 2_+^{1+2k} is the central product of k copies of D_8;
- 2_-^{1+2k} is the central product of Q_8 with $k-1$ copies of D_8;
- 2^{a+b} is a 2-group P such that $Z(P) \cong 2^a$ and $P/Z(P) \cong 2^b$;
- $[2^n]$ is an unspecified group of order 2^n; and
- $H{:}K$, $H{\cdot}K$, and $H.K$ are extensions (split, unsplit, or indeterminate, respectively) with kernel H and quotient K.

THEOREM 9.1. *If L is one of the sporadic simple groups, then $L \in \mathfrak{L}^{\geq 2}(2)$.*

PROOF. When L is one of the groups $\mathbf{M_{11}}$, $\mathbf{M_{12}}$, $\mathbf{J_1}$, or $\mathbf{O'N}$, then $\mathrm{rk}_2(L) \leq 3$ [**GLS3**, §5.6]. Hence $\mathfrak{R}^{\geq 2}(L\,;2) = \varnothing$ by Proposition 4.6(b), and so $L \in \mathfrak{L}^{\geq 2}(2)$ in all of these cases by Proposition 4.2.

The remaining sporadic groups are considered individually. We recall now (without repeating it each time when used in the proof) that $\mathrm{rk}(E) \geq 4$ for any $E \in \mathfrak{E}^{\geq 2}(L\,;2)$.

$\mathbf{L = M_{22}}$ **or** $\mathbf{M_{23}}$: We have the following inclusions with odd index:

$$M_{22} \leq M_{23} \geq M_{21}{:}2 \cong P\Sigma L_3(4),$$

where $P\Sigma L_3(4)$ is the extension of $PSL_3(4)$ by the field automorphism. Hence all three of these groups have isomorphic Sylow 2-subgroups. Any elementary abelian 2-subgroup of rank 4 in $P\Sigma L_3(4)$ is contained in $PSL_3(4)$, and any Sylow 2-subgroup of $PSL_3(4)$ contains exactly two such subgroups.

Identify L as a subgroup of M_{24}: the subgroup of elements which fix one or two points under the action on a set X of order 24. Fix $S \in \mathrm{Syl}_2(L)$, and let $V_1, V_2 \leq S$ be the two elementary abelian subgroups of rank four. We take V_1 to be the subgroup whose normalizer in M_{24} is the octad group $V_1{:}A_8$, where V_1 acts freely on 16 points in X and A_8 permutes the remaining 8 points in the obvious way (cf. [**Gr3**, 6.8]). Restriction to the subgroups fixing one or two points shows that $\mathrm{Aut}_{M_{22}}(V_1) \cong A_6$ and $\mathrm{Aut}_{M_{23}}(V_1) \cong A_7$. Hence $V_1 \notin \mathfrak{E}^{\geq 2}(L\,;2)$ by Proposition

4.6(e), and thus $\mathfrak{R}^{\ge 2}(L\,;2)_{\not\ge V_2} = \varnothing$ since the only possible element of $\mathfrak{R}^{\ge 2}(L\,;2)$ is V_2 itself.

Either $\mathrm{Aut}_L(V_2) \cong \mathrm{Aut}_L(V_1)$, in which case $V_2 \notin \mathfrak{E}^{\ge 2}(L\,;2)$ and so $\mathfrak{E}^{\ge 2}(L\,;2) = \varnothing$; or else $\mathrm{Aut}_L(V_2) \not\cong \mathrm{Aut}_L(V_1)$, V_1 and V_2 are not $\mathrm{Aut}(L)$-conjugate, and hence both are weakly closed in S with respect to $\mathrm{Aut}(L)$. So in either case, $L \in \mathfrak{L}^{\ge 2}(2)$ by Proposition 4.2. (In fact, it is well known that $\mathrm{Aut}_{M_{22}}(V_2) \cong \Sigma_5$ and $\mathrm{Aut}_{M_{23}}(V_2) \cong 3 \times \Sigma_5$, and using this one can show that $V_2 \in \mathfrak{R}^2(L\,;2)$ in both cases.)

$\mathbf{L = M_{24}}$ **or He**: We refer to [**A4**, §39–42] for details of the structure of these groups. In both cases, there is an involution $z \in L$ such that $C_L(z) \cong 2_+^{1+6}{:}L_3(2)$, the centralizer of a transvection in $L_5(2)$, and this centralizer has odd index in L. To handle elements in this group, we fix $V \cong (\mathbb{F}_2)^5$ with basis $\{v_1, \dots, v_5\}$, and set $V_i = \langle v_1, \dots, v_i \rangle$. We identify $H = C_L(z)$ with the group of automorphisms of V which leave V_1 and V_4 invariant.

For $1 \le i < j \le 5$, let $e_{ij} \in H$ be the element which sends $v_j \mapsto v_i + v_j$ and is the identity on the other basis elements. Thus $z = e_{15}$. Let S be the subgroup generated by all e_{ij} for $i < j$; this is a Sylow 2-subgroup of H and hence of L. For each $1 \le i \le 4$, let $U_i \le S$ be the subgroup of automorphisms which are the identity on V_i and on V/V_i. By [**A4**, Lemma 39.1(3)] (or by the argument given below), U_2 and U_3 are the only subgroups of S of rank six.

Assume $E \in \mathfrak{E}^{\ge 2}(L\,;2)$; we can assume $E \le S \le H$. In particular, $\mathrm{rk}(E) \ge 4$, and $e_{15} \in E$ since E is pivotal and $\langle e_{15} \rangle = Z(H)$. For each involution $u \in E$, we write $K(u) = \mathrm{Ker}(u - \mathrm{Id})$ and $I(u) = \mathrm{Im}(u - \mathrm{Id})$; thus $I(u) \le K(u) \le V$. In particular, $I(z) = V_1$ and $K(z) = V_4$.

Assume first that for some $u, v \in E$, $I(v) \not\le K(u)$; i.e., $(u - \mathrm{Id})(v - \mathrm{Id}) \ne 0$. Then $u|_{I(v)}$ and $v|_{I(u)}$ must be (nonidentity) involutions, which implies that $\dim(I(u)) = \dim(I(v)) = 2$ and $\dim(I(u) \cap I(v)) = 1$. Similarly, $\dim(K(u) + K(v)) = 4$, so $\dim(K(u) \cap K(v)) = 2$; and the three subspaces $K(u) \cap K(v)$, $I(u)$, $I(v)$ are linearly independent modulo $I(u) \cap I(v)$. Also, since both of these commute with z, we have $V_1 \le K(u) \cap K(v)$ and $I(u) + I(v) \le V_4$. In all cases, we can choose a basis $\{\xi_0, \xi_u, \xi_v, \xi_1, \xi_2\}$ for V such that $I(u) = \langle \xi_0, \xi_u \rangle$, $I(v) = \langle \xi_0, \xi_v \rangle$, $K(u) = \langle \xi_0, \xi_u, \xi_1 \rangle$, $K(v) = \langle \xi_0, \xi_v, \xi_1 \rangle$, $(u - \mathrm{Id})$ sends $\xi_v \mapsto \xi_0$ and $\xi_2 \mapsto \xi_u$, and $(v - \mathrm{Id})$ sends $\xi_u \mapsto \xi_0$ and $\xi_2 \mapsto \xi_v$ (recall $[u, v] = 1$). We are thus reduced (up to conjugacy in H) to one of the following situations:

- $I(u) \cap I(v) = V_1$ and $K(u) + K(v) = V_4$:
  $I(u) = \langle v_1, v_2 \rangle$, $I(v) = \langle v_1, v_3 \rangle$, $K(u) = \langle v_1, v_2, v_4 \rangle$, $K(v) = \langle v_1, v_3, v_4 \rangle$; $u = e_{13}e_{25}$, $v = e_{12}e_{35}$, $E \le C_H(\langle u, v \rangle) = \langle e_{13}e_{25}, e_{12}e_{35}, e_{15}, e_{14}, e_{45} \rangle \cong 2^2 \times D_8$
- $I(u) \cap I(v) = V_1$ and $K(u) + K(v) \ne V_4$:
  $I(u) = \langle v_1, v_2 \rangle$, $I(v) = \langle v_1, v_3 \rangle$, $K(u) = \langle v_1, v_2, v_5 \rangle$, $K(v) = \langle v_1, v_3, v_5 \rangle$; $u = e_{13}e_{24}$, $v = e_{12}e_{34}$, $E = C_H(\langle u, v \rangle) = \langle e_{13}e_{24}, e_{12}e_{34}, e_{15}, e_{14} \rangle \cong 2^4$
- $I(u) \cap I(v) \ne V_1$, and hence $K(u) = I(u) + V_1$ and $K(v) = I(v) + V_1$:
  $I(u) = \langle v_2, v_3 \rangle$, $I(v) = \langle v_2, v_4 \rangle$, $K(u) = \langle v_1, v_2, v_3 \rangle$, $K(v) = \langle v_1, v_2, v_4 \rangle$; $u = e_{24}e_{35}$, $v = e_{23}e_{45}$, $E = C_H(\langle u, v \rangle) = \langle e_{24}e_{35}, e_{23}e_{45}, e_{15}, e_{25} \rangle \cong 2^4$

This leaves the following four possibilities for $E \le C_H(\langle u,v\rangle)$ of rank 4, as described in the following list. In each case, N_i is a subgroup of $N_S(E_i)$ whose elements are

$$
\begin{aligned}
E_1 &= \langle e_{15}, e_{14}, e_{12}e_{35}, e_{13}e_{25}\rangle & N_1 &= \langle e_{12}, e_{13}, e_{45}\rangle\\
E_2 &= \langle e_{15}, e_{45}, e_{12}e_{35}, e_{13}e_{25}\rangle & N_2 &= \langle e_{12}, e_{13}, e_{14}\rangle\\
E_3 &= \langle e_{15}, e_{14}, e_{12}e_{34}, e_{13}e_{24}\rangle & N_3 &= \langle e_{24}, e_{25}, e_{34}, e_{35}\rangle\\
E_4 &= \langle e_{15}, e_{25}, e_{23}e_{45}, e_{24}e_{35}\rangle & N_4 &= \langle e_{13}, e_{14}, e_{23}, e_{24}\rangle
\end{aligned}
$$

independent in $\mathrm{Aut}_S(E_i)$. Thus in all cases, $\mathrm{Aut}_S(E)$ has rank ≥ 3, so the Sylow 2-subgroups of $\mathrm{Aut}_L(E)$ are neither dihedral nor semidihedral. Hence $E \notin \mathfrak{E}^{\ge 2}(L\,;2)$ by Proposition 4.6(c).

