Reduced fusion systems over 2-groups of sectional rank at most 4

Bob Oliver

Author address:

Universit Paris 13, Sorbonne Paris Cit, LAGA, UMR 7539 du CNRS, 99, Av. J.-B. Clment, 93430 Villetaneuse, France.

E-mail address: bobol@math.univ-paris13.fr

Contents

Abstract

We classify all reduced, indecomposable fusion systems over finite 2-groups of sectional rank at most 4. The resulting list is very similar to that by Gorenstein and Harada of all simple groups of sectional 2-rank at most 4. But our method of proof is very different from theirs, and is based on an analysis of the essential subgroups which can occur in the fusion systems.

2010 *Mathematics Subject Classification.* Primary 20D20. Secondary 20D05, 20E32, 20E45.

Key words and phrases. finite groups, finite simple groups, Sylow subgroups, fusion.

B. Oliver is partially supported by UMR 7539 of the CNRS, and by project ANR BLAN08-2_338236, HGRT.

Introduction

A *saturated fusion system* $\mathcal{F}$ over a finite p-group S is a category whose objects are the subgroups of S, whose morphisms are injective homomorphisms between subgroups, and where the morphism sets satisfy certain axioms first formulated by Puig and motivated by the properties of conjugacy relations among p-subgroups of a finite group. In particular, for each finite group G and each Sylow p-subgroup $S \leq G$, the category $\mathcal{F}_S(G)$ whose objects are the subgroups of G and whose morphisms are those homomorphisms induced by conjugation in G is a saturated fusion system over S. We refer to Puig's paper [**Pg**], and to [**AKO**] and [**Cr**], for more background details on saturated fusion systems.

A saturated fusion system $\mathcal{F}$ is *reduced* if it contains no nontrivial normal p-subgroups, and no proper normal subsystems of p-power index or of index prime to p. All of these concepts are defined by analogy with finite groups; the precise definitions are given in Section 1.2. The class of reduced fusion systems is larger than that of simple fusion systems, although a reduced fusion system which is not simple has to be fairly large. We refer to main theorems in [**AOV1**] for the motivation for defining this class.

The *sectional p-rank* of a finite group G is the largest possible value of $\mathrm{rk}(P/Q)$, where $Q \trianglelefteq P \leq G$ are p-subgroups and P/Q is elementary abelian. When G is a p-group, we just call this the sectional rank, and denote it $r(G)$. In their book which appeared in 1974, Gorenstein and Harada [**GH**] gave a classification of all finite simple groups whose sectional 2-rank is at most 4.

A fusion system is *indecomposable* if it is not isomorphic to a product of fusion systems over smaller p-groups. The following theorem, where we list all reduced, indecomposable fusion systems over finite 2-groups of sectional rank at most 4, is the main result of this paper. We refer to the end of the introduction for the notation used for certain central products and semidirect products. When q is a prime power and $n \geq 2$, $UT_n(q)$ denotes the group of upper triangular matrices over $\mathbb{F}_q$ with 1's on the diagonal. Also, we write $L_n^+(q) = PSL_n(q)$ and $L_n^-(q) = PSU_n(q)$.

A fusion system is *simple* if it contains no nontrivial proper normal subsystems. We refer to [**AKO**, Definition I.6.1] for the precise definition of a normal subsystem. Here, we just note that a reduced fusion system $\mathcal{F}$ over S is simple if S contains no nontrivial proper subgroup strongly closed in $\mathcal{F}$.

Theorem A. *Let $\mathcal{F}$ be a reduced, indecomposable fusion system over a nontrivial 2-group S of sectional rank at most 4. Then one of the following holds.*

(1) *$S \cong D_{2^k}$ for some $k \geq 3$, and $\mathcal{F}$ is isomorphic to the fusion system of $L_2^{\pm}(q)$ (when $v_2(q \pm 1) = k$).*

(2) *$S \cong SD_{2^k}$ for some $k \geq 4$, and $\mathcal{F}$ is isomorphic to the fusion system of $L_3^{\pm}(q)$ (when $v_2(q \pm 1) = k - 2$).*

(3) *$S \cong C_{2^k} \wr C_2$ for some $k \geq 2$, and $\mathcal{F}$ is isomorphic to the fusion system of $L_3^{\pm}(q)$ (when $v_2(q \mp 1) = k$).*

(4) *$S \cong (C_{2^k} \times C_{2^k}) \overset{-1,t}{\rtimes} C_2^2$ for some $k \geq 2$, and $\mathcal{F}$ is isomorphic to the fusion system of $G_2(q)$ (when $v_2(q \pm 1) = k$), or of M_{12} (if $k = 2$).*

(5) *$S \cong (D_{2^k} \times_{C_2} D_{2^k}) \overset{t}{\rtimes} C_2 \cong (Q_{2^k} \times_{C_2} Q_{2^k}) \overset{t}{\rtimes} C_2$ for some $k \geq 3$, and $\mathcal{F}$ is isomorphic to the fusion system of $PSp_4(q)$ (when $v_2(q^2 - 1) = k$), or of $GL_4(2) \cong A_8$ (if $k = 3$).*

(6) *$S \cong D_{2^k} \wr C_2$ for some $k \geq 3$, and $\mathcal{F}$ is isomorphic to the fusion system of $L_4^{\pm}(q)$ (when $v_2(q \pm 1) = k - 1$), or of A_{10} (if $k = 3$).*

(7) *$S \cong SD_{2^k} \wr C_2$ for some $k \geq 4$, and $\mathcal{F}$ is isomorphic to the fusion system of $L_5^{\pm}(q)$ (when $v_2(q \pm 1) = k - 2$).*

(8) *S contains a normal subgroup $T \cong UT_3(4)$, where $[S{:}T] \leq 4$ and $\mathrm{Aut}_S(T)$ is generated by field and/or graph automorphisms; and $\mathcal{F}$ is isomorphic to the fusion system of $PSL_3(4)$, $L_4^{\pm}(q)$ for $q \equiv \pm 5 \pmod 8$, M_{22}, M_{23}, McL, J_2, J_3, or Ly.*

Conversely, if G is any of the groups listed in (1)–(8) and $S \in \mathrm{Syl}_2(G)$, then $\mathcal{F}_S(G)$ is indecomposable and reduced, and is in fact simple.

Certain simple groups with sectional 2-rank 4, such as those with abelian Sylow 2-subgroup, do not appear in the above list because their fusion system is not reduced. (See Proposition 1.12(b) for more detail.) A few other simple groups, such as A_7 and M_{11}, fail to appear because their fusion system is isomorphic to that of another simple group in the list.

It will be convenient to have names for some of the classes of 2-groups which appear in the statement of Theorem A. See the end of the introduction for an explanation of the notation used, especially that used for semidirect products.

Definition 0.1. Fix a finite 2-group S.

- $S \in \mathcal{D}$ if $S \cong D_{2^n}$ for some $n \geq 3$.
- $S \in \mathcal{Q}$ if $S \cong Q_{2^n}$ for some $n \geq 3$.
- $S \in \mathcal{S}$ if $S \cong SD_{2^n}$ for some $n \geq 4$.
- $S \in \mathcal{W}$ if $S \cong C_{2^n} \wr C_2$ for some $n \geq 2$.
- $S \in \mathcal{V}$ if $S \cong \Delta \wr C_2$ or $S \cong (\Delta \times_{C_2} \Delta) \overset{t}{\rtimes} C_2$ for some $\Delta \in \mathcal{D}$ or $\Delta \in \mathcal{S}$.
- $S \in \mathcal{G}$ if $S = (C_{2^n} \times C_{2^n}) \overset{t,\lambda}{\rtimes} C_2^2$, for $n \geq 2$, and for $\lambda = -1$ or $\lambda = 2^{n-1} - 1$ (the latter only if $n \geq 3$). (When $\lambda = -1$, these are all of type $G_2(q)$ for odd q.)
- $S \in \mathcal{U}$ if there is $T \trianglelefteq S$ such that $T \cong UT_3(4)$, and $C_{S/Z(T)}(T/Z(T)) = T/Z(T)$.
- Juxtaposition of these symbols denotes union; e.g., $\mathcal{DSQ}$ is the family of 2-groups which are (nonabelian) dihedral, semidihedral, or quaternion.

- If $\mathcal{C}$ is among the classes listed above, then $S \in \mathcal{C}{\times}\mathcal{C}$ if $S = S_1 \times S_2$ where $S_1, S_2 \in \mathcal{C}$.

Note that $UT_4(2) \cong (D_8 \times_{C_2} D_8) \overset{t}{\rtimes} C_2 \in \mathcal{V}$ (Lemma C.4).

Proof of Theorem A. Fix a reduced, indecomposable fusion system $\mathcal{F}$ over a 2-group S of sectional rank at most 4. By the results of Section 3, summarized in Theorem 3.1, $S \in \mathcal{DSWGUV}$ or S has type $\mathrm{Aut}(M_{12})$. If S has type $\mathrm{Aut}(M_{12})$, then by Proposition 4.3, there are no reduced fusion systems over S.

By [**BMO**, Theorem A(d)], if $\mathcal{F}$ is the fusion system (at the prime 2) of $PSU_n(q)$ for some odd prime power q, then it is also the fusion system of $PSL_n(q')$ for any q' such that $\overline{\langle q' \rangle} = \overline{\langle -q \rangle}$ as closed subgroups of $\mathbb{Z}_2^\times$. Hence for each statement in Theorem A about fusion systems of $PSL_n^\pm(q)$, it suffices to handle the linear case.

When $S \in \mathcal{DSW}$, $\mathcal{F}$ is as described in (1)–(3) by [**AOV1**, Propositions 4.3 & 4.4] and [**AOV2**, Proposition 3.1]. When $S \in \mathcal{G}$, $\mathcal{F}$ is as in (4) by Proposition 4.2; and when $S \in \mathcal{V}$ (cases (5)–(7), and including the case $S \cong UT_4(2)$) by Propositions 5.1, 5.5, and 5.6.

Assume $S \in \mathcal{U}$: an extension of $UT_3(4)$ as described above. The isomorphism classes in $\mathcal{U}$ are listed in Lemma 6.2(a). Reduced fusion systems over 2-groups of type M_{22} or J_2 (denoted S_ϕ and S_θ in Lemma 6.2) are listed in [**OV**, Theorems 4.8 & 5.11]. The remaining cases are handled in Propositions 6.4, 6.5, and 6.6.

Conversely, assume G is one of the simple groups listed in (1)–(8), fix $S \in \mathrm{Syl}_2(G)$, and set $\mathcal{F} = \mathcal{F}_S(G)$. Then $O^2(\mathcal{F}) = \mathcal{F}$ and $O_2(\mathcal{F}) = 1$ by Proposition 1.12(a,b). Also, by [**Gd1**, Theorem A] and [**Ft**, Theorem 1], S has no proper subgroups strongly closed in G. Hence $\mathcal{F}$ is indecomposable, and if it has any proper normal subsystems, they must be over S, and hence contain $O^{2'}(\mathcal{F})$ by [**AOV1**, Lemma 1.26]. So $\mathcal{F}$ is reduced and simple if $O^{2'}(\mathcal{F}) = \mathcal{F}$.

If $S \in \mathcal{DS}$ or $S \in \mathcal{W}$, then $\mathrm{Aut}(S)$ is a 2-group by point (b) or (a), respectively, in Corollary A.10. If $S \in \mathcal{V}$ and $S \ncong UT_4(2)$, or if S has type Ly, then $\mathscr{Y}(S) \neq \varnothing$ (see Definition 2.1) and hence $\mathrm{Aut}(S)$ is a 2-group by Corollary 2.5. If $S \in \mathcal{G}$, then $\mathrm{Aut}(S)$ is a 2-group by Proposition 4.2. Hence $O^{2'}(\mathcal{F}) = \mathcal{F}$ in all of these cases by Proposition 1.12(c), and $\mathcal{F}$ is reduced and simple.

If S is of type M_{22} or J_2, then $\mathcal{F}$ is reduced by [**AOV1**, Proposition 4.5]. If $S \cong UT_4(2)$ or $UT_3(4)$, then $\mathcal{F}$ is reduced by Proposition 5.1 or 6.4, respectively. □

The main idea behind our proof of Theorem A is to analyze and classify reduced fusion systems by studying their essential subgroups. These are subgroups whose automorphisms generate the fusion system (see Definition 1.1 and Proposition 1.6), and we refer to Theorem 3.1 for a brief summary of results in Section 3 describing them. The main tools used for handling essential subgroups are Bender's classification of finite groups with strongly 2-embedded subgroups [**Be**, Satz 1], and Goldschmidt's classification of amalgams of index $(3,3)$ [**Gd2**, Theorem A].

It is unclear to me whether or not this paper, when combined with the deep group theoretic results classifying finite simple groups having Sylow 2-subgroups in certain families, gives a shorter proof of the Gorenstein-Harada theorem than that in [**GH**]. In any case, that is not our goal here. Our proof of Theorem A is organized very differently from that by Gorenstein and Harada, by setting focus on the essential subgroups in the fusion systems rather than on the centralizers of involutions, and we hope that this approach can give some new insight into the classification of these groups.

The paper is organized as follows. Section 1 is mostly a review of background results on fusion systems. The properties of certain families of subgroups of 2-groups are studied in Section 2, and this is applied in Section 3 to describe the (potential) essential subgroups and prove Theorem 3.1. This is then followed by three chapters dealing with fusion systems over the families $\mathcal{G}$, $\mathcal{V}$, and $\mathcal{U}$, respectively. Fusion systems over groups in the families $\mathcal{DSW}$ were studied in the earlier papers [**AOV1**] and [**AOV2**]. Background results on groups and actions on groups are then collected in the appendices.

Notation and terminology: Most of the notation used here is standard among group theorists. For a prime p, "p-group" always means a finite p-group. For a group G, $Z_i(G)$ denotes the i-th term in the upper central series for G; thus $Z_1(G) = Z(G)$ and $Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G))$. Also, $G^\# = G \smallsetminus \{1\}$, and $I(G)$ is the set of involutions in G (elements of order 2). When G and H are finite groups, and Z is identified as a subgroup of $Z(G)$ and of $Z(H)$, then $G \times_Z H$ denotes the central product:

$$G \times_Z H = (G \times H)/\{(z, z^{-1}) \mid z \in Z\}.$$

When G is a finite group and S is a 2-group, S is "of type G" if S is isomorphic to a Sylow 2-subgroup of G. Also, C_n, D_n, Q_n, and SD_n denote cyclic, dihedral, quaternion, and semidihedral groups of order n, and $2_+^{1+4} = Q_8 \times_{C_2} Q_8 \cong D_8 \times_{C_2} D_8$ and $2_-^{1+4} = D_8 \times_{C_2} Q_8$. When $H \leq G$ is a subgroup, we write

$$\langle H^G \rangle = \langle H^g \mid g \in G \rangle$$

for the normal closure of H in G.

As perhaps less standard notation, for a group G, we set

$$G^{\mathrm{ab}} = G/[G,G]\,,$$

the abelianization of G; and let

$$[\alpha] = \alpha{\cdot}\mathrm{Inn}(G) \in \mathrm{Out}(G)$$

denote the class of $\alpha \in \mathrm{Aut}(G)$.

When A is a finite abelian group, B is cyclic, and $\lambda \in \mathbb{Z}$ is prime to $|A|$, we let $A \overset{\lambda}{\rtimes} B$ denote the semidirect product in which a generator of B acts on A via $a \mapsto a^\lambda$. When A is any group and B is cyclic, then $(A \times A) \overset{t}{\rtimes} B$ denotes the semidirect product where a generator of B exchanges the two factors A, and similarly for $(A \times_Z A) \overset{t}{\rtimes} B$ when $Z \leq Z(A)$. Similarly, when A is abelian, $(A \times A) \overset{\lambda,t}{\rtimes} C_2^2$ is the semidirect product where one generator of C_2^2 acts via $g \mapsto g^\lambda$ and the other acts by exchanging the factors.

When $q = 2^k$ and $n \geq 2$, $UT_n(q) \in \mathrm{Syl}_2(SL_n(q))$ denotes the subgroup of strict upper triangular natrices. For $1 \leq i < j \leq n$ and $a \in \mathbb{F}_q$, $e_{ij}^a \in UT_n(q)$ is the elementary matrix whose unique nonzero off-diagonal entry is a in position (i,j). When $q = 2$, we write $e_{ij} = e_{ij}^1$.

I would like very much to thank Andy Chermak for first telling me about Goldschmidt's classification of amalgams. That was when I became convinced that this project should be possible. I would also like to thank the referee for going through the paper in great detail and making many very helpful suggestions.

CHAPTER 1

Background on fusion systems

A *saturated fusion system* over a p-group S is a category $\mathcal{F}$ whose objects are the subgroups of S, and where for each $P, Q \le S$, $\mathrm{Mor}_{\mathcal{F}}(P,Q)$ is a set of injective homomorphisms from P to Q which includes all morphisms induced by conjugation in S, and which satisfies a set of axioms which are described, for example, in [**AKO**, § I.2], [**BLO2**, Definition 1.2], or [**Cr**, Definition 4.11]. We write $\mathrm{Hom}_{\mathcal{F}}(P,Q) = \mathrm{Mor}_{\mathcal{F}}(P,Q)$ to emphasize that the morphisms are all homomorphisms.

The following terminology for subgroups in a fusion system will be used frequently. Recall that a subgroup $H < G$ is *strongly p-embedded* if $p \big| |H|$, and $p \nmid |H \cap {}^gH|$ for $g \in G \smallsetminus H$.

Definition 1.1. Fix a prime p, a p-group S, and a saturated fusion system $\mathcal{F}$ over S. Let $P \le S$ be any subgroup.

- Let $P^{\mathcal{F}}$ denote the set of subgroups of S which are $\mathcal{F}$-conjugate (isomorphic in $\mathcal{F}$) to P. Similarly, $g^{\mathcal{F}}$ denotes the $\mathcal{F}$-conjugacy class of an element $g \in S$.
- P is *fully normalized* in $\mathcal{F}$ (*fully centralized* in $\mathcal{F}$) if $|N_S(P)| \ge |N_S(R)|$ ($|C_S(P)| \le |C_S(R)|$) for each $R \in P^{\mathcal{F}}$.
- P is *fully automized* in $\mathcal{F}$ if $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$.
- P is *$\mathcal{F}$-centric* if $C_S(P') = Z(P')$ for all P' which is $\mathcal{F}$-conjugate to P.
- P is *$\mathcal{F}$-essential* if P is $\mathcal{F}$-centric and fully normalized in $\mathcal{F}$, and $\mathrm{Out}_{\mathcal{F}}(P)$ contains a strongly p-embedded subgroup. Let $\mathbf{E}_{\mathcal{F}}$ denote the set of all $\mathcal{F}$-essential subgroups of S.
- P is *central in* $\mathcal{F}$ if every morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$ in $\mathcal{F}$ extends to a morphism $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(PQ,PR)$ such that $\overline{\varphi}|_P = \mathrm{Id}_P$.
- P is *normal in* $\mathcal{F}$ if every morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$ in $\mathcal{F}$ extends to a morphism $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(PQ,PR)$ such that $\overline{\varphi}(P) = P$.
- For any $\varphi \in \mathrm{Aut}(S)$, ${}^{\varphi}\mathcal{F}$ denotes the fusion system over S defined by
$$\mathrm{Hom}_{{}^{\varphi}\mathcal{F}}(P,Q) = \varphi \circ \mathrm{Hom}_{\mathcal{F}}(\varphi^{-1}(P), \varphi^{-1}(Q)) \circ \varphi^{-1} \qquad \text{(all } P, Q \le S\text{)}$$

By analogy with finite groups, the maximal normal p-subgroup of a saturated fusion system $\mathcal{F}$ is denoted $O_p(\mathcal{F})$. Also, for any $P \le S$, $N_{\mathcal{F}}(P) \subseteq \mathcal{F}$ is the largest fusion subsystem over $N_S(P)$ in which P is normal. If P is fully normalized in $\mathcal{F}$, then $N_{\mathcal{F}}(P)$ is a saturated fusion system by, e.g., [**AKO**, Theorem I.5.5].

Since we will have frequent need to refer to the "Sylow axiom" and the "extension axiom" for a saturated fusion system, we state them here in the form of a proposition. (These conditions are used to define saturation in [**BLO2**] and other papers.)

PROPOSITION 1.2 ([**AKO**, Proposition I.2.5]). *A fusion system $\mathcal{F}$ over a p-group S is saturated if and only if the following two conditions hold.*

(I) (Sylow axiom) *If $P \le S$ is fully normalized, then P is fully centralized and fully automized.*

(II) (Extension axiom) *For each $P, Q \le S$ and $\varphi \in \mathrm{Iso}_{\mathcal{F}}(P,Q)$ such that Q is fully centralized, if we set $N_\varphi = \{g \in N_S(P) \,|\, {}^{\varphi}c_g \in \mathrm{Aut}_S(Q)\}$, then φ extends to some $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$.*

PROPOSITION 1.3. *Let $\mathcal{F}$ be a saturated fusion system over a p-group S.*

(a) *For each $P \le S$, and each $R \in P^{\mathcal{F}}$ which is fully normalized in $\mathcal{F}$, there is $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_S(P), N_S(R))$ such that $\varphi(P) = R$.*

(b) *If $Q < P \le S$ are such that Q is characteristic in P, Q is fully normalized in $\mathcal{F}$, and P is fully normalized in $N_{\mathcal{F}}(Q)$, then P is fully normalized in $\mathcal{F}$.*

(c) *Assume $Q \trianglelefteq P \le S$, where P is fully normalized in $\mathcal{F}$, $N_S(Q) = N_S(P)$, and*

$$N_S(\varphi(Q)) \cap N_S(\varphi(N_S(P))) \le N_S(\varphi(P)) \quad \forall\, \varphi \in \mathrm{Hom}_{\mathcal{F}}(N_S(P), S). \tag{1.1}$$

Then Q is also fully normalized in $\mathcal{F}$.

PROOF. **(a)** See, e.g., [**AKO**, Lemma I.2.6(c)].

(b) By (a), there are $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_S(P), S)$ and $\psi \in \mathrm{Hom}_{\mathcal{F}}(N_S(\varphi(Q)), N_S(Q))$ such that $\varphi(P)$ is fully normalized in $\mathcal{F}$ and $\psi(\varphi(Q)) = Q$. Also, $\varphi(N_S(P)) \le N_S(\varphi(P)) \le N_S(\varphi(Q))$ since Q is characteristic in P.

Set $\chi = \psi\varphi$; then $\chi \in \mathrm{Hom}_{N_{\mathcal{F}}(Q)}(N_S(P), N_S(Q))$ since $\chi(Q) = Q$. Since P is fully normalized in $N_{\mathcal{F}}(Q)$ (and since $N_S(P) \le N_S(Q)$), $\chi(N_S(P)) = N_S(\chi(P))$. Since $\psi(N_S(\varphi(P))) \le N_S(\chi(P))$, this proves that $\varphi(N_S(P)) = N_S(\varphi(P))$, so P is fully normalized in $\mathcal{F}$ since $\varphi(P)$ is.

(c) By (a), there is $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_S(Q), S)$ such that $\varphi(Q)$ is fully normalized. If Q is not fully normalized, then $N_S(\varphi(Q)) > \varphi(N_S(Q)) = \varphi(N_S(P))$. Hence by Lemma A.1(a), $N_S(\varphi(Q)) \cap N_S(\varphi(N_S(P))) > \varphi(N_S(P))$. Together with (1.1), this shows that $N_S(\varphi(P)) > \varphi(N_S(P))$, contradicting the assumption that P is fully normalized in $\mathcal{F}$. □

The next theorem is a special (much weaker) version of the model theorem, first shown in [**BCGLO1**]. That theorem says that if $\mathcal{F}$ is a saturated fusion system and $Q \trianglelefteq \mathcal{F}$ is normal and centric, then there is a unique "model" G for $\mathcal{F}$: a unique group G which realizes the fusion system $\mathcal{F}$ and contains Q as normal centric subgroup.

THEOREM 1.4 ([**AKO**, Proposition III.5.8(a)]). *Let $\mathcal{F}$ be a saturated fusion system over a p-group S, and let $Q \le S$ be an $\mathcal{F}$-centric subgroup which is fully normalized in $\mathcal{F}$. There is a finite group M such that $N_S(Q) \in \mathrm{Syl}_p(M)$, $Q \trianglelefteq M$, $C_M(Q) \le Q$, and $M/Q \cong \mathrm{Out}_M(Q) = \mathrm{Out}_{\mathcal{F}}(Q)$.*

PROOF. Since Q is $\mathcal{F}$-centric, it is normal and centric in the normalizer fusion system $N_{\mathcal{F}}(Q)$. Hence $N_{\mathcal{F}}(Q)$ is *constrained* in the sense of [**BCGLO1**, § 4] or [**AKO**, § I.4]. So by the model theorem [**BCGLO1**, Proposition 4.3] or [**AKO**, Theorem I.4.9(a)], there is a finite group M (a "model" for $N_{\mathcal{F}}(Q)$) which satisfies the above conditions (and also $\mathcal{F}_{N_S(Q)}(M) \cong N_{\mathcal{F}}(Q)$). □

The following lemma on automorphisms will also be useful.

LEMMA 1.5. *Let $\mathcal{F}$ be a fusion system over a p-group S. Let $Q \trianglelefteq P \leq S$ be a pair of subgroups both fully normalized in $\mathcal{F}$, such that Q is $\mathcal{F}$-centric and normalized by $\mathrm{Aut}_{\mathcal{F}}(P)$. Set*

$$\mathrm{Out}(P,Q) = N_{\mathrm{Aut}(P)}(Q)/\mathrm{Inn}(P) = \{\alpha \in \mathrm{Aut}(P) \,|\, \alpha(Q) = Q\}/\mathrm{Inn}(P)\,,$$

and let

$$R\colon \mathrm{Out}(P,Q) \longrightarrow N_{\mathrm{Out}(Q)}(\mathrm{Out}_P(Q))/\mathrm{Out}_P(Q)$$

be the homomorphism

$$R\big([\alpha]\big) = [\alpha|_Q]{\cdot}\mathrm{Out}_P(Q)\,.$$

Here, $[\alpha] \in \mathrm{Out}(P)$ denotes the class of $\alpha \in \mathrm{Aut}(P)$. Then the following hold.

(a) *R sends $\mathrm{Out}_{\mathcal{F}}(P)$ isomorphically to $N_{\mathrm{Out}_{\mathcal{F}}(Q)}(\mathrm{Out}_P(Q))/\mathrm{Out}_P(Q)$.*

(b) *Assume that $p = 2$, and that either $Z(Q)$ has exponent 2 and P/Q acts freely on some basis of $Z(Q)$, or that $|Z(Q)| = |P/Q| = 2$. If $\Gamma \leq \mathrm{Out}(P,Q)$ is any subgroup such that $R(\Gamma) = N_{\mathrm{Out}_{\mathcal{F}}(Q)}(\mathrm{Out}_P(Q))/\mathrm{Out}_P(Q)$ and $\mathrm{Out}_S(P) \in \mathrm{Syl}_p(\Gamma)$, then $\Gamma = \mathrm{Out}_{\mathcal{F}}(P)$.*

PROOF. By [**OV**, Lemma 1.2], R is well defined and $\mathrm{Ker}(R) \cong H^1(P/Q; Z(Q))$. In particular, $\mathrm{Ker}(R)$ is a p-group since $Z(Q)$ is a p-group. Also, $\mathrm{Out}_{\mathcal{F}}(P) \leq \mathrm{Out}(P,Q)$ since $\mathrm{Aut}_{\mathcal{F}}(P)$ normalizes Q.

(a) By the extension axiom (and since $C_S(Q) \leq Q$ and Q is fully normalized), R sends $\mathrm{Out}_{\mathcal{F}}(P)$ onto $N_{\mathrm{Out}_{\mathcal{F}}(Q)}(\mathrm{Out}_P(Q))/\mathrm{Out}_P(Q)$. Also, $\mathrm{Out}_S(P) \in \mathrm{Syl}_p(\mathrm{Out}_{\mathcal{F}}(P))$ since P is fully normalized, so $\mathrm{Ker}(R|_{\mathrm{Out}_{\mathcal{F}}(P)}) \leq \mathrm{Out}_S(P)$ since $\mathrm{Ker}(R)$ is a p-group. Hence if $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$ and $[\alpha] \in \mathrm{Ker}(R)$, then $\alpha = c_g$ for some $g \in N_S(P)$, $g \in PC_S(Q) = P$ since $[\alpha|_Q] \in \mathrm{Out}_P(Q)$ and Q is $\mathcal{F}$-centric, and thus $[\alpha] = 1$ in $\mathrm{Out}_{\mathcal{F}}(P)$. So $R|_{\mathrm{Out}_{\mathcal{F}}(P)}$ is injective.

(b) If $Z(Q)$ has exponent 2, and the conjugation action of P/Q permutes freely some basis for $Z(Q)$, then R is injective by [**OV**, Corollary 1.3], and the result is immediate.

If $|P/Q| = |Z(Q)| = 2$, then each element in $\mathrm{Ker}(R)$ is represented by some $\alpha \in \mathrm{Aut}(P)$ such that $\alpha|_Q = \mathrm{Id}$, and $\alpha(g) \in gZ(Q)$ for all $g \in P{\smallsetminus}Q$. Thus $|\mathrm{Ker}(R)| \leq 2$, and in particular, $\mathrm{Ker}(R) \leq Z(\mathrm{Out}(P,Q))$. By (a), R sends $\mathrm{Out}_{\mathcal{F}}(P)$ isomorphically onto $N_{\mathrm{Out}_{\mathcal{F}}(Q)}(\mathrm{Aut}_P(Q))/\mathrm{Out}_P(Q)$. By a similar argument, for $\Gamma \leq \mathrm{Out}(P,Q)$ as in (b), R sends Γ isomorphically onto $N_{\mathrm{Out}_{\mathcal{F}}(Q)}(\mathrm{Aut}_P(Q))/\mathrm{Out}_P(Q)$. Since $\mathrm{Ker}(R)$ is central,

$$\mathrm{Out}(P,Q) = \mathrm{Ker}(R) \times \mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Ker}(R) \times \Gamma.$$

In particular, $\mathrm{Out}_{\mathcal{F}}(P)$ and Γ have the same p'-elements. By assumption, $\mathrm{Out}_S(P)$ is a Sylow p-subgroup of both $\mathrm{Out}_{\mathcal{F}}(P)$ and Γ, and hence $\mathrm{Out}_{\mathcal{F}}(P) = \Gamma$. □

1.1. Essential subgroups in fusion systems

Recall that $\mathbf{E}_{\mathcal{F}}$ denotes the set of $\mathcal{F}$-essential subgroups of a fusion system $\mathcal{F}$. We begin with Alperin's fusion theorem for fusion systems, in the form originally proven by Puig.

PROPOSITION 1.6 ([**AKO**, Theorem I.3.5]). *Let $\mathcal{F}$ be a saturated fusion system over a p-group S. Then each morphism in $\mathcal{F}$ is a composite of restrictions of automorphisms in* $\mathrm{Aut}_{\mathcal{F}}(S)$*, and in* $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))$ *for* $P \in \mathbf{E}_{\mathcal{F}}$.

LEMMA 1.7. *Let $\mathcal{F}$ be a saturated fusion system over a p-group S, and assume* $P \in \mathbf{E}_{\mathcal{F}}$*. Then* $O_p(\mathrm{Out}_{\mathcal{F}}(P)) = 1$*, and* $\mathrm{Out}_{\mathcal{F}}(P)$ *acts faithfully on* $P/\mathrm{Fr}(P)$.

PROOF. Since $\mathrm{Out}_{\mathcal{F}}(P)$ has a strongly p-embedded subgroup, $O_p(\mathrm{Out}_{\mathcal{F}}(P)) = 1$ (cf. [**AKO**, Proposition A.7(c)]). The kernel of the action of $\mathrm{Aut}_{\mathcal{F}}(P)$ on $P/\mathrm{Fr}(P)$ is a p-group by Lemma A.9, so $\mathrm{Out}_{\mathcal{F}}(P)$ acts faithfully since $O_p(\mathrm{Aut}_{\mathcal{F}}(P)) = \mathrm{Inn}(P)$. □

The next two results give some necessary conditions for a subgroup to be essential. They were in fact proven in [**OV**] as conditions for a subgroup to be "critical", but by [**OV**, Proposition 3.2], a subgroup of S which is $\mathcal{F}$-essential for some saturated fusion system over S is a critical subgroup of S.

LEMMA 1.8 ([**OV**, Lemma 3.4]). *Let $\mathcal{F}$ be a saturated fusion system over a p-group S. Let $P < S$, let Θ be a characteristic subgroup in P, and assume there is* $g \in N_S(P) \smallsetminus P$ *such that*

(i) $[g, P] \le \Theta{\cdot}\mathrm{Fr}(P)$*, and*

(ii) $[g, \Theta] \le \mathrm{Fr}(P)$.

Then $c_g \in O_p(\mathrm{Aut}(P))$*, and hence* $P \notin \mathbf{E}_{\mathcal{F}}$.

The proof of Lemma 1.8 is based on the fact that $O_p(\mathrm{Out}_{\mathcal{F}}(P)) = 1$ (Lemma 1.7). The next proposition is based on Bender's classification [**Be**, Satz 1] of groups with strongly 2-embedded subgroups.

PROPOSITION 1.9 ([**OV**, Proposition 3.3(c)]). *Let $\mathcal{F}$ be a saturated fusion system over a 2-group S. Fix* $P \in \mathbf{E}_{\mathcal{F}}$*, and let k be such that* $|N_S(P)/P| = 2^k$*. Then* $\mathrm{rk}(P/\mathrm{Fr}(P)) \ge 2k$.

1.2. Reduced fusion systems

We now consider the class of *reduced* fusion systems, as defined in [**AOV1**]. First recall the following definitions from [**BCGLO2**].

DEFINITION 1.10. Let $\mathcal{F}$ be a saturated fusion system over a p-group S.

(a) The *focal subgroup* of $\mathcal{F}$ is the subgroup

$$\mathfrak{foc}(\mathcal{F}) \stackrel{\text{def}}{=} \langle s^{-1}t \,|\, s,t \in S,\ t \in s^{\mathcal{F}} \rangle = \langle s^{-1}\alpha(s) \,|\, s \in P \le S, \alpha \in \mathrm{Aut}_{\mathcal{F}}(P) \rangle.$$

(b) The *hyperfocal subgroup* of $\mathcal{F}$ is the subgroup

$$\mathfrak{hyp}(\mathcal{F}) = \langle s^{-1}\alpha(s) \,|\, s \in P \le S, \alpha \in O^p(\mathrm{Aut}_{\mathcal{F}}(P)) \rangle.$$

For any saturated fusion subsystem $\mathcal{F}_0 \subseteq \mathcal{F}$ over a subgroup $S_0 \le S$,

(c) $\mathcal{F}_0$ has *p-power index* in $\mathcal{F}$ if $S_0 \ge \mathfrak{hyp}(\mathcal{F})$, and $\mathrm{Aut}_{\mathcal{F}_0}(P) \ge O^p(\mathrm{Aut}_{\mathcal{F}}(P))$ for all $P \le S_0$; and

(d) $\mathcal{F}_0$ has *index prime to p* in $\mathcal{F}$ if $S_0 = S$, and $\mathrm{Aut}_{\mathcal{F}_0}(P) \ge O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))$ for all $P \le S$.

By [**BCGLO2**, Theorem 4.3], each fusion system $\mathcal{F}$ over a p-group S contains a unique minimal saturated fusion subsystem $O^p(\mathcal{F})$ (over $\mathfrak{hyp}(\mathcal{F})$) of p-power index, and $O^p(\mathcal{F}) = \mathcal{F}$ if and only if $\mathfrak{hyp}(\mathcal{F}) = S$. By [**BCGLO2**, Theorem 5.4], each such $\mathcal{F}$ contains a unique minimal saturated fusion subsystem $O^{p'}(\mathcal{F})$ (over S) of index prime to p, and $O^{p'}(\mathcal{F}) = \mathcal{F}$ if and only if $\mathrm{Aut}_{O^{p'}(\mathcal{F})}(S) = \mathrm{Aut}_{\mathcal{F}}(S)$.

DEFINITION 1.11. A *reduced fusion system* is a saturated fusion system $\mathcal{F}$ such that $O_p(\mathcal{F}) = 1$, $O^p(\mathcal{F}) = \mathcal{F}$, and $O^{p'}(\mathcal{F}) = \mathcal{F}$.

For any saturated fusion system $\mathcal{F}$, the *reduction* of $\mathcal{F}$ is the fusion system $\mathfrak{red}(\mathcal{F})$ which is defined as follows: first set $\mathcal{F}_0 = C_{\mathcal{F}}(O_p(\mathcal{F}))/Z(O_p(\mathcal{F}))$, and then let $\mathfrak{red}(\mathcal{F}) \subseteq \mathcal{F}_0$ be the minimal subsystem which can be obtained by alternately taking $O^p(-)$ and $O^{p'}(-)$. A certain concept of "tameness" for fusion systems is defined in [**AOV1**], and the main results there state that a reduced fusion system is tame if and only if it is not the reduction of any exotic fusion system. Thus Theorem A, together with the result that all of the fusion systems listed in the theorem are tame (to be shown in later papers), imply that all fusion systems over 2-groups of sectional rank at most 4 are realizable.

In many, but not all cases, the 2-fusion system of a simple group is reduced. The following proposition, which is based on a theorem of Goldschmidt, is an attempt to make this statement more precise.

PROPOSITION 1.12. *Let G be a finite simple group. Fix $S \in \mathrm{Syl}_2(G)$, and set $\mathcal{F} = \mathcal{F}_S(G)$. Then*

(a) $O^2(\mathcal{F}) = \mathcal{F}$*;*

(b) $O_2(\mathcal{F}) = 1$ *if S is nonabelian and G is not isomorphic to a unitary group $PSU_3(2^n)$ ($n \geq 2$) nor to a Suzuki group $Sz(2^{2n+1})$ ($n \geq 1$); and*

(c) $O^{2'}(\mathcal{F}) = \mathcal{F}$ *if* $\mathrm{Out}_G(S) = 1$ *(in particular, if* $\mathrm{Aut}(S)$ *is a 2-group).*

Thus $\mathcal{F}_S(G)$ is reduced whenever the assumptions in (b) and (c) hold.

PROOF. **(a)** By the focal subgroup theorem for groups (cf. [**G**, Theorem 7.3.4] or [**Sz2**, Theorem 5.2.8]), $\mathfrak{foc}(\mathcal{F}) = [G,G] \cap S$. Hence $\mathfrak{foc}(\mathcal{F}) = S$ since $[G,G] = G$, so $\mathfrak{hyp}(\mathcal{F}) = S$ and hence $O^2(\mathcal{F}) = \mathcal{F}$ by [**AOV1**, Theorem 1.22(a)] or [**AKO**, Corollary I.7.5]. (See also Proposition 1.14(b).)

(b) Assume $O_2(\mathcal{F}) \neq 1$, and set $A = Z(O_2(\mathcal{F})) \neq 1$. Then $A \trianglelefteq \mathcal{F}$ (cf. [**AKO**, Proposition I.4.4], or Lemma 1.15 below), and hence is strongly closed in S with respect to G. Since G is simple, G is the normal closure of A in G. By a theorem of Goldschmidt [**Gd1**, Theorem A], either S is abelian, or $G \cong PSU_3(2^n)$ or $Sz(2^{2n+1})$.

(c) See [**BCGLO2**, Theorem 5.4] or [**AKO**, Theorem I.7.7(a,b)]. □

Note that $\mathrm{Out}_G(S) = 1$ whenever $N_G(S) = S$. Of course, $O^{2'}(\mathcal{F}_S(G)) = \mathcal{F}_S(G)$ in many cases when $\mathrm{Out}_G(S) \neq 1$, but it seems to be very difficult to find more general conditions which imply this.

1.3. The focal subgroup

We now list some conditions on a 2-group S, or on a saturated fusion system $\mathcal{F}$ over S, which imply that $\mathcal{F}$ (or all saturated fusion systems over S) have proper subsystems of 2-power index. All of these are based on Proposition 1.14, which says that this is equivalent to showing that $\mathfrak{foc}(\mathcal{F}) < S$. So we need techniques for computing the focal subgroup, or for showing that it is properly contained in S. The following definitions are useful when doing this.