We are left with the case where $W \overset{\text{def}}{=} \langle I(u) \,|\, u \in E\rangle$ is contained in $W' \overset{\text{def}}{=} \bigcap_{u\in E} K(u)$. If $W = W'$, then either $W = V_1$ or V_4 and $E = U_1$ or U_4, or $\dim(W) = 2,3$ and (up to H-conjugacy) $E \le U_i$ for $i = 2,3$. In general, if $u \in E$ and $g \in C_H(E)$, then $g(I(u)) = I(u)$ and $g(K(u)) = K(u)$. Hence if $W = W' = V_i$ for $i = 2,3$, then every $g \in C_H(E)$ induces the identity on W and on V/W. In other words, $C_H(E) = U_i$ in this case, hence $C_L(E) = U_i$ since $H = C_L(e_{15})$ and $e_{15} \in E$; and thus $E = U_i$ since E is pivotal (Proposition 4.4(c)). By [**A4**, Lemma 40.5], $\mathrm{Aut}_L(E) \cong L_4(2)$ or $2^3{:}L_3(2)$ when $E = U_1$ or U_4 ($\mathrm{rk}(E) = 4$), and $\mathrm{Aut}_L(E) \cong L_3(2) \times \Sigma_3$ or $3{\cdot}\Sigma_6$ when $E = U_2$ or U_3 ($\mathrm{rk}(E) = 6$). Since none of these automorphism groups has dihedral or semidihedral Sylow 2-subgroups, $E \notin \mathfrak{E}^{\ge 2}(L\,;2)$ by Proposition 4.6(c).

The only remaining case is that where $\dim(W) = 2$ and $\dim(W') = 3$, and hence where E is H-conjugate to $U_2 \cap U_3$. Thus the only subgroup of S which could lie in $\mathfrak{E}^{\ge 2}(L\,;2)$ is $U_2 \cap U_3$. Since $C_L(U_2 \cap U_3) = C_H(U_2 \cap U_3) = U_2U_3$, it follows that $\mathfrak{R}^{\ge 2}(L\,;2)_{\not\ge U_2U_3} = \varnothing$. Since U_2U_3 is weakly closed in S with respect to $\mathrm{Aut}(L)$ (U_2 and U_3 are the only rank six subgroups of S), Proposition 4.2 now implies that $L \in \mathfrak{L}^{\ge 2}(2)$. (In fact, one can show that in all three cases $L = M_{24}$, $L = \mathrm{He}$, and $L = L_5(2)$, $U_2 \cap U_3 \in \mathfrak{E}^2(L\,;2)$ and $U_2U_3 \in \mathfrak{R}^2(L\,;2)$.)

$\mathbf{L = J_2}$: This group contains two conjugacy classes of involutions, of which those of type **2A** are in the centers of Sylow subgroups. By [**FR1**, §3], the elements of type **2A** in any elementary abelian $E \le L$ form a subgroup, and there are no **2A**- or **2B**-pure subgroups of rank 3. Thus any elementary abelian 2-subgroup $E \le L$ contains an $N_{\mathrm{Aut}(L)}(E)$-invariant subgroup $E_0 \le E$ (generated by the elements of type **2A**) such that $\mathrm{rk}(E_0) \le 2$ and $\mathrm{rk}(E/E_0) \le 2$. Hence $\mathfrak{E}^{\ge 2}(L\,;2) = \varnothing$ by Proposition 4.6(c), and $L \in \mathfrak{L}^{\ge 2}(2)$ by Proposition 4.2.

$\mathbf{L = Co_3}$ **or** $\mathbf{L = HS}$: By [**Fi**, §4], there are two conjugacy classes of involutions in Co_3, of which those in the center of a Sylow 2-subgroup are of type **2A** with centralizer $2Sp_6(2)$, and those of type **2B** have centralizer $2 \times M_{12}$. (These are denoted (2_1) and (2_2), respectively, in [**Fi**].) By [**Fi**, Lemma 4.7], this group $2Sp_6(2)$ has two conjugacy classes of noncentral involutions whose centralizers have different orders, and hence which project to different classes in $Sp_6(2)$. In other words, if x, y are commuting involutions and x has type **2A**, then y and xy are conjugate in $C_L(x)$, and in particular have the same type in L. This shows that in any elementary abelian 2-subgroup $E \le L$, the elements of type **2A** together with the identity form a subgroup of E.

By [**PW**, Lemma 2.2 & §4], there are two conjugacy classes of involutions in HS, of which those in the center of a Sylow 2-subgroup have type **2A** and centralizer $(2^{1+4}_+ \times_{C_2} C_4){\cdot}\Sigma_5$, and the others have type **2B** with centralizer $2 \times \mathrm{Aut}(A_6)$. Also, HS is contained as a subgroup of Co_3 (see [**A4**, §§23–24]). Since the order of the centralizer of a **2A**-element in HS does not divide the order of the centralizer of a **2B**-element in Co_3, the inclusion must send involutions of type **2A** in HS to involutions of type **2A** in Co_3. If it sent all involutions in HS to **2A**-elements in Co_3, then a Sylow 2-subgroup of Co_3 would contain an index 2 subgroup (a Sylow subgroup of HS) all of whose involutions have type **2A**, so there would be no **2B**-pure subgroup of rank 2 in Co_3, which would contradict [**Fi**, Lemma 5.10]. Thus involutions of type **2B** in HS get sent to involutions of type **2B** in Co_3.

Now assume $E \in \mathfrak{E}^{\ge 2}(L\,;2)$, for $L = Co_3$ or HS, and let $E_0 \le E$ be the subgroup generated by type **2A** involutions. Then $E_0 \ne 1$, since E contains the center of a Sylow 2-subgroup (Proposition 4.4(e)), and it is clearly $N_{\mathrm{Aut}(L)}(E)$-invariant. Thus $\mathrm{rk}(E_0) \ge 4$ or $\mathrm{rk}(E/E_0) \ge 4$ by Proposition 4.6(b). Since $\mathrm{rk}_2(Co_3) = 4$ [**GLS3**, p.305], this means that $E = E_0$ is a rank 4 **2A**-pure subgroup. If $L = Co_3$, then by [**Fi**, Lemma 5.9], L has a unique class of such subgroups, and $\mathrm{Aut}_L(E) \cong A_8 \cong GL_4(2)$ for any such E. If $L = HS$, then by [**PW**, Lemma 4.1], $\mathrm{Aut}_L(E) \cong \Sigma_6$ for any such E. In both cases, the Sylow 2-subgroups of $\mathrm{Aut}_L(E)$ are neither dihedral nor semidihedral, and hence $E \notin \mathfrak{E}^{\ge 2}(L\,;2)$ by Proposition 4.6(c). Thus $\mathfrak{R}^{\ge 2}(L\,;2) = \varnothing$, and $L \in \mathfrak{L}^{\ge 2}(2)$ by Proposition 4.2.

$\mathbf{L = McL}$ **or** $\mathbf{L = Ly}$: By [**GLS3**, p.308], $\mathrm{rk}_2(L) = 4$ (see also the discussion in [**Fi**, §5] when $L = \mathrm{McL}$). By [**Fi**, Lemma 5.2] (when $L = \mathrm{McL}$) or [**W5**, §2] (when $L = \mathrm{Ly}$), $\mathrm{Aut}_L(E) \cong A_7$ for every elementary abelian 2-subgroup $E \le L$ of rank 4, and so $E \notin \mathfrak{E}^{\ge 2}(L\,;2)$ by Proposition 4.6(e). Thus $\mathfrak{E}^{\ge 2}(L\,;2) = \varnothing$, and hence $L \in \mathfrak{L}^{\ge 2}(2)$.

$\mathbf{L = F_5 = HN}$: We refer to [**NW**, §3.1] for the following information about L. There are two conjugacy classes of involutions in L, types **2A** and **2B**. For any elementary abelian 2-subgroup $E \le L$, the function $\mathfrak{q}\colon E \longrightarrow \mathbb{F}_2$, defined by $\mathfrak{q}(v) = 1$ if $v \in \mathbf{2A}$ and $\mathfrak{q}(v) = 0$ otherwise, is quadratic.

Assume $E \in \mathfrak{E}^{\ge 2}(L\,;2)$. Set $E_0 = E \cap E^{\perp}$ (with respect to the quadratic form $\mathfrak{q}$), and $E_1 = \mathrm{Ker}(\mathfrak{q}|_{E_0})$. Clearly, these subgroups are both $N_{\mathrm{Aut}(L)}(E)$-invariant, and $\mathrm{rk}(E_0/E_1) \le 1$. So either $\mathrm{rk}(E_1) \ge 4$ or $\mathrm{rk}(E/E_0) \ge 4$.