DEFINITION 1.13. Let $\mathcal{F}$ be a saturated fusion system over a p-group S. For each $P \leq S$, define

$$\mathrm{Aut}^*_{\mathcal{F}}(P) = \begin{cases} O^p(\mathrm{Aut}_{\mathcal{F}}(P)) & \text{if } P = S \\ O^p(O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))) & \text{if } P < S. \end{cases}$$

Set $\mathfrak{foc}(\mathcal{F}, P) = \big\langle [\mathrm{Aut}^*_{\mathcal{F}}(P), P]^S \big\rangle$: the normal closure in S of $[\mathrm{Aut}^*_{\mathcal{F}}(P), P]$.

For example, if $P < S$ and $\mathrm{Aut}_{\mathcal{F}}(P) \cong \Sigma_3 \times C_3$, then $\mathrm{Aut}^*_{\mathcal{F}}(P) \cong C_3$.

Recall that for any group P and any $H \leq \mathrm{Aut}(P)$, $[H, P]$ is normal in P (cf. [**G**, Theorem 2.2.1]). Thus $\mathfrak{foc}(\mathcal{F}, S) = [\mathrm{Aut}^*_{\mathcal{F}}(S), S]$, and $[\mathrm{Aut}^*_{\mathcal{F}}(P), P] \trianglelefteq P$ for each P. So when $P < S$, $\mathfrak{foc}(\mathcal{F}, P)$ is the subgroup generated by all $[\mathrm{Aut}^*_{\mathcal{F}}(Q), Q]$ for Q S-conjugate to P.

PROPOSITION 1.14. *The following hold for any saturated fusion system $\mathcal{F}$ over a p-group S.*

(a) *Each morphism in $\mathcal{F}$ is a composite of restrictions of morphisms in* $\mathrm{Inn}(S)$ *and in* $\mathrm{Aut}^*_{\mathcal{F}}(P)$ *for $P = S$ or $P \in \mathbf{E}_{\mathcal{F}}$, and*

$$\mathfrak{foc}(\mathcal{F}) = \big\langle [S,S], \mathfrak{foc}(\mathcal{F}, P) \,\big|\, P \in \mathbf{E}_{\mathcal{F}} \cup \{S\} \big\rangle\,.$$

(b) $O^p(\mathcal{F}) = \mathcal{F} \iff \mathfrak{foc}(\mathcal{F}) = S \iff S = \big\langle \mathfrak{foc}(\mathcal{F}, P) \,\big|\, P \in \mathbf{E}_{\mathcal{F}} \cup \{S\} \big\rangle$. *In particular, these all hold if $\mathcal{F}$ is reduced.*

(c) *If $P \leq S$, and $\Gamma \leq$* $\mathrm{Aut}^*_{\mathcal{F}}(P)$ *is such that* $\mathrm{Aut}^*_{\mathcal{F}}(P) \leq \mathrm{Inn}(P)\Gamma$, *then*

$$[\mathrm{Aut}^*_{\mathcal{F}}(P), P] = [\Gamma, P].$$

PROOF. **(a)** By Proposition 1.6, each morphism in $\mathcal{F}$ is a composite of restrictions of automorphisms in $\mathrm{Aut}_{\mathcal{F}}(S) = \mathrm{Aut}^*_{\mathcal{F}}(S)\mathrm{Inn}(S)$, and in $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P)) = \mathrm{Aut}^*_{\mathcal{F}}(P)\mathrm{Aut}_S(P)$ for $P \in \mathbf{E}_{\mathcal{F}}$. Hence $\mathcal{F}$ is generated by restrictions of automorphisms in $\mathrm{Inn}(S)$ and in $\mathrm{Aut}^*_{\mathcal{F}}(P)$ for $P \in \mathbf{E}_{\mathcal{F}} \cup \{S\}$, and

$$\mathfrak{foc}(\mathcal{F}) = \big\langle [S,S], [\mathrm{Aut}^*_{\mathcal{F}}(P), P] \,\big|\, P \in \mathbf{E}_{\mathcal{F}} \cup \{S\} \big\rangle\,.$$

Since this is clearly normal in S (each subgroup S-conjugate to an essential subgroup is essential), we can replace the commutators $[\mathrm{Aut}^*_{\mathcal{F}}(P), P]$ by their normal closures $\mathfrak{foc}(\mathcal{F}, P)$.

(b) The first equivalence is shown in [**AOV1**, Theorem 1.22(a)] or [**AKO**, Corollary I.7.5]. The second follows from (a), since for $U \leq S$, $U[S,S] = S$ implies $U = S$ (cf. [**G**, Theorems 5.1.1 & 5.1.3]). The last statement follows from the definition of a reduced fusion system.

(c) Assume $\Gamma \leq \mathrm{Aut}^*_{\mathcal{F}}(P) \leq \mathrm{Inn}(P)\Gamma$. Then

$$\mathrm{Aut}^*_{\mathcal{F}}(P) = O^p(\mathrm{Inn}(P)\Gamma) = \langle {}^{\alpha}\Gamma \,|\, \alpha \in \mathrm{Inn}(P) \rangle:$$

by definition when $P = S$, and since $\mathrm{Inn}(P)\Gamma \leq O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))$ when $P < S$. Also, $\mathrm{Inn}(P)\Gamma$ normalizes $[\Gamma, P]$ and hence acts on $P/[\Gamma, P]$ (cf. [**G**, Theorem 2.2.1(iii)]), and Γ acts on $P/[\Gamma, P]$ via the identity. Since $\mathrm{Aut}^*_{\mathcal{F}}(P)$ is the normal closure of Γ in $\mathrm{Inn}(P)\Gamma$, it also acts trivially on $P/[\Gamma, P]$, and so $[\mathrm{Aut}^*_{\mathcal{F}}(P), P] = [\Gamma, P]$. □

We next note three consequences of Proposition 1.14. The first one provides a simple condition for showing that a subgroup is normal in a fusion system, and is a slightly strengthened version of [**AKO**, Proposition I.4.5].

LEMMA 1.15. *Let $\mathcal{F}$ be a saturated fusion system over a p-group S. Fix a normal subgroup $Q \trianglelefteq S$. Then Q is normal in $\mathcal{F}$ if and only if for each $P \in \mathbf{E}_{\mathcal{F}} \cup \{S\}$, $P \geq Q$ and $\mathrm{Aut}^*_{\mathcal{F}}(P)$ normalizes Q.*

PROOF. The condition is clearly necessary for Q to be normal. Conversely, if $P \geq Q$ and $\mathrm{Aut}^*_{\mathcal{F}}(P)$ normalizes Q for each $Q \in \mathbf{E}_{\mathcal{F}} \cup \{S\}$, then by Proposition 1.14(a) (and since $Q \trianglelefteq S$), each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P_1, P_2)$ extends to some $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(P_1Q, P_2Q)$ such that $\overline{\varphi}(Q) = Q$, so $Q \trianglelefteq \mathcal{F}$. □

LEMMA 1.16. *Let $\mathcal{F}$ be a saturated fusion system over a p-group S.*

(a) *If $P < S$ is not fully normalized in $\mathcal{F}$, then there are $R \in \mathbf{E}_{\mathcal{F}}$ and $\alpha \in \mathrm{Aut}^*_{\mathcal{F}}(R)$, such that $R \geq N_S(P)$ and $\alpha(P)$ is not S-conjugate to P.*

(b) *Assume $P < S$ and $S_0 \trianglelefteq S$ are such that $S_0 \geq [S,S]$ and $[\mathrm{Aut}_{\mathcal{F}}(P), P] \nleq S_0$. Then there is $R \in \mathbf{E}_{\mathcal{F}} \cup \{S\}$ such that $R \geq Q$ for some $Q \in P^{\mathcal{F}}$ and $\mathfrak{foc}(\mathcal{F}, R) \nleq S_0$.*

PROOF. **(a)** Assume $P < S$ is not fully normalized. By Proposition 1.3(a), there is $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_S(P), S)$ such that $|N_S(\varphi(P))| > |N_S(P)|$. By Proposition 1.14(a), there are a sequence of subgroups $R_1, \ldots, R_m \in \mathbf{E}_{\mathcal{F}} \cup \{S\}$, automorphisms $\alpha_i \in \mathrm{Aut}_{\mathcal{F}}(R_i)$, and restrictions β_i of α_i, such that $\varphi = \beta_m \circ \cdots \circ \beta_1$. If $\alpha_1(P)$ is S-conjugate to P, then we can replace R_1 by S and α_1 by an element of $\mathrm{Inn}(S)$, without changing $\varphi(P)$. If $R_1 = S$, then we can define $R_i^* = \alpha_1^{-1}(R_i)$ and $\alpha_i^* = \alpha_1^{-1}\alpha_i\alpha_1$, and get a shorter sequence $R_2^*, \ldots, R_m^*$ without changing $|N_S(\varphi(P))|$. We can thus arrange that $R_1 \in \mathbf{E}_{\mathcal{F}}$ and $\alpha_1(P)$ not be S-conjugate to P. This proves (a), with $(R, \alpha) = (R_1, \alpha_1)$.

(b) Choose $\beta \in \mathrm{Aut}_{\mathcal{F}}(P)$ and $g \in P$ such that $\beta(g)g^{-1} \notin S_0$. By Proposition 1.14(a), there are $\mathcal{F}$-essential subgroups $R_1, \ldots, R_m$ each of which contains a subgroup $\mathcal{F}$-conjugate to P, and automorphisms $\gamma_i \in \mathrm{Aut}^*_{\mathcal{F}}(R_i)$ or (if $R_i = S$) $\gamma_i \in \mathrm{Inn}(S)$, such that $\beta = \gamma'_m \circ \cdots \circ \gamma'_1$ where γ'_i is a restriction of γ_i. Set $g_i = \gamma'_i \circ \cdots \circ \gamma'_1(g)$ (and $g_0 = g$). Hence $\beta(g)g^{-1} = g_m g_0^{-1} \notin S_0$. So there is $1 \leq i \leq m$ such that $g_i g_{i-1}^{-1} \notin S_0$, and $\gamma_i \notin \mathrm{Inn}(S)$ ($\gamma_i \in \mathrm{Aut}^*_{\mathcal{F}}(R_i)$) since $[\gamma_i, R_i] \nleq [S,S]$. Thus $\mathfrak{foc}(\mathcal{F}, R_i) \nleq S_0$. □

LEMMA 1.17. *Let S be a 2-group such that $S/[S,S] \cong C_{2^n} \times A$ where A has exponent at most 2^{n-1}. Set $S_0 = \{g \in S \mid g^{2^{n-1}} \in [S,S]\}$, and let $\mathcal{F}$ be a reduced fusion system over S. Then there are subgroups $P \in \mathbf{E}_{\mathcal{F}}$ and $Q \trianglelefteq P$ such that $P/Q \cong C_{2^n} \times C_{2^n}$ and $P \nleq S_0$. Furthermore, for any $R \trianglelefteq P$ such that $R \leq S_0$ and $P/R \cong C_{2^n}$, there are $g \in R$ and $\alpha \in \mathrm{Aut}^*_{\mathcal{F}}(P)$ such that $\alpha(g) \in P \smallsetminus S_0$ and hence $R\langle \alpha(g) \rangle = P$.*

PROOF. By definition, S_0 is characteristic in S. Also, $[S{:}S_0] = 2$ since by hypothesis, $S/[S,S]$ contains no subgroup $C_{2^n} \times C_{2^n}$. Hence

$$\mathfrak{foc}(\mathcal{F}, S) = \big\langle [\mathrm{Aut}^*_{\mathcal{F}}(S), S]^S \big\rangle \leq S_0.$$

By Proposition 1.14(b) (and since $\mathcal{F}$ is reduced), there is a subgroup $P \in \mathbf{E}_{\mathcal{F}}$ such that $\mathfrak{foc}(\mathcal{F}, P) \nleq S_0$. Set

$$P_0 = \{g \in P \mid g^{2^{n-1}} \in [P,P]\} \leq S_0\,.$$

Thus P_0 is characteristic in P, and $P_0 \leq S_0$. Since $\mathrm{Aut}^*_{\mathcal{F}}(P) = O^2(O^{2'}(\mathrm{Aut}_{\mathcal{F}}(P)))$ is generated by automorphisms of odd order, there is $\alpha \in \mathrm{Aut}^*_{\mathcal{F}}(P)$ of odd order such that $[\alpha, P] \nleq S_0$, and hence such that α acts nontrivially on P/P_0. So P/P_0 must be noncyclic (Corollary A.10(a)). If $a, b \in P$ are elements whose classes are distinct of order 2 in P/P_0, then the classes of $a^{2^{n-1}}$ and $b^{2^{n-1}}$ are distinct of order 2 in $P/[P,P]$. Thus $P/[P,P]$ has a subgroup $\langle [a],[b]\rangle \cong C_{2^n} \times C_{2^n}$, and hence a quotient group isomorphic to $C_{2^n} \times C_{2^n}$.

Now assume $R \leq P \cap S_0$ is such that $R \trianglelefteq P$ and $P/R \cong C_{2^n}$. Since $[\alpha, P] \nleq S_0$, there is $h \in P \cap S_0$ such that $\alpha(h) \notin S_0$. Since P/R is cyclic and $[S{:}S_0] = 2$, $P = R\langle \alpha(h)\rangle$, and $P \cap S_0 = R\langle \alpha(h^2)\rangle$. Thus there is $m \in \mathbb{Z}$ such that $g \overset{\text{def}}{=} h\alpha(h^{2m}) \in R$, and $\alpha(g) \in \alpha(h)\mathrm{Fr}(P) \subseteq P \smallsetminus S_0$. So $P = R\langle \alpha(g)\rangle$. □

As examples of how Lemma 1.17 can be applied, there are no reduced fusion systems over either of the groups $C_2^4 \rtimes C_4$ (where C_4 acts freely on a basis) or $(C_4 \times C_4) \rtimes C_4$ (where C_4 acts via the matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in GL_2(\mathbb{Z}/4)$): neither group has a subquotient isomorphic to $C_4 \times C_4$.

The next proposition gives another way to handle the focal subgroup of a fusion system.

Proposition 1.18. *Let $\mathcal{F}$ be a saturated fusion system over a 2-group S.*

(a) *Set $S_0 = \Omega_1(Z(S))$ and $S_1 = S_0 \cap [S,S]$, and assume $|S_0/S_1| = 2$. Then for $g \in S_0 \smallsetminus S_1$, $g \notin \mathfrak{foc}(\mathcal{F})$.*

(b) *Let $U \trianglelefteq S$ be such that $\mathrm{Aut}_{\mathcal{F}}(S)$ normalizes U, and $U \leq [P,P]$ for each $P < S$ of index 2. Assume $g \in S \smallsetminus [S,S]$ is such that $[g,S] \leq U$, $g^2 \in U$, and each $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ sends the coset $g[S,S]$ to itself. Then $g \notin \mathfrak{foc}(\mathcal{F})$.*

(c) *Let $U \trianglelefteq S$ be such that $\mathrm{Aut}_{\mathcal{F}}(S)$ normalizes U, and $U \leq \mathrm{Fr}(P)$ for each $P < S$ of index 2. Assume $g \in S \smallsetminus \mathrm{Fr}(S)$ is such that $[g,S] \leq U$, $g^2 \in U$, and each $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ sends the coset $g\mathrm{Fr}(S)$ to itself. Then $g \notin \mathfrak{foc}(\mathcal{F})$.*

In any of these cases, $\mathcal{F}$ is not reduced.

Proof. Point (a) is shown in [**AKO**, Corollary I.8.5].

To prove (b), we refer to [**AKO**, § I.8] for some of the properties of the transfer homomorphism $\mathrm{trf}_{\mathcal{F}}\colon S \longrightarrow S/[S,S]$ for a saturated fusion system $\mathcal{F}$ over S. In particular, $\mathrm{Ker}(\mathrm{trf}_{\mathcal{F}}) \geq \mathfrak{foc}(\mathcal{F})$. Let $g \in S$ be as above, and let $[g] \in S^{\mathrm{ab}} = S/[S,S]$ be its class. By assumption, $[g] \neq 1$.

For $P < S$, let $\mathrm{trf}_P^S\colon S^{\mathrm{ab}} \longrightarrow P^{\mathrm{ab}}$ be the usual transfer homomorphism (cf. [**AKO**, Lemma I.8.1(b)]). If $[S{:}P] = 2$, then $\mathrm{trf}_P^S([g]) = [gxgx^{-1}]$ for any choice of $x \in S \smallsetminus P$: this follows from the construction in [**AKO**] upon taking coset representatives $\{1, x\}$. Since $gxgx^{-1} \in g^2[g,S] \subseteq U \leq [P,P]$ by assumption, $\mathrm{trf}_P^S([g]) = 1$. Since this holds for each $P < S$ of index 2, $\mathrm{trf}_P^S([g]) = 1$ for each $P < S$ since transfers compose (cf. [**AKO**, Lemma I.8.1(d)]).

By assumption, for each $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$, $\alpha([g]) = [g]$. So by [**AKO**, Proposition I.8.4(a)], $\mathrm{trf}_{\mathcal{F}}(g) = [g]^k \neq 1$, where $k = |\mathrm{Out}_{\mathcal{F}}(S)|$ is odd. Thus $\mathrm{trf}_{\mathcal{F}}(g) \neq 1$, so $g \notin \mathfrak{foc}(\mathcal{F})$ since $\mathrm{Ker}(\mathrm{trf}_{\mathcal{F}}) \geq \mathfrak{foc}(\mathcal{F})$, and $\mathfrak{foc}(\mathcal{F}) < S$. By Proposition 1.14(b), $\mathcal{F}$ is not reduced.

The proof of (c) is similar, but carried out by regarding $\mathrm{trf}_{\mathcal{F}}$ as a homomorphism to $S/\mathrm{Fr}(S)$, and replacing P^{ab} by $P/\mathrm{Fr}(P)$ for each $P \leq S$. □

CHAPTER 2

Normal dihedral and quaternion subgroups

The definitions and results in this chapter will be applied in Section 3, when analyzing certain essential subgroups (those of index 2 in their normalizer). Recall that $r(S)$ denotes the sectional rank of a 2-group S.

Definition 2.1. Let S be a 2-group with $r(S) \leq 4$.

(a) A (nonabelian) dihedral or quaternion subgroup $Q \leq S$ will be called *strongly automized* if two of the three subgroups of index 2 in Q are $N_S(Q)$-conjugate.

(b) $\mathscr{X}(S) = \{Q \trianglelefteq S \,|\, Q \in \mathcal{DQ} \text{ and is strongly automized}\}$.

(c) $\mathscr{Y}_0(S)$ is the set of all $Y_0 \leq S$ such that $Y_0 \cong C_2^4$, $2_{\pm}^{1+4}$, or $Q_8 \times Q_8$, and $N_S(Y_0)/\mathrm{Fr}(Y_0) \cong D_8 \wr C_2$.

(d) $\mathscr{Y}(S) = \{\langle (Y_0)^S \rangle \,|\, Y_0 \in \mathscr{Y}_0(S)\}$: the set of all normal closures in S of subgroups in $\mathscr{Y}_0(S)$.

The sets $\mathscr{X}(S)$ and $\mathscr{Y}(S)$ will play a central role in the next chapter (see Theorem 3.1 and Proposition 3.9), when identifying and characterizing essential subgroups. Most of this chapter is aimed at describing 2-groups for which one of these sets is nonempty. The next definition will be used later in this chapter, but is placed here for easier reference.

Definition 2.2. Let S be a 2-group such that $\mathscr{Y}(S) \neq \varnothing$ (and hence $\mathscr{Y}_0(S) \neq \varnothing$). Fix a subgoup $Y_0 \in \mathscr{Y}_0(S)$.

(a) Set

$$\mathscr{A}_S^+(Y_0) = \{\Gamma \leq \mathrm{Out}(Y_0) \,|\, \Gamma \geq \mathrm{Aut}_S(Y_0) \text{ and } \Gamma \cong SO_4^+(2) \cong \Sigma_3 \wr C_2\}$$
$$\mathscr{A}_S^-(Y_0) = \{\Gamma \leq \mathrm{Out}(Y_0) \,|\, \Gamma \geq \mathrm{Aut}_S(Y_0) \text{ and } \Gamma \cong SO_4^-(2) \cong \Sigma_5\}$$
$$\mathscr{A}_S(Y_0) = \mathscr{A}_S^+(Y_0) \cup \mathscr{A}_S^-(Y_0)\,.$$

(b) Let $\mathscr{U}_S(Y_0)$ be the set of unordered pairs $\{U_1, U_2\}$ of subgroups of Y_0 such that
- for $i = 1, 2$, $U_i \trianglelefteq Y_0$, and $U_i \cong C_2^2$ or Q_8;
- $[U_1, U_2] \leq U_1 \cap U_2 \leq \mathrm{Fr}(U_1)$ and $Y_0 = U_1U_2$; and
- each element of $\mathrm{Aut}_S(Y_0)$ either normalizes U_1 and U_2 or exchanges them.

(c) Elements $\Gamma \in \mathscr{A}_S(Y_0)$ and $\{U_1, U_2\} \in \mathscr{U}_S(Y_0)$ are *compatible* if each $\alpha \in \mathrm{Aut}(U_i)$ $(i = 1, 2)$ extends to some $\overline{\alpha} \in \mathrm{Aut}(Y_0)$ such that $[\overline{\alpha}] \in \Gamma$.

When $Y_0 \in \mathscr{Y}_0(S)$, and $\mathcal{F}$ is a reduced fusion system over S, we will show that $\mathrm{Out}_{\mathcal{F}}(Y_0) \in \mathscr{A}_S(Y_0)$ (Propositions 3.9(a) and 3.11(b.1)). Thus this set contains the “candidates” for $\mathrm{Out}_{\mathcal{F}}(Y_0)$. The sets $\mathscr{U}_S(Y_0)$ and the compatibility relation will be used to help identify $\mathrm{Out}_{\mathcal{F}}(Y_0)$ among the elements of $\mathscr{A}_S(Y_0)$ (Proposition 3.11(b.2)), and also when determining the list of all essential subgroups in $\mathcal{F}$ (see Proposition 3.11 and Lemma 5.2). We will see in Lemma 2.9 that this compatibility relation defines a bijection between $\mathscr{U}_S(Y_0)$ and $\mathscr{A}_S^{\pm}(Y_0)$.

LEMMA 2.3. *Let $P < S$ be 2-groups, where $P \cong D_8 \wr C_2$ and $r(S) \leq 4$. Then $Z(S) = Z(P) \cong C_2$, $|N_S(P)/P| = 2$, and $N_S(P)/Z(S) \cong D_8 \wr C_2$. If $V \trianglelefteq P$ and $V \cong C_2^4$, then $N_S(V) = P$.*

PROOF. Let $Q < P$ be the unique subgroup isomorphic to 2_+^{1+4} (see Lemma C.5(a)). Then $N_S(P) \leq N_S(Q)$. Since $r(S) \leq 4 = r(Q/Z(Q))$, $C_S(Q) \leq Q$ by Lemma A.6(a). Thus $Z(S) = Z(Q) = Z(P)$, and the homomorphism

$$\mathrm{cj}\colon N_S(Q)/Z(S) = N_S(Q)/Z(Q) \longrightarrow \mathrm{Aut}(Q) \cong \mathrm{Aut}(Q_8) \wr C_2 \cong \Sigma_4 \wr C_2$$

induced by conjugation is injective. Also, $|N_S(P)/Z(S)| > |P/Z(P)| = 2^6$ since $N_S(P) > P$ by Lemma A.1(a). Thus $N_S(P)/Z(S) \cong D_8 \wr C_2$, a Sylow 2-subgroup of $\mathrm{Aut}(Q)$. Also, $N_S(Q) = N_S(P)$, and $|N_S(P)/P| = 2$.

Assume $V \trianglelefteq P$ and $V \cong C_2^4$. Then $V \cap Q$ is one of six subgroups of Q isomorphic to C_2^3 (Lemma C.5(a) again), these subgroups are permuted transitively by $\mathrm{Aut}(Q)$, so none is normalized by a Sylow 2-subgroup of $\mathrm{Aut}(Q)$. Hence $V \cap Q \ntrianglelefteq N_S(P)$, so $V \ntrianglelefteq N_S(P)$. If $N_S(V) > P$, then $N_{N_S(P)}(V) > P$ by Lemma A.1(a), $V \trianglelefteq N_S(P)$ since $|N_S(P)/P| = 2$, and we just saw this is impossible. So $N_S(V) = P$. □

Recall that $Z_i(G)$ denotes the i-th term in the upper central series for G. Thus $Z_0(G) = 1$, $Z_1(G) = Z(G)$, and $Z_i(G)/Z_{i-1}(G) = Z(G/Z_{i-1}(G))$.

LEMMA 2.4. *Let S be a 2-group such that $r(S) \leq 4$ and $\mathscr{Y}(S) \neq \varnothing$. Let m be such that $|S| = 2^m$. Then the following hold.*

(a) *For each $0 \leq i \leq m-5$, $|Z_i(S)| = 2^i$. Also, $S/Z_{m-7}(S) \cong D_8 \wr C_2$, $S/[S,S] \cong C_2^3$, and*

$$P \trianglelefteq S \quad\Longrightarrow\quad P \geq Z_{m-5}(S) \ \text{ or } \ P = Z_i(S) \text{ for some } 0 \leq i \leq m-6. \tag{2.1}$$

For each $Y_0 \in \mathscr{Y}_0(S)$, $\mathrm{Fr}(Y_0) \trianglelefteq S$.

(b) *For each $Y \in \mathscr{Y}(S)$, $Y \geq Z_{m-5}(S)$, the image of Y in $S/[S,S]$ has order 2, and $Y/Z_{m-7}(S) \cong C_2^4$ or 2_+^{1+4}. If Y is the normal closure of $Y_0 \in \mathscr{Y}_0(S)$, then $[Y{:}Y_0] = 2^k$ for even k. There are at most two subgroups in $\mathscr{Y}(S)$ of index 8 in S and at most one subgroup of index 4 in S. If $Y_1, Y_2 \in \mathscr{Y}(S)$ and $Y_1 \neq Y_2$, then $Y_1 \nleq Y_2$.*

(c) *For each $0 \leq i \leq m-7$, $Z_{i+2}(S)/Z_i(S) \cong C_2^2$, and for each $0 \leq i \leq m-8$, $Z_{i+3}(S)/Z_i(S) \cong C_4 \times C_2$.*

(d) *If $Y \in \mathscr{Y}(S)$ has index 4 in S, then $m \geq 8$ and $Y/Z_{m-8}(S) \cong D_8 \times D_8$.*

PROOF. Fix a subgroup $Y_0 \in \mathscr{Y}_0(S)$, and let $Y \in \mathscr{Y}(S)$ be its normal closure in S. Let $j = 0, 1, 2$ be such that $2^j = |\mathrm{Fr}(Y_0)|$. We first claim that $\mathrm{Fr}(Y_0) = Z_j(S)$. This is clear when $j = 0$, and follows from Lemma A.6(a) when $j = 1$.

Assume $j = 2$, and hence $Y_0 \cong Q_8 \times Q_8$. Set $\overline{Y}_0 = Y_0/Z(Y_0)$, and $\overline{X} = XZ(Y_0)/Z(Y_0)$ for $X \leq Y_0$. Fix $U_1, U_2 \trianglelefteq Y_0$ such that $U_i \cong Q_8$ and $Y_0 = U_1 \times U_2$. Let $z_i \in Z(U_i)$ be a generator, and set $z = z_1 z_2$. For $\overline{g} = gZ(Y_0) \in (\overline{Y}_0)^{\#}$ and $i = 1, 2$, $g^2 = z_i$ if and only if $\overline{g} \in \overline{U}_i$. Hence under the (faithful) action of $\mathrm{Out}_S(Y_0) \cong D_8$ on $\overline{Y}_0$, each element either normalizes the $\overline{U}_i$ or exchanges them, and there is some $g \in N_S(Y_0)$ which exchanges them. Hence ${}^g z_1 = z_2$, and $Z(N_S(Y_0)) = \langle z \rangle < Z(Y_0)$. So $\mathrm{Fr}(Y_0) = Z_2(S)$ by Lemma A.6(b).

Write $Z_i = Z_i(S)$ for short (for all $i \geq 0$). Let

$$N_S(Y_0) = N_0 < N_1 < N_2 < \cdots < N_r = S$$

be such that $N_i = N_S(N_{i-1})$ for $i > 0$.

(a) We just showed that $\mathrm{Fr}(Y_0) = Z_j(S) \trianglelefteq S$.

By definition of $\mathscr{Y}_0(S)$, $N_0/Z_j \cong D_8 \wr C_2$. If $N_0 < S$, then by Lemma 2.3 (applied to the inclusion $N_0/Z_j < S/Z_j$), $|Z_{j+1}/Z_j| = |Z(S/Z_j)| = 2$, $|N_1/N_0| = 2$, and $N_1/Z_{j+1} \cong D_8 \wr C_2$. Upon repeating this procedure, we see that for all $1 \leq i \leq r$,

$$|Z_{j+i}/Z_{j+i-1}| = 2, \quad |N_i/N_{i-1}| = 2, \quad \text{and} \quad N_i/Z_{j+i} \cong D_8 \wr C_2. \tag{2.2}$$

Since $|S| = |N_r| = 2^m$, $|Z_{j+r}| = 2^{m-7} = 2^{j+r}$, and thus

$$j + r = m - 7. \tag{2.3}$$

In particular, $S/Z_{m-7} \cong D_8 \wr C_2$. Since $|Z_2(D_8 \wr C_2)| = 4$, $|Z_{m-5}| = 2^{m-5}$. Point (2.1) now follows from Lemma A.2. In particular, $[S, S] > Z_{m-7}$, so $S/[S, S] \cong C_2^3$ (the abelianization of $D_8 \wr C_2$).

(c) By Lemma A.6(a), $C_{S/Z_i}(N_0/Z_i) \leq N_0/Z_i$ for all $i \leq j+2$ (i.e., all i such that $r(N_0/Z_i) = 4$). Since $Z_2(D_8 \wr C_2) \cong C_2^2$, $|Z_i(N_0)| = 2^i = |Z_i|$ for such i, and thus

$$Z_i(N_0) = Z_i \quad \text{for all} \quad i \leq j + 2.$$

If $Y_0 \cong Q_8 \times Q_8$, then $Z_1(N_0) \cong C_2$, $Z_2 = Z_2(N_0) = Z(Y_0) \cong C_2^2$, and $Z_3 = Z_3(N_0) \cong C_4 \times C_2$ since all elements of order 2 in Y_0 are in its center.

If $Y_0 \cong 2_-^{1+4}$, then $Y_0/Z(Y_0) = Y_0/Z_1$ has 5 involutions which lift to involutions in Y_0 (Lemma C.2(a)). Four of these are permuted by $\mathrm{Out}_{N_0}(Y_0) \cong D_8$ while the fifth is fixed. Hence $Z_2 = Z_2(N_0) \cong C_2^2$, and $Z_3 = Z_3(N_0) \cong C_4 \times C_2$ since there are no involutions in $Z_3 \smallsetminus Z_2$.

If $Y_0 \cong 2_+^{1+4}$, then $Y_0/Z(Y_0) = Y_0/Z_1$ has a basis $\{a_1, a_2, a_3, a_4\}$ such that each of the subgroups $\langle a_1, a_2 \rangle$ and $\langle a_3, a_4 \rangle$ both lifts to a quaternion subgroup of Y_0, and such that $\mathrm{Aut}_{N_0}(Y_0/Z_1) \cong D_8$ is generated by the permutations $(1\,2)$, $(3\,4)$, and $(1\,3)(2\,4)$ (with respect to this indexing). Thus $Z_2/Z_1 = \langle a_1a_2a_3a_4 \rangle$, so $Z_2 \cong C_2^2$; and $Z_3/Z_1 = \langle a_1a_2, a_3a_4 \rangle$, so $Z_3 \cong C_4 \times C_2$.

Now assume $Y_0 \cong C_2^4$. Since $N_1/Z_1 \cong D_8 \wr C_2$ and $r((N_1/Z_2)/Z(N_1/Z_2)) = 4$, $C_{S/Z_2}(N_1/Z_2) \leq N_1/Z_2$ by Lemma A.6(a). Hence $Z_3 \leq N_1$. Also, $N_0 \cong D_8 \wr C_2$, so $Z_2 = Z_2(N_0) \cong C_2^2$. If $N_0 < S$ (if $m \geq 8$), then for $x \in N_1 \smallsetminus N_0$, c_x exchanges Y_0 with the other normal subgroup in N_0 isomorphic to C_2^4 (see Lemma C.5(a), and recall that $N_S(Y_0) = N_0$ by Lemma 2.3). Hence c_x acts on $Z_3(N_0) \cong C_2 \times D_8$ by exchanging the two subgroups C_2^3, and so $Z_3 = Z_3(N_1) \cong C_4 \times C_2$.

Thus $Z_2 \cong C_2^2$ in all cases, and $Z_3 \cong C_4 \times C_2$ if $m \geq 8$. For each $1 \leq i \leq m-8$, $Z_{i+2}/Z_i \cong C_2^2$ and $Z_{i+3}/Z_i \cong C_4 \times C_2$ by a similar argument applied to $N_{i-j}/Z_i < S/Z_i$ if $i \geq j$ (recall $N_{i-j}/Z_i \cong D_8 \wr C_2$ by (2.2) and $i - j = r + (i + 7 - m) < r$ by (2.3)), or to $N_0/Z_1 < S/Z_1$ if $i = 1$ and $j = 2$. If $i = m - 7$, then $S/Z_i \cong D_8 \wr C_2$ by (a), and $Z_{i+2}/Z_i \cong Z_2(D_8 \wr C_2) \cong C_2^2$.

(b) Let Y_i be the normal closure of Y_0 in N_i, and set $Y = Y_r$: the normal closure of Y_0 in S. We claim that for each $i \le m-7-j$,

$$Y_i > Z_{j+i} \qquad \text{and} \qquad Y_i/Z_{j+i} \cong \begin{cases} C_2^4 & \text{if } i \text{ is even} \\ 2_+^{1+4} & \text{if } i \text{ is odd.} \end{cases} \tag{2.4}$$

This holds by definition when $i = 0$. If i is even and $Y_i/Z_{j+i} \cong C_2^4$, then by (2.2), $N_i/Z_{j+i} \cong D_8 \wr C_2$ is the normalizer of Y_i/Z_{j+i} in S/Z_{j+1} by the last statement in Lemma 2.3, so Y_i/Z_{j+i} is N_{i+1}-conjugate to the other normal subgroup C_2^4 in N_i/Z_{j+i} (see Lemma C.5(a)). Thus $Y_{i+1}/Z_{j+i} \cong D_8 \times D_8$, and $Y_{i+1}/Z_{j+i+1} \cong D_8 \times_{C_2} D_8 \cong 2_+^{1+4}$. If i is odd and $Y_i/Z_{j+i} \cong 2_+^{1+4}$, then since this is the only subgroup of N_i/Z_{j+i} of this isomorphism type (Lemma C.5(a) again), $Y_i \trianglelefteq N_{i+1}$. Hence in this case, $Y_{i+1} = Y_i$ and $Y_{i+1}/Z_{i+j+1} \cong C_2^4$. This proves (2.4).

In particular, $[Y{:}Y_0] = |Y_r|/2^{4+j} = |Y_r/Z_{r+j}|{\cdot}2^{r-4}$ is always an even power of 2.

When $i = r$, so $N_i = S$, and $i = m-7-j$ by (2.3), (2.4) implies that $Y > Z_{m-7}$, and that $Y/Z_{m-7} \cong C_2^4$ or 2_+^{1+4}. Since $S/Z_{m-7} \cong D_8 \wr C_2$ contains exactly two normal subgroups isomorphic to C_2^4 and one isomorphic to 2_+^{1+4} (Lemma C.5(a)), $\mathscr{Y}(S)$ contains at most two subgroups of index 8 and at most one of index 4 (and none of any other index). Also, since none of these three subgroups of $D_8 \wr C_2$ is contained in any other by Lemma C.5(a), no member of $\mathscr{Y}(S)$ is contained in any other member.

(d) If $Y \in \mathscr{Y}(S)$ and $[S{:}Y] = 4$, then $Y > Y_0$ since $[S{:}Y_0] \ge [N_0{:}Y_0] = 8$. So $N_0 = N_S(Y_0) < S$, and $2^m = |S| \ge 2{\cdot}|N_0| = 2^{8+j}$. Also, $N_{m-8-j}/Z_{m-8} \cong D_8 \wr C_2$, as seen in the proof of (a). Let $\overline{Y} < N_{m-8-j}$ be such that $\overline{Y}/Z_{m-8} \cong D_8 \times D_8$. Then $\overline{Y}/Z_{m-7} \cong D_8 \times_{C_2} D_8 \cong 2_+^{1+4}$, and $\overline{Y} = Y$ since there is a unique such subgroup in S/Z_{m-7}. Thus $Y/Z_{m-8} \cong D_8 \times D_8$. □

As one example, set $S = \langle a_1, b_1, a_2, b_2, t\rangle \cong D_{2^n} \wr C_2$, with the presentation of Notation 5.4. Then $\mathscr{Y}(S) = \{Y_1, Y_2, Y_3\}$, where $Y_1 = \langle a_1^2, b_1, a_2^2, b_2\rangle \cong D_{2^{n-1}} \times D_{2^{n-1}}$, $Y_2 = \langle a_1^2, a_1b_1, a_2^2, a_2b_2\rangle \cong D_{2^{n-1}} \times D_{2^{n-1}}$, and $Y_3 = \langle a_1a_2, a_1^2, b_1b_2, t\rangle \cong Q_{2^n} \times_{C_2} Q_{2^n}$.

COROLLARY 2.5. *If S is a 2-group such that $r(S) \le 4$ and $\mathscr{Y}(S) \ne \varnothing$, then $\mathscr{X}(S) = \varnothing$ and* $\mathrm{Aut}(S)$ *is a 2-group.*

PROOF. Let m be such that $2^m = |S|$. By Lemma 2.4(a), there is a sequence of subgroups $1 < Z_1 < Z_2 < \ldots < Z_{m-7} < S$ characteristic in S such that $|Z_i| = 2^i$ for each i and $S/Z_{m-7} \cong D_8 \wr C_2$. Since $\mathrm{Aut}(D_8 \wr C_2)$ is a 2-group by Corollary A.10(c), $\mathrm{Aut}(S)$ is a 2-group by Lemma A.9.

If $R \in \mathscr{X}(S)$, then by definition, $R \trianglelefteq S$, is dihedral or quaternion of order at least 8, and is strongly automized in S. By Lemma 2.4(a), either $R = Z_i(S)$ for some $i \le m-5$, or $R \ge Z_{m-5}(S)$. If $m \ge 8$, this is impossible since $Z_3(S) \cong C_4 \times C_2$ by Lemma 2.4(c).

If $m = 7$, then R contains $Z_2(S) \cong C_2^2$ (Lemma 2.4(c) again) as a normal subgroup. Hence $R \cong D_8$. Since $Z_2(S) < R$ is normal in S, this contradicts the assumption that R is strongly automized. □

We next look at conditions which imply that a subgroup $Y \trianglelefteq S$ lies in $\mathscr{Y}(S)$.

LEMMA 2.6. *Let S be a 2-group with $r(S) \le 4$. Assume $Y = \Theta_1\Theta_2 \trianglelefteq S$, where $\{\Theta_1, \Theta_2\}$ is an S-conjugacy class, $\Theta_i \cong D_{2^k}$ ($k \ge 3$) or Q_{2^k} ($k \ge 4$) and is strongly automized in S for $i = 1, 2$, $[\Theta_1, \Theta_2] \le \Theta_1 \cap \Theta_2 \le Z(S)$, and $\Theta_1 \cap \Theta_2 = 1$ if $\Theta_i \in \mathcal{D}$. Then the conjugation action of S/Y on $Y/[Y, Y] \cong C_2^4$ permutes transitively a basis, and one of the following holds.*

(a) *If $S/Y \cong C_2^2$ or D_8, then $Y \in \mathscr{Y}(S)$, and*

$$\{Y_0 \in \mathscr{Y}_0(S) \mid Y_0 < Y\} = \{U_1U_2 \mid U_i < \Theta_i,\ U_i \cong C_2^2 \text{ or } Q_8\}, \tag{2.5}$$

where this set consists of one S-conjugacy class if $S/Y \cong D_8$ and two classes if $S/Y \cong C_2^2$. Also, if $Y_0 = U_1U_2 \in \mathscr{Y}_0(S)$ as in (2.5), then $\{U_1, U_2\} \in \mathscr{U}_S(Y_0)$.

(b) *If $S/Y \cong C_4$, then $S/[S, S] \cong C_4 \times C_2$ and $\mathscr{Y}(S) = \varnothing$.*

PROOF. By the 3-subgroup lemma [**G**, Theorem 2.2.3], and since $[\Theta_1, \Theta_2] \le Z(S)$,

$$\bigl[[\Theta_1, \Theta_1], \Theta_2\bigr] = 1 \qquad \text{and} \qquad \bigl[[\Theta_2, \Theta_2], \Theta_1\bigr] = 1. \tag{2.6}$$

Set $Z_* = Z(\Theta_1)Z(\Theta_2)$. Then $Y/Z_* \cong (\Theta_1/Z(\Theta_1)) \times (\Theta_2/Z(\Theta_2))$, so $Z(Y) \le Z_2(\Theta_1)Z_2(\Theta_2)$. If $z_1z_2 \in Z(Y)$, where $z_i \in Z_2(\Theta_i) \le [\Theta_i, \Theta_i]$, then $[z_i, \Theta_i] = [z_1z_2, \Theta_i] = 1$ ($i = 1, 2$) by (2.6), so $z_1z_2 \in Z(\Theta_1)Z(\Theta_2)$. Thus $Z(Y) = Z_*$, and $|Z(Y)| \le 4$. Also, $Z(S) \le Z(Y)$ by Lemma A.6(a) and since $r(Y/Z(Y)) = 4$. If $|Z(Y)| = 4$, then $Z(Y) = Z(\Theta_1) \times Z(\Theta_2)$, and $Z(S) < Z(Y)$ since Θ_1 and Θ_2 are S-conjugate. Thus $|Z(S)| = 2$ in all cases, and $Z(S) = Z(Y) = Z(\Theta_1) = Z(\Theta_2)$ if $\Theta_1 \cap \Theta_2 \ne 1$.

We first check that

$$U_1 \le \Theta_1, \quad U_2 \le \Theta_2, \quad U_i \cong C_2^2 \text{ (if } \Theta_i \in \mathcal{D}) \text{ or } Q_8 \text{ (if } \Theta_i \in \mathcal{Q}) \quad \Longrightarrow \quad U_1U_2 \cong C_2^4,\ 2_+^{1+4},\ 2_-^{1+4}, \text{ or } Q_8 \times Q_8. \tag{2.7}$$

This is clear whenever $[U_1, U_2] = 1$ (recall that $\Theta_1 \cap \Theta_2 = 1$ if $\Theta_i \in \mathcal{D}$). If $[\Theta_1, \Theta_2] = [U_1, U_2] = Z(S)$, then $U_i \cong Q_8$, $|C_{U_1}(U_2)| \ge |U_1 \cap [\Theta_1, \Theta_1]| = 4$ by (2.6), and so $U_1U_2 \cong 2_-^{1+4}$ by Lemma C.2(a).

For each $i = 1, 2$, let $Q_{i1}, Q_{i2} < \Theta_i$ be the two noncyclic subgroups of index 2. Set $\mathbf{Q} = \{Q_{ij} \mid i, j = 1, 2\}$. We first claim that the conjugation action of S/Y on $\mathbf{Q}$ is faithful. Assume otherwise: then there is $x \in S \smallsetminus Y$ such that $x \in N_S(Q_{ij})$ for each i, j. Fix $U_i \le Q_{i1}$ as in (2.7). Each subgroup of Q_{i1} which is isomorphic to U_i is Θ_i-conjugate to U_i, and $\mathrm{Aut}_S(U_i) = \mathrm{Aut}_{\Theta_i}(U_i) \in \mathrm{Syl}_2(\mathrm{Aut}(U_i))$. Hence there are elements $x_i \in \Theta_i$ ($i = 1, 2$) such that $c_x|_{U_i} = c_{x_i}|_{U_i}$. Upon replacing x by $xx_1^{-1}x_2^{-1}$, we can assume that $[x, U_1U_2] \le [\Theta_1, \Theta_2]$. If $[\Theta_1, \Theta_2] = Z(S)$, then $U_i \cong Q_8$, and U_1U_2 is extraspecial of order 2^5 by (2.7). So $c_x|_{U_1U_2} \in \mathrm{Inn}(U_1U_2)$ in all cases. But this is impossible, since U_1U_2 is centric in S by Lemma A.6(a) (and since $r(U_1U_2) = 4$ by (2.7) again).

Thus S/Y acts faithfully on $\mathbf{Q}$. Also, Q_{i1} is S-conjugate to Q_{i2} (Θ_i is strongly automized) and Θ_1 is S-conjugate to Θ_2. Hence S/Y acts transitively on $\mathbf{Q}$, and $S/Y \cong C_2^2$, C_4, or D_8. Furthermore, each Q_{ij} has image in Y^{ab} of order 2, the involutions in these images form a basis for $Y^{\mathrm{ab}} \cong \Theta_1^{\mathrm{ab}} \times \Theta_2^{\mathrm{ab}} \cong C_2^4$, and thus S/Y permutes this basis transitively.

(a) Assume $S/Y \cong C_2^2$ or D_8. Fix indices $j_1, j_2 \in \{1, 2\}$. There are elements $g, h \in S \smallsetminus Y$ such that ${}^gQ_{1j_1} = Q_{2j_2}$ and ${}^hQ_{i1} = Q_{i2}$ for $i = 1$ and $i = 2$, and such that $g^2, h^2 \in Y$. In particular, c_g exchanges Θ_1 and Θ_2, and $\langle c_g, c_h \rangle$ acts freely and transitively on $\mathbf{Q}$. Set $g^2 = d_1d_2$ where $d_i \in \Theta_i$. Then $d_2 \equiv gd_1g^{-1} \pmod{\Theta_1 \cap \Theta_2}$ since $[g, d_1d_2] = 1$, so

$$(gd_1^{-1})^2 = gd_1^{-1}gd_1^{-1} \equiv d_2^{-1}g^2d_1^{-1} \equiv 1 \pmod{\Theta_1 \cap \Theta_2}.$$

Upon replacing g by gd_1^{-1}, we can arrange that $g^2 \in \Theta_1 \cap \Theta_2 \le Z(Y)$.

Choose $U_1^* \le Q_{1j_1}$ such that $U_1^* \cong C_2^2$ if $\Theta_1 \in \mathcal{D}$ or $U_1^* \cong Q_8$ if $\Theta_1 \in \mathcal{Q}$. Set $U_2^* = {}^gU_1^* \le Q_{2j_2}$ and $Y_0^* = U_1^*U_2^*$. Since $g^2 \in Z(Y)$, $g \in N_S(Y_0^*)$. By (2.7), $Y_0^* \cong C_2^4$, 2_+^{1+4}, 2_-^{1+4}, or $Q_8 \times Q_8$.

Set $N_i = N_{\Theta_i}(U_i^*)$, so that $N_1N_2 = N_Y(Y_0^*)$ and $N_1N_2/\mathrm{Fr}(Y_0^*) \cong D_8 \times D_8$. For each $x \in N_S(\Theta_1) \smallsetminus Y$, c_x exchanges Q_{i1} with Q_{i2} for either or both $i = 1, 2$, and hence cannot normalize Y_0^*. Thus $N_S(Y_0^*) = N_1N_2\langle g \rangle$ for g as above. Also, ${}^gN_i = N_{3-i}$ and $g^2 \in \mathrm{Fr}(Y_0^*)$, so $N_S(Y_0^*)/\mathrm{Fr}(Y_0^*) \cong D_8 \wr C_2$. Hence $Y_0^* \in \mathscr{Y}_0(S)$ in this case, and $Y \in \mathscr{Y}(S)$ since it is the normal closure of Y_0^* in S. Moreover, for any $U_1 < Q_{1j_1}$ and $U_2 < Q_{2j_2}$ isomorphic to U_1^* and U_2^*, $U_1U_2 \in \mathscr{Y}_0(S)$ since it is Y-conjugate to Y_0^*. Since $j_1, j_2 \in \{1, 2\}$ were arbitrary, this proves that the right hand side in (2.5) is contained in the left hand side.

Set $\mathscr{Y}_0 = \{Y_0 \in \mathscr{Y}_0(S) \,|\, Y_0 < Y\}$. Since no subgroup in $\mathscr{Y}(S)$ is contained in any other (Lemma 2.4(b)), Y is the normal closure of each $Y_0 \in \mathscr{Y}_0$. For each $Y_0 \in \mathscr{Y}_0$, $[Y{:}Y_0]$ is an even power of 2 by Lemma 2.4(b). So if $|Y| = 2^m$ for even m, then $\Theta_1 \cap \Theta_2 = 1$, and $Y_0 \cong C_2^4$ or $Q_8 \times Q_8$. If $Y_0 \cong C_2^4$, then its images under projection to each Θ_i have order at most 4, hence have order exactly 4, so $U_i = Y_0 \cap \Theta_i \cong C_2^2$ for $i = 1, 2$ (and $\Theta_i \in \mathcal{D}$). If $Y_0 \cong Q_8 \times Q_8$, then a similar argument shows that $U_i = Y_0 \cap \Theta_i \cong Q_8$ for $i = 1, 2$ and $\Theta_i \in \mathcal{Q}$. So (2.5) holds in this case. Also, $Y_0 = U_1 \times U_2$, each element of $\mathrm{Aut}_S(Y_0)$ normalizes U_1 and U_2 or exchanges them since each element of S normalizes or exchanges the Θ_i, and so $\{U_1, U_2\} \in \mathscr{U}_S(Y_0)$ (Definition 2.1(e)).

If $|Y|$ is an odd power of 2, then $Z(S) = Z(Y_0) = Z(Y) = \Theta_1 \cap \Theta_2$, $\Theta_i \in \mathcal{Q}$, and the hypotheses of the lemma hold after replacing S, Y, and Θ_i by $S/Z(S)$, $Y/Z(S)$, and $\Theta_i/Z(S)$. For each $Y_0 \in \mathscr{Y}_0(S)$ contained in Y, $|Y_0| = 2^5$ since $[Y{:}Y_0]$ is an even power of 2, so $Y_0/Z(S) \cong C_2^4$, $Y_0/Z(S) \in \mathscr{Y}_0(S/Z(S))$, and $Y_0 = U_1U_2$ for some $U_i < \Theta_i$ with $U_i \cong Q_8$. This finishes the proof of (2.5), and $\{U_1, U_2\} \in \mathscr{U}_S(Y_0)$ by an argument similar to that used in the last paragraph.

By (2.5), there are exactly four Y-conjugacy classes $\mathscr{Y}_0^{ij} \subseteq \mathscr{Y}_0$ $(i, j \in \{1, 2\})$, where $\mathscr{Y}_0^{ij}$ is the set of those U_1U_2 such that $U_1 < Q_{1i}$ and $U_2 < Q_{2j}$. If $S/Y \cong C_2^2$, then $S = Y\langle g, h\rangle$ where g and h are as defined above, and $\mathscr{Y}_0^{11} \cup \mathscr{Y}_0^{22}$ and $\mathscr{Y}_0^{12} \cup \mathscr{Y}_0^{21}$ are the two S-conjugacy classes in $\mathscr{Y}_0$. If $S/Y \cong D_8$, then there is also $a \in S$ such that ${}^aQ_{11} = Q_{12}$ and ${}^aQ_{21} = Q_{21}$, so these two sets are S-conjugate.

(b) Assume $S/Y \cong C_4$. Since S/Y permutes the basis $\mathcal{B}$ transitively, $Y/[S, S] = Y/[S, Y] \cong C_2$, and $S/[S, S] \cong C_4 \times C_2$ since if $S/[S, S]$ were cyclic then S would be cyclic. Thus $\mathscr{Y}(S) = \varnothing$ by Lemma 2.4(a). □

The next lemma can be regarded as a converse to Lemma 2.6(a), but with the extra (necessary) hypothesis (2.8) added.

Lemma 2.7. *Let S be a 2-group with $r(S) \le 4$ and $\mathscr{Y}(S) \ne \varnothing$. Choose $Y_0 \in \mathscr{Y}_0(S)$, and let $Y \in \mathscr{Y}(S)$ be its normal closure in S. Fix $\{U_1, U_2\} \in \mathscr{U}_S(Y_0)$, and assume that*

$$U_1 \text{ is not } S\text{-conjugate to any other subgroup of } U_1Z_2(S). \tag{2.8}$$

Then there is an S-conjugacy class $\{\Theta_1, \Theta_2\}$ of subgroups such that $U_i \le \Theta_i$, $Y = \Theta_1\Theta_2$, and

$$\begin{aligned} Y_0 \cong C_2^4 &\implies \Theta_i \in \mathcal{D} \text{ or } \Theta \cong C_2^2, \text{ and } Y = \Theta_1 \times \Theta_2, \\ Y_0 \cong 2_{\pm}^{1+4} &\implies \Theta_i \in \mathcal{Q} \text{ and } [\Theta_1, \Theta_2] \le \Theta_1 \cap \Theta_2 = Z(S), \\ Y_0 \cong Q_8 \times Q_8 &\implies \Theta_i \in \mathcal{Q} \text{ and } Y = \Theta_1 \times \Theta_2. \end{aligned}$$

PROOF. If $Y_0 = Y$, then the lemma holds (with $\Theta_i = U_i$) by definition of $\mathscr{U}_S(Y_0)$. So assume $Y > Y_0$.

Case 1: Assume that $Y_0 \cong C_2^4$ or $Q_8 \times Q_8$, and thus that $Y_0 = U_1 \times U_2$. Set $S_0 = C_S(Z_2(S))$. Since $|Z_2(S)| = 4$ by Lemma 2.4(a), $[S{:}S_0] = 2$. We prove the lemma in this case by induction on $|S|$.

Set $N_0 = N_S(Y_0)$ and $\widehat{S} = YN_0$. By Lemma 2.3 and since $Y > Y_0$, the two normal subgroups of $N_0/\mathrm{Fr}(Y_0) \cong D_8 \wr C_2$ isomorphic to C_2^4 are $N_S(N_0)$-conjugate and hence are both contained in $Y/\mathrm{Fr}(Y_0)$. Hence $[N_0{:}Y \cap N_0] \le 2$. Also, $Y = \langle (Y_0)^S \rangle \le S_0$ since $Y_0 \le S_0 \trianglelefteq S$, and $N_0 \not\le S_0$ since U_1 and U_2 are N_0-conjugate (by definition of $\mathscr{U}_S(Y_0)$) but not S_0-conjugate. Thus $[\widehat{S}{:}Y] = [N_0{:}Y \cap N_0] = 2$, and $Y = \widehat{S} \cap S_0 = C_{\widehat{S}}(Z_2(\widehat{S}))$ (where $Z_2(\widehat{S}) = Z_2(S)$ by Lemma 2.4(a) again). Let $\widehat{Y}$ be the normal closure of Y_0 in $\widehat{S}$ (hence that in Y). Then $N_{\widehat{S}}(Y_0) = N_0$, so $Y_0 \in \mathscr{Y}_0(\widehat{S})$ and $\widehat{Y} \in \mathscr{Y}(\widehat{S})$. Also, $\{U_1, U_2\} \in \mathscr{U}_{\widehat{S}}(Y_0)$, and (2.8) holds in $\widehat{S}$.

Now, $\widehat{S} < S$, since $[S{:}Y] \ge 4$ by Lemma 2.4(b) while $[\widehat{S}{:}Y] = 2$. So by the induction hypothesis, $\widehat{Y} = \widehat{\Theta}_1 \times \widehat{\Theta}_2$, where $\{\widehat{\Theta}_1, \widehat{\Theta}_2\}$ is an $\widehat{S}$-conjugacy class, $U_i \le \widehat{\Theta}_i \trianglelefteq Y$, and $\widehat{\Theta}_i \in \mathcal{DQ}$ or $\widehat{\Theta}_i = U_i \cong C_2^2$.

Let $\mathscr{P}_i$ be the S_0-conjugacy class of $\widehat{\Theta}_i$, and set $\mathscr{P} = \mathscr{P}_1 \cup \mathscr{P}_2$. Then $Y = \langle \mathscr{P} \rangle$, since Y is the normal closure in S of $Y_0 = U_1U_2$ and hence that of $\widehat{\Theta}_1\widehat{\Theta}_2 \ge Y_0$. Since $\widehat{\Theta}_i \trianglelefteq Y$, the hypotheses of Lemma B.6 hold with Y in the role of S. By that lemma, there are subgroups $\Theta_i \trianglelefteq Y$ such that $\Theta_i \le \langle \mathscr{P}_i \rangle \le \Theta_i Z_2(S)$, $\Theta_i \in \mathcal{DSQ}$, and $Y = \Theta_1 \times \Theta_2$.

Set $\mathbf{P}_1 = \langle \mathscr{P}_1 \rangle$ for short. If $\Theta_1 = \mathbf{P}_1$, then $\Theta_1 \trianglelefteq S_0$, and hence $\{\Theta_1, \Theta_2\}$ is an S-conjugacy class which satisfies the conditions in the lemma. So assume otherwise: assume $\mathbf{P}_1 = \Theta_1 Z_2(S) = \Theta_1 \times Z(S)$. Then $\widehat{\Theta}_1 \trianglelefteq \mathbf{P}_1$ with index 4 (recall $\widehat{\Theta}_1 \trianglelefteq Y$), so $\widehat{\Theta}_1 \ge [\mathbf{P}_1, \mathbf{P}_1] = \mathrm{Fr}(\mathbf{P}_1)$ with index 2. Hence for each $Q \in \mathscr{P}_1$, the image of Q in $\mathbf{P}_1/\mathrm{Fr}(\mathbf{P}_1) \cong C_2^3$ has order 2. So $|\mathscr{P}_1| \ge 3$, and $|\mathscr{P}_1| = 4$ since $|\mathscr{P}_1| = [S_0{:}N_S(\widehat{\Theta}_i)] \le [S_0{:}Y] = 4$.

The image in $\mathbf{P}_1/Z(S)$ of each $Q \in \mathscr{P}_1$ is one of the two noncyclic subgroups of index 2. Hence there is $Q \in \mathscr{P}_1$ such that $Q \ne \widehat{\Theta}_1$ and $QZ_2(S) = \widehat{\Theta}_1 Z_2(S)$. Let $g \in S_0$ be such that $Q = {}^g\widehat{\Theta}_1$. Since $\widehat{\Theta}_1$ is the normal closure of U_1 in $\mathbf{P}_1 = \Theta_1 \times Z(S)$, there is $x \in \Theta_1$ such that ${}^g({}^xU_1) \not\le \widehat{\Theta}_1$. Let $y \in \Theta_1$ be such that ${}^{gx}U_1 \le ({}^yU_1)Z_2(S)$. Then $U_1 \ne {}^{y^{-1}gx}U_1 \le U_1Z_2(S)$. Since this contradicts (2.8), we now conclude that $\Theta_1 = \mathbf{P}_1 = \langle \mathscr{P}_1 \rangle$. Thus $\Theta_1 \in \mathcal{D}$ or $\Theta_1 \in \mathcal{Q}$ (depending on whether $U_1 \cong C_2^2$ or Q_8), and the lemma holds in this case.

Case 2: Now assume that $Y_0 \cong 2_{\pm}^{1+4}$. Set $Z = Z(S)$, $\overline{S} = S/Z$, and $\overline{X} = XZ/Z$ for each $X \le S$. The hypotheses of the lemma hold for $\overline{Y}_0 < \overline{Y} < \overline{S}$, where $\{\overline{U}_1, \overline{U}_2\} \in \mathscr{U}_{\overline{S}}(\overline{Y}_0)$ and $\overline{U}_i \cong C_2^2$. In particular, (2.8) holds since $\overline{U}_1 Z_2(\overline{S}) = \overline{U}_1 Z(\overline{S}) = \overline{U_1 Z_2(S)}$. So by Case 1, there is an $\overline{S}$-conjugacy class $\{\overline{\Theta}_1, \overline{\Theta}_2\}$ such that $\overline{Y} = \overline{\Theta}_1 \times \overline{\Theta}_2$ and $\overline{\Theta}_i \in \mathcal{D}$. Let $\Theta_i \trianglelefteq Y$ be the preimage of $\overline{\Theta}_i \trianglelefteq \overline{Y}$.

By construction, $Y = \Theta_1\Theta_2$, and $[\Theta_1, \Theta_2] \le \Theta_1 \cap \Theta_2 \le Z$. So the only thing to check is that $\Theta_i \in \mathcal{Q}$ for $i = 1, 2$. Since $\Theta_i > U_i \cong Q_8$ and $Z = [U_i, U_i]$, $(\Theta_i)^{\text{ab}} \cong (\overline{\Theta}_i)^{\text{ab}} \cong C_2^2$. So $\Theta_i \in \mathcal{DSQ}$ by Proposition B.2, and $\Theta_i \in \mathcal{Q}$ since it is generated by subgroups conjugate to $U_i \in \mathcal{Q}$ (recall $Y = \langle (U_1U_2)^S \rangle = \langle (U_i)^S \rangle$). □

The following example helps illustrate why condition (2.8) is needed in the last lemma. It also shows that $\mathscr{U}_S(Y_0)$ can be empty for $Y_0 \in \mathscr{Y}_0(S)$ when $Y_0 \cong Q_8 \times Q_8$.

Example 2.8. Fix $n \ge 4$. Set $S = \langle a_1, b_1, a_2, b_2, t \rangle = \Delta_1\Delta_2\langle t \rangle$, where for $i = 1, 2$, $\Delta_i = \langle a_i, b_i \rangle \cong Q_{2^n}$, $|a_i| = 2^{n-1}$, and $|b_i| = 4$. Also, $[b_1, b_2] = [a_1, a_2] = 1$, $[a_1, b_2] = [a_2, b_1] = b_1^2 b_2^2 \in Z(S)$, $t^2 = 1$, ${}^t a_1 = a_2$, and ${}^t b_1 = b_2$. In particular, $S/\langle b_1^2, b_2^2 \rangle \cong D_{2^{n-1}} \wr C_2$.

Set $U_i = \langle a_i^{2^{n-3}}, b_i \rangle \cong Q_8$ $(i = 1, 2)$, and set $Y_0 = U_1U_2 \cong Q_8 \times Q_8$. Then $N_S(Y_0)/\langle b_1^2, b_2^2 \rangle \cong D_8 \wr C_2$, so $Y_0 \in \mathscr{Y}_0(S)$. Also, $\langle (Y_0)^S \rangle = \Theta_1 \times \Theta_2 \in \mathscr{Y}(S)$, where $\Theta_i = \langle a_i^2, b_i \rangle \cong Q_{2^{n-1}}$ for $i = 1, 2$. If $n \ge 5$, then $\{U_1, U_2\} \in \mathscr{U}_S(Y_0)$, but (2.8) does not hold in this case, and the conclusion of Lemma 2.7 also fails to hold since $\{\Theta_1, \Theta_2\}$ is not an S-conjugacy class. If $n = 4$ (so $Y = Y_0$), then $\mathscr{U}_S(Y_0) = \varnothing$.

We are now ready to look at the sets $\mathscr{A}_S(Y_0)$ of Definition 2.2, and the compatibility relation defined there between elements of $\mathscr{A}_S(Y_0)$ and $\mathscr{U}_S(Y_0)$.

Lemma 2.9. *Let S be a 2-group such that $r(S) \le 4$ and $\mathscr{Y}(S) \ne \varnothing$. Then the following hold for each $Y_0 \in \mathscr{Y}_0(S)$.*

(a) $Y_0 \cong C_2^4$, $2_\pm^{1+4}$, *or* $Q_8 \times Q_8$, $\text{Out}_S(Y_0) \cong D_8$, *and the action of* $\text{Out}_S(Y_0)$ *on* $Y_0/\text{Fr}(Y_0) \cong C_2^4$ *is faithful and permutes a basis.*

(b) *For each $\Gamma \in \mathscr{A}_S(Y_0)$, there is a unique pair $\{U_1, U_2\} \in \mathscr{U}_S(Y_0)$ which is compatible with Γ. If, furthermore, $U \trianglelefteq Y_0$ is such that $U \cong C_2^2$ or Q_8, $|N_{\text{Aut}_S(Y_0)}(U)| = 4$, and each $\alpha \in \text{Aut}(U)$ extends to $\overline{\alpha} \in \text{Aut}(Y_0)$ such that $[\overline{\alpha}] \in \Gamma$, then $U \in \{U_1, U_2\}$.*

(c) *If $Y_0 \cong C_2^4$, 2_+^{1+4}, or $Q_8 \times Q_8$, then for each $\{U_1, U_2\} \in \mathscr{U}_S(Y_0)$, there is a unique subgroup $\Gamma \in \mathscr{A}_S^+(Y_0)$ which is compatible with $\{U_1, U_2\}$.*
If $Y_0 \cong C_2^4$ or 2_-^{1+4}, then for each $\{U_1, U_2\} \in \mathscr{U}_S(Y_0)$, there is a unique subgroup $\Gamma \in \mathscr{A}_S^-(Y_0)$ which is compatible with $\{U_1, U_2\}$.

Proof. **(a)** By Definition 2.1, $N_S(Y_0)/\text{Fr}(Y_0) \cong D_8 \wr C_2$, where $Y/\text{Fr}(Y_0) \cong C_2^4$. Hence $\text{Out}_S(Y_0) \cong N_S(Y_0)/Y_0 \cong D_8$, and its action on $Y_0/\text{Fr}(Y_0)$ is faithful and permutes a basis.

(b,c) We consider separately the cases $\Gamma \cong \Sigma_3 \wr C_2$ and $\Gamma \cong \Sigma_5$.

Case 1: $\Gamma \cong SO_4^+(2) \cong \Sigma_3 \wr C_2$. If $\Gamma \le \text{Out}(Y_0)$ and $\Gamma \cong \Sigma_3 \wr C_2$, then there are exactly two subgroups $H_1, H_2 < \Gamma$ of order 3 with $\text{rk}([H_i, Y_0/\text{Fr}(Y_0)]) = 2$. (If $H < \Gamma$ is any other subgroup of order 3, then $\text{rk}([H_i, Y_0/\text{Fr}(Y_0)]) = 4$.) Let $\alpha_i \in \text{Aut}(Y_0)$ be of order 3 such that $[\alpha_i] \in \text{Out}(Y_0)$ generates H_i, and set $U_i = [\alpha_i, Y_0]$. Then $\{U_1, U_2\} \in \mathscr{U}_S(Y_0)$ and is compatible with Γ.

If $U \le Y_0$ is as in point (b), then the condition on extending automorphisms implies that $U = [\alpha, Y_0]$ for some $\alpha \in \text{Aut}(Y_0)$ of order 3 such that $[\alpha] \in \Gamma$. Since U_1 and U_2 are independent of the choice of $\langle \alpha_i \rangle$ (any two choices are conjugate by an element of $\text{Inn}(Y_0)$), $U \in \{U_1, U_2\}$.

Conversely, if $\{U_1, U_2\} \in \mathscr{U}_S(Y_0)$, then let $\Gamma \le \text{Out}(Y_0)$ be the group of (classes of) all automorphisms of Y_0 which either normalize the U_i or exchange them. (Note that $[U_1, U_2] = 1$ since $Y_0 \not\cong 2_-^{1+4}$.) Then $\Gamma \cong \Sigma_3 \wr C_2$, $\Gamma \geq \mathrm{Out}_S(Y_0)$, and hence $\Gamma \in \mathscr{A}_S^+(Y_0)$ is the unique element which is compatible with $\{U_1, U_2\}$. This proves (c).

Case 2: $\boldsymbol{\Gamma \cong SO_4^-(2) \cong \Sigma_5}$. Assume first that $Y_0 \cong C_2^4$. Fix a basis $\mathcal{B} = \{e_1, e_2, e_3, e_4\}$ for Y_0 which is permuted transitively by $\mathrm{Aut}_S(Y_0)$, ordered so that $\mathrm{Aut}_S(Y_0)$ contains the transpositions $(e_1\, e_2)$ and $(e_3\, e_4)$. Set $z = e_1e_2e_3e_4$, the generator of $C_V(\mathrm{Aut}_S(Y_0))$, and set $e_i' = e_iz$ for $i = 1, 2, 3, 4$. Since $\mathrm{Aut}_S(Y_0)$ permutes the fifteen involutions in V in orbits of length $4, 4, 4, 2, 1$, and the orbit $\{e_1e_3, e_1e_4, e_2e_3, e_2e_4\}$ is not a basis, $\mathcal{B}' = \{e_1', e_2', e_3', e_4'\}$ is the only other basis which is permuted transitively by $\mathrm{Aut}_S(Y_0)$. Let $\Gamma, \Gamma' \leq \mathrm{Aut}(Y_0)$ be the subgroups of automorphisms which permute the sets $\mathcal{B} \cup \{z\}$ and $\mathcal{B}' \cup \{z\}$, respectively. Set $e_0 = e_0' = z$ for convenience.

For any $\Delta \in \mathscr{A}_S^-(Y_0)$, by Proposition D.1(d), Y_0 is the orthogonal module for Δ, since otherwise it cannot contain $\mathrm{Aut}_S(Y_0)$. Thus Δ acts on $Y_0^\#$ with an orbit of length 5 the product of whose elements is the identity. This orbit contains four elements permuted transitively by $\mathrm{Aut}_S(Y_0)$ and which generate Y_0 (since the Δ-action is irreducible) and one which is fixed. Thus Δ permutes one of the sets $\mathcal{B} \cup \{z\}$ or $\mathcal{B}' \cup \{z\}$. It follows that $\mathscr{A}_S^-(Y_0) = \{\Gamma, \Gamma'\}$.

Assume $U < Y_0$ is such that $U \cong C_2^2$, $|N_{\mathrm{Aut}_S(Y_0)}(U)| = 4$, and each $\alpha \in \mathrm{Aut}(U)$ of order 3 extends to $\overline{\alpha} \in \Gamma < \mathrm{Aut}(Y_0)$ of order 3. Since $\mathrm{rk}([\beta, Y_0]) = 2$ for each $\beta \in \Gamma \cong \Sigma_5$ of order 3, $U = [\overline{\alpha}, Y_0]$. Also, $\overline{\alpha}$ permutes cyclically e_i, e_j, e_k for some triple $i, j, k \in \{0, 1, 2, 3, 4\}$ of distinct indices, and hence $U = \langle e_ie_j, e_ie_k\rangle$. The only such triples of indices which are normalized by a subgroup of index 2 in $\mathrm{Aut}_S(Y_0)$ are $\{e_0, e_1, e_2\}$ and $\{e_0, e_3, e_4\}$, so $U \in \{U_1, U_2\}$ where $U_1 = \langle e_1', e_2'\rangle$ and $U_2 = \langle e_3', e_4'\rangle$. Thus $\{U_1, U_2\}$ is the only pair in $\mathscr{U}_S(Y_0)$ which is compatible with Γ, and the U_i are the only subgroups which satisfy the hypotheses in the second statement in (b). Similarly, $\bigl\{\langle e_1, e_2\rangle, \langle e_3, e_4\rangle\bigr\}$, is the only pair in $\mathscr{U}_S(Y_0)$ which is compatible with Γ'. This proves (b) and (c) when $Y_0 \cong C_2^4$.

Now assume that $Y_0 \cong 2_-^{1+4}$, and set $Z = Z(Y_0)$ for short. Then $\Gamma = \mathrm{Out}(Y_0)$ is the unique element of $\mathscr{A}_S(Y_0)$. Let $a_0, a_1, \ldots, a_4 \in Y_0$ be such that $\{a_iZ \,|\, 0 \leq i \leq 4\}$ are the five cosets of noncentral involutions in Y_0 (Lemma C.2(a)). Then $a_0a_1a_2a_3a_4 \in Z$. Each element of order 4 in Y_0 lies in a_ia_jZ for some unique pair of distinct indices i, j. So each quaternion subgroup has the form $U = \langle a_ia_j, a_ka_\ell\rangle$ for indices $i \neq j$ and $k \neq \ell$, and $\{i, j\} \cap \{k, \ell\} \neq \varnothing$ since otherwise $a_ia_ja_ka_\ell$ has order 2. Thus $U = \langle a_ia_j, a_ia_k\rangle$ for some triple of distinct indices i, j, k, and $U = [\alpha, Y_0]$ for any $\alpha \in \mathrm{Aut}(Y_0)$ of order 3 which permutes cyclically the cosets a_iZ, a_jZ, a_kZ. Thus there are exactly ten quaternion subgroups in Y_0. Since $\mathrm{Out}_S(Y_0) \cong D_8$ fixes one of the cosets a_iZ and permutes the other four transitively, it permutes the ten quaternion subgroups in two orbits of length 4 and one of length 2. Thus there is a unique pair $\{U_1, U_2\} \in \mathscr{U}_S(Y_0)$ (the orbit of length 2). Since $U_i = [\alpha_i, Y_0]$ $(i = 1, 2)$ for some $\alpha_1, \alpha_2 \in \mathrm{Aut}(Y_0)$ of order 3, this pair is compatible with Γ. □

We now look at 2-groups S for which $\mathscr{X}(S) \neq \varnothing$.

LEMMA 2.10. *Fix a 2-group S and a subgroup $\Delta \in \mathscr{X}(S)$. Let $A \trianglelefteq \Delta$ be the cyclic subgroup of index 2 (the one which is normal in S if $\Delta \cong Q_8$), fix a generator $a \in A$, and set $A_0 = \langle a^2\rangle$. Let $\Delta_0 \leq \Delta$ be dihedral or quaternion of order 8, and set $T = C_S(\Delta_0)$.*

(a) $[S{:}T\Delta]=2$, *and for all* $g\in S\smallsetminus T\Delta$, ${}^gb=a^jb$ *for some odd* j. *Also,* $TA_0\trianglelefteq S$, $S/TA_0\cong D_8$, *and* $TA/TA_0=Z(S/TA_0)$.

(b) *There is* $x\in S\smallsetminus T\Delta$ *such that* ${}^xa=a^{1+4\ell}$ *for some* $\ell\in\mathbb{Z}$, ${}^xb=ab$, *and* $x^2=a^it$ *for* $t\in T$ *and* i *odd. Also,* $A=[S,\Delta]$.

(c) *If* $y\in TAx$, *where* x *is as in (b), then either* $|y|>|a|$, *or* $|y|=|a|$, $Z(\Delta)\le \mathrm{Fr}(T)$, *and* $\langle y\rangle\cap A=1$.

Proof. **(a)** Most of this was shown in [**AOV2**, Lemma B.3], but we give a slightly different argument here. Recall that by Definition 2.1, $\Delta\in\mathscr{X}(S)$ implies that $\Delta\trianglelefteq S$, $\Delta\in\mathcal{DQ}$, and Δ is strongly automized in S. Also, $A\trianglelefteq S$: by assumption if $\Delta\cong Q_8$, and since A is characteristic in Δ if $\Delta\not\cong Q_8$.

Set $B=\langle a^4\rangle$. Then $\Delta/B\cong D_8$, and $\mathrm{Aut}(\Delta/B)=\langle\alpha,\beta\rangle\cong D_8$ where

$$\alpha\colon \begin{cases}\bar a\mapsto\bar a\\ \bar b\mapsto\overline{ab}\end{cases}\qquad\text{and}\qquad \beta\colon\begin{cases}\bar a\mapsto\bar a^{-1}\\ \bar b\mapsto\bar b\end{cases}$$

(Here, $\bar g\in\Delta/B$ denotes the class of $g\in\Delta$). Consider the homomorphism

$$\psi\colon S\longrightarrow \mathrm{Aut}(\Delta/B)\cong D_8$$

induced by conjugation. For $t\in T=C_S(\Delta_0)$, ${}^tb=b$ and ${}^ta=a^{4i+1}$ for some i, so $t\in\mathrm{Ker}(\psi)$. Thus $TA_0\le\mathrm{Ker}(\psi)$. Conversely, if $g\in\mathrm{Ker}(\psi)$, then ${}^gb=a^{4j}b$ and ${}^ga=a^{4k+1}$ for some $j,k\in\mathbb{Z}$, so $[ga^{2m},b]=1$ whenever $m\in\mathbb{Z}$ is such that $m(4k+1)\equiv -j \pmod{|a|}$. Also, $[ga^{2m},a]\in B$, so $ga^{2m}\in T$ and $g\in TA_0$. This proves that $\mathrm{Ker}(\psi)=TA_0$ and hence that $TA_0\trianglelefteq S$.

Since $\psi(\Delta)=\psi(T\Delta)=\mathrm{Inn}(\Delta/B)=\langle\alpha^2,\beta\rangle$, and since there is $x\in S$ such that $\psi(x)\notin\mathrm{Inn}(\Delta/B)$ (Δ is strongly automized), ψ is onto. Thus $S/TA_0\cong\mathrm{Aut}(\Delta/B)\cong D_8$. Also, $[S{:}T\Delta]=|\mathrm{Out}(\Delta/B)|=2$, and $Z(S/TA_0)=TA/TA_0$. For $g\in S\smallsetminus T\Delta$, $\psi(g)\notin\mathrm{Inn}(\Delta/B)$ and hence ${}^gb=a^jb$ for odd j.

(b) Choose $x\in\psi^{-1}(\alpha)$. Thus $x\in S\smallsetminus T\Delta$, ${}^xb=a^mb$ for some $m\equiv 1 \pmod 4$, and upon replacing x by an appropriate element of xA_0, we can arrange that ${}^xb=ab$ (and still $\psi(x)=\alpha$). Also, $\psi(x^2)=\alpha^2=\psi(a)$, so $x^2\in aTA_0$. Since α acts via the identity on A/B, we have ${}^xa=a^{1+4\ell}$ for some $\ell\in\mathbb{Z}$.

Thus $A=\langle[x,b]\rangle\le[S,\Delta]$. Conversely, $[S,\Delta]\le A$ since $\Delta/A\le Z(S/A)$.

(c) Assume $y\in TAx$. Then TA_0y has order 4 in $S/TA_0\cong D_8$ since $y\in TA_0x$ or $y\in TA_0ax$, so $y^2=a^kt$ for $t\in T$ and k odd. Set $2^n=|a|$. Since $t\in C_S(\Delta_0)$, ${}^ta=a^{1+4m}$ for some m. Then $y^4=a^k({}^ta)^kt^2=a^{2k(1+2m)}t^2$. Upon iterating this procedure, we get that $y^{2^{n-1}}=a^{\pm 2^{n-2}}t^{2^{n-2}}\ne 1$, and so $y^{2^n}=a^{2^{n-1}}t^{2^{n-1}}$. Thus either $|y|>2^n=|x|$, or $|y|=2^n$, $t^{2^{n-1}}=a^{2^{n-1}}$, and thus $Z(\Delta)=\langle a^{2^{n-1}}\rangle\le\mathrm{Fr}(T)$. In the latter case, $y^{2^{n-1}}\notin A$ since $t^{2^{n-2}}\in T\smallsetminus Z(\Delta)$, so $\langle y\rangle\cap A=1$. □

The next lemma provides a necessary condition on certain 2-groups S with $\mathscr{X}(S)\ne\varnothing$ for there to be a nontrivial automorphism of odd order.

Lemma 2.11. *Fix a 2-group* S *with* $r(S)\le 4$, *and a normal dihedral subgroup* $\Delta\trianglelefteq S$ *with* $|\Delta|\ge 8$. *Set* $Z=Z(\Delta)$, *let* $\Delta_0\le\Delta$ *be dihedral of order* 8, *and assume* Z *is a direct factor of* $C_S(\Delta_0)$. *Let* $1\ne G\le\mathrm{Aut}(S)$ *be a subgroup of odd order, and set* $\widehat{T}=[G,S]$. *Then*

(a) $\widehat{T}\trianglelefteq S$, $[\widehat{T},\Delta]=\widehat{T}\cap\Delta=1$, *and* $[S{:}\widehat{T}\Delta]\le 2$; *and*

(b) $|G|=3$, *and* $\widehat{T}\cong Q_8$ *or* $C_{2^k}\times C_{2^k}$ *for some* $k\ge 1$.