By [**NW**, p.365], there are exactly two conjugacy classes of **2B**-pure subgroups of rank 2 in L. If $x \in \mathbf{2B}$, then $C_L(x) \cong 2^{1+8}_+{\cdot}(A_5 \times A_5){:}2$; and the two types are represented by $V_1 = \langle x, a\rangle$ and $V_2 = \langle x, b\rangle$, where $a \in O_2(C_L(x))$, and b is a diagonal involution in $A_5 \times A_5$. Also, V_1 is contained in a unique **2B**-pure subgroup of rank 3 having the property that all of its rank 2 subgroups have the same type as V_1. Hence we can assume E_1 contains V_2. By [**NW**, p.365] again, V_2 and E_1 are both contained in a unique extraspecial subgroup $X \cong 2^{1+8}_+$ with $N_L(X)/X \cong (A_5 \times A_5){:}2$ (but where $Z(X) \not\le V_2$). Thus $N_L(E) \le N_L(E_1) \le N_L(X)$, so $E \le Z(X) \cong C_2$ by Proposition 4.9(a), and this contradicts the assumption $\mathrm{rk}(E_1) \ge 4$.

We are left with the possibility $\text{rk}(E/E_0) \geq 4$. Choose $E' \leq E$ such that $E = E' \times E_0$; then $\mathfrak{q}|_{E'}$ is nondegenerate and $\text{rk}(E') \geq 4$. Also, E' contains a **2A**-pure subgroup E'' of rank two, and $C_L(E'') \cong 2^2 \times A_8 \leq A_{12} \leq L$. Inside A_{12}, the involutions whose support have order 4 or 12 are of type **2A**, while those whose support have order 8 are of type **2B**. Using this, one sees that either $\text{rk}(E') = 6$ and $C_L(E') = E'$, or $\text{rk}(E') = 4$ and $C_L(E') \cong 2^6$ or $2^4 \times A_4$. Thus in all of these cases, E is contained in a unique subgroup $\bar{E}$ of rank 6 (on which $\mathfrak{q}$ is nondegenerate), and $N_L(E) \leq N_L(\bar{E})$. Also, $\text{Aut}_L(\bar{E}) = \Omega(\bar{E}, \mathfrak{q}) \cong \Omega_6^-(2)$, and so $\bar{E} \notin \mathfrak{E}^{\geq 2}(L;2)$ by Proposition 4.6(c) (Sylow 2-subgroups of $\text{Aut}_L(\bar{E})$ are neither dihedral nor semidihedral). If $\text{rk}(E) = 5$, then either $\mathfrak{q}|_{E_0} \neq 0$ and $\text{Aut}_L(E) \cong \Omega_5(2) \cong \Sigma_6$, which again contradicts Proposition 4.6(c); or $\mathfrak{q}|_{E_0} = 0$ and $O_2(\text{Aut}_L(E)) \neq 1$, in which case E is not pivotal. Finally, if $\text{rk}(E) = 4$ and $\mathfrak{q}$ is nondegenerate on E, then we have just seen that the Sylow 2-subgroups of $C_L(E)$ are isomorphic to 2^6, and so E cannot be pivotal.

This shows that $\mathfrak{E}^{\geq 2}(L;2) = \varnothing$, and thus that $L \in \mathfrak{L}^{\geq 2}(2)$ by Proposition 4.2.

In all of the remaining cases, the proof that $L \in \mathfrak{L}^{\geq 2}(2)$ will be based on a list of maximal 2-local subgroups of L — or in some cases, a list of proper subgroups of L (not necessarily 2-local) which contain all 2-local subgroups up to conjugacy. We label these subgroups H_n for $n = 1, 2, \ldots$, and set $V_n = Z(O_2(H_n))$. The goal is to show that $\mathfrak{R}^{\geq 2}(L;2) = \varnothing$; unless we set $Q = V_n$ for some n, in which case we show that Q is weakly closed in some (any) Sylow 2-subgroup which contains it, and that $\mathfrak{R}^{\geq 2}(L;2)_{\ngeq Q} = \varnothing$. In either case, Proposition 4.2 then implies that $L \in \mathfrak{L}^{\geq 2}(2)$.

Fix a subgroup $P \in \mathfrak{R}^{\geq 2}(L;2)$, and set $E = \Omega_1(Z(P))$. Thus $N_L(P) \leq N_L(E) \leq H_n$ for some n. Also, $P \geq O_2(H_n)$ and $E \cap V_n \neq 1$ by Proposition 4.9(a), and hence

$$\text{If } O_2(H_n) \text{ is centric in } H_n, \text{ and } N_L(E) \leq H_n, \text{ then } E \leq V_n. \qquad (*)$$

This will frequently be used below, without reference. Also, by Proposition 4.6(b,e),

If $O_2(H_n)$ is centric in H_n, and either $\text{rk}(V_n) \leq 3$, or $\text{rk}(V_n) = 4$ and $\text{Aut}_{H_n}(V_n)$ is isomorphic to A_5, $GL_2(4)$, A_6, Σ_6, or A_7, then $N_L(E) \nleq H_n$. (1)

Finally, if $\text{Aut}_L(V_n) = \text{Aut}(V_n)$, then $\text{Aut}_L(E) = \text{Aut}(E)$, and so by Proposition 4.7(a) we get:

$$\text{If } O_2(H_n) \text{ is centric in } H_n, \text{ and } \text{Aut}_L(V_n) = \text{Aut}(V_n), \text{ then } N_L(E) \nleq H_n. \qquad (2)$$

Points (1) and (2) will frequently be referred to below.

We use the Atlas notation **2A**, **2B**, etc. for the conjugacy classes of involutions in L. Recall that by Proposition 4.4(e), $E \geq \Omega_1(Z(S))$ for some $S \in \text{Syl}_2(L)$, and thus E contains elements from each conjuacy class of involutions represented in $Z(S)$.

$\mathbf{L = J_3}$: By [**FR2**, §2], L contains three conjugacy classes of maximal 2-local subgroups: $H_1 \cong 2_-^{1+4}{:}A_5$, $H_2 \cong 2^4{:}GL_2(4)$, and $H_3 \cong 2^{2+4}{:}(3 \times \Sigma_3)$. In all three cases, $O_2(H_i)$ is centric in H_i, and $N_L(E) \leq H_i$ is impossible by (1): either because $V_i = Z(O_2(H_i))$ has rank ≤ 3, or because $\text{rk}(V_i) = 4$ and $\text{Aut}_L(V_i) \cong GL_2(4)$.

$\mathbf{L = Suz}$: We refer to [**W2**] for the following information. There are two conjugacy classes of elements of order 2 in L, of which those of type **2A** are in the centers of Sylow 2-subgroups. Also, in any elementary abelian subgroup $E \le L$, the involutions of type **2A** together with the identity form a subgroup of E. Hence $N_L(E)$ is contained in the normalizer of some **2A**-pure subgroup; and hence by [**W2**, §2.4], in one of the three groups $H_1 \cong 2_-^{1+6}.U_4(2)$, $H_2 \cong 2^{2+8}{:}(A_5 \times \Sigma_3)$, or $H_3 \cong 2^{4+6}{:}3A_6$. In all of these cases, $O_2(H_i)$ is centric in H_i, and $N_L(E) \not\le H_i$ by (1): either because $\mathrm{rk}(V_i) \le 3$ (when $i = 1, 2$), or because $\mathrm{rk}(V_3) = 4$ and $\mathrm{Aut}_L(V_3) \cong A_6$.

$\mathbf{L = Ru}$: There are two conjugacy classes of involutions, of which those of type **2A** lie in the centers of Sylow 2-subgroups. Thus E contains elements of type **2A** (Proposition 4.4(e)). By [**W3**, §2.4], the involutions of type **2A** in E together with the identity form a subgroup $E_0 \le E$, and hence $N_L(E) \le N_L(E_0)$.

By [**W3**, §2.4–2.5], the normalizer of each **2A**-pure elementary abelian subgroup of L is conjugate to a subgroup of one of three subgroups H_n listed below. In all cases, V_i is **2A**-pure and $O_2(H_i)$ is centric in H_i, so $E = E_0 \le V_i$ by $(*)$. (In particular, $V_3 \cong 2^6$ is **2A**-pure by [**W3**, Lemma 1], where V_3 is denoted R_1.)

The first two subgroups in the list, $H_1 \cong 2{\cdot}2^{4+6}{:}\Sigma_5$ ($V_1 \cong 2$) and $H_2 \cong 2^{3+8}{:}L_3(2)$ ($V_2 \cong 2^3$), cannot contain $N_L(E)$ by (1). That $\mathrm{rk}(V_1) = 1$ follows either from the description in [**A3**, 12.12], or directly from the commutator relations listed in [**P2**, Lemma 12] (where $O_2(H_1)$ is the subgroup generated by the eleven elements $z, t, v, w, w_1, x_1, x_2, a, b, c, d$). This leaves only the following subgroup to consider:

- $N_L(E) \le \mathbf{H_3} \cong \mathbf{2^6{\cdot}G_2(2)}$, $E \le V_3 \cong 2^6$. We just saw that $N_L(E)$ is not conjugate to a subgroup of H_1 or H_2. Let $\mathcal{T}$ be the conjugacy class of the subgroups of rank 2 in V_2. Let D be the "diagram" of E in the sense of [**W3**]: the graph with one node for each involution in E, and an edge connecting two nodes whenever the elements generate a subgroup in $\mathcal{T}$. The automorphism group $\mathrm{Aut}_L(E)$ acts on D (and $\mathrm{Out}(L) = 1$ in this case). There cannot be any $\mathrm{Aut}_L(E)$-invariant node or triangle in D, since this would imply an $\mathrm{Aut}_L(E)$-invariant element or subgroup in $\mathcal{T}$, hence that $N_L(E) \le H_1$ or H_2, contradicting our assumption on E.