PROOF. Let $A \trianglelefteq \Delta$ be the cyclic subgroup of index 2, let $a \in A$ be a generator, and fix $b \in \Delta_0 \smallsetminus A$. Set $A_0 = \langle a^2 \rangle$ and $Z = Z(\Delta)$. Set $T = C_S(\Delta_0)$. By assumption, there is $T_0 < T$ such that $T = T_0 Z$ and $T_0 \cap Z = 1$. For each $t \in T$, since T centralizes the subgroup of order 4 in A, ${}^t a = a^{1+4k}$ for some k. Thus $[T, A] \leq \langle a^4 \rangle$.

Step 1: We first show that for each $\alpha \in G$,

$$\alpha(A_0) = A_0 \qquad \text{and} \qquad \alpha(T\Delta) = T\Delta\,. \tag{2.9}$$

By Lemma B.7, $\alpha(Z) = Z$. If $A_0 > Z$ (if $|A| > 4$), then by Lemma B.7 applied to S/Z, α sends the subgroup of order 4 in A to itself. Upon iterating this procedure, we get that $\alpha(A_0) = A_0$.

If Δ is not strongly automized, then for each $g \in S$, $[g, b] \in A_0$, so $[ga^i, b] = 1$ for some i, and $ga^i \in T$ or $ga^i b \in T$. Thus $S = T\Delta$, and $\alpha(T\Delta) = T\Delta$ trivially.

Now assume that Δ is strongly automized in S. By Lemma 2.10(a,b), $[S{:}T\Delta] = 2$, and there is $x \in S \smallsetminus T\Delta$ such that ${}^x a = a^{1+4\ell}$ for some ℓ and ${}^x b = ab$. If $|\Delta| \geq 16$, then $TA\langle x \rangle$ is the centralizer of $\Delta_0 \cap A_0$ (the subgroup of order 4 in A_0), so $\alpha(TA\langle x \rangle) = TA\langle x \rangle$.

If $|\Delta| = 8$, then $T = C_S(\Delta) \trianglelefteq S$ and $T\Delta \cong T_0 \times \Delta$. Since $A \leq [S, S] \leq TA$ by Lemma 2.10(b), $\alpha(a) \in TA$, and hence

$$[TA, \alpha(a)] \leq [TA, TA] \cap \alpha([S, a]) \leq T_0 \cap A_0 = 1\,.$$

Thus $TA \leq C_S(\alpha(a)) = \alpha(C_S(a)) = \alpha(TA\langle x \rangle)$. Let $t \in T$ and $i \in \mathbb{Z}$ be such that $\alpha(a) = ta^i$. Then $a^2 = \alpha(a^2) = t^2 a^{2i}$, so i is odd, and $b \notin C_S(\alpha(a))$. Thus $\alpha(TA\langle x \rangle) = TA\langle x \rangle$ or $TA\langle bx \rangle$, the same holds for $\alpha^i(TA\langle x \rangle)$ for all i, and since $|\alpha|$ is odd, we get $\alpha(TA\langle x \rangle) = TA\langle x \rangle$.

Thus $\alpha(TA\langle x \rangle) = TA\langle x \rangle$, independently of $|\Delta|$. Hence

$$\begin{aligned}[\alpha(TA), S] &= [\alpha(TA), \alpha(TA\langle x \rangle)][\alpha(TA), \alpha(b)] \\ &\leq [TA\langle x \rangle, TA\langle x \rangle]\alpha([TA, b]) \leq (TA_0)\alpha(A_0) = TA_0.\end{aligned}$$

Since $Z(S/TA_0) = TA/TA_0$ by Lemma 2.10(a) again, this proves that $\alpha(TA) = TA$. Also, α induces the identity on $S/TA \cong C_2^2$ since it has odd order and sends the class of x to itself, and hence $\alpha(T\Delta) = T\Delta$. This finishes the proof of (2.9).

Step 2: Now, $T\Delta/\langle a^4 \rangle \cong T_0 \times D_8$ (recall that $T_0 \cap \Delta = 1$ and $[T_0, \Delta] = [T, \Delta] \leq \langle a^4 \rangle$). By (2.9), each $\alpha \in G^\#$ induces an automorphism of $T\Delta/\langle a^4 \rangle$ which sends $A_0/\langle a^4 \rangle = Z(\Delta/\langle a^4 \rangle)$ to itself. So by the Krull-Schmidt theorem (Theorem A.8(b)), $\alpha(TA_0/\langle a^4 \rangle) = TA_0/\langle a^4 \rangle$. Thus $\alpha(TA_0) = TA_0$. Also, $\alpha(TA) = TA$ since A_0 is in the Frattini subgroup of TA but not those of $TA_0\langle b \rangle$ or $TA_0\langle ab \rangle$. To summarize,

$$\alpha(TA_0) = TA_0, \quad \alpha(TA) = TA, \quad \text{and} \quad \alpha(T\Delta) = T\Delta\,. \tag{2.10}$$

Now, $\alpha|_{A_0} = \mathrm{Id}$ since $\mathrm{Aut}(A_0)$ is a 2-group by Corollary A.10(a). Since $\alpha \neq \mathrm{Id}$ and $|\alpha|$ is odd, (2.10) together with Lemma A.9 imply that the automorphism of TA_0/A_0 induced by α is nontrivial. Since $r(T_0) \leq r(S) - r(D_8) \leq 2$ (recall $T_0 \times \Delta_0 \leq S$), $T_0 \cong TA_0/A_0$ is metacyclic by Lemma B.1(a). By Lemma B.1(c), either $T_0 \cong C_{2^n} \times C_{2^n}$ for some $n \geq 1$, or $T_0 \cong Q_8$.

By Lemma A.9 and since $T_0/\mathrm{Fr}(T_0) \cong C_2^2$, $|\mathrm{Aut}(T_0)| = 3{\cdot}2^m$ for some m. So $|G| = 3$, since G acts faithfully on $TA_0/A_0 \cong T_0$, and G acts via the identity on S/TA_0 by (2.10) and Lemma A.9. So $\widehat{T} \overset{\text{def}}{=} [G, S] \leq TA_0$, and hence $\widehat{T} = [G, TA_0]$ (see [**G**, Theorem 5.3.6]). Also, $\widehat{T}A_0 = TA_0$, since $[G, TA_0/A_0] = TA_0/A_0$ in either case ($TA_0/A_0 \cong C_{2^n} \times C_{2^n}$ or Q_8).

Since G centralizes A_0, $\widehat{T} = [G,S]$ also centralizes A_0. Thus $[TA_0, A_0] = [\widehat{T}A_0, A_0] = 1$. Also,

$$\widehat{T} \cap A_0 = [G, TA_0] \cap C_{TA_0}(G) \cap A_0 \leq [TA_0, TA_0] \cap A_0 \leq [T,T] \cap A_0 = 1\,,$$

where the second relation holds by [**G**, Theorem 5.2.3]. Finally, $\widehat{T} \trianglelefteq S$ (cf. [**G**, Theorem 2.2.1(iii)]), and hence

$$\begin{aligned} [\widehat{T}, \Delta] \leq \widehat{T} \cap \Delta &= (\widehat{T} \cap TA_0) \cap \Delta \\ &= \widehat{T} \cap ((T \cap \Delta)A_0) = \widehat{T} \cap (C_\Delta(\Delta_0)A_0) = \widehat{T} \cap A_0 = 1. \end{aligned}$$

□

CHAPTER 3

Essential subgroups in 2-groups of sectional rank at most 4

We now analyze the different possibilities for $\mathcal{F}$-essential subgroups when $\mathcal{F}$ is a saturated fusion system over a 2-group S with $r(S) \leq 4$. It will be convenient to use the following shorthand to refer to the different "types" of $\mathcal{F}$-essential subgroups $R < S$ which can occur. We say that R has

type (I) when $|N_S(R)/R| \geq 4$,

type (II) when $|N_S(R)/R| = 2$ and R is not normal in S, or

type (III) when $|S/R| = 2$.

We let $\mathbf{E}_{\mathcal{F}}^{(\mathrm{I})}$, $\mathbf{E}_{\mathcal{F}}^{(\mathrm{II})}$, and $\mathbf{E}_{\mathcal{F}}^{(\mathrm{III})}$ denote the sets of $\mathcal{F}$-essential subgroups of types (I), (II), and (III), respectively, so that $\mathbf{E}_{\mathcal{F}} = \mathbf{E}_{\mathcal{F}}^{(\mathrm{I})} \cup \mathbf{E}_{\mathcal{F}}^{(\mathrm{II})} \cup \mathbf{E}_{\mathcal{F}}^{(\mathrm{III})}$. For each $Y \trianglelefteq S$,

$$\mathbf{E}_{\mathcal{F}}(Y) = \bigl\{ P \in \mathbf{E}_{\mathcal{F}} \bigm| \mathfrak{foc}(\mathcal{F}, P) = Y \bigr\},$$

$\mathbf{E}_{\mathcal{F}}^{(\mathrm{I})}(Y) = \mathbf{E}_{\mathcal{F}}^{(\mathrm{I})} \cap \mathbf{E}_{\mathcal{F}}(Y)$, etc.

The results in this chapter are summarized in the following theorem, formulated in terms of the sets $\mathscr{X}(S)$ and $\mathscr{Y}(S)$ of Definition 2.1.

THEOREM 3.1. *Let $\mathcal{F}$ be a reduced, indecomposable fusion system over a 2-group S such that $r(S) \leq 4$.*

(a) *If $R \in \mathbf{E}_{\mathcal{F}}^{(\mathrm{I})}$, then $R \cong C_2^4$ or 2_-^{1+4}, and $S \cong UT_4(2) \in \mathcal{V}$ or $S \in \mathcal{U}$.*

(b) *If $R \in \mathbf{E}_{\mathcal{F}}^{(\mathrm{II})}$, then either*

  (b.1) *$\mathfrak{foc}(\mathcal{F}, R) \in \mathscr{X}(S)$, R is as in Lemma 3.8(a), and $S \in \mathcal{DSWG}$; or*

  (b.2) *$\mathfrak{foc}(\mathcal{F}, R) \in \mathscr{Y}(S)$ and $S \in \mathcal{UV}$; or*

  (b.3) *$\mathfrak{foc}(\mathcal{F}, R) \notin \mathscr{Y}(S)$, $\mathscr{X}(S) = \varnothing$, $\mathfrak{foc}(\mathcal{F}, R) \cong C_2^4$ or $UT_3(4)$, R is as in Lemma 3.7(b), and $S \in \mathcal{U}$.*

(c) *If $\mathbf{E}_{\mathcal{F}} = \mathbf{E}_{\mathcal{F}}^{(\mathrm{III})}$ (i.e., $\mathbf{E}_{\mathcal{F}}^{(\mathrm{I})} = \mathbf{E}_{\mathcal{F}}^{(\mathrm{II})} = \varnothing$), then $S \cong D_8$, $C_4 \wr C_2$, or $UT_4(2)$, or S has type M_{12} or $\mathrm{Aut}(M_{12})$, or $S \in \mathcal{U}$.*

PROOF. **(a)** If $R \in \mathbf{E}_{\mathcal{F}}^{(\mathrm{I})}$, then by Lemma 3.3, $|N_S(R)/R| = 4$. By Proposition 3.5, $N_S(R)/R \not\cong C_4$. The result thus follows from Proposition 3.4.

(b) Assume $R \in \mathbf{E}_{\mathcal{F}}^{(\mathrm{II})}$. If $\mathfrak{foc}(\mathcal{F}, R) \in \mathscr{Y}(S)$, then $S \in \mathcal{UV}$ by Proposition 3.12, and (b.2) holds.

If $\mathfrak{foc}(\mathcal{F}, R) \notin \mathscr{Y}(S)$, then by Lemma 3.6(b), we are in one of the following two situations:

- If 3.6(b.i) holds, then we are in the situation of Lemma 3.7(b): $S \in \mathcal{U}$, $\mathscr{X}(S) = \varnothing$, $\mathfrak{foc}(\mathcal{F}, R) \cong C_2^4$ or $UT_3(4)$, and (b.3) holds.
- If 3.6(b.ii) holds, then (since $\mathcal{F}$ is reduced and $\mathfrak{foc}(\mathcal{F}, R) \notin \mathscr{Y}(S)$) we are in the situation of Lemma 3.8(a). In particular, $\mathfrak{foc}(\mathcal{F}, R) \in \mathscr{X}(S)$, so $S \in \mathcal{DSWG}$ by Proposition 3.14, and (b.1) holds.

(c) This is shown in Proposition 3.15. □

In this chapter, in addition to proving Theorem 3.1, we also collect more detailed information about the essential subgroups: information which will be useful in later chapters when analyzing the fusion systems themselves.

The following lemma limits the possibilities for $\mathcal{F}$-automorphism groups of essential subgroups of type (II) or (III).

LEMMA 3.2. *Fix a* 2*-group* S *with* $r(S) \leq 4$*, a saturated fusion system* $\mathcal{F}$ *over* S*, and a subgroup* $R \in \mathbf{E}_{\mathcal{F}}$*. Assume that either* $R \in \mathbf{E}_{\mathcal{F}}^{(\mathrm{II})}$*, or* $R \in \mathbf{E}_{\mathcal{F}}^{(\mathrm{III})}$ *and* $|\mathbf{E}_{\mathcal{F}}^{(\mathrm{III})}| \geq 2$*. Then* $\mathrm{Out}_{\mathcal{F}}(R) \cong \Sigma_3$*,* $\Sigma_3 \times C_3$*, or* $(C_3 \times C_3) \overset{-1}{\rtimes} C_2$*.*

PROOF. Assume otherwise. By Lemma 1.7, $\mathrm{Out}_{\mathcal{F}}(R)$ acts faithfully on the quotient $R/\mathrm{Fr}(R)$, and hence is isomorphic to a subgroup of $GL_4(2) \cong A_8$. By the Sylow axiom, $|\mathrm{Out}_{\mathcal{F}}(R)| = 2m$ for some odd m, so $|O_{2'}(\mathrm{Out}_{\mathcal{F}}(R))| = m$ by Burnside's normal p-complement theorem [**G**, Theorem 7.4.3]. By Proposition D.1(a), and since $\mathrm{Out}_{\mathcal{F}}(R)$ is not isomorphic to one of the groups listed above, $\mathrm{Out}_{\mathcal{F}}(R)$ is isomorphic to a subgroup of $C_{15} \overset{2}{\rtimes} C_4$ or $(C_3 \times A_5) \rtimes C_2$, and in particular, contains a subgroup $\mathrm{Aut}_{\mathcal{F}}^0(R)/\mathrm{Inn}(R) \cong D_{10}$. By Lemma D.8, $\mathrm{Fr}(R) \leq Z(R)$.

If $R \in \mathbf{E}_{\mathcal{F}}^{(\mathrm{II})}$, set $S_0 = N_S(R)$, fix $x \in N_S(S_0) \smallsetminus S_0$ such that $x^2 \in S_0$, and set $Q = {}^xR$ and $\mathrm{Aut}_{\mathcal{F}}^0(Q) = c_x \mathrm{Aut}_{\mathcal{F}}^0(R) c_x^{-1}$. Then $S_0 = RQ$ since $[S_0{:}R] = 2$. If $R \in \mathbf{E}_{\mathcal{F}}^{(\mathrm{III})}$ (thus $[S{:}R] = 2$) and $|\mathbf{E}_{\mathcal{F}}^{(\mathrm{III})}| \geq 2$, choose $Q \in \mathbf{E}_{\mathcal{F}}^{(\mathrm{III})}$ different from R, and choose $\mathrm{Aut}_{\mathcal{F}}^0(Q) \leq \mathrm{Aut}_{\mathcal{F}}(Q)$ such that $\mathrm{Aut}_{\mathcal{F}}^0(Q)/\mathrm{Inn}(Q) \cong \Sigma_3$ or D_{10}. In either case, let $T \trianglelefteq R \cap Q < R$ be the largest subgroup which is normalized by $\mathrm{Aut}_{\mathcal{F}}^0(R)$ and by $\mathrm{Aut}_{\mathcal{F}}^0(Q)$. Since $T < R$ and $\mathrm{Out}_{\mathcal{F}}^0(R) \cong D_{10}$ acts irreducibly on $R/\mathrm{Fr}(R) \cong C_2^4$, $T \leq \mathrm{Fr}(R) \leq Z(R)$, and hence $C_S(T) \geq R$.

Thus T is not centric in S. This situation is impossible by [**AOV2**, Theorem 4.5 or 4.6(a)]: $\mathrm{Out}_{\mathcal{F}}(R)$ cannot contain D_{10} when T (as defined here) is not centric in S. □

3.1. Essential subgroups of index 4 in their normalizer

We begin with a very general lemma on essential subgroups of index 4 in their normalizer, and then make it more explicit in two propositions.

LEMMA 3.3. *Let* $\mathcal{F}$ *be a saturated fusion system over a* 2*-group* S*. Assume* $R \leq S$ *is an* $\mathcal{F}$*-essential subgroup with* $|N_S(R)/R| \geq 4$ *and* $\mathrm{rk}(R/\mathrm{Fr}(R)) \leq 4$*. Then* $\mathrm{rk}(R/\mathrm{Fr}(R)) = 4$*,* $\mathrm{Out}_{\mathcal{F}}(R)$ *acts faithfully on* $R/\mathrm{Fr}(R)$*, and one of the following holds: either*

(a) $N_S(R)/R \cong C_4$ *permutes freely some basis of* $R/\mathrm{Fr}(R)$*; or*

(b) $N_S(R)/R \cong C_2^2$ *permutes freely some basis of* $R/\mathrm{Fr}(R)$*; or*

(c) $N_S(R)/R \cong C_2^2$ *acts on* $R/\mathrm{Fr}(R)$ *with* $\mathrm{rk}\big(C_{R/\mathrm{Fr}(R)}(N_S(R)/R)\big) = 2$*.*

Also, $\mathrm{Out}_{\mathcal{F}}(R) \cong C_{15} \overset{2}{\rtimes} C_4$, $C_5 \overset{2}{\rtimes} C_4$, *or* $(C_3 \times C_3) \rtimes C_4$ *in case* (a)*;* $\mathrm{Out}_{\mathcal{F}}(R) \cong A_5$ *and* $R/\mathrm{Fr}(R)$ *is its orthogonal module in case* (b)*; and* $\mathrm{Out}_{\mathcal{F}}(R) \cong A_5$ *or* $C_3 \times A_5$ *and* $R/\mathrm{Fr}(R)$ *is its* $L_2(4)$*-module in case* (c)*.*

Proof. By Proposition 1.9, $\mathrm{rk}(R/\mathrm{Fr}(R)) \geq 4$. Since the opposite inequality holds by assumption, $\mathrm{rk}(R/\mathrm{Fr}(R)) = 4$. Also, $\mathrm{Out}_S(R) \cong N_S(R)/R \cong C_2^2$ or C_4. By Lemma 1.7, $\mathrm{Out}_{\mathcal{F}}(R)$ acts faithfully on $R/\mathrm{Fr}(R)$, and hence is isomorphic to a subgroup of $GL_4(2) \cong A_8$.

Set $\Gamma = \mathrm{Out}_{\mathcal{F}}(R)$. If $\mathrm{Out}_S(R) \cong C_4$, then $|\Gamma/O_{2'}(\Gamma)| = 4$ by Burnside's normal p-complement theorem (cf. [**G**, Theorem 7.4.3]). The involution in $\mathrm{Out}_S(R)$ is not central in Γ since Γ has a strongly 2-embedded subgroup, so $\Gamma \cong C_{15} \overset{2}{\rtimes} C_4$, $C_5 \overset{2}{\rtimes} C_4$, or $(C_3 \times C_3) \rtimes C_4$ by Proposition D.1(a). In either of the first two cases, $\mathrm{Out}_S(R)$ acts on $R/\mathrm{Fr}(R) \cong C_2^4$ via the Galois action on $\mathbb{F}_{16}$, hence permutes a basis by the Hilbert normal basis theorem. If $O_{2'}(\Gamma) \cong C_3 \times C_3$, then $\mathrm{Out}_S(R)$ acts by exchanging the two irreducible factors in $R/\mathrm{Fr}(R)$, and acts on each by exchanging the elements in a basis.

If $\mathrm{Out}_S(R) \cong C_2^2$, then $\Gamma/O_{2'}(\Gamma) \cong A_5$ by Bender's theorem on groups with strongly 2-embedded subgroups [**Be**, Satz 1] and since $|\mathrm{Out}(A_5)| = 2$. Hence by Proposition D.1(a,d), $\Gamma \cong A_5$ or $A_5 \times C_3$, and $R/\mathrm{Fr}(R)$ is its orthogonal module (in case (b)) or $L_2(4)$-module (case (c)). □

We now deal separately with the cases where $N_S(R)/R \cong C_2^2$ or C_4. As in Proposition D.1(d), when $V \cong \mathbb{F}_2^4$ is an A_5- or Σ_5-module, we call it the "$L_2(4)$-module" if $V|_{A_5}$ is the natural module for $SL_2(4) \cong A_5$, and the "orthogonal module" if it is the natural module for $\Omega_4^-(2) \cong A_5$.

Proposition 3.4. *Let $\mathcal{F}$ be a saturated fusion system over a 2-group S with $r(S) \leq 4$. Assume $R \in \mathbf{E}_{\mathcal{F}}$ is such that $N_S(R)/R \cong C_2^2$. Then $R \cong C_2^4$ or 2_-^{1+4}, $|S| \leq 2^7$, and either $R \trianglelefteq S$ and $S/\mathrm{Fr}(R) \cong UT_4(2)$, or $R \cong C_2^4$ and $N_S(R) \cong UT_3(4)$. If $\mathcal{F}$ is reduced, then $S \cong UT_4(2)$ or $S \in \mathcal{U}$.*

Proof. By Lemma 3.3, $\mathrm{Out}_{\mathcal{F}}(R) \cong A_5$ or $C_3 \times A_5$ and acts faithfully on $R/\mathrm{Fr}(R) \cong C_2^4$. By Theorem 1.4, there is a finite group G such that $N_S(R) \in \mathrm{Syl}_2(G)$, $R \trianglelefteq G$, $C_G(R) \leq R$, and $G/R \cong \mathrm{Out}_G(R) = O^3(\mathrm{Out}_{\mathcal{F}}(R)) \cong A_5$. Then $R \cong C_2^4$ or 2_-^{1+4} by Lemma D.4.

Since any nontrivial extension of C_2^4 by A_5 splits by [**GH**, Lemma II.2.6], $G/\mathrm{Fr}(R)$ splits over $R/\mathrm{Fr}(R)$. Hence by Lemmas C.4(a) and C.7, $N_S(R)/\mathrm{Fr}(R)$ is isomorphic to $C_2 \wr C_2^2 = C_2^4 \rtimes C_2^2 \cong UT_4(2)$ (if $R/\mathrm{Fr}(R)$ is the orthogonal module for A_5), or to $UT_3(4)$ (if $R/\mathrm{Fr}(R)$ is the $L_2(4)$-module).

Case 1: Assume $N_S(R)/\mathrm{Fr}(R) \cong UT_3(4)$. If $R \cong 2_-^{1+4}$, then $\mathrm{Out}(R) \cong \Sigma_5$, and this group acts on $R/\mathrm{Fr}(R)$ as the orthogonal module, contradicting our assumption. Thus $R \cong C_2^4$. If $R \trianglelefteq S$, then $S \cong UT_3(4) \in \mathcal{U}$, so assume $R \ntrianglelefteq S$.

Set $T = N_S(R)$. Since $T \cong UT_3(4)$ contains exactly two subgroups Q, R isomorphic to C_2^4, each element of $N_S(T) \smallsetminus T$ exchanges them, and thus T has index 2 in its normalizer. Set $S_0 = N_S(T)$. Then $S_0 \in \mathcal{U}$ (see Definition 0.1), so T is characteristic in S_0 by Lemma C.9. Hence $S = S_0 \in \mathcal{U}$ by Lemma A.1(b), and $|S| = 2^7$.

Case 2: Assume $N_S(R)/\mathrm{Fr}(R) \cong UT_4(2)$. Since $\mathrm{Fr}(R) = 1$ or $Z(N_S(R))$, $\mathrm{Fr}(R)$ is characteristic in $N_S(R)$. Also, $R/\mathrm{Fr}(R) \cong C_2^4$ is the unique abelian subgroup of rank 4 in $N_S(R)/\mathrm{Fr}(R)$ (Lemma C.4(a)), so R is characteristic in $N_S(R)$, and $S = N_S(R)$ by Lemma A.1(b). Thus $R \trianglelefteq S$, and $S/\mathrm{Fr}(R) \cong UT_4(2)$.

Now assume $\mathcal{F}$ is reduced; we must show that $S \cong UT_4(2)$ or $S \in \mathcal{U}$. If $R \cong C_2^4$, then $S \cong S/\mathrm{Fr}(R) \cong UT_4(2)$, so assume $R \cong 2_-^{1+4}$. Set $Z = \langle z \rangle = Z(R) = \mathrm{Fr}(R)$. Let $U < S$ be such that U/Z is the unique subgroup of $S/Z \cong UT_4(2)$ isomorphic to 2_+^{1+4} (Lemma C.4(b)). Then $Z(U) = [U,U] \cong C_2^2$ by Lemma D.3 (or by explicit computations), so U is special of type 2^{2+4}. Also, there is $\alpha \in \mathrm{Aut}_G(S)$ of order 3 since $N_G(S)/R \cong A_4$, α acts nontrivially on $U/(R\cap U)$ and on $R \cap U$, and trivially on $Z(U)$ since it fixes $Z = Z(S)$. Thus $C_{U/Z(U)}(\alpha) = 1$.

By Lemma D.2, either $U \cong UT_3(4)$, or $U \cong Q_8 \times Q_8$, or $U/\langle x \rangle \cong 2_+^{1+4}$ for exactly two of the involutions $x \in Z(U)$. Fix $z' \in Z(U) \smallsetminus Z$. Since $U/\langle z \rangle \cong 2_+^{1+4}$ and $U/\langle z' \rangle \cong U/\langle zz' \rangle$ (z' and zz' are S-conjugate), the last case is impossible.

If $U \cong Q_8 \times Q_8$, then $I(U) \subseteq Z(U) \le R$. By Lemma C.4(c), $I(S/Z) \subseteq (R/Z) \cup (U/Z)$, so $I(S) \subseteq R$. All noncentral involutions in $R \cong 2_-^{1+4}$ are $\mathcal{F}$-conjugate to each other since $\mathrm{Aut}_{\mathcal{F}}(R) \cong A_5$ (see Lemma C.2(a)). Since $Z(\mathcal{F}) = 1$, they are all $\mathcal{F}$-conjugate to z, and so $z' \in z^{\mathcal{F}}$. By the extension axiom and since $C_S(z') = U$, there is $\varphi \in \mathrm{Hom}_{\mathcal{F}}(U,S)$ such that $\varphi(z') = z$. Then $\varphi(U)/Z \cong U/\langle z' \rangle \cong Q_8 \times C_2^2$ is a subgroup of index 2 in $S/Z \cong UT_4(2)$, which is impossible by Lemma C.4(c).

Thus if $\mathcal{F}$ is reduced, then $U \cong UT_3(4)$, and $S \in \mathcal{U}$. □

We next consider essential subgroups R such that $N_S(R)/R \cong C_4$.

PROPOSITION 3.5. *Let $\mathcal{F}$ be a saturated fusion system over a 2-group S with $r(S) \le 4$. Assume $R \in \mathbf{E}_{\mathcal{F}}$ with $N_S(R)/R \cong C_4$. Then $R \trianglelefteq S$ (so $S/R \cong C_4$), $|R| \le 2^6$, and $\mathcal{F}$ is not reduced.*

PROOF. By Theorem 1.4, there is a finite group G such that $N_S(R) \in \mathrm{Syl}_2(G)$, $R \trianglelefteq G$, $C_G(R) \le R$, and $G/R \cong \mathrm{Out}_G(R) = \mathrm{Out}_{\mathcal{F}}(R)$. By Lemma 3.3, $G/R \cong \mathrm{Out}_{\mathcal{F}}(R)$ acts faithfully on $R/\mathrm{Fr}(R) \cong C_2^4$, and is isomorphic to $C_5 \overset{2}{\rtimes} C_4$, $C_{15} \overset{2}{\rtimes} C_4$, or $(C_3 \times C_3) \rtimes C_4$. So by Lemma D.8 (if G/R contains a subgroup isomorphic to $C_5 \rtimes C_4$) or Lemma D.7 (if $G/R \cong (C_3 \times C_3) \rtimes C_4$), we have

$$R \cong C_2^4,\ 2_\pm^{1+4},\ Q_8 \times Q_8,\ \text{or is of type } PSU_3(4). \tag{3.1}$$

Set $S_0 = N_S(R)$ for short. By Lemma A.6(a,b), either $R \cong C_2^4$; or $Z(R) = Z(S_0)$; or $Z(R) > Z(S_0)$, $|Z(R)| = 4$, $R \cong Q_8 \times Q_8$ or is of type $PSU_3(4)$, and $Z(R) = Z_2(S_0)$ since all involutions in R are central (cf. [**Sz2**, Lemma 6.4.27(iii)] when R is of type $PSU_3(4)$). Thus $\mathrm{Fr}(R)$ is characteristic in S_0 in all cases. By Lemma 3.3(a), $S_0/\mathrm{Fr}(R) \cong C_2 \wr C_4 = C_2^4 \rtimes C_4$ where $S_0/R \cong C_4$ permutes freely some basis of $R/\mathrm{Fr}(R)$. So by Lemma A.4(b), $R/\mathrm{Fr}(R)$ is the only abelian subgroup of rank 4 in $S_0/\mathrm{Fr}(R)$, and hence R is characteristic in $S_0 = N_S(R)$. Thus $R \trianglelefteq S$ ($S_0 = S$) by Lemma A.1(b). In particular, $S^{\mathrm{ab}} \cong (C_2 \wr C_4)^{\mathrm{ab}} \cong C_2 \times C_4$.

It remains to show that $\mathcal{F}$ is not reduced. Assume otherwise. By Lemma 1.17, there are subgroups $Q \trianglelefteq P \le S$ such that $S = RP$ and $P/Q \cong C_4 \times C_4$. Furthermore, by the same lemma, there are elements $g, h \in P$ and $\alpha \in \mathrm{Aut}_{\mathcal{F}}^*(P)$ such that $h = \alpha(g)$, $g \in R$, and $S = R\langle h \rangle$. In particular, $g^2 \notin [P,P]$ since $h^2 \notin R \ge [S,S]$.

Now, $4 \le |h| = |g| \le 4$, so $|h| = |g| = 4$ and R has exponent 4. Thus R is nonabelian. Set $T = [h^2, R]$. For $x \in R$, $[x, h^2]^{-1} = [h^2, x] = {}^{h^2}[x, h^2]$ since $h^4 = 1$. Thus c_{h^2} inverts T, and T is abelian. Also, $T \trianglelefteq R$ by [**G**, Theorem 2.1(iii)], and its image in $R/\mathrm{Fr}(R)$ has rank 2 (since c_h permutes freely a basis for $R/\mathrm{Fr}(R)$). By inspection of the list in (3.1), $T \geq \mathrm{Fr}(R)$.

If $g \in T$, then $g^2 = [g, h^2] \in [P, P]$, which is impossible by the above remarks. Thus $g \notin T$. Upon regarding $R/Z(R)$ as an $\mathbb{F}_2[\langle h \rangle] \cong \mathbb{F}_2[C_4]$-module, we have

$$R/Z(R) \cong \mathbb{F}_2[h]/(h^4 - 1) = \mathbb{F}_2[h]/(h-1)^4.$$

Since the polynomial ring $\mathbb{F}_2[X]$ is a PID, $R/Z(R)$ contains a unique $\mathbb{F}_2[\langle h \rangle]$-submodule of rank k for each $0 \leq k \leq 4$ (generated by $(h-1)^{4-k}$ under the above identification). Since $g \notin T$, the subgroup generated by g and iterated conjugates with h has rank at least 3 in $R/Z(R)$, so the image of Q in $R/Z(R)$ has rank at least 2, and $Q \geq T$. But this is impossible, since $Q \geq Z(R)$ and $g^2 \notin Q$. We conclude that $\mathcal{F}$ is not reduced. □

3.2. Essential pairs of type (II)

Let $\mathcal{F}$ be a saturated fusion system over a 2-group S. An *$\mathcal{F}$-essential pair of type* (II) in $\mathcal{F}$ is a pair of subgroups (R_1, R_2) such that

- $R_1, R_2 \in \mathbf{E}_{\mathcal{F}}^{(\mathrm{II})}$,
- $N_S(R_1) = N_S(R_2) = R_1R_2 < S$, and
- $R_2 = {}^xR_1$ for some $x \in N_S(R_1R_2) \smallsetminus R_1R_2$ where $x^2 \in R_1R_2$.

The $\mathcal{F}$-essential pairs play a key role when describing fusion systems containing essential subgroups of type (II).

We first show that each $\mathcal{F}$-essential subgroup of type (II) lies in an $\mathcal{F}$-essential pair of type (II), and prove some of the basic properties of such pairs.

LEMMA 3.6. *Let $\mathcal{F}$ be a saturated fusion system over a 2-group S with $r(S) \leq 4$, and assume $R_1 \in \mathbf{E}_{\mathcal{F}}$ is of type* (II)*. Then there is a subgroup $R_2 \in \mathbf{E}_{\mathcal{F}}$ of type* (II) *such that (R_1, R_2) is an $\mathcal{F}$-essential pair of type* (II)*. Set $R = R_1R_2 = N_S(R_1) = N_S(R_2)$, and let $x \in N_S(R) \smallsetminus R$ be such that $x^2 \in R$ and ${}^xR_1 = R_2$.*

(a) *For $i = 1, 2$, $O^{2'}(\mathrm{Out}_{\mathcal{F}}(R_i)) \cong \Sigma_3$ or $(C_3 \times C_3) \rtimes C_2$. There are subgroups*

$$\mathrm{Aut}^0_{\mathcal{F}}(R_i) \leq \mathrm{Aut}_{\mathcal{F}}(R_i) \quad \text{and} \quad \mathrm{Out}^0_{\mathcal{F}}(R_i) = \mathrm{Aut}^0_{\mathcal{F}}(R_i)/\mathrm{Inn}(R_i) \leq \mathrm{Out}_{\mathcal{F}}(R_i)$$

such that $\mathrm{Out}_S(R_i) \leq \mathrm{Out}^0_{\mathcal{F}}(R_i) \cong \Sigma_3$ and $c_x \mathrm{Aut}^0_{\mathcal{F}}(R_i) c_x^{-1} = \mathrm{Aut}^0_{\mathcal{F}}(R_{3-i})$.

(b) *For $\mathrm{Aut}^0_{\mathcal{F}}(R_i)$ as in* (a)*, let $T \leq R_1 \cap R_2$ be the largest subgroup normalized by $\mathrm{Aut}^0_{\mathcal{F}}(R_1)$ and by $\mathrm{Aut}^0_{\mathcal{F}}(R_2)$. Then $x \in N_S(T)$, and either*

 (b.i) *$T = R_1 \cap R_2$, $C_S(T) \leq T$, and $N_S(T) = R\langle x \rangle$; or*

 (b.ii) *$T < R_1 \cap R_2$ and $C_R(T) \nleq T$.*

(c) *For $\mathrm{Out}^0_{\mathcal{F}}(R_i)$ as in* (a)*, there are groups $G_1 > R < G_2$ and an isomorphism $\beta \in \mathrm{Iso}(G_1, G_2)$, such that $[G_i{:}R] = 3$, $R_i \trianglelefteq G_i$, $\mathrm{Out}^0_{\mathcal{F}}(R_i) = \mathrm{Out}_{G_i}(R_i) \cong G_i/R_i$ $(i = 1, 2)$, and $\beta|_R = c_x|_R$.*

PROOF. Set $R = N_S(R_1)$, choose $x \in N_S(R) \smallsetminus R$ such that $x^2 \in R$, and set $R_2 = {}^xR_1$. Then $R_2 \in \mathbf{E}_{\mathcal{F}}^{(\mathrm{II})}$, $N_S(R_2) = {}^xR = R$, and $R = R_1R_2$ since the R_i are distinct of index 2 in R. Thus (R_1, R_2) is an $\mathcal{F}$-essential pair of type (II).

(a,c) The first statement in (a) follows from Lemma 3.2, and the others immediately from that. Point (c) follows from the model theorem (Theorem 1.4); we refer to [**AOV2**, Theorem 4.6] for more details. The uniqueness in the choice of T implies that $x \in N_S(T)$.

(b) If $C_S(T) \nleq T$, then $T < R_1 \cap R_2$ by [**AOV2**, Theorem 4.6(a.ii)]. So it remains to prove that

$$C_R(T) \le T \quad \Longrightarrow \quad T = R_1 \cap R_2 \text{ and } N_S(T) = R\langle x\rangle. \tag{3.2}$$

If $C_R(T) \le T$, then by [**AOV2**, Theorem 4.6(b)], $\mathrm{Out}_S(T) \cong N_S(T)/T$ acts faithfully on $T/\mathrm{Fr}(T)$. Since $\mathrm{rk}(T/\mathrm{Fr}(T)) \le 4$, $N_S(T)/T$ is isomorphic to a subgroup of $GL_4(2) \cong A_8$. In particular, $N_S(T)/T$ contains no elements of order 8.

Set $R_{12} = R_1 \cap R_2$ for short. Then $R/R_{12} = (R_1/R_{12}) \times (R_2/R_{12}) \cong C_2^2$, c_x exchanges the two factors R_i/R_{12}, and hence $R\langle x\rangle/R_{12} \cong D_8$. If $N_S(T) > R\langle x\rangle$, then there is $g \in N_{N_S(T)}(R\langle x\rangle) \smallsetminus R\langle x\rangle$ such that $g^2 \in R\langle x\rangle$ (Lemma A.1). Since $g \notin R = N_S(R_i)$ $(i = 1, 2)$, $g \notin N_S(R)$, and hence ${}^g(R/R_{12}) \ne R/R_{12}$. In other words, $R\langle x\rangle/R_{12} \cong D_8$ is strongly automized in $R\langle x, g\rangle/R_{12}$. By Lemma 2.10(c), applied with $R\langle x, g\rangle/R_{12}$ and $R\langle x\rangle/R_{12}$ in the role of S and $\Delta = T\Delta$, $R\langle x, g\rangle/R_{12}$ contains an element of order 8, which is a contradiction. (In fact, $R\langle x, g\rangle/R_{12} \cong D_{16}$ or SD_{16}.)

Thus $N_S(T) = R\langle x\rangle$. Now assume $T < R_{12}$. By the maximality of T, the amalgam $(G_1/T > R_{12}/T < G_2/T)$ of (c) is primitive of index $(3,3)$ as defined in [**Gd2**]. Hence it is one of those in Goldschmidt's list (Table 1 and Theorem A in [**Gd2**]). Since $G_1/T \cong G_2/T$ and $|R_i/T| \ge 4$, we have $G_i/T \cong \Sigma_4 \times C_2^k$, $R/T \cong D_8 \times C_2^k$, and $R_i/T \cong C_2^{2+k}$ for some $k = 0, 1$.

Choose elements $y_i \in R_i \smallsetminus R_{12}$ $(i = 1, 2)$. Thus $y_i^2 \in T$ (since $R_i/T \cong C_2^{2+k}$), and $(xy_1)^2 \equiv y_1y_2 \pmod{R_{12}}$. So $(y_1y_2)^2 \equiv [y_1, y_2] \not\equiv 1 \pmod{T}$, and xy_1T has order 8 in $R\langle x\rangle/T$, which is impossible. This finishes the proof of (3.2). □

Recall (Definition 1.13) that for a fusion system $\mathcal{F}$ over a p-group S, we set $\mathrm{Aut}^*_{\mathcal{F}}(R) = O^p(O^{p'}(\mathrm{Aut}_{\mathcal{F}}(R)))$ if $R < S$ and $\mathrm{Aut}^*_{\mathcal{F}}(S) = O^p(\mathrm{Aut}_{\mathcal{F}}(S))$, and let $\mathfrak{foc}(\mathcal{F}, R)$ be the normal closure in S of $[\mathrm{Aut}^*_{\mathcal{F}}(R), R]$.

In the next two lemmas, we examine $\mathcal{F}$-essential pairs of the two types described in points (b.i) and (b.ii) of Lemma 3.6. A priori, this type depends on the choice of subgroups $\mathrm{Aut}^0_{\mathcal{F}}(R_i) \le \mathrm{Aut}_{\mathcal{F}}(R_i)$, but we will see in each case that $\mathrm{Aut}_{\mathcal{F}}(R_i) \cong \Sigma_3$ or $C_3 \times \Sigma_3$, and hence that this choice is unique.

Lemma 3.7. *Let $\mathcal{F}$ be a saturated fusion system over a 2-group S with $r(S) \le 4$. Let (R_1, R_2) be an $\mathcal{F}$-essential pair of type* (II), *choose subgroups $\Sigma_3 \cong \mathrm{Out}^0_{\mathcal{F}}(R_i) \le \mathrm{Out}_{\mathcal{F}}(R_i)$ as in Lemma 3.6(a), and assume we are in the situation of Lemma 3.6(b.i). Thus $T = R_1 \cap R_2$ in the notation of that lemma. Then one of the following holds.*

(a) *If $\mathfrak{foc}(\mathcal{F}, R_i) \in \mathscr{Y}(S)$, then $T \in \mathscr{Y}_0(S)$, $[\mathrm{Aut}^*_{\mathcal{F}}(R_i), R_i] \le T \le \mathfrak{foc}(\mathcal{F}, R_i)$, $\mathrm{Aut}_{\mathcal{F}}(T) \in \mathscr{A}_S(T)$ (Definition 2.2), and either*
   - *$\mathrm{Out}_{\mathcal{F}}(T) \cong \Sigma_3 \wr C_2$, and $T \cong C_2^4$, 2^{1+4}_+, or $Q_8 \times Q_8$; or*
   - *$\mathrm{Out}_{\mathcal{F}}(T) \cong \Sigma_5$, $T \cong C_2^4$ or 2^{1+4}_-, and $T/\mathrm{Fr}(T)$ is the orthogonal module for $\mathrm{Out}_{\mathcal{F}}(T)$.*

(b) *If* $\mathfrak{foc}(\mathcal{F}, R_1) \notin \mathscr{Y}(S)$*, then* $T \cong C_2^4$ *and is the* $L_2(4)$*-module for* $\mathrm{Out}_{\mathcal{F}}(T) \cong \Sigma_5$ *or* $(A_5 \times C_3) \rtimes C_2$*. Also,* $S \in \mathcal{U}$*,* $[\mathrm{Aut}^*_{\mathcal{F}}(R_1), R_1] = T$*, and* $\mathfrak{foc}(\mathcal{F}, R_1) = T$ *or* $\mathfrak{foc}(\mathcal{F}, R_1) \cong UT_3(4)$*. If* $\mathscr{Y}(S) \neq \varnothing$*, then* $|S| = 2^8$*,* $S/Z(S) \cong D_8 \wr C_2$*, and* $\mathfrak{foc}(\mathcal{F}, R_1)/Z(S) \cong 2^{1+4}_+$*.*

Also, in all cases, $\mathscr{X}(S) = \varnothing$*, and for* $i = 1, 2$*,* $\mathrm{Out}_{R_i}(T) \nleq O^2(\mathrm{Out}_{\mathcal{F}}(T))$*, and* $\mathrm{Out}_{\mathcal{F}}(R_i) \cong \Sigma_3$*.*

PROOF. Set $R = R_1R_2$, let $(G_1 > R < G_2)$ be an amalgam as in Lemma 3.6(c), and set $\mathrm{Out}^0_{\mathcal{F}}(R_i) = \mathrm{Out}_{G_i}(R_i) \leq \mathrm{Out}_{\mathcal{F}}(R_i)$. Thus $R = N_S(R_1) = N_S(R_2)$, $[G_i{:}R] = 3$ and $\mathrm{Out}^0_{\mathcal{F}}(R_i) \cong \Sigma_3$ $i = 1, 2)$. Let $x \in N_S(R) \smallsetminus R$ be such that $x^2 \in R$ and $R_2 = {}^xR_1$. By assumption, $T = R_1 \cap R_2$ is normal in both G_1 and G_2.

By Lemma 3.6(b), T is centric in S and $\mathrm{Out}_S(T) \cong N_S(T) = R\langle x \rangle$. If $T^* \in T^{\mathcal{F}}$ is fully normalized, then there is $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_S(T), N_S(T^*))$ such that $\varphi(T) = T^*$ (Proposition 1.3(a)), $N_S(T^*) = \varphi(N_S(T))$ by the same lemma applied to $\varphi(R_i)$, $\varphi(R)$, $\varphi(x)$, etc. Thus T is fully normalized in $\mathcal{F}$. So by Lemma 1.5(a),

$$\mathrm{Out}_{\mathcal{F}}(R_i) \cong N_{\mathrm{Out}_{\mathcal{F}}(T)}(\mathrm{Out}_{R_i}(T))/\mathrm{Out}_{R_i}(T) \quad \text{for } i = 1, 2. \tag{3.3}$$

If $\mathrm{Out}_{R_i}(T) \leq O^2(\mathrm{Out}_{\mathcal{F}}(T))$ for some $i = 1, 2$, then

$$\mathrm{Out}_{R_i}(T) \leq [\mathrm{Out}_{\mathcal{F}}(T), \mathrm{Out}_{\mathcal{F}}(T)],$$

and by the focal subgroup theorem for groups [**G**, Theorem 7.3.4], $\mathrm{Out}_{R_i}(T)$ is conjugate in $\mathrm{Out}_{\mathcal{F}}(T)$ to the center of $\mathrm{Out}_S(T) \cong D_8$. Hence R_i is $\mathcal{F}$-conjugate to the third subgroup $R_3 < R$ of index 2 containing T. Since $R_3 \trianglelefteq R\langle x \rangle$, this would contradict the assumption that R_1 is $\mathcal{F}$-essential (hence fully normalized). Thus $\mathrm{Out}_{R_i}(T) \nleq O^2(\mathrm{Out}_{\mathcal{F}}(T))$.

Let $x \in \mathrm{Out}(T)$ be such that $\langle x \rangle = \mathrm{Out}_{R_1}(T)$. Thus $x \notin O^2(\mathrm{Out}_{\mathcal{F}}(T))$. By (3.3) and since $O_2(\mathrm{Out}_{\mathcal{F}}(R_i)) = 1$ (Lemma 1.7), $O_2(C_{\mathrm{Out}_{\mathcal{F}}(T)}(x)) = \langle x \rangle$. Hence by Proposition D.1(f), $\mathrm{Out}_{\mathcal{F}}(T) \cong \Sigma_3 \wr C_2$, Σ_5, or $\Gamma L_2(4) \cong (A_5 \times C_3) \rtimes C_2$. So in all cases, by (3.3) and since $\mathrm{Out}_{\mathcal{F}}(R_i)$ is not a 2-group, we have $\mathrm{Out}_{\mathcal{F}}(R_i) \cong \Sigma_3$.

By Theorem 1.4, there is a finite group G such that $N_S(T) \in \mathrm{Syl}_2(G)$, $T \trianglelefteq G$, $C_G(T) \leq T$, and $\mathrm{Out}_{\mathcal{F}}(T) = \mathrm{Out}_G(T) \cong G/T$. If $\mathrm{Out}_{\mathcal{F}}(T) \cong \Sigma_3 \wr C_2$, then by Lemma D.7, $T \cong C_2^4$, 2^{1+4}_+, or $Q_8 \times Q_8$. If $\mathrm{Out}_{\mathcal{F}}(T) \cong \Sigma_5$ or $\Gamma L_2(4)$, then by Lemma D.4, $T \cong C_2^4$ or 2^{1+4}_-. Finally, if $T \cong C_2^4$ and $\mathrm{Out}_{\mathcal{F}}(T) \cong \Sigma_5$, then $\mathrm{Out}_{\mathcal{F}}(T)$ acts on C_2^4 as either the $L_2(4)$-module or the orthogonal module.

In all cases, $G/\mathrm{Fr}(T)$ splits as a semidirect product

$$G/\mathrm{Fr}(T) \cong (T/\mathrm{Fr}(T)) \rtimes \mathrm{Out}_{\mathcal{F}}(T) : \tag{3.4}$$

by [**GH**, Lemma II.2.6] when $\mathrm{Out}_{\mathcal{F}}(T) \cong \Sigma_5$ or $\Gamma L_2(4)$, and by [**AOV2**, Lemma A.8] when $\mathrm{Out}_{\mathcal{F}}(T) \cong \Sigma_3 \wr C_2$.

We now consider the individual cases.

(a) Assume that either $\mathrm{Out}_{\mathcal{F}}(T) \cong \Sigma_3 \wr C_2$, or $\mathrm{Out}_{\mathcal{F}}(T) \cong \Sigma_5$ and $T/\mathrm{Fr}(T)$ is the orthogonal module for $\mathrm{Out}_{\mathcal{F}}(T)$. We will show that $\mathfrak{foc}(\mathcal{F}, R_i) \in \mathscr{Y}(S)$, that $\mathscr{X}(S) = \varnothing$, and prove the other claims in (a) which have not already been shown.

If $\mathrm{Out}_{\mathcal{F}}(T) \cong \Sigma_5$, then by Proposition D.1(d), $T/\mathrm{Fr}(T)$ is generated by an $\mathrm{Out}_{\mathcal{F}}(T)$-orbit of length 5, and hence $\mathrm{Out}_S(T)$ permutes a basis. If $\mathrm{Out}_{\mathcal{F}}(T) \cong \Sigma_3 \wr C_2$, let $V_1, V_2 < T/\mathrm{Fr}(T)$ be the irreducible components for the action of the Sylow 3-subgroup, choose bases for each V_i permuted by $N_{\mathrm{Out}_S(T)}(V_i)$, and the union of these two bases is a basis for $T/\mathrm{Fr}(T)$ permuted by $\mathrm{Out}_S(T)$. In either case, $N_S(T)/\mathrm{Fr}(T)$ splits over $T/\mathrm{Fr}(T)$ by (3.4), and hence $N_S(T)/\mathrm{Fr}(T) \cong D_8 \wr C_2$.

Since $T \cong C_2^4$, 2_+^{1+4}, 2_-^{1+4}, or $Q_8 \times Q_8$, this proves that $T \in \mathscr{Y}_0(S)$. Also, $\mathrm{Out}_{\mathcal{F}}(T) \in \mathscr{A}_S(T)$ since it is isomorphic to Σ_5 or $\Sigma_3 \wr C_2$.

If $\mathrm{Out}_{\mathcal{F}}(T) \cong \Sigma_3 \wr C_2$, then the $\mathrm{Out}_{R_i}(T)$ are distinct noncentral subgroups of order 2 which are conjugate in $\mathrm{Out}_{\mathcal{F}}(T)$. Choose $\alpha_i \in \mathrm{Aut}_{\mathcal{F}}(T)$ of order 3 such that $\big[[\alpha_i], \mathrm{Out}_{R_i}(T)\big] = 1$ in $\mathrm{Out}_{\mathcal{F}}(T)$. By the extension axiom (and since T is fully normalized), each α_i extends to $\overline{\alpha}_i \in \mathrm{Aut}^*_{\mathcal{F}}(R_i)$. Then $[\alpha_1] \neq [\alpha_2]$ in $\mathrm{Out}_{\mathcal{F}}(T)$, so

$$T \geq [\mathrm{Aut}^*_{\mathcal{F}}(R_1), R_1][\mathrm{Aut}^*_{\mathcal{F}}(R_2), R_2] \geq [\alpha_1, T][\alpha_2, T] = [\langle \alpha_1, \alpha_2 \rangle, T] = T.$$

Hence $[\mathrm{Aut}^*_{\mathcal{F}}(R_i), R_i] \leq T$, and $\mathfrak{foc}(\mathcal{F}, R_i) = \langle T^S \rangle \in \mathscr{Y}(S)$.

Now assume that $\mathrm{Out}_{\mathcal{F}}(T) \cong \Sigma_5$. Identify these groups in such a way that $\mathrm{Out}_{R_1}(T) = \langle (1\,2) \rangle$ and $\mathrm{Out}_{R_2}(T) = \langle (3\,4) \rangle$ (recall that $\mathrm{Out}_{R_i}(T) \notin O^2(\mathrm{Out}_{\mathcal{F}}(T))$). Let $\alpha_1, \alpha_2 \in \mathrm{Aut}_{\mathcal{F}}(T)$ be elements of order 3 such that $[\alpha_1] = (3\,4\,5)$ and $[\alpha_2] = (1\,2\,5)$. Then $\langle \alpha_1, \alpha_2 \rangle \cong A_5$, so $[\alpha_1, T][\alpha_2, T] = T$.

For $i = 1, 2$, α_i normalizes $\mathrm{Aut}_{R_i}(T)$, so by the extension axiom (and since T is fully normalized), α_i extends to $\overline{\alpha}_i \in \mathrm{Aut}_{\mathcal{F}}(R_i)$. In particular, $\overline{\alpha}_i \in \mathrm{Aut}^*_{\mathcal{F}}(R_i)$, so

$$T \geq [\mathrm{Aut}^*_{\mathcal{F}}(R_1), R_1][\mathrm{Aut}^*_{\mathcal{F}}(R_2), R_2] \geq [\alpha_1, T][\alpha_2, T] = T.$$

Hence $[\mathrm{Aut}^*_{\mathcal{F}}(R_i), R_i] \leq T$, and $\mathfrak{foc}(\mathcal{F}, R_i) = \langle T^S \rangle \in \mathscr{Y}(S)$.

In either case, $\mathscr{X}(S) = \varnothing$ by Corollary 2.5 and since $\mathscr{Y}(S) \neq \varnothing$.

(b) Assume $T \cong C_2^4$ is the $L_2(4)$-module for $\mathrm{Out}_{\mathcal{F}}(T) = \Sigma L_2(4)$ or $\Gamma L_2(4)$. We must show that $\mathfrak{foc}(\mathcal{F}, R_1) \notin \mathscr{Y}(S)$, and prove the other claims in (b).

Let $G \geq N_S(T)$ be as in (3.4), let $G_0 \trianglelefteq G$ be such that $G_0 > T$ and $G_0/T \cong SL_2(4)$, and let $\widehat{T} \leq N_S(T)$ be such that $\widehat{T} \in \mathrm{Syl}_2(G_0)$. Thus $\widehat{T}$ is a semidirect product of C_2^4 by C_2^2 where C_2^2 acts as $UT_2(4)$. So $\widehat{T} \cong UT_3(4)$ by Lemma C.7(a), $\widehat{T}/Z(\widehat{T})$ is centric in $N_S(T)/Z(\widehat{T})$, and hence $N_S(T) \in \mathcal{U}$.

Set $S_0 = N_S(T) = R\langle x \rangle$ for short. By (3.4) again, S_0 splits as a semidirect product of C_2^4 by D_8. If $S > S_0$, then $\mathrm{Out}_S(S_0)$ exchanges the two subgroups of $\widehat{T} \cong UT_3(4)$ isomorphic to C_2^4, and hence $[N_S(S_0){:}S_0] = 2$ and $N_1 \overset{\text{def}}{=} N_S(S_0) \in \mathcal{U}$. Hence $\widehat{T}$ is characteristic in N_1 by Lemma C.9, and $S = N_1 \in \mathcal{U}$ by Lemma A.1(b). Thus $|S| \leq 2^8$. Also, $[\mathrm{Aut}^*_{\mathcal{F}}(R_i), R_i] = T$, and so $\mathfrak{foc}(\mathcal{F}, R_i) = T$ (if $T \trianglelefteq S$) or $\widehat{T}$.

If $\mathscr{Y}(S) \neq \varnothing$, then $S > S_0$ since $|S_0| = 2^7 = |D_8 \wr C_2|$ and $S_0 \not\cong D_8 \wr C_2$. Hence $|S| = 2^8$, and $S/Z(S) \cong D_8 \wr C_2$ by Lemma 2.4(a). Also, $T \notin \mathscr{Y}_0(S)$ (hence $T \notin \mathscr{Y}(S)$) since $S_0 = N_S(T) \not\cong D_8 \wr C_2$. Also, $\widehat{T} \notin \mathscr{Y}_0(S)$ since $|\widehat{T}| = 2^6$ and $\widehat{T} \not\cong Q_8 \times Q_8$, and for any $Q < \widehat{T}$ of index 2, $|Z(Q)| \geq 4$ and hence Q is not extraspecial. Thus no subgroup of $\widehat{T}$ lies in $\mathscr{Y}_0(S)$, and $\widehat{T} \notin \mathscr{Y}(S)$. Also, $\widehat{T}/Z(S) \cong 2_+^{1+4}$ (since $UT_3(4)/Z \cong 2_+^{1+4}$ for each $Z < Z(UT_3(4))$ of order 2).

It remains to show that $\mathscr{X}(S) = \varnothing$. Assume otherwise: assume $Q \in \mathscr{X}(S)$. Thus $Q \in \mathcal{DQ}$, $Q \trianglelefteq S$, and Q is strongly automized in S. Since $Z_2(S) = Z_2(S_0) = Z(\widehat{T})$ has order 4, $Q \geq Z(\widehat{T})$ by Lemma A.2(b). Thus $C_2^2 \cong Z(\widehat{T}) \trianglelefteq Q$, so $Q \cong D_8$, and (since $Z(\widehat{T}) \trianglelefteq S$) Q is not strongly automized in S, a contradiction. □

We next look at essential pairs of type (II) where $T < R_1 \cap R_2$ in the notation of Lemma 3.6(b). The starting point for doing this is the description in [**AOV2**, Theorem 4.6].

Lemma 3.8. *Let $\mathcal{F}$ be a saturated fusion system over a 2-group S with $r(S) \leq 4$. Let (R_1, R_2) be an $\mathcal{F}$-essential pair of type* (II), *choose subgroups $\Sigma_3 \cong \mathrm{Out}^0_{\mathcal{F}}(R_i) \leq \mathrm{Out}_{\mathcal{F}}(R_i)$ as in Lemma 3.6(a), and assume we are in the situation of Lemma*

3.6(b.ii). Thus $T < R_1 \cap R_2$ in the notation of Lemma 3.6(b). Set $R = R_1R_2 = N_S(R_1) = N_S(R_2)$.

Set $U_i = [\mathrm{Aut}^*_{\mathcal{F}}(R_i), R_i] \trianglelefteq R$ and $U = U_1U_2$. Set $W = \mathrm{Fr}(U)$, $S_* = N_S(W)$, and let Δ be the normal closure of U in S_*. Then either

(a) *$S_* = S$ and $\mathfrak{foc}(\mathcal{F}, R_1) = \Delta \in \mathscr{X}(S)$; or*

(b) *$S_* < S$, $N_S(\Delta) = S^*$, $[S{:}S_*] = 2$, $\mathfrak{foc}(\mathcal{F}, R_1) = \Delta\Delta^*$ where $\Delta^* \neq \Delta$, $\{\Delta, \Delta^*\}$ is an S-conjugacy class, $[\Delta, \Delta^*] \leq \Delta \cap \Delta^* \leq Z(S)$, and $\Delta \cap \Delta^* = 1$ if $\Delta \in \mathcal{D}$. In this case, if $\mathcal{F}$ is reduced or $\mathscr{Y}(S) \neq \varnothing$, then $\mathfrak{foc}(\mathcal{F}, R_1) \in \mathscr{Y}(S)$.*

Also, the following hold in both cases.

(c) *Either*
  (c.1) *$U_1 \cong U_2 \cong C_2^2$, $U \cong D_8$, $\Delta \in \mathcal{D}$, and $W = Z(U) = Z(\Delta) \cong C_2$; or*
  (c.2) *$U_1 \cong U_2 \cong Q_8$, $U \cong Q_{16}$, $\Delta \in \mathcal{Q}$, and $W = Z_2(U) = Z_2(\Delta) \cong C_4$.*

(d) *For $i = 1, 2$, U_i is fully normalized in $\mathcal{F}$, and $R_i = U_iC_S(U_i)$. If $U_i \cong C_2^2$, then it is a direct factor of R_i.*

(e) *For $i = 1, 2$, $\mathrm{Out}_{\mathcal{F}}(R_i) \cong \Sigma_3$ or $C_3 \times \Sigma_3$.*

Proof. Let $G_1 > R < G_2$ be as in Lemma 3.6(c). Thus $R_i \trianglelefteq G_i$ and $\mathrm{Out}^0_{\mathcal{F}}(R_i) = \mathrm{Out}_{G_i}(R_i) \cong \Sigma_3$ for $i = 1, 2$. Set $\widehat{U}_i = R \cap O^2(G_i) \trianglelefteq R$ and $\widehat{U} = \widehat{U}_1\widehat{U}_2$. We first prove (e), and then use that to show that $\widehat{U}_i = U_i$ and $\widehat{U} = U$.

(e) By [**AOV2**, Theorem 4.6(a)], there is a subgroup $T^\bullet < R_1 \cap R_2$ such that $R_i = \widehat{U}_iT^\bullet$ and $R = \widehat{U}T^\bullet$, and such that either

$$\widehat{U}_i \cong C_2^2,\ \widehat{U} \cong D_8, \text{ and } [T^\bullet, \widehat{U}] = T^\bullet \cap \widehat{U} = 1; \text{ or} \tag{3.5}$$

$$\widehat{U}_i \cong Q_8,\ \widehat{U} \cong Q_{16}, \text{ and } [T^\bullet, \widehat{U}] \leq T^\bullet \cap \widehat{U} = Z(\widehat{U}). \tag{3.6}$$

In particular, since $[T^\bullet, \widehat{U}] \leq [\widehat{U}_i, \widehat{U}_i]$ in both cases,

$$[R, R_i] = [\widehat{U}T^\bullet, \widehat{U}_iT^\bullet] = [\widehat{U}, \widehat{U}_i][T^\bullet, T^\bullet] \leq [\widehat{U}, \widehat{U}_i]\mathrm{Fr}(R_i)\,.$$

So $\mathrm{rk}([R, R_i/\mathrm{Fr}(R_i)]) \leq \mathrm{rk}([\widehat{U}, \widehat{U}_i/\mathrm{Fr}(\widehat{U}_i)]) = 1$. Hence $\mathrm{Out}_{\mathcal{F}}(R_i) \ncong (C_3 \times C_3) \overset{-1}{\rtimes} C_2$, and by Lemma 3.2, $\mathrm{Out}_{\mathcal{F}}(R_i) \cong \Sigma_3$ or $C_3 \times \Sigma_3$.

In particular, $O^{2'}(\mathrm{Aut}_{\mathcal{F}}(R_i)) = \mathrm{Aut}_{G_i}(R_i)$, and hence

$$\mathrm{Aut}^*_{\mathcal{F}}(R_i) = O^2(\mathrm{Aut}_{G_i}(R_i)) = \mathrm{Aut}_{O^2(G_i)}(R_i).$$

For any $Q \in \mathrm{Syl}_3(G_i)$, $Q \leq O^2(G_i) \leq R_iQ$ (recall $R_i \trianglelefteq G_i$ and $G_i/R_i \cong \Sigma_3$). Thus $\mathrm{Aut}_Q(R_i) \leq \mathrm{Aut}^*_{\mathcal{F}}(R_i) \leq \mathrm{Inn}(R_i)\mathrm{Aut}_Q(R_i)$. So

$$\widehat{U}_i = R \cap O^2(G_i) = R \cap (Q[Q, R_i]) = [Q, R_i] = [\mathrm{Aut}^*_{\mathcal{F}}(R_i), R_i] = U_i\,,$$

where the fourth equality holds by Proposition 1.14(c) (applied with $\mathrm{Aut}_Q(R_i)$ in the role of Γ).

(c) By Lemma 3.6(b), T is not centric in S. So points (c.1) and (c.2) follow from [**AOV2**, Theorem 4.6(a)] and since $U_i = \widehat{U}_i$. Note that since $\Delta \trianglelefteq S_*$ and $[S{:}S_*] = 2$, the S-conjugacy class of Δ has order at most 2, and hence is $\{\Delta, \Delta^*\}$.

(d) By [**AOV2**, Theorem 4.6(a)] again, $T^\bullet \leq C_{S_*}(U_1) \leq T^\bullet U_1$. So $R_1 = T^\bullet U_1 = U_1C_{S_*}(U_1)$. Since $W = \mathrm{Fr}(U) \leq U_1$, $C_S(U_1) \leq N_S(W) = S_*$, so $C_S(U_1) = C_{S_*}(U_1)$, and $R_1 = U_1C_S(U_1)$. By Lemma 3.6(c), there are $\beta \in \mathrm{Iso}(G_1, G_2)$ and $x \in N_S(R)$ such that $\beta(R) = R$, $\beta|_R = c_x|_R$, and $\beta(R_1) = {}^xR_1 = R_2$. Then

${}^xU_1 = \beta(U_1) = U_2$, and so $R_2 = {}^xR_1 = U_2C_S(U_2)$. If $U_i \cong C_2^2$ $(i = 1, 2)$, then $U \cap T^\bullet = 1$ by (3.5), and hence U_i is a direct factor of $R_i = T^\bullet U_i$.

For $i = 1, 2$, $R = UT^\bullet = UR_i \le N_S(U_i) \le N_S(U_iC_S(U_i)) = N_S(R_i) = R$, so $N_S(U_i) = N_S(R_i)$. Since R_i is fully normalized, U_i is also fully normalized by Proposition 1.3(c), applied with $U_i \trianglelefteq R_i$ in the role of $Q \trianglelefteq P$. Note that since $R = N_S(R_i)$, (1.1) takes the form $N_S(\varphi(U_i)) \cap N_S(\varphi(R)) \le N_S(\varphi(R_i))$ for each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(R, S)$, and this holds since $R_i = U_iC_R(U_i)$.

(a,b) By (e), $\mathfrak{foc}(\mathcal{F}, R_1)$ is the normal closure of U_1 in S. Also, $U_1 = R \cap O^2(G_1)$ and $U_2 = R \cap O^2(G_2)$ are S-conjugate by the conditions on G_1 and G_2 in Lemma 3.6(c). Since Δ is the normal closure of $U = U_1U_2$ in S_*, this shows that $\mathfrak{foc}(\mathcal{F}, R_1)$ is the normal closure of Δ in S.

If $S_* = S$, then $\Delta \trianglelefteq S$, so $\mathfrak{foc}(\mathcal{F}, R_1) = \Delta$. Also, Δ is fully automized in S since it is the normal closure of $U_1 < \Delta$, so $\Delta \in \mathscr{X}(S)$.

Assume for the rest of the proof that $S_* < S$. Thus $\Delta \ntrianglelefteq S$. Also, $S_* \le N_S(\Delta) \le N_S(W) = S_*$ since $\Delta \trianglelefteq S_*$ by construction and W is characteristic in Δ. Thus $S_* = N_S(W)$. If $\Delta \in \mathcal{D}$, then $S_* = C_S(Z(\Delta))$, and $[S{:}S_*] = 2$ by Lemma B.4.

Now assume $\Delta \in \mathcal{Q}$, and set $S_0 = C_S(Z(\Delta))$. Thus $S_* \le S_0 \le S$, and $[S_0{:}S_*] \le 2$ by Lemma B.4 applied to $\Delta/Z(\Delta) < S_0/Z(\Delta)$. If $S_0 = S_*$, then upon applying the lemma again, we get that $[S{:}S_*] = 2$. If $[S_0{:}S_*] = 2$, and $\widehat{\Delta}$ is the normal closure of Δ in S_0, then $\widehat{\Delta}/Z(\Delta) \cong (\Delta/Z(\Delta)) \times (\Delta/Z(\Delta))$ by Lemma B.3, so $r(\widehat{\Delta}) = 4$, and Lemma B.4 applied to $\widehat{\Delta} < S$ implies that $S = S_0$ and hence $[S{:}S_*] = 2$.

Thus in both cases, $[S{:}N_S(\Delta)] = 2$, so $\mathfrak{foc}(\mathcal{F}, R_1) = \Delta\Delta^*$ where $\Delta^* = {}^g\Delta$ for any $g \in S \smallsetminus S_*$. Since $S_* = N_S(W)$, $Z(\Delta) \ne Z(\Delta^*)$ if $\Delta \in \mathcal{D}$, and $Z_2(\Delta) \ne Z_2(\Delta^*)$ if $\Delta \in \mathcal{Q}$. So $[\Delta, \Delta^*] \le \Delta \cap \Delta^* \le Z(\Delta)$ by Lemma B.3 (applied with $\Delta/Z(\Delta)$ and $\Delta^*/Z(\Delta)$ in the role of P and Q if $\Delta \in \mathcal{Q}$ and $Z(\Delta) = Z(\Delta^*)$), and $\Delta \cap \Delta^* = 1$ if $\Delta \in \mathcal{D}$.

Assume $\Delta\Delta^* \notin \mathscr{Y}(S)$; we must show that $\mathscr{Y}(S) = \varnothing$ and $\mathcal{F}$ is not reduced. Set $Y = \Delta\Delta^*$. By Lemma 2.6 and since $Y \notin \mathscr{Y}(S)$, $\mathscr{Y}(S) = \varnothing$, $S^{\mathrm{ab}} \cong C_4 \times C_2$, $S/Y \cong C_4$, and the action of S/Y on $Y^{\mathrm{ab}} \cong C_2^4$ permutes freely a basis. Also, $Y < S_* < S$ since $S_* \ge \Delta\Delta^*$ and $[S{:}S_*] = 2$.

Assume $\mathcal{F}$ is reduced; we must find a contradiction. By Lemma 1.17, there are subgroups $Q \trianglelefteq P \in \mathbf{E}_{\mathcal{F}}$ such that $P/Q \cong C_4 \times C_4$, and such that $[\mathrm{Aut}^*_{\mathcal{F}}(P), P]$ surjects onto $S/Y \cong C_4$. Then $|N_S(P)/P| \le 4$ by Lemma 3.3, $N_S(P)/P \ncong C_2^2$ by Proposition 3.4 and since P^{ab} is not elementary abelian, and $N_S(P)/P \ncong C_4$ by Proposition 3.5 and since $\mathcal{F}$ is reduced. Thus $P \notin \mathbf{E}^{(\mathrm{I})}_{\mathcal{F}}$. Since $\mathscr{Y}(S) = \varnothing$ by Lemma 2.4(a) ($S^{\mathrm{ab}} \ncong C_2^3$) and $P \ncong C_2^4$, P is not in a pair of the type described in Lemma 3.7. By (e) and since $[\mathrm{Aut}^*_{\mathcal{F}}(P), P]$ surjects on to S/Y, P is not in a pair of the type described in this lemma. Thus $P \notin \mathbf{E}^{(\mathrm{II})}_{\mathcal{F}}$, and hence $[S{:}P] = 2$.

Fix $g \in P \smallsetminus S_*$, choose $a_1, b_1 \in \Delta$ such that $[\Delta{:}\langle a_1 \rangle] = 2$ and $b_1 \in \Delta \smallsetminus \langle a_1 \rangle$, and set $a_2 = {}^ga_1, b_2 = {}^gb_1 \in \Delta^*$. Then ${}^gb_2 = {}^{g^2}b_1 = a_1^ib_1$, where i is odd since c_g permutes freely a basis of Y^{ab}. Also, $b_1b_2^{-1} \in [S, S]$, so $b_1b_2 \in \mathrm{Fr}(S) \le P$, and ${}^g(b_1b_2)(b_1b_2)^{-1} = (b_2a_1^ib_1)(b_1b_2)^{-1} \equiv a_1^i \pmod{[\Delta, \Delta^*] \le Z(\Delta)}$. Thus $a_1, a_2 \in [P, P]$, so $P/[P, P] = \langle [g], [b_1b_2] \rangle \cong C_4 \times C_2$ contains no subgroup $C_4 \times C_4$, which contradicts our original assumption. Hence this situation is impossible. □

We now summarize the results in Section 3.1 and in the last two lemmas, as they apply when $\mathscr{Y}(S) \neq \varnothing$.

PROPOSITION 3.9. *Let $\mathcal{F}$ be a saturated fusion system over a 2-group S such that $r(S) \leq 4$ and $\mathscr{Y}(S) \neq \varnothing$. Let $Z_* \trianglelefteq S$ be the unique normal subgroup such that $S/Z_* \cong D_8 \wr C_2$. Let $Y_1, Y_2, Y_3 \trianglelefteq S$ be the distinct normal subgroups such that $Y_1/Z_* \cong Y_2/Z_* \cong C_2^4$ and $Y_3/Z_* \cong 2_+^{1+4}$.*

(a) $\mathrm{Out}_{\mathcal{F}}(S) = 1$, *and* $\mathbf{E}_{\mathcal{F}} = \mathbf{E}_{\mathcal{F}}(Y_1) \cup \mathbf{E}_{\mathcal{F}}(Y_2) \cup \mathbf{E}_{\mathcal{F}}(Y_3)$. *If $\mathcal{F}$ is reduced, then $\mathbf{E}_{\mathcal{F}}(Y_i) \neq \varnothing$ for each $i = 1, 2, 3$.*

(b) *If $Y_i \in \mathscr{Y}(S)$ (some $i = 1, 2, 3$), then $\mathbf{E}_{\mathcal{F}}(Y_i) \subseteq \mathbf{E}_{\mathcal{F}}^{(\mathrm{II})}$. Also, each $R \in \mathbf{E}_{\mathcal{F}}(Y_i)$ is in an $\mathcal{F}$-essential pair of the form described in Lemma 3.7(a) or 3.8(b).*

(c) *If $Y_i \notin \mathscr{Y}(S)$ (some $i = 1, 2, 3$) and $R \in \mathbf{E}_{\mathcal{F}}(Y_i)$, then $i = 3$, and either*

  (c.1) *$R \in \mathbf{E}_{\mathcal{F}}^{(\mathrm{II})}$, $|R| = 2^5$, $Y_3 \cong UT_3(4)$, and R is in an $\mathcal{F}$-essential pair of the form described in Lemma 3.7(b); or*

  (c.2) *$R \in \mathbf{E}_{\mathcal{F}}^{(\mathrm{III})}$, $R > Y_3$, and $Y_3 \cong 2_+^{1+4}$, $Q_8 \times Q_8$, or $UT_3(4)$.*

PROOF. By definition of $\mathscr{Y}(S)$, $|S| \geq 2^7$, with equality only if $S \cong D_8 \wr C_2$. Also, $(D_8 \wr C_2)^{\mathrm{ab}} \cong C_2^3$, $D_8 \wr C_2$ contains no subgroup isomorphic to 2_-^{1+4} by Lemma C.5(a), and contains none isomorphic to $UT_3(4)$ by Lemma C.5(b). So by Propositions 3.4 and 3.5, $\mathbf{E}_{\mathcal{F}}^{(\mathrm{I})} = \varnothing$. By Lemma 2.4(b) and Corollary 2.5, $\mathscr{Y}(S) \subseteq \{Y_1, Y_2, Y_3\}$, and $\mathrm{Aut}(S)$ is a 2-group (hence $\mathrm{Aut}_{\mathcal{F}}^*(S) = 1$).

By Corollary 2.5, $\mathscr{X}(S) = \varnothing$. So if $R \in \mathbf{E}_{\mathcal{F}}^{(\mathrm{II})}$, then by Lemma 3.6(b.1,b.2), R is in an $\mathcal{F}$-essential pair of the form described in Lemma 3.7(a) or 3.8(b) (if $\mathfrak{foc}(\mathcal{F}, R) \in \mathscr{Y}(S)$), or in Lemma 3.7(b). In the latter case, $|R| = 2^5$, $\mathfrak{foc}(\mathcal{F}, R) = Y_3 \notin \mathscr{Y}(S)$, $Y_3 \cong UT_3(4)$, and we are in the situation of (c.1).

We claim that

$$R \in \mathbf{E}_{\mathcal{F}} \text{ of type (III)} \implies R > Y_3,\ \mathfrak{foc}(\mathcal{F}, R) = Y_3,\ Y_3 \notin \mathscr{Y}(S), \text{ and}$$
$$Y_3 \cong 2_+^{1+4},\ Q_8 \times Q_8, \text{ or } UT_3(4). \tag{3.7}$$

Once this has been shown, points (b) and (c) then follow. Also, for each $R \in \mathbf{E}_{\mathcal{F}}$, $\mathfrak{foc}(\mathcal{F}, R) = Y_i$ for some $i = 1, 2, 3$, and hence $\mathbf{E}_{\mathcal{F}} = \bigcup_{i=1}^{3} \mathbf{E}_{\mathcal{F}}(Y_i)$. Finally, if $\mathcal{F}$ is reduced, then $S = \langle \mathfrak{foc}(\mathcal{F}, R) \mid R \in \mathbf{E}_{\mathcal{F}} \cup \{S\} \rangle$ by Proposition 1.14(b), $\mathfrak{foc}(\mathcal{F}, S) = 1$ since $\mathrm{Aut}_{\mathcal{F}}^*(S) = 1$, and so $\mathbf{E}_{\mathcal{F}}(Y_i) \neq \varnothing$ for each $i = 1, 2, 3$ since no two of the Y_i generate S.

It remains to prove (3.7). Assume $R \in \mathbf{E}_{\mathcal{F}}^{(\mathrm{III})}$. In particular, $[S{:}R] = 2$ and $\mathrm{Aut}(R)$ is not a 2-group. Set $|S| = 2^m$, and for each $0 \leq i \leq m-5$, set $Z_i = Z_i(S)$ for short. Let $\mathscr{C}h(R)$ be the set of subgroups characteristic in R.

We first show, for each $1 \leq i \leq m-6$, that

(i) either $Z_i \in \mathscr{C}h(R)$ or $Z_{i-1} \in \mathscr{C}h(R)$; and

(ii) if $Z_i, Z_{i-1} \in \mathscr{C}h(R)$, then $Z_j \in \mathscr{C}h(R)$ for each $j \leq i$.

To prove (i), assume $Z_j \in \mathscr{C}h(R)$ for some $0 \leq j \leq m-8$, and let $P \trianglelefteq S$ be such that $P/Z_j = \Omega_1(Z(R/Z_j))$. Thus $P \in \mathscr{C}h(R)$. By Lemma 2.4(a), and since $|P| \geq 2^{j+1}$, either $P = Z_k$ for $k = j+1, j+2$, or $P \geq Z_{j+3}$. Since P/Z_j is elementary abelian, and $Z_{j+3}/Z_j \cong C_4 \times C_2$ by Lemma 2.4(c), this last case is impossible. Thus $Z_{j+1} \in \mathscr{C}h(R)$ or $Z_{j+2} \in \mathscr{C}h(R)$, and (i) now follows by induction on i.

If $i \geq 2$, Z_i and Z_{i-1} are both characteristic, and Z_{i-2} is not, then $i \geq 3$ ($Z_0 \in \mathscr{C}h(R)$), $Z_{i-3} \in \mathscr{C}h(R)$ by (i), $Z_i/Z_{i-2} \cong C_2^2$ and $Z_i/Z_{i-3} \cong C_4 \times C_2$ by Lemma 2.4(c), so $Z_{i-2}/Z_{i-3} = \mathrm{Fr}(Z_i/Z_{i-3})$ is characteristic in R/Z_{i-3}, which contradicts our assumption. Point (ii) now follows by induction on i.

By (i), either $Z_* = Z_{m-7} \in \mathscr{C}h(R)$ or $Z_{m-6} \in \mathscr{C}h(R)$ (or both). Since $\mathrm{Fr}(R) \trianglelefteq S$, and $[S{:}\mathrm{Fr}(R)] \leq 2^5$ since $r(R) \leq 4$, $\mathrm{Fr}(R) \geq Z_{m-5}(S) > Z_*$ by Lemma 2.4(a) again. If $Z_* \in \mathscr{C}h(R)$, then $\mathrm{Aut}(R/Z_*)$ is not a 2-group by Lemma A.9 and since $\mathrm{Aut}(R)$ is not a 2-group, so $R/Z_* \cong (2_+^{1+4}) \rtimes C_2$ by Lemma C.5(b). Since $Y_3 < S$ is the unique subgroup such that $Y_3/Z_* \cong 2_+^{1+4}$ (Lemma C.5(a)), $R > Y_3$.

If Z_* is not characteristic in R, then $m \geq 8$ since $Z_* \neq 1$, $Z_{m-8}, Z_{m-6} \in \mathscr{C}h(R)$ by (i), and so $Z_{m-5} \notin \mathscr{C}h(R)$ by (ii). Thus $R/Z_{m-6} < S/Z_{m-6} \cong (D_8 \times_{C_2} D_8) \overset{t}{\rtimes} C_2 \cong UT_4(2)$ with index 2, $R/Z_{m-6} \not\cong 2_+^{1+4}$ (that would imply $Z_{m-5} = \mathrm{Fr}(R) \in \mathscr{C}h(R)$), and $\mathrm{Aut}(R/Z_{m-6})$ is not a 2-group by Lemma A.9 (recall $Z_{m-6} \leq \mathrm{Fr}(R)$). So $R > Y_3$ by Lemma C.4(c).