  By [**W3**, §2-4–2.5], for each $E \le V_3$ of rank ≥ 2, either the diagram D contains an $\mathrm{Aut}_L(E)$-invariant node or triangle; or D is a disjoint union of two or more triangles and isolated nodes in which case $C_L(E) = V_3 \cong 2^6$; or $E = V_3$ and $\mathrm{Aut}_L(E) \cong G_2(2) \cong U_3(3){:}2$; or $\mathrm{rk}(E) = 2$ and $N_L(E)/V_3 \cong (\Sigma_3 \times \Sigma_3)$. Since $\mathrm{rk}(E) \ge 4$, this shows that $C_L(E) \cong V_3$, and hence (since E is pivotal) that $E = V_3$. Thus $(E, \mathrm{Aut}_L(E)) \cong (2^6, G_2(2))$, and this contradicts Proposition 6.5.

$\mathbf{L = F_3 = Th}$: By [**W7**, Theorem 2.2], each 2-local subgroup of L is conjugate to a subgroup of either $H_1 \cong 2_+^{1+8}{\cdot}A_9$ or $H_2 \cong 2^5{\cdot}L_5(2)$. By (1) or (2), respectively, neither of these can contain $N_L(E)$.

$\mathbf{L = J_4}$: By [**KW**, §2], there are four conjugacy classes of maximal 2-local subgroups H_i. Of these, (1) shows that neither $H_2 \cong 2^{3+12}.(\Sigma_5 \times L_3(2))$ (with $V_2 \cong 2^3$) nor

$H_3 \cong 2_+^{1+12}\cdot(3M_{22}{:}2)$ can contain $N_L(E)$. It remains to consider the following two cases:

- $N_L(E) \le \mathbf{H_1} \cong \mathbf{2^{10}{:}L_5(2)}$, $E \le V_1 \cong 2^{10}$. If $N_L(E) \le H_1$ and $E \le V_1$, then by Proposition 4.9(a), $\bar{P} \stackrel{\text{def}}{=} P/V_1$ is a radical 2-subgroup of $H_1/V_1 \cong L_5(2)$, and $N_L(P)/P \cong N_{L_5(2)}(\bar{P})/\bar{P}$. Also, $L_5(2)$ acts irreducibly on V_1 [**Ja**, Theorem A], and hence acts as $\Lambda^2(V)$, where V is one of the standard 5-dimensional representations. The only radical subgroups which have fixed subspace on V_1 of rank ≥ 4 are the trivial subgroup, and two subgroups 2^4 with normalizer $2^4{:}L_4(2)$ with fixed subspaces of rank 4 or 6. By Proposition 4.6(c), no such subgroup can be in $\mathfrak{R}^{\ge 2}(L\,;2)$.

- $N_L(E) \le \mathbf{H_4} \cong \mathbf{2^{11}{:}M_{24}}$, $E \le \mathbf{Q} = \mathbf{V_4} \cong \mathbf{2^{11}}$. By [**KW**, Lemma 1.1.2], $Q = V_4$ is the unique subgroup of $N_L(Q) \cong 2^{11}{:}M_{24}$ of rank 11; and this group contains a Sylow 2-subgroup $S \in \mathrm{Syl}_2(L)$. Hence $Q = V_4$ is weakly closed in S with respect to $\mathrm{Aut}(L)$; and we have just shown that $\mathfrak{R}^{\ge 2}(L\,;2)_{\not\ge Q} = \varnothing$.

$\mathbf{L} = \mathbf{Co_1}$: Our main reference for this group is Curtis's paper [**Cu**]. The conjugacy classes of involutions which we call **2A**, **2B**, and **2C** are referred to there as types BD, F, and C, respectively. Also, the subgroups which Curtis calls (0), (a), and (c) are what we call H_7, H_2, and H_3.

By [**Cu**, Theorem 2.1], the normalizer of each elementary abelian 2-subgroup of L is contained in one of seven subgroups, listed here as H_i $(i = 1, \dots, 7)$. Three of these, $H_1 \cong 2_+^{1+8}\cdot\Omega_8^+(2)$, $H_2 \cong 2^{4+12}.(3\Sigma_6 \times \Sigma_3)$ ($V_2 \cong 2^4$ and $\mathrm{Aut}_L(V_2) \cong \Sigma_6$), and $H_3 \cong 2^{2+12}.(\Sigma_3 \times A_8)$, cannot contain $N_L(E)$ by (1).

Set $Q = V_7 \cong 2^{11}$. We claim that Q is weakly closed in S and $\mathfrak{R}^{\ge 2}(L\,;2)_{\not\ge Q} = \varnothing$.

- $N_L(E) \le \mathbf{H_4} \cong \mathbf{Co_2}$ (the stabilizer of a 2-vector). Upon examination of the proof of [**Cu**, Theorem 2.1], this case is seen to occur only when $E = \langle (4), \sigma \rangle$ (in the notation of [**Cu**, p.419]) is a certain **2A**-pure subgroup of rank 4 (rank 5 in $2Co_1$). As noted by Wilson in [**W1**, p.112], this subgroup is contained in a unique rank 5 subgroup $\langle (7), \sigma \rangle$ whose involutions all have type **2A**, and hence its normalizer is contained in the normalizer of that subgroup, which is contained in some subgroup conjugate to H_1. We can thus ignore the case $N_L(E) \le Co_2$.

- $N_L(E) \le \mathbf{H_5} \cong (\mathbf{A_4} \times \mathbf{G_2(4)}).\mathbf{2}$. Then by Lemma 1.5(a,b), $P \cap (A_4 \times G_2(4)) = P_1 \times P_2$ where $P_1 \cong 2^2 \le A_4$ and $P_2 \le G_2(4)$ are radical 2-subgroups. By examination of the two maximal parabolic subgroups $2^{2+8}{:}(3 \times A_5)$ and $2^{4+6}{:}(A_5 \times 3)$ of $G_2(4)$, we see that the only possibility for $E = \Omega_1(Z(P))$ with an irreducible component of rank ≥ 4 (see Proposition 4.6(b)) is
$$E = 2^2 \times 2^4 \le A_4 \times G_2(4).$$
In this case, E lifts to a subgroup of $2Co_1$ isomorphic to $Q_8 \times 2^4$; and $N_L(E)$ is contained in the normalizer of the second factor, whose elements are not of type **2B** (since they lift to involutions in $2Co_1$). Thus by [**Cu**, Lemmas 2.2 & 2.5]

(together with the above remarks about H_4), this normalizer is contained in one of the subgroups H_1, H_2, H_3, or H_7.

- $N_L(E) \le \mathbf{H_6} \cong (\mathbf{A_6} \times \mathbf{U_3(3)}).\mathbf{2}$. Since $\mathrm{rk}_2(A_6) = \mathrm{rk}_2(U_3(3)) = 2$, E has a filtration by $N_L(E)$-invariant subgroups for which the quotients all have rank ≤ 2, and this contradicts Proposition 4.6(b).

- $N_L(E) \le \mathbf{H_7} \cong \mathbf{2^{11}}{:}\mathbf{M_{24}}$, $E \le \mathbf{Q} = \mathbf{V_7} \cong \mathbf{2^{11}}$. By [**A3**, (30.3) & (31.11)], Q is the unique subgroup of H_7 of rank 11, and hence weakly closed (with respect to $\mathrm{Aut}(L)$) in any Sylow 2-subgroup which contains it. We have now shown that $\mathfrak{R}^{\ge 2}(L\,;2)_{\not\ge Q} = \varnothing$.

$\mathbf{L} = \mathbf{Co_2}$: By [**W1**, §3], the normalizer of each elementary abelian 2-subgroup of L is contained in one of seven subgroups H_n listed here.

- $N_L(E) \le \mathbf{H_1} = \mathbf{2^{4+10}}{\cdot}(\mathbf{\Sigma_5} \times \mathbf{\Sigma_3})$, $E \le V_1 \cong 2^4$. Then $E = V_4$ and $\mathrm{Aut}_L(E) \cong \Sigma_5$, and an examination of the first diagram in [**W1**, p.113] shows that $\mathrm{Aut}_L(E)$ acts via the permutation representation (with two orbits of lengths 5 and 10). This contradicts Proposition 4.7(b).

- $N_L(E) \le \mathbf{H_2} = \mathbf{2^{1+8}_+}{:}\mathbf{Sp_6(2)}$, $E \le V_2 \cong 2$. Impossible by (1).

- $N_L(E) \le \mathbf{H_3} = (\mathbf{2^{1+6}_+} \times \mathbf{2^4}).\mathbf{A_8}$, $E \le V_3 \cong 2^5$. Let $E' \le E$ be the intersection of E with the commutator subgroup of $O_2(H_3)$. Then $\mathrm{rk}(E') \le 1$, and hence $\mathrm{rk}(E/E') = 4$. Since $A_8 \cong GL_4(2)$ has the usual action on the factor 2^4 in $O_2(H_3)$, this shows that $C_{H_3}(E) = O_2(H_3)$, and hence $P = O_2(H_3)$ and $E = V_3$. Thus $\mathrm{Aut}_L(E) = GL_4(2)$, which contradicts Proposition 4.7(a).