We have now shown that $R > Y_3$. Also, $Y_3 \in \mathscr{C}h(R)$, since $Y_3/Z_* < S/Z_*$ and $Y_3/Z_{m-6} < S/Z_{m-6}$ are the unique subgroups of their isomorphism type (and Z_* or Z_{m-6} is in $\mathscr{C}h(R)$). Since

$$\begin{aligned} S/\mathrm{Fr}(Y_3) = S/Z_{m-6} &\cong (D_8 \wr C_2)/Z(D_8 \wr C_2) \\ &\cong (D_8 \times_{C_2} D_8) \overset{t}{\rtimes} C_2 \cong UT_4(2) \cong C_2 \wr C_2^2 \end{aligned}$$

(see Lemma C.4(a,b)), $\mathrm{Out}_S(Y_3) \cong S/Y_3 \cong C_2^2$ acts faithfully on $Y_3/\mathrm{Fr}(Y_3) \cong C_2^4$, permuting a basis freely. Hence $\mathrm{Out}_{\mathcal{F}}(Y_3)$ also acts faithfully on $Y_3/\mathrm{Fr}(Y_3)$ (Lemma A.9). Fix $\mathrm{Id} \neq \alpha \in \mathrm{Aut}_{\mathcal{F}}(R)$ of odd order, and set $\alpha_0 = \alpha|_{Y_3} \in \mathrm{Aut}_{\mathcal{F}}(Y_3)$. Then $\big[[\alpha_0], \mathrm{Out}_R(Y_3)\big] = 1$ in $\mathrm{Out}_{\mathcal{F}}(Y_3)$, so these commute as automorphisms of $Y_3/\mathrm{Fr}(Y_3) \cong C_2^4$, which implies that $|\alpha_0| = 3$ and α_0 acts on $Y_3/\mathrm{Fr}(Y_3)$ with trivial fixed component. Thus $\mathfrak{foc}(\mathcal{F}, R)\mathrm{Fr}(Y_3) = Y_3$, so $\mathfrak{foc}(\mathcal{F}, R) = Y_3$ (cf. [**G**, Theorem 5.1.1]).

Now, $Z_{m-6} = \mathrm{Fr}(Y_3) \in \mathscr{C}h(R)$, and hence $Z_{m-j} \in \mathscr{C}h(R)$ for each even $6 \leq j \leq m$ by (i) and (ii). If $m = 7$, then $Y_3 \cong 2_+^{1+4}$. If $m \geq 8$, then $Z(Y_3/Z_{m-8}) \cong C_2^2$ by Lemma D.3, so by Lemma D.2, $Y_3/Z_{m-8} \cong Q_8 \times Q_8$, $UT_3(4)$, or a certain special 2-group of type 2^{2+4} which contains a characteristic subgroup $P/Z_{m-8} \cong C_2^4$. This last case is impossible, since then $P > Z_{m-5}$ by Lemma 2.4(a), while $Z_{m-5}/Z_{m-8} \cong C_4 \times C_2$ by Lemma 2.4(c). If $m \geq 9$, then $Z(Y_3/Z_{m-9}) = Z_{m-6}/Z_{m-9} \cong C_2^3$ by Lemma D.3, which again contradicts Lemma 2.4(c). Thus $m \leq 8$, and Y_3 is as described in (3.7). Finally, $Y_3 \notin \mathscr{Y}(S)$ by Lemma 2.4(d), since $Y_3/Z_{m-8} \not\cong D_8 \times D_8$. □

Recall Definition 2.1(e): for a 2-group S and $Y_0 \in \mathscr{Y}_0(S)$, $\mathscr{U}_S(Y_0)$ is the set of all pairs $\{U_1, U_2\}$ such that $U_i \cong C_2^2$ or Q_8, $[U_1, U_2] \leq U_1 \cap U_2 \leq \mathrm{Fr}(U_1)$, $Y_0 = U_1U_2$, and each element of $\mathrm{Out}_S(Y_0) \cong D_8$ either normalizes the U_i or exchanges them.

LEMMA 3.10. *Let $\mathcal{F}$ be a saturated fusion system over a 2-group S such that $r(S) \leq 4$ and $\mathscr{Y}(S) \neq \varnothing$. Fix $Y_0 \in \mathscr{Y}_0(S)$. Then Y_0 is fully normalized in $\mathcal{F}$, and for each $\{U_1, U_2\} \in \mathscr{U}_S(Y_0)$, U_1 and U_2 are fully normalized in $\mathcal{F}$.*

PROOF. Set $Z = \mathrm{Fr}(Y_0)$ for short. By Lemma 2.4(a), $Z \trianglelefteq S$.

We first prove that Y_0 is fully normalized. By Proposition 1.3(a), there is $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_S(Y_0), S)$ such that $\varphi(Y_0)$ is fully normalized. Set $Y_1 = \varphi(Y_0)$ and $Z_1 = \varphi(Z) = \mathrm{Fr}(Y_1) \trianglelefteq N_S(Y_1)$. Then $\varphi(N_S(Y_0))/Z_1 \cong N_S(Y_0)/Z \cong D_8 \wr C_2$. By

Lemma 2.3, applied with $N_S(Z_1)/Z_1$, $\varphi(N_S(Y_0))/Z_1$, and Y_1/Z_1 in the role of S, P, and V, $N_S(Y_1)/Z_1 = \varphi(N_S(Y_0))/Z_1$, and hence Y_0 is also fully normalized.

Now fix $\{U_1, U_2\} \in \mathscr{U}_S(Y_0)$. It remains to show, for $i = 1, 2$, that U_i is fully normalized. Assume otherwise. Set $M = N_{N_S(Y_0)}(U_i)$. By Lemma 1.16(a), there is an $\mathcal{F}$-essential subgroup $R \geq N_S(U_i) \geq M$. We will show that this is impossible.

By Definition 2.1(e), M/Z has index 2 in $N_S(Y_0)/Z \cong D_8 \wr C_2$, and normalizes the two complementary subgroups $U_1Z/Z \cong U_2Z/Z \cong C_2^2$ in $Y_0/Z \cong C_2^4$. Hence $M = Y_0\langle g_1, g_2\rangle$ where $\mathrm{rk}([g_i, Y_0/Z]) = 1$, so $M/Z \cong D_8 \times D_8$.

By Proposition 3.9 and since $|R| \geq 2^6$, either $\mathfrak{foc}(\mathcal{F}, R) \in \mathscr{Y}(S)$ and R is as described in 3.9(b), or $R \in \mathbf{E}_{\mathcal{F}}^{(\mathrm{III})}$ and is as described in 3.9(c). In the former case, R is in an essential pair of the type described in Lemma 3.7(a) or 3.8(b). We consider these three cases individually.

If R is in a pair as in 3.7(a), then it contains a subgroup $T < R$ of index 2, where $T \in \mathscr{Y}_0(S)$ and hence $T \cong C_2^4$, 2_+^{1+4}, 2_-^{1+4}, or $Q_8 \times Q_8$. Thus

$$T/Z < R/Z \geq M/Z \cong D_8 \times D_8$$

and $[R/Z{:}T/Z] = 2$. If $|T/Z| \leq 2^5$, then $R = M$, and T/Z is isomorphic to a subgroup of index 2 in $D_8 \times D_8$. Hence T surjects onto D_8, which is impossible in all cases. If $|T/Z| \geq 2^6$, then $Z = 1$ and $T \cong Q_8 \times Q_8$, which is also impossible since $Q_8 \times Q_8$ contains only three involutions while each subgroup of index 2 in $D_8 \times D_8$ contains more.

If $R \in \mathbf{E}_{\mathcal{F}}^{(\mathrm{III})}$, then a similar argument applies using Proposition 3.9(c), except that we could have $T \cong UT_3(4)$. In this case, $|R| = 2^7$ and $|S| = 2^8$. By Lemma 2.4(a), $|Z(S)| = 2$ and $S/Z(S) \cong D_8 \wr C_2$. By Proposition 3.9(c) again, $T/Z(S)$ is the unique subgroup of $S/Z(S)$ isomorphic to 2_+^{1+4}.

If $Z = 1$, then $Y_0 \cong C_2^4$, $M \cong D_8 \times D_8$, and $N_S(Y_0) \cong D_8 \wr C_2$. Thus $M/Z(S) = M/Z(N_S(Y_0)) \cong 2_+^{1+4}$. Hence $M/Z(S) = T/Z(S)$, which is impossible since $M \not\cong T$.

If $Z \neq 1$, then $|M| = 2^7 = |R|$, so $M = R$ and $|Z| = 2$. Then $R/Z \cong D_8 \times D_8$, $T/Z = T/Z(S) \cong 2_+^{1+4}$, $R \geq T$, and this is impossible.

If R is in a pair as in 3.8(b), then there is $V \trianglelefteq R$ such that either $V \cong C_2^2$ and is a direct factor of R, or $V \cong Q_8$ and $R = VC_S(V)$. By Lemma A.6(c) (applied with R, V, and M or Y_0 in the role of S, U, and Q), $V \leq M$, and $V \leq Y_0$ if $Z \neq 1$. If $Z = 1$, this is impossible since $M \cong D_8 \times D_8$ contains no subgroup isomorphic to Q_8, and no direct factor isomorphic to C_2^2.

If $Z \neq 1$, then $V \trianglelefteq Y_0$, so $VZ/Z \cong C_2^i$ for $i \leq 2$, and VZ/Z is a direct factor in $M/Z \cong D_8 \times D_8$. Hence $i = 0$, $V \leq Z$, so $V \cong C_2^2$ and is a direct factor in M, which is impossible. □

We now make a more precise analysis, for a fusion system $\mathcal{F}$ over S, of essential subgroups R such that $\mathfrak{foc}(\mathcal{F}, R) \in \mathscr{Y}(S)$.

Recall that $\mathcal{D}$ denotes the class of nonabelian dihedral 2-groups. It will be useful — in the following proposition only — to let $\widehat{\mathcal{D}}$ be the extended class of dihedral 2-groups including the group $D_4 = C_2^2$. Thus $S \in \widehat{\mathcal{D}}$ if $S \in \mathcal{D}$ or $S \cong C_2^2$. By extension of Definition 2.1, a subgroup $P \cong C_2^2$ is *strongly automized* in $S > P$ if $\mathrm{Aut}_S(P) \neq 1$.

PROPOSITION 3.11. *Let $\mathcal{F}$ be a saturated fusion system over a 2-group S such that $r(S) \leq 4$ and $\mathscr{Y}(S) \neq \varnothing$. Fix $Y \in \mathscr{Y}(S)$, and assume that $\mathbf{E}_{\mathcal{F}}(Y) \neq \varnothing$. Let*

$\mathscr{Y}_0$ be the set of all $Y_0 \in \mathscr{Y}_0(S)$ whose normal closure is Y; equivalently, those which are contained in Y. Then the following hold.

(a) *There is a pair of subgroups $\Theta_1, \Theta_1 \trianglelefteq Y$ such that*
  (a.1) *$\{\Theta_1, \Theta_2\}$ is an S-conjugacy class and $\Theta_i \in \widehat{\mathcal{D}}\mathcal{Q}$;*
  (a.2) *$[\Theta_1, \Theta_2] \le \Theta_1 \cap \Theta_2 \le Z(S)$, $\Theta_1 \cap \Theta_2 = 1$ if $\Theta_i \in \widehat{\mathcal{D}}$, and $Y = \Theta_1\Theta_2$; and*
  (a.3) *for each $U \le \Theta_1$ or $U \le \Theta_2$ such that $U \cong C_2^2$ or Q_8, $\mathrm{Aut}_{\mathcal{F}}(U) = \mathrm{Aut}(U)$.*
  Furthermore, Θ_1 and Θ_2 are strongly automized in S.

(b) *Let $\Theta_1, \Theta_2 \trianglelefteq Y$ be as in* (a). *Let $\mathscr{U}_{\mathcal{F}}(Y)$ be the set of all U such that $U \le \Theta_1$ or $U \le \Theta_2$, and $U \cong C_2^2$ or Q_8. Then all subgroups in $\mathscr{U}_{\mathcal{F}}(Y)$ are S-conjugate to each other, and*

$$\mathscr{U}_{\mathcal{F}}(Y) = \{Y_0 \cap \Theta_i \,|\, Y_0 \in \mathscr{Y}_0,\ i = 1, 2\}. \tag{3.8}$$

  For each $Y_0 \in \mathscr{Y}_0$,
  (b.1) *$\mathrm{Out}_{\mathcal{F}}(Y_0) \in \mathscr{A}_S(Y_0)$, so $\mathrm{Out}_{\mathcal{F}}(Y_0) \cong \Sigma_3 \wr C_2$ or Σ_5; and*
  (b.2) *$\{Y_0 \cap \Theta_1, Y_0 \cap \Theta_2\} \in \mathscr{U}_S(Y_0)$ and is the unique element of its isomorphism type compatible with $\mathrm{Out}_{\mathcal{F}}(Y_0)$ in the sense of Definition 2.2(b).*

(c) *Let $\mathscr{U}_{\mathcal{F}}(Y)$ be as in* (b). *For each $U \in \mathscr{U}_{\mathcal{F}}(Y)$, if we set $R = UC_S(U)$, then*
  (c.1) *U is fully normalized in $\mathcal{F}$,*
  (c.2) *$R \in \mathbf{E}_{\mathcal{F}}^{(\mathrm{II})}(Y)$,*
  (c.3) *$\big[\mathrm{Aut}_{\mathcal{F}}^*(R), R\big] = U$, and*
  (c.4) *$\mathrm{Aut}_{\mathcal{F}}^*(R) = O^2\big(\mathrm{Inn}(R)\langle\alpha\rangle\big)$ for some $\alpha \in \mathrm{Aut}_{\mathcal{F}}^*(R)$ of order 3 which normalizes U and induces the identity on R/U, and such that $\alpha|_{C_S(U)} = \mathrm{Id}$ if $U \cong Q_8$.*

Proof. In Step 1, we prove that there are subgroups $U_1, U_2, T, R \le S$ such that

$$T \in \mathscr{Y}_0,\ \{U_1, U_2\} \in \mathscr{U}_S(T),\ \mathrm{Aut}_{\mathcal{F}}(U_1) = \mathrm{Aut}(U_1);\ \text{and} \tag{3.9}$$

$$R = U_1C_S(U_1) \in \mathbf{E}_{\mathcal{F}}^{(\mathrm{II})}(Y)\ \text{and}\ [\mathrm{Aut}_{\mathcal{F}}^*(R), R] = U_1. \tag{3.10}$$

Then, in Step 2, we apply this to prove the proposition.

Step 1: By assumption, $\mathbf{E}_{\mathcal{F}}(Y) \ne \varnothing$. By Proposition 3.9, each $\mathcal{F}$-essential pair (P_1, P_2) of subgroups in $\mathbf{E}_{\mathcal{F}}(Y) = \mathbf{E}_{\mathcal{F}}^{(\mathrm{II})}(Y)$ has the form described in Lemma 3.7(a) or 3.8(b).

Assume first that there is a pair (P_1, P_2) is as described in Lemma 3.8(b). Set $R = P_1$ and $U_1 = [\mathrm{Aut}_{\mathcal{F}}^*(P_1), P_1]$. Then (3.10) holds by Lemma 3.8(b), and hence $\mathrm{Aut}_{\mathcal{F}}^*(U_1) = \mathrm{Aut}(U_1)$. By the same lemma, there is an S-conjugacy class $\{\Delta, \Delta^*\}$ such that $\Delta \in \mathcal{D}\mathcal{Q}$, $|\Delta| \ge 16$ if $\Delta \in \mathcal{Q}$, $Y = \mathfrak{foc}(\mathcal{F}, P_1) = \Delta\Delta^*$, and $[\Delta, \Delta^*] \le \Delta \cap \Delta^* \le Z(S)$. The hypotheses of Lemma 2.6(a) thus hold. By that lemma, for any $U_2 < \Delta^*$ with $U_2 \cong U_1$, $U_1U_2 \in \mathscr{Y}_0$, and $\{U_1, U_2\} \in \mathscr{U}_S(U_1U_2)$. Thus (3.9) also holds in this case.

Now assume (for the rest of Step 1) that there is a pair (P_1, P_2) is as described in Lemma 3.7(a), and none as described in 3.8(b). Set $T = P_1 \cap P_2$. By Lemma

3.7(a), $T \in \mathscr{Y}_0$, $Y = \mathfrak{foc}(\mathcal{F}, P_1)$ is its normal closure in S, and $\mathrm{Out}_{\mathcal{F}}(T) \cong \Sigma_3 \wr C_2$ or Σ_5. In particular, $\mathrm{Out}_{\mathcal{F}}(T) \in \mathscr{A}_S(T)$.

Let $\{U_1, U_2\} \in \mathscr{U}_S(T)$ be the unique element compatible with $\mathrm{Out}_{\mathcal{F}}(T) \in \mathscr{A}_S(T)$. Fix $\beta \in \mathrm{Aut}(U_1)$ of order 3. Since $\{U_1, U_2\}$ is compatible with $\mathrm{Out}_{\mathcal{F}}(T)$ (Definition 2.2(b)), β extends to an automorphism in $\mathrm{Aut}_{\mathcal{F}}(T)$, and in particular, $\beta \in \mathrm{Aut}_{\mathcal{F}}(U_1)$. This proves (3.9).

By Lemma 2.4(a,b), there is a unique normal subgroup $Z_* \trianglelefteq S$ of index 2^7, and there is a unique triple of subgroups $Y_1, Y_2, Y_3 \trianglelefteq S$ such that $Y_1/Z_* \cong Y_2/Z_* \cong C_2^4$ and $Y_3/Z_* \cong 2_+^{1+4}$. By the same lemma, $Y \in \mathscr{Y}(S) \subseteq \{Y_1, Y_2, Y_3\}$.

Set $R = U_1C_S(U_1)$. Set $\mathscr{Y}' = \{Y_1, Y_2, Y_3\} \smallsetminus \{Y\}$ and $S_0 = \mathrm{Fr}(S)\langle \mathscr{Y}' \rangle$. Since the images in $S/\mathrm{Fr}(S) \cong C_2^3$ of the subgroups Y_1, Y_2, Y_3 have rank 1 and are independent (Lemma C.5), $[S{:}S_0] = 2$ and $Y \cap S_0 \le \mathrm{Fr}(S)$.

Since U_1 is fully normalized by Lemma 3.10 and $R = U_1C_S(U_1)$, $\beta \in \mathrm{Aut}_{\mathcal{F}}(U_1)$ extends to $\widehat{\beta} \in \mathrm{Aut}_{\mathcal{F}}(R)$. Since the normal closure of U_1 in S is the normal closure of $U_1U_2 = T$ and hence equal to Y, there is $h \in U_1 \smallsetminus \mathrm{Fr}(S) \subseteq Y \smallsetminus \mathrm{Fr}(S)$ such that $\beta(h) \in \mathrm{Fr}(S)$. Thus $[\mathrm{Aut}_{\mathcal{F}}(R), R] \nleq S_0$, so by Lemma 1.16(b), there is $Q \in \mathbf{E}_{\mathcal{F}}$ such that $Q \ge R^*$ for some $R^* \in R^{\mathcal{F}}$, and $\mathfrak{foc}(\mathcal{F}, Q) \nleq S_0$. Hence $Q \in \mathbf{E}_{\mathcal{F}}(Y)$.

Assume first that $|Q| \ge 4{\cdot}|T|$. If Q is in a pair of type (3.7a), then it contains a subgroup $\widehat{T} \in \mathscr{Y}_0(S)$ with index 2. Also, $|\widehat{T}| > |T|$ and $T \in \mathscr{Y}_0(S)$. By Lemma 2.4(b), $[Y{:}\widehat{T}]$ and $[Y{:}T]$ are both even powers of 2, and hence $\widehat{T} \cong Q_8 \times Q_8$ and $T \cong C_2^4$. By construction, $Q \ge R^* \in R^{\mathcal{F}}$ where $R = U_1C_S(U_1) > T \cong C_2^4$. Since $[Q{:}\widehat{T}] = 2$, $\widehat{T}$ contains a subgroup isomorphic to C_2^3, which is impossible. Thus Q is in an $\mathcal{F}$-essential pair of type (3.8b), contradicting our assumption that there is no such pair.

Thus $|Q| = 2{\cdot}|T|$. Then $Q \in R^{\mathcal{F}}$ and $[R{:}T] = 2$. Since $Q \in \mathbf{E}_{\mathcal{F}}(Y) \subseteq \mathbf{E}_{\mathcal{F}}^{(\mathrm{II})}$ (Proposition 3.9(b)), $|N_S(Q)/Q| = 2$, so R is fully normalized in $\mathcal{F}$ and hence also $\mathcal{F}$-essential. In particular, $\mathrm{Out}_{\mathcal{F}}(R) \cong \Sigma_3$ by Lemma 3.7. Upon replacing $\widehat{\beta}$ by an appropriate power, we can assume that $|\widehat{\beta}| = 3$ in $\mathrm{Aut}_{\mathcal{F}}(R)$, and hence that

$$\widehat{\beta} \in \mathrm{Aut}_{\mathcal{F}}^*(R) = O^2(O^{2'}(\mathrm{Aut}_{\mathcal{F}}(R))) \le \mathrm{Inn}(R)\langle \widehat{\beta} \rangle.$$

By Proposition 1.14(c), $[\mathrm{Aut}_{\mathcal{F}}^*(R), R] = [\widehat{\beta}, R]$. Also, $\widehat{\beta}(T) = T$ by the condition defining T in Lemma 3.6(b), $T/U_1 \cong U_2/(U_1 \cap U_2) \cong C_2^2$ or Q_8, and $R/T \cong C_2$ acts on U_2 as a subgroup of order 2 in $\mathrm{Out}(U_2) \cong \Sigma_3$. Thus no automorphism of T/U_1 of order 3 extends to R/U_1, and $\widehat{\beta}$ induces the identity on R/U_1 by Lemma A.9. This proves that $[\mathrm{Aut}_{\mathcal{F}}^*(R), R] = [\widehat{\beta}, R] = U_1$. Finally, $\mathfrak{foc}(\mathcal{F}, R) = Y$ since Y is the normal closure in S of T and hence of U_1.

This finishes the proof of (3.10).

Step 2: Let U_1, U_2, $T = U_1U_2$, and $R = U_1C_S(U_1)$ be as in (3.9) and (3.10). Since $\{U_1, U_2\} \in \mathscr{U}_S(T)$, U_1 is fully normalized in $\mathcal{F}$ by Lemma 3.10. By (3.9), there is $\beta \in \mathrm{Aut}_{\mathcal{F}}(U_1)$ of order 3. By the extension axiom, β extends to some $\widehat{\beta} \in \mathrm{Aut}_{\mathcal{F}}(R)$.

Set $W = U_1Z_2(S)$. We first check that condition (2.8) in Lemma 2.7 holds. Assume otherwise: then there is $g \in S$ such that $U_1 \ne {}^gU_1 \le W$. In particular, since $W = U_1Z_2(S) = {}^gU_1Z_2(S)$, $g \in N_S(W)$. By (3.10), $U_1 = [\mathrm{Aut}_{\mathcal{F}}^*(R), R] \trianglelefteq N_S(R)$.

If $U_1 \cap U_2 = 1$, then $W = U_1Z(S)$ since $U_1 \cap Z_2(S) \ne 1$, so $WC_S(W) = U_1C_S(U_1) = R$. Hence $g \in N_S(R)$ and $g \notin N_S(U_1)$, a contradiction.

If $U_1 \cap U_2 = Z(S)$, set $Z = Z(S)$, and $\overline{S} = S/Z$, $\overline{U}_1 = U_1/Z$, etc. Then $\overline{R} = \overline{U}_1 C_{\overline{S}}(\overline{U}_1)$ since each element of S which centralizes $\overline{U}_1 = U_1/Z$ acts on $U_1 \cong Q_8$ via an inner automorphism. Since $\overline{W} = \overline{U}_1 Z(\overline{S})$, this shows that $\overline{W} C_{\overline{S}}(\overline{W}) = \overline{R}$, so $g \in N_S(R) \smallsetminus N_S(U_1)$, which again is a contradiction.

Thus condition (2.8) in Lemma 2.7 holds. Let $\{\Theta_1, \Theta_2\}$ be as in that lemma. Let $\mathscr{U}_{\mathcal{F}}(Y)$ be the set of subgroups of the Θ_i isomorphic to C_2^2 or Q_8.

(a) By Lemma 2.7, (a.1) and (a.2) hold, and $\Theta_i \geq U_i$.

If $Y = T$, then $\Theta_i = U_i$ for $i = 1, 2$. By definition of $\mathscr{U}_S(Y)$, there is a subgroup $\Lambda < \mathrm{Out}_S(Y) \cong D_8$ of order 4 which normalizes U_1 and U_2. If Λ induces the identity on the images of U_1 and U_2 in $Y/\mathrm{Fr}(Y)$, then (since $T = U_1U_2$) it induces the identity on $T/\mathrm{Fr}(T)$, which is impossible since $\mathrm{Out}_S(T)$ acts faithfully on this quotient. Thus for each i, there is $g \in N_S(U_i)$ such that c_g acts nontrivially on $U_i/\mathrm{Fr}(U_i) \cong C_2^2$, so $\Theta_i = U_i$ is strongly automized in S.

If $Y > T$, then it is the normal closure of T and hence of U_1 or U_2. Since $\{\Theta_1, \Theta_2\}$ is an S-conjugacy class, $\mathscr{U}_{\mathcal{F}}(Y)$ contains the S-conjugacy class of U_1, and each Θ_i is generated by subgroups in that class. Since Θ_i contains two Θ_i-conjugacy classes of subgroups isomorphic to C_2^2 or Q_8, neither of which generates Θ_i, both conjugacy classes must be S-conjugate to U_1. Thus $\mathscr{U}_{\mathcal{F}}(Y)$ is the S-conjugacy class of U_1, and Θ_1 and Θ_2 are strongly automized in S. Since $\mathrm{Aut}_{\mathcal{F}}(U_1) = \mathrm{Aut}(U_1)$, this also proves (a.3).

(c) We just showed that $\mathscr{U}_{\mathcal{F}}(Y)$ is the S-conjugacy class of U_1. So it suffices to prove (c) when $U = U_1$. Points (c.2) and (c.3) hold for U_1 by (3.10), and we already saw that U_1 is fully normalized.

The image of $\mathrm{Aut}^*_{\mathcal{F}}(R) = O^2(O^{2'}(\mathrm{Aut}_{\mathcal{F}}(R)))$ in $\mathrm{Out}_{\mathcal{F}}(R)$ has order 3 since $\mathrm{Out}_{\mathcal{F}}(R) \cong \Sigma_3$ or $\Sigma_3 \times C_3$ (Lemmas 3.7 and 3.8(e)). Hence $\mathrm{Aut}^*_{\mathcal{F}}(R) \leq \mathrm{Inn}(R)\langle\alpha\rangle \leq O^{2'}(\mathrm{Aut}_{\mathcal{F}}(R))$ for some $\alpha \in \mathrm{Aut}^*_{\mathcal{F}}(R)$ of order 3, so $\mathrm{Aut}^*_{\mathcal{F}}(R) = O^2\bigl(\mathrm{Inn}(R)\langle\alpha\rangle\bigr)$. Also, α normalizes U_1 and is the identity on R/U_1 by (c.3). If $U_1 \cong Q_8$, then α induces the identity on $Z(U_1)$ and on $C_S(U_1)/Z(U_1)$, and hence on $C_S(U_1)$ by Lemma A.9. This proves (c.4).

(b) Fix $Y_0 \in \mathscr{Y}_0$. We have already seen that $\mathscr{U}_{\mathcal{F}}(Y)$ is the S-conjugacy class of U_1. Set $V_i = Y_0 \cap \Theta_i$.

If $Y = T = Y_0 \in \mathscr{Y}_0$, then $\{\Theta_1, \Theta_2\} = \{U_1, U_2\} \in \mathscr{U}_S(Y)$ by assumption. So assume $Y > T$. Then $\Theta_1 > U_1$, so $\Theta_i \cong D_{2^n}$ for $n \geq 3$ or Q_{2^n} for $n \geq 4$. Hence the hypotheses of Lemma 2.6 hold by (a); and (3.8) ($\mathscr{U}_{\mathcal{F}}(Y)$ is the set of all $Y_0 \cap \Theta_i$ for $i = 1, 2$ and $Y_0 \in \mathscr{Y}_0$) follows from Lemma 2.6(a). By the same lemma, $Y_0 = V_1V_2$ and $\{V_1, V_2\} \in \mathscr{U}_S(Y_0)$.

It remains to prove (b.1) and the compatibility statement in (b.2). Recall (Lemma 2.9(a)) that there is a basis of $Y_0/\mathrm{Fr}(Y_0)$ which is permuted by $\mathrm{Out}_S(Y_0) \cong D_8$. Let $B < \mathrm{Out}_S(Y_0)$ be the subgroup generated by products of two disjoint transpositions, and let $\gamma_1, \gamma_2 \in \mathrm{Out}_S(Y_0)$ be the two classes which act as transpositions. (Thus $\mathrm{rk}([\gamma_i, Y_0/\mathrm{Fr}(Y_0)]) = 1$, while $\mathrm{rk}([\beta, Y_0/\mathrm{Fr}(Y_0)]) = 2$ for $\beta \in B^\#$.) In particular, no element in $\mathrm{Out}_S(Y_0) \smallsetminus B$ is $\mathrm{Out}_{\mathcal{F}}(Y_0)$-conjugate to any element in B. By the focal subgroup theorem for groups (see [**G**, Theorem 7.3.4]), $\mathrm{Out}_S(Y_0) \cap [\mathrm{Out}_{\mathcal{F}}(Y_0), \mathrm{Out}_{\mathcal{F}}(Y_0)] \leq B$, and thus $\gamma_1, \gamma_2 \notin O^2(\mathrm{Out}_{\mathcal{F}}(Y_0))$.

Now, $\langle\gamma_1, \gamma_2\rangle$ is the normalizer in $\mathrm{Out}_S(Y_0)$ of V_1 and of V_2, and we can index them such that γ_i acts nontrivially on $V_i/\mathrm{Fr}(V_i)$ and trivially on $V_{3-i}/\mathrm{Fr}(V_{3-i})$. Set $R^* = V_1C_S(V_1)$. By (c), and since $V_1 \in \mathscr{U}_{\mathcal{F}}(Y)$, $R^* \in \mathbf{E}_{\mathcal{F}}(Y)$, and there is $\beta^* \in \mathrm{Aut}_{\mathcal{F}}(R^*)$ of order 3 which normalizes V_1 and acts trivially on R^*/V_1. Thus $[\beta^*|_{Y_0}] \in \mathrm{Out}_{\mathcal{F}}(Y_0)$ is inverted by γ_1, and normalizes (hence centralizes) $\gamma_2 \in \mathrm{Out}_{R^*}(Y_0)$. Since $C_{\mathrm{Out}_S(Y_0)}(\gamma_2) = \langle \gamma_1, \gamma_2 \rangle$, this gives $O_2(C_{\mathrm{Out}_{\mathcal{F}}(Y_0)}(\gamma_2)) = \langle \gamma_2 \rangle$.

Since $\mathrm{Out}_S(Y_0)$ acts faithfully on $Y_0/\mathrm{Fr}(Y_0)$, $\mathrm{Out}_{\mathcal{F}}(Y_0)$ acts faithfully by Lemma A.9. Also, $\mathrm{Out}_{\mathcal{F}}(Y_0) \not\cong \Gamma L_2(4)$ since $\mathrm{Out}_S(Y_0) \cong D_8$ permutes a basis of $Y_0/\mathrm{Fr}(Y_0)$. So by Proposition D.1(f), applied with γ_2 in the role of x, $\mathrm{Out}_{\mathcal{F}}(Y_0) \cong \Sigma_3 \wr C_2$ or Σ_5. Thus $\mathrm{Out}_{\mathcal{F}}(Y_0) \in \mathscr{A}_S(Y_0)$. By (c.4), $\mathrm{Out}_{\mathcal{F}}(Y_0)$ is compatible, in the sense of Definition 2.2(b), with $\{Y_0 \cap \Theta_1, Y_0 \cap \Theta_2\} \in \mathscr{U}_S(Y_0)$. □

Proposition 3.12. *Let $\mathcal{F}$ be a reduced fusion system over a 2-group S such that $r(S) \leq 4$ and $\mathscr{Y}(S) \neq \varnothing$. Then $S \in \mathcal{UV}$.*

Proof. Assume $S \notin \mathcal{U}$ and $S \not\cong D_8 \wr C_2$. By Lemma 2.4(a), $|S| \geq 2^8$, there is a unique normal subgroup $1 \neq Z_* \trianglelefteq S$ of index 2^7, and $S/Z_* \cong D_8 \wr C_2$. Let $Y_1, Y_2, Y_3 \trianglelefteq S$ be the three distinct normal subgroups such that $Y_1/Z_* \cong Y_2/Z_* \cong C_2^4$ and $Y_3/Z_* \cong 2_+^{1+4}$. By Lemma 2.4(b), $\mathscr{Y}(S) \subseteq \{Y_1, Y_2, Y_3\}$.

By Proposition 3.9(a), $\mathbf{E}_{\mathcal{F}} = \mathbf{E}_{\mathcal{F}}(Y_1) \cup \mathbf{E}_{\mathcal{F}}(Y_2) \cup \mathbf{E}_{\mathcal{F}}(Y_3)$, and $\mathbf{E}_{\mathcal{F}}(Y_i) \neq \varnothing$ for each $i = 1, 2, 3$. Also, $Y_3 \not\cong UT_3(4)$ since $S \notin \mathcal{U}$, and $|Y_3| \geq 2^6$ since $|S| \geq 2^8$. So by Proposition 3.9(b,c), for each $i = 1, 2, 3$ and each $R \in \mathbf{E}_{\mathcal{F}}(R_i)$, either

- $Y_i \in \mathscr{Y}(S)$ and $\mathbf{E}_{\mathcal{F}}(Y_i) \subseteq \mathbf{E}_{\mathcal{F}}^{(\mathrm{II})}$, and R is in an $\mathcal{F}$-essential pair as described in Lemma 3.7(a) or 3.8(b); or (3.11)
- $i = 3$, $Y_3 \notin \mathscr{Y}(S)$, $\mathbf{E}_{\mathcal{F}}(Y_3) = \mathbf{E}_{\mathcal{F}}^{(\mathrm{III})}$, $Y_3 \cong Q_8 \times Q_8$, and $[R{:}Y_3] = 2$.

Together with Proposition 3.11(a) (applied when $Y_i \in \mathscr{Y}(S)$), this proves that for each $i = 1, 2, 3$, there are subgroups $\Theta_{i1}, \Theta_{i2} \trianglelefteq Y_i$ where

$$|Y_i| = 2^m \text{ for } m \text{ even} \implies \Theta_{ij} \in \mathcal{DQ},\ Y_i = \Theta_{i1} \times \Theta_{i2} \tag{3.12}$$

$$|Y_i| = 2^m \text{ for } m \text{ odd} \implies \Theta_{ij} \in \mathcal{Q},\ [\Theta_{i1}, \Theta_{i2}] \leq \Theta_{i1} \cap \Theta_{i2} = Z(S)\,. \tag{3.13}$$

(Note that $Y_i \not\cong C_2^4$ since $|S| \geq 2^8$ and $[S{:}Y_i] \leq 2^3$.) Also, by Proposition 3.11(a.1), $\{\Theta_{i1}, \Theta_{i2}\}$ is an S-conjugacy class if $Y_i \in \mathscr{Y}(S)$, and in particular, if $i = 1, 2$.

We claim that

$$\text{for some } i = 1, 2, 3,\ Y_i \in \mathcal{D} \times \mathcal{D}. \tag{3.14}$$

To see this, note first that since $\mathcal{F}$ is reduced, $Z(S) \ntrianglelefteq \mathcal{F}$, so by Lemma 1.15, there is $R \in \mathbf{E}_{\mathcal{F}}$ and $\alpha \in \mathrm{Aut}_{\mathcal{F}}^*(R)$ such that $\alpha(Z(S)) \neq Z(S)$. Let i be such that $R \in \mathbf{E}_{\mathcal{F}}(Y_i)$.

Let $R_0 \leq R$ be any subgroup such that $r(R_0/Z(R_0)) = 4$. By Lemma A.6(a) and since $\mathrm{Aut}_{\mathcal{F}}^*(R)$ does not normalize $Z(S)$, $Z(S) < Z(R) \leq Z(R_0)$. Thus $|Z(R_0)| \geq 4$. If $R_0 \cong Q_8 \times Q_8$, then there is a unique element $z \in Z(R_0)^\#$ such that $z = g^2$ for 9 classes $gZ(R_0) \in (R_0/Z(R_0))^\#$, so $\langle z \rangle$ is characteristic in R_0. If $\mathrm{Aut}_{\mathcal{F}}^*(R)$ normalizes R_0, then it also normalizes $\langle z \rangle$, so $\langle z \rangle \neq Z(S)$. Each element of odd order in $\mathrm{Aut}_{\mathcal{F}}^*(R)$ centralizes the two elements in $Z(R) \smallsetminus \langle z \rangle$, and each element in $\mathrm{Aut}_S(R)$ centralizes $\langle z \rangle$ and $Z(S)$. Thus $\mathrm{Aut}_{\mathcal{F}}^*(R)$ centralizes $Z(R_0) > Z(S)$, which is impossible. We conclude that

there is no $R_0 \leq R$ such that $r(R_0/Z(R_0)) = 4$, and such that either $|Z(R_0)| = 2$, or $R_0 \cong Q_8 \times Q_8$ and is normalized by $\mathrm{Aut}_{\mathcal{F}}^*(R)$. (3.15)

If $R \in \mathbf{E}_{\mathcal{F}}^{(\mathrm{II})}(Y_i)$ is in an essential pair as in Lemma 3.7(a), then there is a subgroup $T < R$ such that $T \in \mathscr{Y}_0(S)$ and $\mathrm{Aut}_{\mathcal{F}}^*(R)$ normalizes T. By (3.15),

$T \not\cong 2^{1+4}_{\pm}$ and $T \not\cong Q_8 \times Q_8$. Thus $T \cong C_2^4$. If $\Theta_{i1}, \Theta_{i2} \in \mathcal{Q}$, then $T \cap \Theta_{i1} \le Z(\Theta_{i1})$, so the image of T under projection to Y_i/Θ_{i1} has rank at least 3, which is impossible since $r(Y_i/\Theta_{i1}) \le r(\Theta_{i2}) = 2$. Thus $\Theta_{i1}, \Theta_{i2} \in \mathcal{D}$, and so $Y_i \in \mathcal{D} \times \mathcal{D}$.

If R is in a pair as in Lemma 3.8(b), then $R = UC_S(U)$ for some $U \cong C_2^2$ or Q_8, and $[\mathrm{Aut}^*_{\mathcal{F}}(R), R] = U$. By Lemma 3.8(b,c), $U \le \Delta$ for some $\Delta \in \mathcal{DQ}$ in a conjugacy class $\{\Delta, \Delta^*\}$ such that $[\Delta, \Delta^*] \le \Delta \cap \Delta^* \le Z(S)$, $|\Delta| \ge 16$ if $\Delta \in \mathcal{Q}$, and $Y_i = \Delta \times \Delta^*$ if $\Delta \in \mathcal{D}$. If $\Delta, \Delta^* \in \mathcal{Q}$, then $U \cong Q_8$, and $R \ge U\Delta^*$ since $\mathrm{Aut}_{\Delta^*}(U) \le \mathrm{Inn}(U)$. Choose $U^* < \Delta^*$ with $U^* \cong Q_8$ and set $R_0 = UU^*$. Then $\mathrm{Aut}^*_{\mathcal{F}}(R)$ normalizes R_0 since $[\mathrm{Aut}^*_{\mathcal{F}}(R), R] = U < R_0$. Either $[U, U^*] = 1$ and $R_0 \cong 2^{1+4}_{+}$ or $Q_8 \times Q_8$; or $[U, U^*] \ne 1$, $[\mathrm{Fr}(\Delta), U^*] = 1$, and hence $R_0 \cong 2^{1+4}_{-}$ by Lemma C.2(a) (with U, U^* in the role of Δ_1, Δ_2). Since this contradicts (3.15), we have $\Delta, \Delta^* \in \mathcal{D}$, and hence $Y_i = \Delta \times \Delta^* \in \mathcal{D} \times \mathcal{D}$ by (3.12) and (3.13).

Finally, if $R \in \mathbf{E}^{(\mathrm{III})}_{\mathcal{F}}(Y_3)$, then $Y_3 \cong Q_8 \times Q_8$, and $\mathrm{Aut}^*_{\mathcal{F}}(R)$ normalizes Y_3 since $\mathfrak{foc}(\mathcal{F}, R) = Y_3$. This again contradicts (3.15), and finishes the proof of (3.14).

Case 1: Assume $|S|$ is an odd power of 2. Then $|Y_1| = |Y_2| = 2^m$ and $|Y_3| = 2^{m+1}$ for some even m. So $Y_i = \Theta_{i1} \times \Theta_{i2}$ for $i = 1, 2$ by (3.12), and $Y_3 \notin \mathcal{D} \times \mathcal{D}$ by (3.13). Hence by (3.14), for $i = 1$ or $i = 2$, $\Theta_{i1} \cong \Theta_{i2} \in \mathcal{D}$.

Set $Z_{ij} = Z(\Theta_{ij})$. Set $S_0 = C_S(Z_2(S))$, where $Z(S) < Z_2(S) \cong C_2^2$ by Lemma 2.4(c). Hence $[S{:}S_0] = 2$. For $i = 1, 2$, $\{\Theta_{i1}, \Theta_{i2}\}$ is an S-conjugacy class by Proposition 3.11(a.1) (and since $Y_i \in \mathscr{Y}(S)$ by (3.11)), so $Z_{i1}Z_{i2} \trianglelefteq S$, and $Z_{i1}Z_{i2} = Z_2(S) \le Z(S_0)$ by Lemma 2.4(a). Thus no element of $\mathrm{Inn}(S_0)$ can exchange Θ_{i1} with Θ_{i2}, so $\Theta_{ij} \trianglelefteq S$ for all $i, j = 1, 2$.

Let $z_1, z_2 \in Z_2(S)$ be the two elements not in $Z(S)$. After renumbering, if necessary, we can assume that $Z_{ij} = \langle z_j \rangle$ for $i, j = 1, 2$. By Lemma B.6, applied with S_0 and $\{\Theta_{ij} \,|\, i, j = 1, 2\}$ in the role of S and $\mathscr{P}$, $S_0 = \Gamma_1 \times \Gamma_2$, where for each $j = 1, 2$, $\Gamma_j \in \mathcal{DSQ}$, $Z(\Gamma_j) = \langle z_j \rangle$, and $\Gamma_j \le \Theta_{1j}\Theta_{2j} \le \Gamma_j Z(\Gamma_{3-j})$. Then $\Gamma_1 \cong \Gamma_2$, and $\Gamma_j \in \mathcal{DS}$ since Θ_{1j} or Θ_{2j} is dihedral. Also, for any $g \in S \smallsetminus S_0$, $S_0 = \Gamma_1 \times {}^g\Gamma_1$ by Lemma B.3 ($Z({}^g\Gamma_1) = \langle {}^g z_1 \rangle = \langle z_2 \rangle$), so upon replacing Γ_2 by ${}^g\Gamma_1$, we can assume that Γ_1 and Γ_2 are S-conjugate.

Fix $g \in S \smallsetminus S_0$, and let $r, s \in \Gamma_1$ be such that $g^2 = r \cdot {}^g s = {}^g s \cdot r$. Then $s = r$ and $[g^2, r] = 1$ since $[g, g^2] = 1$, so

$$(r^{-1}g)^2 = r^{-1}({}^g r^{-1})g^2 = r^{-1}({}^g r^{-1})({}^g r)r = 1\,.$$

Set $h = r^{-1}g$; then $h^2 = 1$, and so $S \cong \Gamma_1 \wr C_2 \in \mathcal{V}$ in this case.

Case 2: Now assume $|S|$ is an even power of 2. Set $Z = \langle z \rangle = Z(S)$. Then $|Y_1| = |Y_2| = 2^m$ for odd m, so by (3.13), for $i = 1, 2$, $\Theta_{i1} \cap \Theta_{i2} = Z(S)$ and $\Theta_{i1}, \Theta_{i2} \in \mathcal{Q}$. An argument similar to that used in Case 1, applied to S/Z, Y_i/Z, and Θ_{ij}/Z, shows that $S/Z(S) \cong D \wr C_2$ for some $D \in \mathcal{D}$. Here, D is dihedral since it is generated by two of the subgroups Θ_{ij}/Z, which are dihedral or C_2^2. Thus $S = \langle a_1, a_2, b_1, b_2, t \rangle$, where $D_i = \langle a_i, b_i \rangle \in \mathcal{DSQ}$ for $i = 1, 2$, $|a_i| \ge 8$, $[D_i{:}\langle a_i \rangle] = 2$, $[D_1, D_2] \le D_1 \cap D_2 = Z$, and $t^2 \in Z$. Upon replacing b_2 by $b_2 z$ or a_2 by $a_2 z$, if necessary, we can arrange that ${}^t a_i = a_{3-i}$ and ${}^t b_i = b_{3-i}$ for $i = 1, 2$.

Choose $w_i \in \langle a_i^2 \rangle$ of order 4. Since $[a_i, D_{3-i}] \le Z$, $[w_i, D_{3-i}] \le [a_i^2, D_{3-i}] = 1$. If $[a_1, b_2] = z$, then $[a_1 w_2, b_2] = 1$. So after replacing a_1 by $a_1 w_2$ and a_2 by $a_2 w_1$ if necessary, we can arrange that $[a_1, b_2] = [a_2, b_1] = 1$. Also, upon replacing b_i by $b_i w_{3-i}$ if necessary, we can arrange that $b_1^2 = b_2^2 = 1$.

To get more information about relations between these generators, we now look more closely at Y_3. We have $Z_* = \langle a_1^4, a_2^4 \rangle$ since it is the unique normal subgroup of S such that $S/Z_* \cong D_8 \wr C_2$. Also, $Y_3/Z_* \cong 2_+^{1+4}$ by definition of Y_3, and hence

$$Y_3 = \langle a_1a_2^{-1}, t, a_1a_2, b_1b_2\rangle.$$

By (3.14) (and since $Y_1, Y_2 \notin \mathcal{D}\times\mathcal{D}$), $Y_3 \in \mathcal{D}\times\mathcal{D}$. Set $A = \langle a_1a_2, a_1a_2^{-1}\rangle = \langle a_1a_2, a_1^2\rangle$. Thus A is abelian by the above remarks ($[a_1^2, a_2] = 1$). By Lemma C.1, A is the unique subgroup of $Y_3 \in \mathcal{D}\times\mathcal{D}$ which is abelian of rank 2 and index 4, and of the three involutions in A, exactly two are squares of elements in $Y_3\smallsetminus A$. Since w_1w_2 and $w_1w_2^{-1}$ are S-conjugate, z is not the square of any $g \in Y_3\smallsetminus A$. Hence $t^2 = 1$, $[b_1, b_2] = (b_1b_2)^2 = 1$, and $[a_1, a_2] = [a_1b_1, a_2b_2] = (a_1b_1a_2b_2)^2 = 1$ (note that $(a_2b_2)^2 = {}^t((a_1b_1)^2) = (a_1b_1)^2 \in Z$). Thus $D_i = \langle a_i, b_i\rangle \in \mathcal{DS}$ (recall $b_i^2 = 1$), $[D_1, D_2] = 1$, $D_1 \cap D_2 = \langle z\rangle$, ${}^tD_1 = D_2$, and so $S = (D_1 \times_{\langle z\rangle} D_2) \rtimes \langle t\rangle \in \mathcal{V}$. $\square$

It remains to look more closely at essential subgroups of the type described in Lemma 3.8(a).

LEMMA 3.13. *Fix a 2-group S with $r(S) \le 4$, and a reduced fusion system $\mathcal{F}$ over S. Assume the following:*

(i) *There is a normal subgroup $\Delta \trianglelefteq S$ which is quaternion of order at least* 16. *Let $A \le \Delta$ be the cyclic subgroup of index* 2, *and fix $b \in \Delta\smallsetminus A$.*

(ii) *There is a subgroup $P \trianglelefteq S$ of index* 2, *and an automorphism $\sigma \in \mathrm{Aut}(P)$ of odd order, such that $A < P$, $b \notin P$, $c_b\sigma c_b^{-1} \equiv \sigma^{-1}$ $(\mathrm{mod}\ \mathrm{Inn}(P))$, $\sigma(A) \cap A = 1$, and such that conjugation by a generator of $\sigma(A)$ exchanges the two noncyclic subgroups of index* 2 *in Δ.*

Then $S \in \mathcal{GW}$.

PROOF. Set $\widehat{A} = A{\cdot}\alpha(A)$. We will prove the following statements:

(a) $\widehat{A} \trianglelefteq S$, $\widehat{A} = A \times \sigma(A)$, $\alpha(\widehat{A}) = \widehat{A}$, $|\sigma| = 3$, $C_{\widehat{A}}(\sigma) = 1$;

(b) $P = \widehat{A}V$ where $V = C_P(\sigma)$ and $\widehat{A} \cap V = 1$;

(c) there is $t \in \widehat{A}b$ such that $c_t\sigma c_t^{-1} = \sigma^{-1} \in \mathrm{Aut}(P)$, $[V, t] = 1$, $t^2 = 1$, $[t, \widehat{A}] = A$, and $\widehat{A}\langle t\rangle \cong A \wr C_2$;

(d) $V \cong P/\widehat{A}$ is cyclic, and there is $\mu \in \mathrm{Hom}(V, (\mathbb{Z}/2^n)^\times)$ (where $2^n = |A|$) such that ${}^xg = g^{\mu(v)}$ for all $x \in V$, $g \in \widehat{A}$; and

(e) μ is injective, $|V| \le 2$, and $2^{n-1} + 1 \notin \mathrm{Im}(\mu)$.

Then by (c) and (e), either $V = 1$ and $S \cong A \wr C_2 \in \mathcal{W}$, or $|V| = 2$, $S = \widehat{A} \rtimes \langle t, V\rangle$ where V acts on $\widehat{A}$ via $(g \mapsto g^\lambda)$ for $\lambda = -1$ or $2^{n-1} - 1$, and $S \in \mathcal{G}$.

(a) Let $\Delta_1, \Delta_2 < \Delta$ be the noncyclic subgroups of index 2 in Δ, and set $S_0 = N_S(\Delta_1) = N_S(\Delta_2)$. Fix a generator $a \in A$, and set $y = \sigma(a)$. Then $y \notin S_0$ by assumption. Since A and $\sigma(A)$ are both normal in P, $[A, \sigma(A)] \le A \cap \sigma(A)$, where $A \cap \sigma(A) = 1$ by assumption. Thus $\widehat{A} \cong A \times A$. Since $[\widehat{A}, b] \le A$, $\widehat{A} \trianglelefteq P\langle b\rangle = S$.

By Lemma A.11 (applied to the action of $\langle[\sigma], [c_b]\rangle$ on P^{ab}), and since $[b, P] \le [b, S] \le A$ is cyclic, $\langle[\sigma], [c_b]\rangle \cong \Sigma_3$ as a subgroup of $\mathrm{Aut}(P^{\mathrm{ab}})$. Also, $\langle\sigma\rangle$ acts faithfully on P^{ab} by Lemma A.9, so σ has order 3 in $\mathrm{Aut}(P)$.

Since $\langle\sigma\rangle \in \mathrm{Syl}_3(\mathrm{Inn}(P)\langle\sigma\rangle)$, there is $h \in P$ such that $c_{hb}\sigma c_{hb}^{-1} = \sigma^{-1}$ in $\mathrm{Aut}(P)$. Set $\widehat{t} = hb$ for short. Then $\sigma(y) = \sigma^{-1}(a) \in {}^{\widehat{t}}(\sigma(A)) \le \widehat{A}$ since $A \trianglelefteq S$ and

$\widehat{A} \trianglelefteq S$. Thus $\sigma(\widehat{A}) = \widehat{A}$. Also, $C_{\widehat{A}}(\sigma) = 1$ since $|\sigma| = 3$ and σ acts nontrivially on $\widehat{A}/\mathrm{Fr}(\widehat{A})$.

(b) Since $[b, P] \le A \le \widehat{A}$, c_b induces the identity on $P/\widehat{A}$. Since $|\sigma|$ is odd and $[\sigma]$ is inverted by $[c_b]$ in $\mathrm{Out}(P)$, σ induces the identity on $(P/\widehat{A})^{\mathrm{ab}}$, and hence on $P/\widehat{A}$ by Lemma A.9. Since $C_{\widehat{A}}(\sigma) = 1$, each coset $g\widehat{A}$ in $P/\widehat{A}$ contains a unique element fixed by σ. Thus $P = \widehat{A}V = V\widehat{A}$, where $V = C_P(\sigma)$, and $\widehat{A} \cap V = 1$.

(c) Recall that $\widehat{t} \in Qb = V\widehat{A}b$. Let $u \in V$ and $t \in \widehat{A}b$ be such that $\widehat{t} = ut$. Then $\sigma c_u \sigma^{-1} = c_{\sigma(u)} = c_u$, so $c_t \sigma c_t^{-1} = c_{u^{-1}\widehat{t}} \sigma c_{u^{-1}\widehat{t}}^{-1} = c_u^{-1}\sigma^{-1}c_u = \sigma^{-1}$ in $\mathrm{Aut}(P)$. Also, $c_t|_{\widehat{A}} = c_b|_{\widehat{A}}$, and c_t induces the identity on $P/\widehat{A} \cong V$ since $[b, P] \le A$. For each $v \in V$, since c_t normalizes $\langle \sigma \rangle$ in $\mathrm{Aut}(P)$, c_t sends $C_{v\widehat{A}}(\sigma) = \{v\}$ to itself. So $[V, t] = 1$. Finally, $[c_{t^2}, \sigma] = 1$ in $\mathrm{Aut}(P)$ since c_t inverts σ, so $t^2 \in C_{\widehat{A}}(\sigma) = 1$.

In particular, $c_b \sigma c_b^{-1}|_{\widehat{A}} = \sigma^{-1}|_{\widehat{A}}$. Also, $a\sigma(a)\sigma^2(a) = ay\sigma(y) \in C_{\widehat{A}}(\sigma) = 1$, so $\sigma(y) = a^{-1}y^{-1}$. Since $c_b(a) = a^{-1}$, c_b sends the σ-orbit $\{a, y, a^{-1}y^{-1}\}$ to the orbit $\{a^{-1}, ay, y^{-1}\}$, and thus ${}^by = ay$. Hence $[b, \widehat{A}] = A$, and $\widehat{A}\langle t \rangle \cong A \wr C_2$ since c_t exchanges y and ay.

(d) For each $x \in V$, c_x commutes in $\mathrm{Aut}(\widehat{A})$ with σ and with $c_b = c_t$ since $[\sigma, V] = [t, V] = 1$. Hence $c_x(A) = A$ since the elements of A are the only ones in $\widehat{A}$ which are inverted by c_b (the two involutions in $\widehat{A} \smallsetminus A$ are exchanged). Also, c_x sends σ-orbits to σ-orbits, and so $c_x(a) = a^i$ and $c_x(y) = y^i$ for some odd i. In other words, there is $\mu \in \mathrm{Hom}(V, (\mathbb{Z}/2^n)^\times)$, where $2^n = |A|$, such that $c_x(g) = g^{\mu(x)}$ for each $x \in V$ and each $g \in \widehat{A}$.

Since $\widehat{A} \trianglelefteq S$, $\langle a, y^2 \rangle = \widehat{A} \cap S_0 \trianglelefteq S$. Also, $[b, S_0] \le \langle a^2 \rangle$, and we just saw that $\bigl[V, \langle a, y^2 \rangle\bigr] \le \langle a^2, y^4 \rangle$. Thus $\langle a, y^2, b, V \rangle / \langle a^2, y^4 \rangle \cong C_2^3 \times V$, and so V is cyclic since $r(S) \le 4$.

(e) Set $Z = Z(\Delta)$. Assume $V \ne 1$ (otherwise there is nothing to prove). Let $v \in V$ be the element of order 2. If $\mu(v) = 1$, then $\Omega_1(Z(S)) = Z\langle v \rangle$, while $\Omega_1(Z(S)) \cap [S, S] = Z$. So $v \notin \mathfrak{foc}(\mathcal{F})$ by Proposition 1.18(a), which contradicts Proposition 1.14(b) ($\mathcal{F}$ is reduced). Thus $\mu(v) \ne 1$, and μ is injective.

Since $S = \widehat{A}V\langle t \rangle$ where $[t, V] = 1$, $[S, S] = [t, \widehat{A}][V, \widehat{A}] \ge A\Omega_1(\widehat{A})$. Hence $C_S([S, S]) \le C_S(\Omega_1(\widehat{A})) = P$, so $C_S([S, S]) \le C_P(A) = \widehat{A}$. Thus $\widehat{A} = C_S([S, S])$ is characteristic in S. So $\mathrm{Aut}(S)$ is a 2-group by Lemma A.9, applied to the chain $\mathrm{Fr}(S) < \widehat{A}\mathrm{Fr}(S) < P < S$ (where $|\widehat{A}\mathrm{Fr}(S)/\mathrm{Fr}(S)| = 2$ since $a, y^2 \in \mathrm{Fr}(S)$). It follows that $\mathrm{Out}_{\mathcal{F}}(S) = 1$.

Set $U = \Omega_1(\widehat{A})$, and let $x \in V$ be a generator. If $|V| \ge 4$, then for each $Q < S$ of index 2, either $Q \ge \widehat{A}$ and $[Q, Q] \ge [x^2, \widehat{A}] \ge U$, or $Q \not\ge \widehat{A}$, there is $x' \in x\widehat{A} \cap Q$, and $[Q, Q] \ge [x', \mathrm{Fr}(\widehat{A})] \ge U$. Also, $\mu(v) = 2^{n-1} + 1$, since it has order 2 and is a square in $(\mathbb{Z}/2^n)^\times$. So $[v, S] = [v, \widehat{A}] = U$, $v^2 = 1$, and $\mathcal{F}$ is not reduced ($v \notin \mathfrak{foc}(\mathcal{F})$) by Proposition 1.18(b) applied with v in the role of g.

Thus $V = \langle v \rangle$ where $v^2 = 1$. Since $n \ge 3$, $U = \Omega_1(\widehat{A}) \le \mathrm{Fr}(\mathrm{Fr}(\widehat{A})) \le \mathrm{Fr}(Q)$ for each $Q < S$ of index 2. If $\mu(v) = 2^{n-1} + 1$, then $[v, S] = [v, \widehat{A}] = U$. So by Proposition 1.18(c), $v \notin \mathfrak{foc}(\mathcal{F})$ and $\mathcal{F}$ is not reduced, a contradiction. □

PROPOSITION 3.14. *Fix a 2-group S with $r(S) \le 4$, and a reduced, indecomposable fusion system $\mathcal{F}$ over S. Assume that there is some $R \in \mathbf{E}_{\mathcal{F}}^{(\mathrm{II})}$ such that $\mathfrak{foc}(\mathcal{F}, R) \in \mathscr{X}(S)$. Then $S \in \mathcal{DSWG}$.*

Proof. By Lemma 3.3 and Propositions 3.4 and 3.5, if $\mathbf{E}_{\mathcal{F}}^{(\mathrm{I})} \neq \varnothing$, then $S \cong UT_4(2)$ or $S \in \mathcal{U}$. Since $\mathscr{X}(UT_4(2)) = \varnothing$ by Lemma C.4(e), and $\mathscr{X}(S) = \varnothing$ for $S \in \mathcal{U}$ by Lemma C.9, $\mathbf{E}_{\mathcal{F}}^{(\mathrm{I})} = \varnothing$.

Since $\mathscr{X}(S) \neq \varnothing$, $\mathscr{Y}(S) = \varnothing$ by Corollary 2.5. So by Lemmas 3.7 and 3.8,

$$(R_1, R_2) \text{ an } \mathcal{F}\text{-essential pair of type (II)} \quad\Longrightarrow\quad \mathfrak{foc}(\mathcal{F}, R_i) \in \mathscr{X}(S), \text{ and } (R_1, R_2) \text{ as in Lemma 3.8.} \tag{3.16}$$

Fix an $\mathcal{F}$-essential pair (R_1, R_2) of type (II), and set $\Delta = \mathfrak{foc}(\mathcal{F}, R_1) \in \mathscr{X}(S)$. By Lemma 3.8(a,c), $\Delta \cong D_{2^n}$ for $n \geq 3$ or $\Delta \cong Q_{2^n}$ for $n \geq 4$. Let $A \trianglelefteq \Delta$ be the cyclic subgroup of index 2, fix a generator $a \in A$, and choose $b \in \Delta \smallsetminus A$. Set $Z = Z(\Delta)$. Let $\Delta_0 < \Delta$ be the subgroup of order 8 which contains b. Set $T = C_S(\Delta_0)$. By Lemma 2.10(a,b), $[S{:}T\Delta] = 2$.

If $\Delta \in \mathcal{D}$, then by Lemma 3.8(c,d), $R_i = U_i C_S(U_i)$ where $U_i \cong C_2^2$ is a direct factor of R_i and $U_1U_2 \cong D_8$. So we can assume $\Delta_0 = U_1U_2$. Thus

$$\Delta \in \mathcal{D} \quad\Longrightarrow\quad \text{there is } T_0 < T \text{ such that } T = T_0 \times Z. \tag{3.17}$$

Throughout the proof, when $P \in \mathbf{E}_{\mathcal{F}} \cup \{S\}$, we write $\Delta_P = \mathfrak{foc}(\mathcal{F}, P)$.

Step 1: Assume $\mathbf{E}_{\mathcal{F}}^{(\mathrm{III})} \neq \varnothing$, and fix $P \in \mathbf{E}_{\mathcal{F}}^{(\mathrm{III})}$. Then $P \geq [S,S]$, and hence $P \geq A$ by Lemma 2.10(b). We will show that either

(i) $P \geq \Delta$ and $\Delta \in \mathcal{Q}$; or

(ii) $P \geq \Delta$, $Z \not\leq \Delta_P \cong C_2^2$ or Q_8, and $\Delta_P \not\leq Z(S)$; or

(iii) $P \not\geq \Delta$, $A < \Delta_P \cong Q_8$, and $|A| = 4$; or

(iv) $P \not\geq \Delta$ and $S \in \mathcal{WG}$.

Case 1: $\boldsymbol{P \geq \Delta}$. Since $\mathrm{Aut}_{\mathcal{F}}(P)$ is generated by $\mathrm{Aut}_S(P)$ and automorphisms of odd order, it normalizes Z by Lemma B.7 (and since $\Delta \trianglelefteq S$).

If $\Delta \in \mathcal{Q}$, then (i) holds. So assume Δ is dihedral, and set $\widehat{T} = [\mathrm{Aut}_{\mathcal{F}}^*(P), P]$. Since $\mathrm{Out}_S(P) \in \mathrm{Syl}_2(\mathrm{Out}_{\mathcal{F}}(P))$ has order 2, $O^2(\mathrm{Out}_{\mathcal{F}}(P))$ has odd order by the Thompson transfer lemma [**Th**, Lemma 5.38(a.i)] or by Burnside's normal p-complement theorem [**G**, Theorem 7.4.3]. Hence $\mathrm{Aut}_{\mathcal{F}}^*(P) \leq \mathrm{Inn}(P){\cdot}H$ for some $H < \mathrm{Aut}_{\mathcal{F}}^*(P)$ of odd order, and $\widehat{T} = [H, P]$ by Proposition 1.14(c).

Since Z is a direct factor in $T = C_S(\Delta_0)$ by (3.17), Lemma 2.11 applies with P, U_1U_2, and H in the roles of S, Δ_0, and G. By that lemma, $|H| = 3$, $\widehat{T} \trianglelefteq S$, $[\widehat{T}, \Delta] = \widehat{T} \cap \Delta = 1$, and either $\widehat{T} \cong C_{2^m} \times C_{2^m}$ (some $m \geq 1$) or $\widehat{T} \cong Q_8$. For $g \in S \smallsetminus P$, c_g normalizes $\mathrm{Aut}_{\mathcal{F}}^*(P) = O^2(O^{2'}(\mathrm{Aut}_{\mathcal{F}}(P)))$, and hence ${}^g\widehat{T} = \widehat{T}$. Thus $\widehat{T} \trianglelefteq S$, and so $\Delta_P = \widehat{T}$. If $\widehat{T} \cong C_{2^m} \times C_{2^m}$, then $\widehat{T}Z < T = C_S(\Delta_0)$ since T is nonabelian, $r(T_0) \leq 2$ since S contains a subgroup isomorphic to $T_0 \times D_8$ and $r(S) \leq 4$, and $T_0 \cong C_{2^m} \wr C_2$ since any extension of $C_{2^m} \times C_{2^m}$ by $\mathrm{Out}_{\mathcal{F}}(P) \cong \Sigma_3$ (with faithful action) is split (cf. [**AOV2**, Lemma A.8]). Hence $m = 1$ since $r(C_4 \wr C_2) = 3$, and $T_0 \cong D_8$. We conclude that $\Delta_P \cong C_2^2$ or Q_8 and $\Delta_P \not\leq Z(S)$, and thus that (ii) holds.

Case 2: $\boldsymbol{P \not\geq \Delta}$. Since $A \leq P$, $b \notin P$ and $A_0 \leq \mathrm{Fr}(P)$. Also, $[b, P] = [b, S] = A$, so $A \not\leq \mathrm{Fr}(P)$ by Lemma 1.8 and since $P \in \mathbf{E}_{\mathcal{F}}$.

Set $\overline{P} = P/\mathrm{Fr}(P)$, and $\overline{X} = X\mathrm{Fr}(P)/\mathrm{Fr}(P)$ for $X \leq P$. Thus $[b, \overline{P}] = \overline{A}$ has rank 1, and $C_{\overline{P}}(b) = \overline{P \cap T\Delta}$. By Lemma A.11, and since $\mathrm{Out}_{\mathcal{F}}(P)$ acts faithfully on $\overline{P}$ (Lemma 1.7), $\mathrm{Out}_{\mathcal{F}}(P) = \Gamma_1 \times \Gamma_2$ where $\Gamma_1 \cong \Sigma_3$ and Γ_2 has odd order, and where $[\Gamma_1, \overline{P}]$ has rank 2 and $[\Gamma_1, \overline{P}] \nleq C_{\overline{P}}(b) = \overline{P \cap T\Delta}$. Hence $\mathrm{Aut}^*_{\mathcal{F}}(P) \le \mathrm{Inn}(P)\langle\sigma\rangle$ for some $\sigma \in \mathrm{Aut}^*_{\mathcal{F}}(P)$ of order 3 for which $[\sigma] \in \Gamma_1$, and such that $\sigma(A) \nleq T\Delta$. Set $y = \sigma(a)$.

By Lemma 2.10(a,b), there is $x \in S \smallsetminus T\Delta$ such that $xax^{-1} = a^{1+4\ell}$ (some $\ell \in \mathbb{Z}$) and $xbx^{-1} = ab$. Also, $\Delta \ncong Q_8$ by Lemma 3.8(a,c). We consider the following two subcases.

Case 2a: $\boldsymbol{P \ngeq \Delta}$ **and** $\boldsymbol{y = \sigma(a) \in TAbx}$**.** Thus $yay^{-1} = a^{4k-1}$ for some $k \in \mathbb{Z}$, and $\langle [y,a] \rangle = \langle a^2 \rangle = A_0$.

Since $A, \sigma(A) \trianglelefteq P$, we have $A_0 = [\sigma(A), A] \le \sigma(A) \cap A$. So $A_0 \le \langle y \rangle$, $[y, A_0] = 1$, and this implies that $A_0 = Z$ and $|A| = 4$. Hence $\Delta \cong D_8$ since $\Delta \ncong Q_8$. Set $Q = \sigma(A)A \trianglelefteq P$. Then $Q = A\langle y \rangle \cong Q_8$ since $yay^{-1} = a^{-1}$ and $|y| = |a| = 4$, and $Q \in \mathscr{X}(S)$ (in particular, $Q \trianglelefteq S$) since $[b,y] = a^{\pm 1}$.

Now, $[T,y] = [T,Q] \le Q \cap T = Z$ since $Q \trianglelefteq S$ and $T = C_S(\Delta) \trianglelefteq S$. We already saw that $[\sigma, P/\mathrm{Fr}(P)]$ has rank 2 and contains the image of $[b,y] = a^{\pm 1}$,and hence is equal to $Q\mathrm{Fr}(P)/\mathrm{Fr}(P)$. So $[\sigma, P] \le Q\mathrm{Fr}(P)$. Since $\sigma^2 \equiv c_b \sigma c_b^{-1}$ (mod $\mathrm{Inn}(P)$), $\sigma^2(a)$ is P-conjugate to $c_b(y^{-1}) \in y^{-1}A \subseteq Q$, so $\sigma(y) = \sigma^2(a) \in Q$ and hence $\sigma(Q) = Q$. Since σ induces the identity on $P/Q\mathrm{Fr}(P)$, it induces the identity on P/Q by Lemma A.9, so $Q = [\sigma, P]$. Hence $Q = [\mathrm{Aut}^*_{\mathcal{F}}(P), P]$ by Proposition 1.14(c) and since $\mathrm{Aut}^*_{\mathcal{F}}(P) \le \mathrm{Inn}(P)\langle\sigma\rangle$. Since $Q \trianglelefteq S$, we conclude that $Q = \mathfrak{foc}(\mathcal{F}, P) = \Delta_P$, and hence that (iii) holds.

Case 2b: $\boldsymbol{P \gneq \Delta}$ **and** $\boldsymbol{y = \sigma(a) \in TAx}$**.** By Lemma 2.10(c), $|y| = |A|$ implies that $Z \le \mathrm{Fr}(T)$, so Δ is quaternion (Lemma 3.8(c,d)), and $\langle y \rangle \cap A = 1$. Also, $|\Delta| \ge 16$ since $\Delta \ncong Q_8$. Hence $S \in \mathcal{WG}$ by Lemma 3.13.

Step 2: From now on, we assume that $S \notin \mathcal{WG}$. We next show that there is an $\mathcal{F}$-essential pair (Q_1, Q_2) of type (II) for which Δ_{Q_1} is dihedral. Assume otherwise; in particular, assume $\Delta \in \mathcal{Q}$. We will show that $Z \trianglelefteq \mathcal{F}$), contradicting the assumption that $\mathcal{F}$ is reduced.

If $P = S$ or $P \in \mathbf{E}^{(\mathrm{III})}_{\mathcal{F}}$, then by (i)–(iv), either $P \ge \Delta$, or $\Delta_P = Q \cong Q_8$ and $Z(Q) = Z$. In either case, each odd order element of $\mathrm{Aut}^*_{\mathcal{F}}(P)$ centralizes Z (by Lemma B.7 when $P \ge \Delta$), and hence $\mathrm{Aut}^*_{\mathcal{F}}(P)$ centralizes Z. If $P \in \mathbf{E}^{(\mathrm{II})}_{\mathcal{F}}$, then by (3.16) and Lemma 3.8(a,c,d), $P = UC_S(U)$ for some $U \cong Q_8$ (since $\Delta_P \in \mathcal{Q}$ by assumption), $[\mathrm{Aut}^*_{\mathcal{F}}(P), P] = U$, and hence $\mathrm{Aut}^*_{\mathcal{F}}(P)$ acts trivially on $C_S(U)$, and in particular on $Z(S) \ge Z$. Since $\mathbf{E}^{(\mathrm{I})}_{\mathcal{F}} = \varnothing$, Lemma 1.15 now implies that $Z \trianglelefteq \mathcal{F}$.

Step 3: We can thus assume (R_1, R_2) was chosen so that $\Delta = \Delta_{R_1}$ is dihedral. We next show that for each $P \in \mathbf{E}_{\mathcal{F}} \cup \{S\}$ with $\Delta_P \ne 1$, either

(v) $\Delta_P \nleq T\Delta$, $\Delta_P \in \mathcal{DQ}$, and $\Delta_P > A$ with index 2; or

(vi) $\Delta_P \le T\Delta$, $\Delta_P \in \mathcal{DQ}$, and $\Delta_P\Delta/\Delta \le Z(T\Delta/\Delta)$; or

(vii) $\Delta_P \le T\Delta$, $\Delta_P \in \mathcal{DQ}$ or $\Delta_P \cong C_2^2$, $\Delta_P \nleq Z(S)$, and $Z \nleq \Delta_P$; or

(viii) $\Delta_P \cong T_0 \cong C_{2^m} \times C_{2^m}$ for some $m \ge 1$.

By points (i)–(iv) and since $\Delta \in \mathcal{D}$ and $S \notin \mathcal{WG}$, each subgroup in $\mathbf{E}^{(\mathrm{III})}_{\mathcal{F}}$ satisfies (v) or (vii). When $P = S$ and $\mathrm{Out}_{\mathcal{F}}(S) \ne 1$, then by the Schur-Zassenhaus theorem (cf. [**G**, Theorem 6.2.1]), there is $1 \ne G < \mathrm{Aut}_{\mathcal{F}}(S)$ of odd order such that $\mathrm{Aut}_{\mathcal{F}}(S) = \mathrm{Inn}(S)G$, and $[\mathrm{Aut}^*_{\mathcal{F}}(S), S] = [G, S]$ by Proposition 1.14(c). By Lemma 2.11, $[G,S] \trianglelefteq S$ (so $\Delta_S = [G,S]$), $\Delta_S \le C_S(\Delta) \le T$, $\Delta_S \cap Z = 1$, and

$[S{:}\Delta_S\Delta] \le 2 = [S{:}T\Delta]$. Hence $T = Z \times \Delta_S$, and $\Delta_S \cong T_0$. By the same lemma, $\Delta_S \cong Q_8$ or $C_{2^m} \times C_{2^m}$, so either (vii) or (viii) holds.

Now assume $P \in \mathbf{E}_{\mathcal{F}}^{(\mathrm{II})}$. Then $\Delta_P \in \mathscr{X}(S)$ by (3.16), and hence $\Delta_P \in \mathcal{DQ}$. If $P \in \mathbf{E}_{\mathcal{F}}^{(\mathrm{II})}$ is such that $\Delta_P \nleq T\Delta$, and $h \in \Delta_P \smallsetminus T\Delta$, then by Lemma 2.10(a), ${}^h b = a^i b$ for some odd i. Hence $\langle [b,h] \rangle = A \le [S, \Delta_P]$. Also, $[S,\Delta_P]$ is cyclic of index 2 in Δ_P since $\Delta_P \in \mathscr{X}(S)$ (is strongly automized). Since $[S,\Delta_P] \le [S,S] \le TA$ and $TA_0 \trianglelefteq TA$ by Lemma 2.10(a), there is no cyclic subgroup in $[S,\Delta_P]$ which strictly contains A, and thus $A = [S,\Delta_P]$ and $|\Delta_P/A| = 2$. So (v) holds in this case.

If $P \in \mathbf{E}_{\mathcal{F}}^{(\mathrm{II})}$ is such that neither (v) nor (vi) holds, then $\Delta_P \le T\Delta$ and $[T,\Delta_P] \nleq \Delta$. We must show that $Z \nleq \Delta_P$. Let $g = ta^ib^j \in \Delta_P$ and $u \in T$ be such that $t \in T$ and $[u,g] \notin \Delta$. Then $[b,g] = a^{-2i} \in [S,\Delta_P]$, $[u,g] \equiv [u,t] \pmod{\langle a^{4i}\rangle \le [S,\Delta_P]}$, and hence $[u,t] \in [S,\Delta_P] \smallsetminus \Delta$. Thus $[S,\Delta_P]$ is cyclic since $\Delta_P \trianglelefteq S$ and $\Delta_P \in \mathcal{DQ}$, and $1 \ne [u,t] \in [T,T] \cap [S,\Delta_P]$. Hence $Z(\Delta_P) \le [T,T]$, and $Z \nleq \Delta_P$ by (3.17).

Step 4: If T is abelian, then by [**AOV2**, Proposition 5.1], $S \in \mathcal{DSW}$. So assume $T = T_0 \times Z$ is nonabelian, and set

$$\mathscr{P} = \bigl\{ \Delta_P = \mathfrak{foc}(\mathcal{F},P) \bigm| P \in \mathbf{E}_{\mathcal{F}} \cup \{S\} \text{ and } \Delta_P \ne 1 \bigr\}.$$

Then $S = \langle \mathscr{P} \rangle$ by Proposition 1.14(b) and since $\mathcal{F}$ is reduced. By (v)–(viii) and since T is nonabelian, for each $P \in \mathbf{E}_{\mathcal{F}} \cup \{S\}$ with $\Delta_P \ne 1$, either $\Delta_P \in \mathcal{DQ}$, or $\Delta_P \cong C_2^2$ and $\Delta_P \nleq Z(S)$.

Since $T = T_0 \times Z$ by (3.17), $T\Delta/\Delta \cong T/Z \cong T_0$ is nonabelian. Hence subgroups satisfying (v) and (vi) cannot generate S, so there is some $\Delta_P \in \mathscr{P}$ which satisfies (vii), and $Z \nleq \Delta_P$. Then by Lemma B.6, $S = S_1 \times S_2$ for $S_1, S_2 \in \mathcal{DSQ}$. So $\mathcal{F}$ is decomposable by [**O1**, Theorem B], contradicting our original assumption. □

3.3. Essential subgroups of index 2 in S

It remains to handle reduced fusion systems all of whose essential subgroups have type (III).

Proposition 3.15. *Let $\mathcal{F}$ be a reduced, indecomposable fusion system over a 2-group S with $r(S) \le 4$, and assume all $\mathcal{F}$-essential subgroups have index 2 in S. Then S is isomorphic to D_8, $UT_4(2)$, or $C_4 \wr C_2$, or $S \in \mathcal{U}$, or S has type M_{12} or $\mathrm{Aut}(M_{12})$.*

Proof. By Lemma 1.15, if $\mathbf{E}_{\mathcal{F}} = \varnothing$, then $S \trianglelefteq \mathcal{F}$, while if $\mathbf{E}_{\mathcal{F}} = \{R\}$, then $R \trianglelefteq \mathcal{F}$. Thus $|\mathbf{E}_{\mathcal{F}}| \ge 2$ since $\mathcal{F}$ is reduced. By Lemma 3.2, for each $R \in \mathbf{E}_{\mathcal{F}}$, $\mathrm{Out}_{\mathcal{F}}(R) \cong \Sigma_3$, $\Sigma_3 \times C_3$, or $(C_3 \times C_3) \overset{-1}{\rtimes} C_2$, and $\mathrm{rk}(R/\mathrm{Fr}(R)) = 4$ if $|\mathrm{Out}_{\mathcal{F}}(R)| = 18$.

Let $\widehat{\mathbf{E}}_{\mathcal{F}}$ be the set of all pairs (R,Γ) for $R \in \mathbf{E}_{\mathcal{F}}$ and $\Gamma \le \mathrm{Aut}_{\mathcal{F}}(R)$ such that

- $\Gamma \ge \mathrm{Inn}(R)$ and $\overline{\Gamma} \overset{\text{def}}{=} \Gamma/\mathrm{Inn}(R) \cong \Sigma_3$; and
- if $\mathrm{Out}_{\mathcal{F}}(R) \cong (C_3 \times C_3) \overset{-1}{\rtimes} C_2$, then $C_{R/\mathrm{Fr}(R)}(O^2(\overline{\Gamma}))$ has rank 2.

Thus each $R \in \mathbf{E}_{\mathcal{F}}$ appears in exactly one pair in $\widehat{\mathbf{E}}_{\mathcal{F}}$, except when $\mathrm{Out}_{\mathcal{F}}(R) \cong (C_3 \times C_3) \overset{-1}{\rtimes} C_2$, in which case it appears in two pairs. By the above remarks,

$$\forall\, R \in \mathbf{E}_{\mathcal{F}}, \quad \mathrm{Aut}^*_{\mathcal{F}}(R) = O^2(O^{2'}(\mathrm{Aut}_{\mathcal{F}}(R))) \le \bigl\langle \Gamma \bigm| (R,\Gamma) \in \widehat{\mathbf{E}}_{\mathcal{F}}. \bigr\rangle \tag{3.18}$$

Set $\widehat{\mathbf{E}}_{\mathcal{F}} = \{(R_i,\Gamma_i) \,|\, i \in I\}$ for some indexing set I.