- $N_L(E) \le \mathbf{H_4} = \mathbf{M_{23}}$. This subgroup arises as (one possible) intersection of a subgroup $2^{11}{\cdot}M_{24} \le Co_1$ with Co_2. From the analysis in [**Cu**, §2], we see that each time the normalizer of an elementary abelian subgroup $E \le Co_1$ was shown to be contained in a subgroup $K \cong 2^{11}{:}M_{24}$, it was contained in such a way that E intersects nontrivially with the rank 11 subgroup. Hence if $N_L(E) \le Co_2$, then $K \cap Co_2$ cannot be isomorphic to M_{23}, and so we can ignore this case.

- $N_L(E) \le \mathbf{H_5} = \mathbf{U_6(2)}{:}\mathbf{2}$. Then $P \in \mathfrak{R}^{\ge 2}(H_5\,;2)$ by Proposition 4.9(b), which is empty by Lemma 6.6. (Note that $\mathrm{Out}(L) = 1$.)

- $N_L(E) \le \mathbf{H_6} = \mathbf{McL}$. Then $\mathrm{rk}(E) = \mathrm{rk}_2(\mathrm{McL}) = 4$, and $\mathrm{Aut}_L(E) \cong A_7$ as described above. This is impossible by Proposition 4.6(e).

- $N_L(E) \le \mathbf{H_7} = \mathbf{2^{10}}{:}\mathbf{M_{22}}{:}\mathbf{2}$, $E \le \mathbf{Q} = \mathbf{V_7} \cong \mathbf{2^{10}}$. By [**A3**, (30.3) & (31.11)], V_7 is the unique rank 10 subgroup of H_7, and hence weakly closed in any Sylow subgroup which contains it. We have just shown that $\mathfrak{R}^{\ge 2}(L\,;2)_{\not\ge Q} = \varnothing$.

$\mathbf{L} = \mathbf{Fi_{22}}$: By [**A5**, (25.7)], for any $S \in \mathrm{Syl}_2(L)$, the set of involutions in S of type **2A** generates a subgroup 2^{10}, which thus is weakly closed in S with respect to $\mathrm{Aut}(L)$. We fix S, and let $Q \cong 2^{10}$ denote this subgroup. We will show that $\mathfrak{R}^{\geq 2}(L\,;2)_{\not\geq Q} = \varnothing$, and also that $\mathfrak{R}^{\geq 2}(\mathrm{Aut}(L)\,;2)_{\not\geq Q} = \varnothing$. The latter will be needed later, when working with the group Fi_{24}'. We set $\Gamma = \mathrm{Aut}(L) = Fi_{22}{:}2$ for short [**A5**, (37.2)]. Throughout the following discussion, we use the term "transposition" to refer to involutions of type **2A**; these all have the property that the product of any two of them has order 2 or 3.

Set $E_0 = E \cap L$. By [**W4**, Proposition 4.4] or [**Fl**], $N_L(E_0) \leq H_n$ and $N_\Gamma(E) \leq \overline{H}_n$ for some H_n and $\overline{H}_n$ $(n = 1, \ldots, 5)$ as described in the following list. More precisely, let $\mathcal{T}$ be the set of transpositions in E. If $\mathcal{T} \neq \varnothing$, then $N_L(E) \leq N_L(\mathcal{T})$, and $N_L(\mathcal{T})$ is conjugate to H_1 if $|\mathcal{T}| = 1$, to H_2 if $|\mathcal{T}| = 2$, and to H_3 or H_5 if $|\mathcal{T}| \geq 3$.

- $N_L(E_0) \leq \mathbf{H_1 = 2U_6(2)}$, $N_\Gamma(E) \leq \overline{H}_1 = 2U_6(2).2$, $E \geq V_1 \cong 2$. If $N_L(E) \leq H_1$ and $N_\Gamma(E) \leq \overline{H}_1$, then P/V_1 is in $\mathfrak{R}^{\geq 2}(U_6(2)\,;2)$ or $\mathfrak{R}^{\geq 2}(U_6(2){:}2\,;2)$ by Proposition 4.9(b); and these sets are empty by Lemma 6.6.

- $N_L(E_0) \leq \mathbf{H_2 = (2 \times 2^{1+8}_{+}{:}U_4(2)){:}2}$, $N_\Gamma(E) \leq \overline{H}_2 = (2 \times 2^{1+8}_{+}{:}U_4(2){:}2){:}2$, $E \leq V_2 \cong 2^2$. Impossible by (1).

- $N_L(E_0) \leq \mathbf{H_3 = 2^{5+8}{:}(\Sigma_3 \times A_6)}$, $N_\Gamma(E) \leq \overline{H}_3 = 2^{5+8}{:}(\Sigma_3 \times \Sigma_6)$, $E \leq V_3 \cong 2^5$. Thus $\mathrm{Aut}_L(E)$ is the stabilizer of the action of A_6 or Σ_6. As described in [**W4**, §§3–4], $V_3 \cong 2^5$ is generated by a unique hexad of transpositions any five of which generate V_3, and hence $\mathrm{Aut}_L(V_3) \cong A_6$ and $\mathrm{Aut}_\Gamma(V_3) \cong \Sigma_6$ act on V_3 via the permutation action on $\mathbb{F}_2^6/\mathbb{F}_2$. By Proposition 4.6(e), E cannot be equal to V_3, nor equal to the index two subgroup of V_3 containing no transpositions (since that is also stabilized by A_6). Thus $E \lneqq V_3$ has rank 4 and $1 \leq |\mathcal{T}| \leq 4$. Also, $\langle \mathcal{T} \rangle$ is an $N_{\mathrm{Aut}(L)}(E)$-invariant subgroup, and must be equal to E since otherwise this would contradict Proposition 4.6(b). This leaves the case where $\mathcal{T}$ is a basis for $E \cong 2^4$, so $\mathrm{Aut}_L(E) = \mathrm{Aut}_\Gamma(E) \cong \Sigma_4$, E is not pivotal (since $O_2(\mathrm{Aut}_L(E)) \neq 1$), and E is in neither $\mathfrak{E}^{\geq 2}(L\,;2)$ nor $\mathfrak{E}^{\geq 2}(\Gamma\,;2)$ by Proposition 4.4(c).

- $N_L(E_0) \leq \mathbf{H_4 = 2^6{:}Sp_6(2)}$, $N_\Gamma(E) \leq \overline{H}_4 = 2^7{:}Sp_6(2)$, $E_0 \leq V_4 \cong 2^6$, and $E \leq \overline{V}_4 \stackrel{\text{def}}{=} Z(O_2(\overline{H}_4)) \cong 2^7$. The action of $H_4/O_2(H_4) \cong Sp_6(2)$ on V_4 is the standard one (see [**W4**, §§3–4]). In particular, E can be neither V_4 nor $\overline{V}_4$ by Proposition 6.5. Thus $P/O_2(H_4)$ or $P/O_2(\overline{H}_4)$ is a nontrivial radical subgroup of $Sp_6(2)$, whose fixed subgroup in V_4 is totally isotropic (with respect to the symplectic form fixed by H_4), and hence of rank ≤ 3 (Proposition 6.4). Hence $\mathrm{rk}(E_0) \leq 3$, and this contradicts Proposition 4.6(b).

- $N_L(E_0) \leq \mathbf{H_5 = 2^{10}{:}M_{22}}$, $N_\Gamma(E) \leq \overline{H}_5 = 2^{10}{:}M_{22}{:}2$, $E \leq \mathbf{Q = V_5 \cong 2^{10}}$. We have already seen that Q is weakly closed in any Sylow 2-subgroup which contains it; and we have just shown that $\mathfrak{R}^{\geq 2}(L\,;2)_{\not\geq Q} = \varnothing$.

$\mathbf{L = Fi_{23}}$: Note that $\mathrm{Out}(L) = 1$ [**A5**, (37.2)]. Fix $S \in \mathrm{Syl}_2(L)$. Then $Z(S) \cong 2^2$, and contains a representative from each of the three classes of involutions in L. Thus by Proposition 4.4(e), E contains involutions from each of the three classes.

By [**A5**, (25.7)], the set of transpositions in S (involutions of type **2A**) generates a subgroup 2^{11}, which thus is weakly closed in S. We let $Q \cong 2^{11}$ denote this subgroup, and will show that $\mathfrak{R}^{\geq 2}(L\,;2)_{\not\geq Q} = \varnothing$.

Let $\mathcal{T}$ be the set of transpositions in E, and set $E_0 = \langle \mathcal{T} \rangle \leq E$. Then E_0 is conjugate to a subgroup of Q. So by [**Fl**, §2], $N_L(E) \leq N_L(E_0)$ is contained in one of the subgroups in the following list. More precisely, if $|\mathcal{T}| = n \leq 3$, then $N_L(E) \leq N_L(\mathcal{T})$, and $N_L(\mathcal{T})$ is conjugate to H_n (the centralizer of the product of the elements in $\mathcal{T}$). If $|\mathcal{T}| \geq 4$, then each subset of $\mathcal{T}$ of order 4 is contained in a unique "heptad" of commuting transpositions, still contained in S, and so the union $\widehat{\mathcal{T}}$ of these heptads is contained in Q, and $N_L(E) \leq N_L(\widehat{\mathcal{T}})$. By [**Fl**, §2], either $\widehat{\mathcal{T}}$ is itself a heptad, in which case $N_L(\widehat{\mathcal{T}})$ is conjugate to H_4; or Q is the unique subgroup in its conjugacy class which contains $\widehat{\mathcal{T}}$, and thus $N_L(\widehat{\mathcal{T}}) \leq N_L(Q) = H_5$.

- $N_L(E) \leq \mathbf{H_1} = \mathbf{C(2A)} \cong \mathbf{2Fi_{22}}$, $E \geq V_1 \cong 2$. By Proposition 4.9(b) (and since $\mathrm{Out}(L) = 1$), $P/V_1 \in \mathfrak{R}^{\geq 2}(Fi_{22}\,;2)$. We have already seen that this implies that P/V_1 contains the subgroup 2^{10} generated by the involutions of type **2A** in some Sylow 2-subgroup of Fi_{22}, and hence that $P \geq Q = V_5$ (up to conjugacy).

- $N_L(E) \leq \mathbf{H_2} = \mathbf{C(2B)} \cong \mathbf{2^2{\cdot}U_6(2).2}$, $V_2 \cong 2^2$. As noted above, this case need occur only if $|\mathcal{T}| = 2$ and $H_2 = N_L(\mathcal{T})$, and thus $V_2 = E_0 \leq E$. Since $\mathrm{Out}(L) = 1$, Proposition 4.9(b) implies that $P/V_2 \in \mathfrak{R}^{\geq 2}(U_6(2){:}2\,;2)$, and this set is empty by Lemma 6.6.

- $N_L(E) \leq \mathbf{H_3} = \mathbf{C(2C)} \cong \mathbf{(2^2 \times 2_+^{1+8}).(3 \times U_4(2)).2}$, $E \leq V_3 \cong 2^3$. Impossible by (1).

- $N_L(E) \leq \mathbf{H_4} \cong \mathbf{2^{6+8}{:}(\Sigma_3 \times A_7)}$, $E \leq V_4 \cong 2^6$. By [**Fl**, §2], $H_4 = N_L(\mathcal{S})$, where $\mathcal{S}$ is a "heptad": a set of seven transpositions whose product is the identity, and the only transpositions in $\langle \mathcal{S} \rangle$. Also, the quotient group A_7 acts by permuting $\mathcal{S}$, and so $\langle \mathcal{S} \rangle = V_4 = Z(O_2(H_4))$. Hence $\mathrm{Aut}_L(E)$ is the stabilizer of E for this permutation action of A_7. Also, $V_4 \notin \mathfrak{E}^{\geq 2}(L\,;2)$ by Proposition 4.6(e), and thus $E \lneqq V_4$. Hence $P/O_2(H_4) \cong P_1 \times P_2$, where $P_1 \leq \Sigma_3$ and $1 \neq P_2 \leq A_7$ are radical 2-subgroups (Lemma 1.5(a)); E is the fixed subgroup of the P_2-action on V_4; and this is impossible since the nontrivial radical 2-subgroups of A_7 all have fixed subgroup on V_4 of rank ≤ 3.

- $N_L(E) \leq \mathbf{H_5} \cong \mathbf{2^{11}{\cdot}M_{23}}$, $E \leq \mathbf{Q} = \mathbf{V_5} \cong \mathbf{2^{11}}$. We have already seen that Q is weakly closed in any Sylow 2-subgroup which contains it; and we have just shown that $\mathfrak{R}^{\geq 2}(L\,;2)_{\not\geq Q} = \varnothing$.

$\mathbf{L = Fi'_{24}}$: By [**A5**, (37.1)], $\mathrm{Out}(L) \cong C_2$, and $\mathrm{Aut}(L) = Fi_{24}$. We write $\Gamma = Fi_{24}$ for short. The group Γ is generated by transpositions: elements in a conjugacy class of involutions in $\Gamma \smallsetminus L$ the product of any two of which has order 2 or 3. By [**A5**, (37.4)], L has two conjugacy classes of involutions: each element of type **2A** is a product of a unique pair of commuting transpositions (its factors), while each element of type **2B** is a product of four commuting transpositions (but not uniquely).

Fix $\widehat{S} \in \mathrm{Syl}_2(\Gamma)$, and set $S = \widehat{S} \cap L \in \mathrm{Syl}_2(L)$. By [**A5**, (25.7)], the set of transpositions in $\widehat{S}$ generates a subgroup $\widehat{Q} \cong 2^{12}$. Set $Q = \widehat{Q} \cap L \cong 2^{11}$. By [**A5**, (34.9)], Q is the Todd module for $N_L(Q)/Q \cong M_{24}$. Hence by [**A3**, 31.11], Q is the unique elementary abelian 2-subgroup of $N_L(Q)$ of rank 11. Since $Q \lhd S$, $N_L(Q) \geq S$, and thus Q is weakly closed in S with respect to $\mathrm{Aut}(L)$. We will show that $\mathfrak{R}^{\geq 2}(L\,;2)_{\not\geq Q} = \varnothing$.

By [**W6**, Theorems D & E] (with corrections in [**LW**, §2]), each 2-local subgroup of L is contained up to conjugacy in one of eight subgroups, labelled here as H_n ($n \leq 8$). In the case of two of these subgroups, $H_2 = N(\mathbf{2B}) \cong 2_+^{1+12}{\cdot}3U_4(3){:}2$ and $H_5 \cong 2^{3+12}.(A_6 \times L_3(2))$, $N_L(E) \not\leq H_n$ by (1). It remains to consider the other cases.

- $N_L(E) \leq \mathbf{H_1} = \mathbf{N(2A)} \cong \mathbf{2Fi_{22}{:}2}$, $E \geq V_1 \cong 2$. Then $N_\Gamma(E) \leq N_\Gamma(H_1)$, since the factors of the generator of V_1 normalize E. By Proposition 4.9(b) again, $P/V_1 \in \mathfrak{R}^{\geq 2}(Fi_{22}{:}2\,;2)$. We have already shown that this implies that P/V_1 contains (up to conjugacy) the subgroup 2^{10} generated by the involutions of type **2A** in any Sylow 2-subgroup of Fi_{22}. Since all involutions in Fi_{22} lift to involutions in $2Fi_{22}$ [**A5**, (23.8)], this shows that P contains a subgroup 2^{11}, which must be conjugate to Q.

- $N_L(E) \leq \mathbf{H_3} \cong \mathbf{2^2{\cdot}U_6(2){:}\Sigma_3}$, $V_3 \cong 2^2$. If $E \not\geq V_3$, then $E \cap V_3 = \langle x \rangle$ for some involution x, and $N_L(E) \leq C_L(x)$ which is conjugate to H_1 or H_2. So we can assume $E \geq V_3$. Also, $V_3 \leq Q$ (up to conjugacy), and all involutions in V_3 are of type **2A**. The factors of the involutions in V_3 all lie in $\widehat{Q}$, and generate a rank three subgroup with just three transpositions, which are permuted by the conjugation action of E and hence normalize E. Thus $N_\Gamma(E) \leq N_\Gamma(H_3)$, and so $P/V_3 \in \mathfrak{R}^{\geq 2}(U_6(2){:}\Sigma_3\,;2)$ by Proposition 4.9(b). But this set is empty by Lemma 6.6.

- $N_L(E) \leq \mathbf{H_4} \cong \mathbf{2^{6+8}{:}(\Sigma_3 \times A_8)}$, $E \leq V_4 \cong 2^6$. Here, $\mathrm{Aut}_L(V_4)$ acts on V_4 via the natural action of $A_8 \cong \Omega_6^+(2)$. This follows from the discussion in [**W6**, p.91], where it is shown that V_4 is the intersection with Fi_{24}' of the subgroup of rank 7 in Fi_{24} generated by the eight transpositions in an octad. Hence $E \lneqq V_4$ by Proposition 4.6(c), since the Sylow 2-subgroups of A_8 are neither dihedral nor semidihedral. So $P/O_2(H_4)$ is a nontrivial radical 2-subgroup of $\Omega_6^+(2)$, and E is the fixed subgroup of its action on V_4. But the fixed subgroup of any such radical 2-subgroup has rank ≤ 3 by Proposition 6.4, and so this case is not possible.

- $N_L(E) \leq \mathbf{H_6} \cong (\mathbf{A_4} \times \mathbf{\Omega_8^+(2){:}3}){:}\mathbf{2}$, $V_6 \cong 2^2$. As in the case $N_L(E) \leq H_3$, we can assume that $E \geq V_6$. Then P is a 2-subgroup of $C_L(V_6) \cong A_4 \times (\Omega_8^+(2){:}3)$, and hence a radical 2-subgroup of $A_4 \times \Omega_8^+(2)$. So by Lemma 1.5(b), $P = V_6 \times P'$, where P' is a radical subgroup of $\Omega_8^+(2)$, and hence of the form $O_2(\mathfrak{P})$ for some parabolic subgroup $\mathfrak{P} \leq \Omega_8^+(2)$ for which $\mathrm{rk}(Z(O_2(\mathfrak{P}))) \geq 4$. The only such subgroups are those of the form $\mathfrak{P} \cong 2^6{:}\Omega_6^+(2)$, which can be described as the subgroup of elements in $\Omega_8^+(2)$ which leave invariant some 1-dimensional isotropic subspace, or some maximal isotropic subspace. This would imply that
$$(E, \mathrm{Aut}_L(E)) \cong (2^2 \times 2^6, (3 \times \Omega_6^+(2)){:}2).$$

By Proposition 6.5, no such group E can be in $\mathfrak{E}^{\geq 2}(L\,;2)$.

- $N_L(E) \leq \mathbf{H_7} \cong \mathbf{2^8{:}\Omega_8^-(2)}$, $E \leq V_7 \cong 2^8$. Then $E \lneqq V_7$ by Proposition 6.5, so P/V_7 is a nontrivial radical 2-subgroup of $H_7/V_7 \cong \Omega_8^-(2)$, and E is the fixed subgroup of its action on V_7. By Proposition 6.4, the fixed subgroup is a totally isotropic subspace of $V_7 \cong \mathbb{F}_2^8$, hence of rank ≤ 3 (since the quadratic form on V_7 has negative type); and thus $E \notin \mathfrak{E}^{\geq 2}(L\,;2)$.

- $N_L(E) \leq \mathbf{H_8} \cong \mathbf{2^{11}{\cdot}M_{24}}$, $E \leq \mathbf{Q} = \mathbf{V_8} \cong \mathbf{2^{11}}$. We have already seen that Q is weakly closed in S with respect to $\mathrm{Aut}(L)$; and have now shown that $\mathfrak{R}^{\geq 2}(L\,;2)_{\not\geq Q} = \varnothing$.

$\mathbf{L} = \mathbf{F_2}$: This group contains four conjugacy classes of involutions. By [**Gr1**, Theorem 3], or by [**Sg**, Theorem 5.6], $\mathrm{Out}(L) = 1$. By [**MS**], every 2-local subgroup of L is conjugate to a subgroup of one of the groups H_n $(n = 1, \ldots, 8)$ described here. In the case of three of these subgroups, $H_1 = N(\mathbf{2B}) \cong 2^{1+22}.Co_2$, $H_2 \cong 2^{2+10+20}.(M_{22}{:}2 \times \Sigma_3)$, and $H_4 \cong 2^{5+5+10+10}.L_5(2)$, $N_L(E) \not\leq H_n$ by (1) or (2). It remains to consider the other five cases.

- $N_L(E) \leq \mathbf{H_3} \cong [\mathbf{2^{35}}].(\mathbf{\Sigma_5} \times \mathbf{L_3(2)})$, $E \leq V_3$. The construction of this subgroup H_3 is described in [**A3**, pp.219–220]. In particular, $H_3 = N_L(W)$ for a certain subgroup $W \leq O_2(H_1) \cong 2^{1+22}$ of rank 3 (and with $Z(H_1) \leq W$), and $H_3/C_L(W) \cong L_3(2)$. Set $Q = O_2(H_1) \cong 2_+^{1+22}$, $\langle z \rangle = Z(Q)$, and $R = O_2(H_3)$ for short. Then $C_L(W) \leq C_L(z) = H_1$ (since $z \in W$), hence $C_Q(W) \lhd C_L(W)$ is a normal 2-subgroup, and so $C_Q(W) \lhd R = O_2(C_L(W))$. Thus $Z(R) \leq C_{H_1}(R) \leq C_{H_1}(C_Q(W))$. Set $W' = C_Q(W)$. Since $Q \cong 2_+^{1+22}$ is extraspecial, $Q \cap Z(R) = C_Q(W') = W$. Also, $[C_{H_1}(W'){:}W] \leq 2$, since the subgroup of elements in $\mathrm{Out}(Q)$ whose restriction to W' is the identity mod $\langle z \rangle$ has order 2, and thus $\mathrm{rk}(V_3/W) \leq 1$. Thus $W \cap E$ is an $N_{\mathrm{Aut}(L)}(E)$-invariant subgroup of rank ≤ 3 and index ≤ 2 in E, and hence $E \notin \mathfrak{E}^{\geq 2}(L\,;2)$. (In fact, $V_3 = W$, shown using the properties of the Co_2-representation on $Q/Z(Q) \cong \mathbb{F}_2^{22}$, as explained to me in detail by Sergey Shpektorov.)

- $N_L(E) \leq \mathbf{H_5} \cong \mathbf{2^{9+16}.Sp_8(2)}$, $E \leq V_5 \cong 2^9$. Then $E \leq V_5 \cong 2^9$, and $P/O_2(H_5)$ is a 2-radical subgroup of $H_5/O_2(H_5) \cong Sp_8(2)$. By the description of this subgroup in [**A3**, pp.218–219], there is a subgroup $V_5' \leq V_5$ of rank 8 upon which $H_5/O_2(H_5)$ acts by preserving a symplectic form $\mathfrak{b}$. Set $E' = E \cap V_5'$; then $E' \lhd N_L(E)$ since $V_5' \lhd H_5$. If $P = O_2(H_5)$, then $E = V_5$ and $\mathrm{Aut}_L(E) \cong Sp_8(2)$, which contradicts Proposition 6.5. Otherwise, $P/O_2(H_5)$ is a nontrivial radical subgroup of $H_5/O_2(H_5) \cong Sp_8(2)$, and by Proposition 6.4, $E' = C_{V_5'}(P)$ is a totally isotropic subgroup of V_5' with respect to $\mathfrak{b}$, and each automorphism of E' is induced by some element of $N_L(P)$. Thus $\mathrm{rk}(E') = 4$, $\mathrm{Aut}_L(E) \cong L_4(2)$ or $2^4{:}L_4(2)$, the first possibility contradicts Proposition 4.7(a), and the second is impossible since E is pivotal $(O_2(\mathrm{Aut}_L(E)) = 1)$.

- $N_L(E) \leq \mathbf{H_6} = \mathbf{N(2A)} \cong \mathbf{2{\cdot}{}^2E_6(2){:}2}$, $P \geq V_6 \cong 2$. By Proposition 4.9(b), $P/V_6 \in \mathfrak{R}^{\geq 2}(H_6/V_6\,;2)$. In particular, $P_0/V_6 \stackrel{\text{def}}{=} (P/V_6) \cap {}^2E_6(2)$ is a radical 2-subgroup of ${}^2E_6(2)$. Of the four maximal parabolic subgroups $\mathfrak{P}_J$ of ${}^2E_6(2)$, there is only one for which $Z(U_J)$ ($U_J = O_2(\mathfrak{P}_J)$) has center of rank ≥ 4. This case arises when J is the set of the two primitive roots corresponding to the extremities in the Dynkin diagram of E_6, and hence $P_0/V_6 = U_J \cong 2^{8+16}$, with normalizer $2^{8+16}{:}SO_8^-(2)$ in ${}^2E_6(2){:}2$. We can thus assume $N_L(P) \leq N_L(E) \leq N_L(P_0)$. By [**A3**, pp.218–219], $P_0 = O_2(H_5)$ and $N_L(P_0) = H_5$ (up to conjugacy), so $N_L(E) \leq H_5$, and we are reduced back to that case.

- $N_L(E) \leq \mathbf{H_7} = \mathbf{N(2C)} \cong \mathbf{(2^2 \times F_4(2)){:}2}$. Then $P \cap (2^2 \times F_4(2)) = V_2 \times P_0$ for some radical 2-subgroup $P_0 \leq F_4(2)$. Thus $P_0 = O_2(\mathfrak{P})$ for some parabolic subgroup $\mathfrak{P} \leq F_4(2)$. Let $I = \{r_1, r_2, r_3, r_4\}$ denote the set of simple roots of F_4, labelled in order along the Dynkin diagram. Set $\mathfrak{P}_i = \mathfrak{P}_{\{r_i\}}$ and $U_i = U_{\{r_i\}}$.

  In particular, $\mathfrak{P}_1$ and $\mathfrak{P}_2$ represent the two classes of maximal parabolic subgroups of $F_4(2)$ up to conjugacy in its automorphism group. Also, $\mathfrak{P}_1/U_1 \cong Sp_6(2) \cong \Omega_7(2)$ acts on $Z(U_1) \cong 2^7$ in the standard way; while $\mathfrak{P}_2/U_2 \cong L_2(2) \times L_3(2)$ acts on $Z(U_2) \cong 2^5$ as a product of an $L_2(2)$-action on 2^2 and an $L_3(2)$-action on 2^3. In both cases, the Sylow 2-subgroups of $\mathfrak{P}_i/U_i$ are neither dihedral nor semidihedral, so P_0 cannot be conjugate to either of these by Proposition 4.6(c).

  Thus $P_0 \gneqq U_i$, $Z(P_0) \leq Z(U_i)$, and $N_{F_4(2)}(P_0) \lneqq \mathfrak{P}_i$ (up to conjugacy) for some i. But in either case ($i = 1$ or 2), E has no $N_L(E)$-irreducible components of rank ≥ 4 (any nontrivial radical subgroup of $Sp_6(2)$ fixes an isotropic subgroup of rank ≤ 3 by Proposition 6.4), and thus $P \notin \mathfrak{R}^{\geq 2}(L\,;2)$ by Proposition 4.6(b).

- $N_L(E) \leq \mathbf{H_8} = \mathbf{N(2C^2)} \cong \mathbf{\Sigma_4 \times {}^2F_4(2)}$. The radical 2-subgroups of ${}^2F_4(2)$ have centers of rank one or two (see [**W3**] or [**P1**]), and hence $E \notin \mathfrak{E}^{\geq 2}(L\,;2)$ and $P \notin \mathfrak{R}^{\geq 2}(L\,;2)$.

$\mathbf{L = F_1}$: By [**Gr1**, Theorem 3], or by [**GMS**, Theorem 5.10], $\mathrm{Out}(L) = 1$. By [**MS**], every 2-local subgroup of L is conjugate to a subgroup of one of seven subgroups H_n ($n = 1, \ldots, 7$) described here. In the case of four of these subgroups, $H_3 = N(\mathbf{2B}) \cong 2^{1+24}.Co_1$, $H_4 \cong 2^{2+11+22}.(M_{24} \times \Sigma_3)$, $H_5 \cong 2^{3+6+12+18}.(L_3(2) \times 3\Sigma_6)$, and $H_6 \cong 2^{5+10+20}.(L_5(2) \times \Sigma_3)$, $N_L(E) \nleq H_n$ by (1) or (2). It remains to consider the other three cases.

- $N_L(E) \leq \mathbf{H_1} = \mathbf{N(2A)} \cong \mathbf{2{\cdot}F_2}$, $E \geq V_1 \cong 2$. By Proposition 4.9(b), $P/V_1 \in \mathfrak{R}^{\geq 2}(F_2\,;2)$, and we have already shown that this last set is empty.

- $N_L(E) \leq \mathbf{H_2} = \mathbf{N(2A^2)} \cong \mathbf{2^2{\cdot}{}^2E_6(2){:}\Sigma_3}$, $V_2 \cong 2^2$. We can assume that $E \geq V_2$, since otherwise $N_L(E)$ is contained in the centralizer of $V_2 \cap E = \langle x \rangle$ for some $x \in \mathbf{2A}$ (and is thus conjugate to a subgroup of H_1). Then $(P/V_2) \cap {}^2E_6(2)$ is a radical 2-subgroup of ${}^2E_6(2)$. By the same reasoning as that used for the subgroup $H_6 \leq F_2$, we are reduced to the case where $N_L(E) \leq N_L(P_0)$ for a certain maximal parabolic subgroup $N_{H_2}(P_0)/V_2$ of ${}^2E_6(2){:}\Sigma_3$, $N_L(P_0)$ is conjugate to H_7 (by the construction in [**A3**, pp.218–219] again), and we are reduced to considering that case.

- $N_L(E) \leq \mathbf{H_7} \cong \mathbf{2^{10+16}.\Omega_{10}^+(2)}$, $E \leq V_7 \cong 2^{10}$. Then $P \geq O_2(H_7)$, and $P/O_2(H_7)$ is a radical 2-subgroup of $H_7/O_2(H_7) \cong \Omega_{10}^+(2)$. By the description in

[**A3**, pp.218–219], $H_7/O_2(H_7) \cong \Omega_{10}^+(2)$ acts on V_7 in the canonical way, preserving a quadratic form $\mathfrak{q}$. If $P = O_2(H_7)$, then $E = V_7$ and $\mathrm{Aut}_L(E) \cong \Omega_{10}^+(2)$, and this contradicts Proposition 6.5. Otherwise, $P/O_2(H_7)$ is a nontrivial radical subgroup of $H_7/O_2(H_7)$, and by Proposition 6.4, $E = C_{V_7}(P)$ is a totally isotropic subgroup of V_7 with respect to $\mathfrak{q}$ and $\mathrm{Aut}_L(E) = \mathrm{Aut}(E)$. But this contradicts Proposition 4.7(a).

This finishes the proof of Theorem 9.1. □

CHAPTER 10

Computations of $\lim^1(\mathcal{Z}_G)$

In this chapter, we summarize what we know about the groups $\varprojlim^1(\mathcal{Y}^{\Gamma}_{\widetilde{L}})$, with indications of the proof in some cases. In all cases, $\varprojlim^0(\mathcal{Y}^{\Gamma}_{\widetilde{L}}) = 0$ by Glauberman's Z^*-theorem [**Gl**]. Using this, the following proposition follows by essentially the same argument as that used to prove Proposition 4.2.

PROPOSITION 10.1. *Fix a finite simple group L and $S \in \mathrm{Syl}_2(L)$. Assume $Q \leq S$ is 2-centric in L and weakly closed in S with respect to* $\mathrm{Aut}(L)$*, and that it has the property that $\mathfrak{R}^{\geq 2}(L\,;p)_{\ngeq Q} = \varnothing$, and that no subgroup in $\mathfrak{R}^1(L\,;p)_{\ngeq Q}$ is contained in any other subgroup in this set. Let $\widetilde{L}$ be a quasisimple group such that $Z(\widetilde{L})$ is a 2-group and $\widetilde{L}/Z(\widetilde{L}) \cong L$, and let $\Gamma \leq \mathrm{Aut}(\widetilde{L})$ be a subgroup which contains* $\mathrm{Inn}(\widetilde{L}) \cong L$*. Then for any set $P_1, \ldots, P_k$ of Γ-conjugacy class representatives for subgroups in $\mathfrak{R}^1(L\,;p)_{\ngeq Q}$,*

$$\varprojlim{}^1(\mathcal{Y}^{\Gamma}_{\widetilde{L}}) \cong \mathrm{Ker}\Big[\bigoplus_{i=1}^{k} \Lambda^1(N_\Gamma(P_i), N_L(P_i); \mathcal{Y}^{\Gamma}_{\widetilde{L}}(P_i)) \longrightarrow\!\!\!\!\rightarrow H^0(N_\Gamma(Q); \mathcal{Y}^{\Gamma}_{\widetilde{L}}(Q))\Big]$$

for some surjection between these two groups.

When $L \cong A_n$, then

$$\varprojlim_{\mathcal{O}_2(\Gamma)}{}^1(\mathcal{Y}^{\Gamma}_{\widetilde{L}}) \cong \begin{cases} \mathbb{Z}/2 & \text{if } \widetilde{L} \cong A_n,\ \Gamma \leq \Sigma_n,\ n \equiv 2,3 \pmod 4 \\ 0 & \text{otherwise;} \end{cases}$$

When $\Gamma \leq \Sigma_n$, this is shown via an easy modification of the proof of Theorem 5.1. (The case $L = A_6 \cong PSL_2(9)$ and $\Gamma \nleq \Sigma_6$ must be handled separately.) Recall that we write $E_{2^k} \leq \Sigma_n$ for the elementary abelian group 2^k acting with one free orbit, and that $Q \leq A_n$ denotes the product of $[n/4]$ copies of E_4. Then $\mathfrak{R}^1(A_n\,;2)_{\ngeq Q}$ is empty if $n \equiv 0,1 \pmod 4$; and contains the conjugacy class of the group $A_n \cap \Big(S' \times E_2^{\times 3}\Big)$ for some $S' \in \mathrm{Syl}_2(\Sigma_{n-6})$ if $n \equiv 2,3 \pmod 4$. So $A_n \in \mathfrak{L}^1(2)$ in the first case by Proposition 4.2, and one gets the above computations using Proposition 10.1 in the second case.

Note, however, that $\varprojlim^1(\mathcal{Z}_{\Sigma_n}) = 0$ for all n, even in the cases when $\varprojlim^1(\mathcal{Y}^{\Sigma_n}_{A_n}) \cong \mathbb{Z}/2$. This follows from the observation that $\varprojlim^0(\mathcal{Z}_{\Sigma_n}/\mathcal{Y}^{\Sigma_n}_{A_n}) \cong \mathbb{Z}/2$ when $n \equiv 2,3 \pmod 4$.

When L is of Lie type in characteristic two, then by Theorem 6.2, $L \in \mathfrak{L}^1(2)$ except when $L = L_3(2) \cong L_2(7)$. When L is of Lie type in odd characteristic, and not isomorphic to $E_7(q)$ or $E_8(q)$, then $L \in \mathfrak{L}^1(2)$ except for the following cases:

- $\varprojlim^1(\mathcal{Y}_L^\Gamma) \cong \mathbb{Z}/2$ if $L \cong PSL_2(q) \cong \Omega_3(q)$, $q \equiv \pm 1 \pmod 8$, $\Gamma \le \mathrm{Aut}_{fg}(L)$
- $\varprojlim^1(\mathcal{Y}_L^\Gamma) \cong \mathbb{Z}/2$ if $L \cong PSL_4(q)$, $q \equiv 3 \pmod 4$, $\Gamma \le \mathrm{Aut}_{fg}(L)$
- $\varprojlim^1(\mathcal{Y}_L^\Gamma) \cong \mathbb{Z}/2$ if $L \cong PSU_4(q)$, $q \equiv 1 \pmod 4$, $\Gamma \le \mathrm{Aut}_{fg}(L)$
- $\varprojlim^1(\mathcal{Y}_L^\Gamma) \cong \mathbb{Z}/2$ if $L \cong \Omega_{2n}^{\mathrm{ns}}(q)$, $n \ge 3$, $\Gamma \le \mathrm{Aut}_{fg}(L)$.

Here, $\mathrm{Aut}_{fg}(L)$ denotes the group generated by inner, field, and graph automorphisms of L. Also, $\Omega_{2n}^{\mathrm{ns}}(q) = \Omega(\mathbb{F}_q^{2n}, \mathfrak{q})$ where $\mathfrak{q}$ is a quadratic form with nonsquare discriminant. The case $L = PSL_2(q)$ is shown using Proposition 1.6 (or an easy modification of the proposition and its proof). In all other cases, there is a weakly closed subgroup Q such that $\mathfrak{R}^1(L;2)_{\not\ge Q}$ contains at most one conjugacy class, or in certain cases two L-conjugacy classes which are $\mathrm{Aut}(L)$-conjugate. The results then follow from Propositions 4.2 and 10.1.

The sporadic groups all lie in $\mathfrak{L}^1(2)$, with the possible exception of the baby monster and the monster. This is shown, either by modifying and extending the arguments used in Chapter 9, or by using lists of radical 2-subgroups of these groups such as those published by Yoshiara [**Y1**] [**Y2**] and by An & O'Brien [**AO**].

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