For each $J \subseteq I$, set $R_J = \bigcap_{j\in J} R_j$, and let $T_J \leq R_J$ be the largest subgroup normalized by Γ_j for each $j \in J$. Also, set

$$\Gamma_J = \bigl\langle \gamma|_{T_J} \bigm| \gamma \in \Gamma_j \text{ for some } j \in J \bigr\rangle \leq \mathrm{Aut}_{\mathcal{F}}(T_J)\,.$$

If T_J is centric in S, then by Theorem 1.4, there is a group G_J such that $S \in \mathrm{Syl}_2(G_J)$, $T_J \trianglelefteq G_J$, $C_{G_J}(T_J) \leq T_J$, and $\mathrm{Aut}_{G_J}(T_J) = \Gamma_J$. Thus $G_J/T_J \cong \Gamma_J/\mathrm{Inn}(T_J)$. When $J = \{i,j\}$, we write $R_{ij} = R_J$, $T_{ij} = T_J$, etc., and similarly with sets of one, three, or four indices. Note that if $R_i = R_j$, then $T_{ij} = R_{ij} = R_i$ since $R_i = R_j$ is normal in G_i and in G_j. By the maximality condition on T_{ij},

for distinct $i,j \in I$, $\bigl(G_i/T_{ij} > S/T_{ij} < G_j/T_{ij}\bigr)$ is a primitive amalgam of the type classified by Goldschmidt in [**Gd2**, Theorem A]. (3.19)

Case 1: Assume $T_{ij} = R_{ij}$ for each $i,j \in I$ with $i \neq j$. Then for each i, Γ_i normalizes R_{ij} for each $j \neq i$, and hence normalizes their intersection R_I. Also, $\mathrm{Aut}_{\mathcal{F}}(S)$ sends R_I to itself since it permutes the $\mathcal{F}$-essential subgroups. Thus R_I is normalized by $\mathrm{Aut}_{\mathcal{F}}(S)$ and (by (3.18)) by $\mathrm{Aut}^*_{\mathcal{F}}(R)$ for each $R \in \mathbf{E}_{\mathcal{F}}$, so $R_I \trianglelefteq \mathcal{F}$ by Lemma 1.15. Since $\mathcal{F}$ is reduced, $R_I = 1$, so S is elementary abelian, which implies $S \trianglelefteq \mathcal{F}$.

Case 2: Assume, for some $i \neq j$, that T_{ij} is not centric in S. Set $T = T_{ij} \trianglelefteq S$. By [**AOV2**, Theorem 4.5], there is a subgroup $U \trianglelefteq S$, and a finite completion Γ of the amalgam $(G_i/T > S/T < G_j/T)$, such that $[S{:}TU] \leq 2$, $[U,T] = 1$, $|U \cap T| \leq 2$, $S/T \in \mathrm{Syl}_2(\Gamma)$, and U and Γ are one of the pairs in Table 3.1:

if $T\cap U = 1$:	if $\lvert T \cap U\rvert = 2$:	if $S = TU$:	if $[S{:}TU] = 2$:
U	U	Γ	Γ
D_8	Q_{16}	A_6	Σ_6
$C_4 \wr C_2$	does not occur	$U_3(3)$	$\mathrm{Aut}(U_3(3))$
$(C_4\times C_4) \overset{t,-1}{\rtimes} C_2^2$	type $2M_{12}$	M_{12}	$\mathrm{Aut}(M_{12})$

TABLE 3.1

Assume $U \cong D_8$. If $S = UT$, then $S \cong U \times T$. If $[S{:}UT] = 2$, then $S/T \cong D_8 \times C_2$ (i.e., it has type Σ_6) by the above table; and since U and T are both normal in S, $S = U \times T\langle x\rangle$ for some x. In either case, by [**O1**, Theorem B], $\mathcal{F}$ is isomorphic to a product of reduced fusion systems one of which over $U \cong D_8$. Hence $S \cong D_8$ ($T = 1$) since $\mathcal{F}$ is indecomposable.

Assume $U \cong Q_{16}$, and fix $a,b \in U$ such that $|a| = 8$ and $b \notin \langle a\rangle$. Then $a^4 \in Z(S)$ since $U \trianglelefteq S$. By Lemma B.7, $\alpha(a^4) = a^4$ for each $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ of odd order. By Lemma 1.15 and (3.18) and since $O_2(\mathcal{F}) = 1$, there is $k \in I$ and $\alpha \in \Gamma_k \leq \mathrm{Aut}_{\mathcal{F}}(R_k)$ of order 3 such that $\alpha(a^4) \neq a^4$. By Lemma B.7 (with R_k in the role of S), $R_k \cap U$ must be abelian. Since $[U{:}R_k \cap U] = 2$, $R_k \cap U = \langle a\rangle$, so $b \notin R_k$, and $b\alpha(a^4)b^{-1} = \alpha^{-1}(a^4)$ since $c_b\alpha c_b \equiv \alpha^{-1} \pmod{\mathrm{Inn}(R_k)}$ by definition of $\widehat{\mathbf{E}}_{\mathcal{F}}$. In particular, $\alpha(a^4) \notin C_S(U) \geq T$, so the image of $\alpha(a)$ in $S/T \cong D_8 \times C_2$ has order 8, which is impossible.

If $U \cong C_4 \wr C_2$, then $r(U) = 3$, so $r(T) = 1$ and hence T is cyclic. If $T = 1$, then either $S = U \cong C_4 \wr C_2$, or S is a Sylow 2-subgroup of $\mathrm{Aut}(SU_3(3))$ and hence of type M_{12} (cf. [**Gd2**, Table 1]). If $T \neq 1$, then $\Omega_1(Z(S)) = \Omega_1(Z(U)Z(T)) \cong C_2^2$ (recall $U \trianglelefteq S$). If $\Omega_1(T) \nleq [S,S]$, then there are no reduced fusion systems over S by Proposition 1.18(a). If $T \neq 1$ and $\Omega_1(T) \leq [S,S]$, let $T_0 < T$ be the subgroup of index 2; then $[S{:}TU] = 2$, S/U is nonabelian since $T \cap [S,S] \neq 1$, so $S = TU\langle x\rangle$ where $x^2 \in T_0U$, and $r(S/T_0) = r(S/T) + r(T/T_0) = 5$ since S/T has type M_{12} ($r(S/T) = 4$).

Now assume U is of type M_{12} or $2M_{12}$. Then $T \leq U$ since M_{12} has sectional 2-rank 4. So S is a Sylow 2-subgroup in M_{12} or $\mathrm{Aut}(M_{12})$, or in a 2-fold central extension of one of those groups. We must eliminate this last possibility.

Assume S contains a subgroup $S_0 \leq S$ of type $2M_{12}$, and let $Z = \langle z\rangle \leq Z(S)$ be the subgroup in the center of $2M_{12}$. Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{r}, \boldsymbol{t} \in S_0$ be elements whose classes $a, b, r, t \in S_0/Z$ satisfy the presentation of Notation 4.1 (with $n = 2$). Set $\widehat{Q} = \langle z, \boldsymbol{a}^2, \boldsymbol{ab}, \boldsymbol{r}, \boldsymbol{t}\rangle < S_0$; thus $\widehat{Q}/Z \cong 2_+^{1+4}$, and $Z(\widehat{Q}) = \langle z, \boldsymbol{a}^2\boldsymbol{b}^2\rangle \cong C_2^2$ by Lemma D.3. There is $\alpha \in \mathrm{Aut}_{\mathcal{F}}(\widehat{Q})$ of order 3 such that $\alpha(z) = z$ (hence $\alpha|_{Z(\widehat{Q})} = \mathrm{Id}$) and $C_{\widehat{Q}/Z(\widehat{Q})}(\alpha) = 1$ (cf. [**A2**, Lemma 5.3(2)]). Let $O_1, \ldots, O_5$ be the orbits of the α-action on $(\widehat{Q}/Z(\widehat{Q}))^{\#}$, where (upon letting $\bar{a}, \bar{b}, \bar{r}, \bar{t} \in S/Z(\widehat{Q})$ denote the classes of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{r}, \boldsymbol{t}$) $O_1 = \{\bar{a}\bar{b}^{-1}, \bar{b}^2\bar{t}, \bar{a}\bar{b}\bar{t}\}$ and $O_2 = \{\bar{a}\bar{b}, \bar{a}\bar{b}^{-1}\bar{r}\bar{t}, \bar{b}^2\bar{r}\bar{t}\}$ are the images of the two quaternion subgroups in $\widehat{Q}/Z$, and $\bar{a}^2 \in O_3$, $\bar{r}\bar{t} \in O_4$, and $\bar{a}\bar{b}\bar{r}\bar{t} \in O_5$ (the products of $\bar{a}\bar{b}^{-1} \in O_1$ with the three elements in O_2). The elements in each O_i lift to elements of $\widehat{Q}$ with the same square $x_i \in Z(\widehat{Q})$ (since $\alpha|_{Z(\widehat{Q})} = \mathrm{Id}$), where $x_1, x_2 \in \boldsymbol{a}^2\boldsymbol{b}^2Z$, $x_3 = 1$ since $\boldsymbol{a}^2$ is $\mathcal{F}$-conjugate to $\boldsymbol{a}^2\boldsymbol{b}^2$ or $\boldsymbol{a}^2\boldsymbol{b}^2z$, and $x_4 = x_5 \in Z$ since rt is S-conjugate to $abrt = {}^a(rt)$. This information suffices to show that $\widehat{Q} \ncong UT_3(4)$ and $\widehat{Q} \ncong Q_8 \times Q_8$. So either $x_4 = x_5 = 1$ and $\widehat{Q} \cong (\widehat{Q}/Z) \times Z$; or $[\widehat{Q}, \widehat{Q}] = Z(\widehat{Q})$ and by Lemma D.2, $\widehat{Q}/\langle x\rangle \cong 2_+^{1+4}$ for exactly two of the three involutions $x \in Z(\widehat{Q})^{\#}$. In either case, $\widehat{Q}/\langle \boldsymbol{a}^2\boldsymbol{b}^2\rangle \ncong \widehat{Q}/\langle \boldsymbol{a}^2\boldsymbol{b}^2z\rangle$, so $\boldsymbol{a}^2\boldsymbol{b}^2$ and $\boldsymbol{a}^2\boldsymbol{b}^2z$ are not S-conjugate, and hence $Z(S) = Z(\widehat{Q}) = \langle \boldsymbol{a}^2\boldsymbol{b}^2, z\rangle$.

If S/Z has type M_{12} (i.e., $S = S_0$), then $\widehat{Q}/Z(S)$ is the unique abelian subgroup of $S/Z(S)$ of rank 4. If S/Z has type $\mathrm{Aut}(M_{12})$ ($|S/S_0| = 2$), then an outer automorphism of M_{12} acts on $\widehat{Q}/Z(\widehat{Q})$ by exchanging the classes $\bar{a}\bar{b}$ and $\bar{a}\bar{b}^{-1}$, hence exchanging O_1 and O_2. (See the description of the extension amalgam in [**Gd2**, (3.8)], or Proposition 4.3(b) below.) Thus $\mathrm{Out}_S(\widehat{Q}) \cong C_2^2$ permutes freely a basis for $\widehat{Q}/Z(S) \cong C_2^4$, and again $\widehat{Q}/Z(S)$ is the unique abelian subgroup of $S/Z(S)$ of rank 4 (Lemma A.4(b)). So in either case, $\widehat{Q}$ is characteristic in S.

If z is $\mathcal{F}$-conjugate to $\boldsymbol{a}^2\boldsymbol{b}^2$ or $\boldsymbol{a}^2\boldsymbol{b}^2z$, then by the extension axiom (and since $\mathrm{Out}_{\mathcal{F}}(S)$ has odd order), there is $\beta \in \mathrm{Aut}_{\mathcal{F}}(S)$ of odd order which permutes cyclically the involutions in $Z(S) = Z(\widehat{Q})$. Then $\beta(\widehat{Q}) = \widehat{Q}$, which is impossible since one of the elements in $\boldsymbol{a}^2\boldsymbol{b}^2Z$ is a square in $\widehat{Q}$ and the other is not. So by Lemma 1.15 and since $Z(\mathcal{F}) = 1$, there must be $R \in \mathbf{E}_{\mathcal{F}}$ (of index 2) and $\gamma \in \mathrm{Aut}_{\mathcal{F}}(R)$ such that $\gamma(z) \neq z$. In particular, $Z(R) > Z(S)$, and by Lemma A.3, $R = C_S(V)$ where $V = \Omega_1(Z_2(S)) = \langle z, \boldsymbol{a}^2, \boldsymbol{b}^2\rangle \cong C_2^3$. But each element in $V \smallsetminus Z$ is $\mathcal{F}$-conjugate to $\boldsymbol{a}^2\boldsymbol{b}^2$ or $\boldsymbol{a}^2\boldsymbol{b}^2z$, so none of them can be conjugate to z. Hence this situation is impossible.

Case 3: Now assume that T_{ij} is centric in S for each $i, j \in I$. By [**AOV2**, Lemma 4.2(e)], $\mathrm{Out}_{G_i}(T_{ij}) \cong G_i/T_{ij}$ and $\mathrm{Out}_{G_j}(T_{ij}) \cong G_j/T_{ij}$ both act faithfully on $T_{ij}/\mathrm{Fr}(T_{ij})$ for all i, j. Since $r(T_{ij}) \leq 4$, this implies that G_i/T_{ij} is isomorphic to a subgroup of $GL_4(2)$, and hence that S/T_{ij} contains no element of order 8.

Recall that $(G_i/T_{ij} > S/T_{ij} < G_j/T_{ij})$ is a Goldschmidt amalgam by (3.19). If $R_i \neq R_j$, then $|S/T_{ij}| \geq |S/R_{ij}| = 4$, and $|S/T_{ij}| \geq 8$ if $T_{ij} < R_{ij}$. So from the list in [**Gd2**, Table 1] of possible amalgams (and since $C_4 \wr C_2$ does contain elements of order 8), we see that $S/T_{ij} \cong C_2$ (if $R_i = R_j$), C_2^2 (type G_1^3, when $T_{ij} = R_{ij}$), D_8 (type G_2^2 or G_3), or $D_8 \times C_2$ (type G_3^1). Also, when $S/T_{ij} \cong D_8$, then $R_i/T_{ij} = O_2(G_i/T_{ij}) \cong C_2^2$, and similarly for R_j/T_{ij}, regardless of whether the amalgam has type G_2^2 or G_3.

Assume $S/T_{ij} \cong D_8 \times C_2$. Set $\overline{\Gamma}_{ij} = \Gamma_{ij}/\mathrm{Inn}(T_{ij})$ for short. Then $(G_i/T_{ij} > S/T_{ij} < G_j/T_{ij})$ is the Σ_6-amalgam, and hence $O^2(\overline{\Gamma}_{ij})$ has index 2 in $\overline{\Gamma}_{ij}$. By Proposition D.1(g), $\overline{\Gamma}_{ij}$ contains a subgroup A_6, A_7, or $GL_3(2)$ with index 2; and hence $\overline{\Gamma}_{ij} \cong \Sigma_6$ since this is the only possible extension of one of these groups with Sylow 2-subgroup $D_8 \times C_2$. (Neither $A_6 \times C_2$ nor $GL_3(2) \times C_2$ is contained in $GL_4(2) \cong A_8$.) But this is impossible by Lemma D.6.

Thus $S/T_{ij} \cong D_8$ whenever $T_{ij} < R_{ij}$, and by Case 1, there is at least one such pair $i, j \in I$. Fix such i, j, and set $T = T_{ij}$. By Proposition D.1(g), and since $O^2(\overline{\Gamma}_{ij}) = \overline{\Gamma}_{ij}$ by the focal subgroup theorem, $\overline{\Gamma}_{ij} \leq \mathrm{Out}_{\mathcal{F}}(T)$ is isomorphic to A_7, A_6, or $PSL_2(7) \cong GL_3(2)$. Hence by [**GH**, Theorem II.B], $T \cong C_2^3$, C_2^4, or $C_2^3 \times C_4$. (Recall that $\mathrm{Out}(2_-^{1+4}) \cong \Sigma_5$.)

If $T = T_{ij} \cong C_2^3$, then $\Gamma_{ij} = \mathrm{Aut}_{G_{ij}}(T_{ij}) \cong GL_3(2)$. By Lemma D.5(a), $S \cong UT_4(2)$ or S has type M_{12}. So we can assume $T \not\cong C_2^3$.

If $T = T_{ij} \cong C_4 \times C_2^3$, then $\Gamma_{ij} \cong GL_3(2) \cong \mathrm{Aut}(T)/O_2(\mathrm{Aut}(T))$ (Lemma A.9). Set $V = \Omega_1(T) \cong C_2^4$ and $Z = \mathrm{Fr}(T)$. By Lemma D.5(c), the Γ_{ij}-action on V is decomposable, that on T/Z is indecomposable, and $G/[G, V] \cong C_4 \times_{C_2} SL_2(7)$. Since $Z(\mathcal{F}) = 1$, there is $Q \leq S$ of index at most 2, together with $\beta \in \mathrm{Aut}_{\mathcal{F}}(Q)$, such that $\beta(Z) \neq Z$. We can assume that $Q = C_S(\beta(Z))$ (otherwise $\beta(Z) \leq Z(S)$ and we can take $Q = S$). Then $\beta(Z) \leq Z(Q)$, and since $[T{:}C_T(g)] \geq 4$ for each involution $g \in S \smallsetminus T$ ($\mathrm{rk}([g, T/Z]) = 2$ by Lemma D.5(b)), $\beta(Z) \leq V$. Hence $T \leq C_S(\beta(Z)) = Q$. The image in $S/T \cong D_8$ of any abelian subgroup of $S/[G, V] \cong C_4 \times_{C_2} Q_{16}$ is cyclic, so the image of $\beta(T)$ is cyclic, hence has order at most 2 since $\beta(Z) \leq T$. Thus $|T \cap \beta(T)| = 2^4$, $\mathrm{Fr}(T \cap \beta(T)) \leq \mathrm{Fr}(T) \cap \mathrm{Fr}(\beta(T)) = 1$, so $T \cap \beta(T) = V$ and hence $\beta(V) = V$. Then $\beta(T) = T$ since $T = C_Q(V)$, which is impossible since $\beta(\mathrm{Fr}(T)) \neq \mathrm{Fr}(T)$.

Thus $T \not\cong C_4 \times C_2^3$. So we can now assume, for $i, j \in I$, that

$$T_{ij} < R_{ij} \implies S/T_{ij} \cong D_8,\ R_i/T_{ij} \cong R_j/T_{ij} \cong C_2^2,\ \text{and } T_{ij} \cong C_2^4 \text{ and } \Gamma_{ij} \cong A_6,\ A_7, \text{ or } GL_3(2). \tag{3.20}$$

Case 3a: Assume there is a unique subgroup $T \trianglelefteq S$ such that $S/T \cong D_8$ and $T = T_{ij}$ for some (possibly more than one) pair of indices $i, j \in I$. We just showed that $T \cong C_2^4$. Set $T_0 = T \cap R_I$. Then $R_I \geq \mathrm{Fr}(S)$ since $|S/R_i| = 2$ for each $i \in I$, so $T_0 \geq T \cap \mathrm{Fr}(S) \geq [T, S]$, and $[T, S] \neq 1$ since T is centric in S by assumption.

Now, $\mathrm{Aut}_{\mathcal{F}}(S)$ normalizes T by its uniqueness, and it normalizes T_0 since it permutes the R_i. By Lemma 1.15 and (3.18), and since $T_0 \ntrianglelefteq \mathcal{F}$ ($\mathcal{F}$ is reduced), there is $k \in I$ such that Γ_k does not normalize T_0. Then Γ_k and R_k have the following properties:

(a) <u>Γ_k does not normalize $T \cap R_k$:</u> Since Γ_k does not normalize the subgroup $T_0 = \bigcap_{\ell \in I}(T \cap R_{k\ell})$, there is ℓ such that Γ_k does not normalize $T \cap R_{k\ell}$. Either

$R_{k\ell} > T_{k\ell} = T$, in which case Γ_k does not normalize $T = T \cap R_k$, or $R_{k\ell} = T_{k\ell}$ is normalized by Γ_k in which case $T \cap R_k$ is not.

(b) $\underline{R_{ik} = T_{ik} \text{ and } R_{jk} = T_{jk}}$: If $R_{ik} > T_{ik}$, then $T_{ik} = T = T \cap R_k$ is normalized by Γ_k, contradicting (a). By a similar argument, $R_{jk} = T_{jk}$.

(c) $\underline{\Gamma_k \text{ does normalize } R_{ijk}}$: Since Γ_k normalizes $T_{ik} = R_{ik}$ and $T_{jk} = R_{jk}$ by (b), it also normalizes $R_{ijk} = R_{ik} \cap R_{jk}$.

(d) $\underline{\Gamma_{ij} \text{ normalizes } T \cap R_k}$: By (b), Γ_i normalizes $T_{ij} \cap T_{ik} = T \cap R_{ik} = T \cap R_k$ and Γ_j normalizes $T_{ij} \cap T_{jk} = T \cap R_{jk} = T \cap R_k$.

(e) $\underline{R_{ijk} \text{ is nonabelian, and } |R_{ijk}/(T \cap R_k)| = 2}$: Since $T \not\geq \mathrm{Fr}(S)$ while $R_{ijk} \geq \mathrm{Fr}(S)$, $R_{ijk} > T \cap R_k$, and so $|R_{ijk}/(T \cap R_k)| = |R_{ij}/T| = 2$. Since $\Gamma_{ij} \leq \mathrm{Aut}(T)$ normalizes $T \cap R_k$ by (d), and since $O_2(\Gamma_{ij}) = 1$ by (3.20), Γ_{ij} acts faithfully on $T \cap R_k$ by Lemma A.9. Hence for $x \in R_{ijk} \smallsetminus (T \cap R_k)$, $[x, T \cap R_k] \neq 1$, and so R_{ijk} is nonabelian.

Since $\Gamma_k \cong \Sigma_3$ acts on R_{ijk} without normalizing $T \cap R_k$ by (a) and (c), where R_{ijk} is nonabelian by (e), it must permute $T \cap R_k$ in an orbit of three different elementary abelian subgroups of index 2 in R_{ijk}. But this is impossible: R_{ijk} would be a semidirect product of $T \cap R_k \cong C_2^m$ ($m = 3, 4$) with C_2, $|[R_{ijk}, R_{ijk}]| = 2$ by Lemma A.4(a), and so $R_{ijk} \cong D_8 \times C_2^{m-2}$ contains only two elementary abelian subgroups of index 2.

Case 3b: Thus there are (at least) two distinct subgroups $V, W < S$ such that $V = T_{ij} < R_{ij}$ and $W = T_{k\ell} < R_{k\ell}$ for some $i \neq j$, $k \neq \ell$ in I. By (3.20), $S/V \cong S/W \cong D_8$ and $V \cong W \cong C_2^4$. Set $X = V \cap W$.

If $[V{:}X] = 2$, then $VW/V \trianglelefteq S/V \cong D_8$ implies that $VW = R_{ij}$, and similarly, $VW = R_{k\ell}$. Since there are only three subgroups of index 2 in S containing VW, $\{R_i, R_j\} \cap \{R_k, R_\ell\} \neq \varnothing$, and we can assume $R_i = R_k$. Then $R_i/V \cong R_i/W \cong C_2^2$ by (3.20), so $R_i/X \cong C_2^3$. Hence $R_i > VW > V > X \geq \mathrm{Fr}(R_i)$, where $[X{:}\mathrm{Fr}(R_i)] \leq 2$, and these cannot all be normalized by Γ_i (Lemma A.9). Hence $\Gamma_k \neq \Gamma_i$, so $\mathrm{Out}_{\mathcal{F}}(R_i) \cong (C_3 \times C_3) \rtimes C_2$ and $\mathrm{rk}(R_i/\mathrm{Fr}(R_i)) = 4$. Also, $V/\mathrm{Fr}(R_i)$ and $W/\mathrm{Fr}(R_i)$ have rank 2 and are normalized by $\Gamma_i/\mathrm{Inn}(R_i)$ and $\Gamma_k/\mathrm{Inn}(R_i)$, respectively, where $C_{R_i/\mathrm{Fr}(R_i)}(O^2(\Gamma_i))$ and $C_{R_i/\mathrm{Fr}(R_i)}(O^2(\Gamma_k))$ have rank 2 by definition of $\widehat{\mathbf{E}}_{\mathcal{F}}$. This implies that $V/\mathrm{Fr}(R_i)$ and $W/\mathrm{Fr}(R_i)$ are equal or complementary, and since $[V{:}V \cap W] = 2$, they must be equal. This contradicts our assumption that $V \neq W$.

Thus $[V{:}X] \geq 4$. Since $VW/V \leq S/V \cong D_8$, we have $VW/V \cong V/X \cong W/X \cong C_2^2$ and so $[S{:}VW] = 2$. Since R_i and R_j are the unique subgroups of S such that $R_i/V \cong C_2^2 \cong R_j/V$, $VW = R_i$ or R_j. Thus there is an automorphism of VW of order 3 which normalizes V. Also, $[V, W] = X$, since otherwise $r(VW) \geq 5$. So $VW \cong UT_3(4)$ by Lemma C.7(b), and hence $S \in \mathcal{U}$. □

CHAPTER 4

Fusion systems over 2-groups of type $G_2(q)$

Throughout this chapter, we will be working with 2-groups $S \in \mathcal{G}$, using the following notation for elements and subgroups of S.

NOTATION 4.1. For some $n \geq 2$ and some $\lambda = -1$ or (if $n \geq 3$) $2^{n-1} - 1$, $S = S_{n,\lambda} = \langle a, b, r, t \rangle$, where $A \overset{\text{def}}{=} \langle a, b \rangle \cong C_{2^n} \times C_{2^n}$, $\langle r, t \rangle \cong C_2^2$, $rar^{-1} = a^{\lambda}$, $rbr^{-1} = b^{\lambda}$, $tat^{-1} = b$, $tbt^{-1} = a$. Set

$$\Delta_1 = \big\langle ab^{-1}, a^{2^{n-1}}t \big\rangle \cong Q_{2^{n+1}} \qquad U_1 = \big\langle (ab^{-1})^{2^{n-2}}, a^{2^{n-1}}t \big\rangle \cong Q_8 \qquad P_1 = U_1\Delta_2$$

$$\Delta_2 = \big\langle ab^{-\lambda}, a^{2^{n-1}}rt \big\rangle \cong Q_{2^{n+1}} \qquad U_2 = \big\langle (ab)^{2^{n-2}}, a^{2^{n-1}}rt \big\rangle \cong Q_8 \qquad P_2 = U_2\Delta_1$$

$$\mathscr{P}_i = \{{}^gP_i \,|\, g \in S\} \quad (i = 1, 2) \qquad Q = \Delta_1\Delta_2 \qquad A_+ = A\langle r \rangle\,.$$

Define $\tau_1 \in \operatorname{Aut}(P_1)$, $\tau_2 \in \operatorname{Aut}(P_2)$, and $\sigma \in \operatorname{Aut}(A_+)$, each of order 3, by setting $\tau_i|_{C_S(U_i)} = \operatorname{Id}$, $\sigma(r) = r$, and letting τ_i act on U_i and σ on A as follows:

$$\begin{array}{lccccccc}
\tau_1|_{U_1}: & (ab^{-1})^{2^{n-2}} & \mapsto & b^{2^{n-1}}t & \mapsto & (ab)^{2^{n-2}}t & \mapsto & (ab^{-1})^{2^{n-2}} \\
\tau_2|_{U_2}: & (ab)^{2^{n-2}} & \mapsto & b^{2^{n-1}}rt & \mapsto & (ab^{-1})^{2^{n-2}}rt & \mapsto & (ab)^{2^{n-2}} \\
\sigma|_A: & a & \mapsto & b & \mapsto & a^{-1}b^{-1} & \mapsto & a
\end{array}$$

Thus when $n = 2$, $U_i = \Delta_i$ and $P_1 = P_2 = Q$. Note that $[\Delta_1, \Delta_2] = 1$ if $\lambda = -1$, and $[\Delta_1, \Delta_2] = \langle (ab)^{2^{n-1}} \rangle = Z(S)$ if $\lambda = -1 + 2^{n-1}$. So in either case, for $i = 1, 2$, $\operatorname{Aut}_{\Delta_{3-i}}(P_i) \leq \operatorname{Inn}(P_i)$ and hence $P_i = U_iC_S(U_i)$.

PROPOSITION 4.2. *Assume $S = S_{n,\lambda}$ is as in Notation 4.1. Then $\operatorname{Aut}(S)$ is a 2-group. If $\mathcal{F}$ is a reduced fusion system over S, then*

(a) $\lambda = -1$;

(b) $\mathbf{E}_{\mathcal{F}} = \{A_+\} \cup \mathscr{P}_1 \cup \mathscr{P}_2$ *(so $\mathbf{E}_{\mathcal{F}} = \{Q, A_+\}$ if $n = 2$); and*

(c) *either $n = 2$, $\operatorname{Out}_{\mathcal{F}}(Q) = \big\langle [\tau_1\tau_2^{-1}], [c_a] \big\rangle \cong \Sigma_3$, and $\mathcal{F}$ is isomorphic to the fusion system of M_{12}; or*

$n = 2$, $\operatorname{Out}_{\mathcal{F}}(Q) = \big\langle [\tau_1], [\tau_2], [c_a] \big\rangle \cong (C_3 \times C_3) \rtimes C_2$, and $\mathcal{F}$ is isomorphic to the fusion system of $G_2(q)$ for each $q \equiv \pm 3$ (mod 8); or

$n \geq 3$, $\operatorname{Aut}_{\mathcal{F}}(P_i) = \langle \tau_i, \operatorname{Aut}_S(P_i) \rangle$ for $i = 1, 2$, and $\mathcal{F}$ is isomorphic to the fusion system of $G_2(q)$ for each odd prime power q such that $v_2(q^2 - 1) = n+1$.

PROOF. By Lemma A.4(b), and since $|S/[S, S]| = 8$, A is the unique abelian subgroup of index 4 in S. Hence $\operatorname{Aut}(S)$ is a 2-group by Lemma A.9, applied to the chain $\operatorname{Fr}(S) < A < A_+ < S$ of characteristic subgroups. So $\operatorname{Out}_{\mathcal{F}}(S) = 1$.

We claim that

$$\mathscr{X}(S) = \big\{\Delta_1, \Delta_2, \langle ab^{-1}, t \rangle, \langle ab^{-\lambda}, rt \rangle \big\}\,. \qquad (4.1)$$

Fix some $\Delta \in \mathscr{X}(S)$. By Definition 2.1, $\Delta = \langle \mathcal{C} \rangle = B \cup \{\mathcal{C}\}$, where B is cyclic and $\mathcal{C}$ is an S-conjugacy class of elements of order 2 (if $\Delta \in \mathcal{D}$) or 4 (if $\Delta \in \mathcal{Q}$). Also, $\mathcal{C} \nsubseteq A$ since Δ is nonabelian. If $\mathcal{C} \subseteq Ar$, then $a^{2^{n-1}}, b^{2^{n-1}} \in B = A \cap \Delta$, which is impossible since B is cyclic. If $\mathcal{C} \subseteq At$, then since each S-conjugacy class in At has the form $\{(ab^{-1})^i gt \,|\, i \in \mathbb{Z}\}$ for some $g \in A$, and since $(a^i t)^2 = (ab)^i$, $\mathcal{C}$ is the conjugacy class of t or of $a^{2^{n-1}}t$, and $\Delta = \langle ab^{-1}, t\rangle$ or $\langle ab^{-1}, a^{2^{n-1}}t\rangle = \Delta_1$. By a similar argument, if $\mathcal{C} \subseteq Art$, then $\Delta = \langle ab^{-\lambda}, rt\rangle$ or $\langle ab^{-\lambda}, a^{2^{n-1}}rt\rangle = \Delta_2$.

(b) When $n = 2$ (hence $\lambda = -1$), $\mathbf{E}_{\mathcal{F}} = \{Q, A_+\}$ by [**AOV2**, Proposition 3.2]. (This also follows upon making minor changes to the argument below.)

Assume $n \geq 3$. If $V < S$ is elementary abelian of rank 4, then $V \cap A = \Omega_1(A) = \langle a^{2^{n-1}}, b^{2^{n-1}}\rangle$ and $VA = S$, so $V \cap At \neq \varnothing$, which is impossible since V is abelian. Hence S has rank 3, and in particular, $S \notin \mathcal{UV}$.

Thus by Theorem 3.1(a,b), $\mathbf{E}_{\mathcal{F}}^{(\mathrm{I})} = \varnothing$; and $R \in \mathbf{E}_{\mathcal{F}}^{(\mathrm{II})}$ implies that $\mathfrak{foc}(\mathcal{F}, R) \in \mathscr{X}(S)$ and R is as in Lemma 3.8(a,c,d,e). By that lemma, if $R \in \mathbf{E}_{\mathcal{F}}^{(\mathrm{II})}$, then there are $V < \Delta \in \mathscr{X}(S)$ such that $V \cong C_2^2$ or Q_8, $R = VC_S(V)$, and $\Delta = \mathfrak{foc}(\mathcal{F}, R)$. Also, if $V \cong C_2^2$, then V is a direct factor in R. In this last case, V is S-conjugate to $\langle (ab)^{2^{n-1}}, t\rangle$ or $\langle (ab)^{2^{n-1}}, rt\rangle$, neither of which is a direct factor in its centralizer. Thus $V \cong Q_8$ and is S-conjugate to U_1 or U_2, and R is S-conjugate to P_1 or P_2. Also, by definition of U_i in the statement of Lemma 3.8,

$$\text{for } i = 1, 2,\ P_i \in \mathbf{E}_{\mathcal{F}} \implies [\mathrm{Aut}^*_{\mathcal{F}}(P_i), P_i] = U_i \text{ and } \mathfrak{foc}(\mathcal{F}, P_i) = \Delta_i. \tag{4.2}$$

Now assume $R \in \mathbf{E}_{\mathcal{F}}^{(\mathrm{III})}$. Thus $[S{:}R] = 2$, so $\langle ab, a^2\rangle = \mathrm{Fr}(S) \leq R$, and $\mathrm{Aut}(R)$ is not a 2-group. Set $A_0 = \mathrm{Fr}(S) = \langle ab, a^2\rangle$. If $R \ngeq A$, then $R = A_0\langle g, h\rangle$ for some $g \in Ar$ and $h \in At$, $|[g, A_0]| = \frac{1}{4}|A_0| \geq 8$, $|[h, A_0]| \geq 2^{n-1} \geq 4$, and $|[gh, A_0]| \geq 4$. So by Lemma A.4(b), A_0 is the unique abelian subgroup of index 4 in R. Also, $\mathrm{Aut}(A_0)$ is a 2-group (Corollary A.10(a)), $A_0\langle g\rangle$ is characteristic in R, so $\mathrm{Aut}(R)$ is a 2-group by Lemma A.9. Thus $R \geq A$. If $R = A\langle t\rangle$ or $R = A\langle rt\rangle$, then $R/[R, R] \cong C_{2^n} \times C_2$, and $\mathrm{Aut}(R)$ is a 2-group by Corollary A.10(a) again. Thus $R = A_+ = A\langle r\rangle$.

This proves that $\mathbf{E}_{\mathcal{F}} \subseteq \{A_+\} \cup \mathscr{P}_1 \cup \mathscr{P}_2$. By (4.2), if $P_i \in \mathbf{E}_{\mathcal{F}}$, then $\mathfrak{foc}(\mathcal{F}, P_i) = \Delta_i$. If $A_+ \in \mathbf{E}_{\mathcal{F}}$, then $[\mathrm{Aut}^*_{\mathcal{F}}(A_+), A_+] = \mathfrak{foc}(\mathcal{F}, A_+) \leq A$ since $A < A_+$ is characteristic of index 2. Since no two of the subgroups A, Δ_1, Δ_2 generate S, $\mathbf{E}_{\mathcal{F}} = \{A_+\} \cup \mathscr{P}_1 \cup \mathscr{P}_2$.

(a) Assume $\lambda = -1 + 2^{n-1}$ (and hence $n \geq 3$). Set $U_2^* = {}^aU_2$, $P_2^* = U_2^* C_S(U_2^*) = {}^aP_2$, and $\tau_2^* = c_a \tau c_a^{-1} \in \mathrm{Aut}_{\mathcal{F}}(P_2^*)$. Then

$$[U_1, U_2^*] = \bigl[\bigl\langle (ab^{-1})^{2^{n-2}}, a^{2^{n-1}}t\bigr\rangle, \bigl\langle (ab)^{2^{n-2}}, ab^{-\lambda}a^{2^{n-1}}rt\bigr\rangle\bigr] = \bigl\langle (ab)^{2^{n-1}}\bigr\rangle = Z(S)\,,$$

so $\mathrm{Aut}_{U_2^*}(U_1) \leq \mathrm{Inn}(U_1)$ and $\mathrm{Aut}_{U_1}(U_2^*) \leq \mathrm{Inn}(U_2^*)$. It follows that $U_1 \leq P_2^*$ and $U_2^* \leq P_1$.

Set $Q_0 = U_1 U_2^* \cong 2_-^{1+4}$ (Lemma C.2(a)). Since $\Delta_1\Delta_2/Z(S) \cong D_{2^n} \times D_{2^n}$, and since each $g \in S \smallsetminus \Delta_1\Delta_2$ acts on each $\Delta_i/Z(S) \cong D_{2^n}$ by exchanging the two noncyclic subgroups of index 2, $N_S(Q_0) = N_{\Delta_1\Delta_2}(Q_0) = N_{\Delta_1}(U_1)N_{\Delta_2}(U_2)$ and hence $|N_S(Q_0)/Q_0| = 4$. By Lemma 1.16(a), and since no essential subgroup contains $N_S(Q_0)$ by (b), Q_0 is fully normalized in $\mathcal{F}$. Also, $\tau_1|_{Q_0}, \tau_2^*|_{Q_0} \in \mathrm{Aut}_{\mathcal{F}}(Q_0)$. For $g \in N_{\Delta_1}(U_1) \smallsetminus U_1$, $c_g|_{U_1} \notin \mathrm{Inn}(U_1)$ while $c_g|_{U_2^*} \in \mathrm{Inn}(U_2^*)$. Hence $\mathrm{Out}_{\mathcal{F}}(Q_0) = \mathrm{Out}(Q_0) \cong \Sigma_5$ by Lemma C.2(b). Since this contradicts the Sylow axiom, we conclude that $\lambda \neq -1 + 2^{n-1}$.

(c) Since A is the unique abelian subgroup of index 2 in A_+, the quotient group $\mathrm{Out}(A_+)/O_2(\mathrm{Out}(A_+))$ injects into $\mathrm{Aut}(A/\mathrm{Fr}(A)) \cong \Sigma_3$ by Lemma A.9. Thus $\mathrm{Out}_{\mathcal{F}}(A_+) \cong \Sigma_3$ since A_+ is essential, and $\mathrm{Out}_{\mathcal{F}}(A_+) = \langle [\gamma], [c_t] \rangle$ for some $\gamma \in \mathrm{Aut}_{\mathcal{F}}(A_+)$ of order 3. Set $\gamma' = [\gamma, c_t] \in \mathrm{Aut}_{\mathcal{F}}(A_+)$; then $[\gamma'] = [\gamma]^{-1}$ in $\mathrm{Out}_{\mathcal{F}}(A_+)$ and $\langle \gamma', c_t \rangle \le \mathrm{Aut}_{\mathcal{F}}(A_+)$ is dihedral. So upon replacing γ by some appropriate power of γ', we can assume that $\langle \gamma, c_t \rangle \cong \Sigma_3$ as a subgroup of $\mathrm{Aut}_{\mathcal{F}}(A_+)$. Also, $C_A(\gamma) = 1$ since $C_{\Omega_1(A)}(\gamma) = 1$.

Set $g = ab$. Then $g{\cdot}\gamma(g){\cdot}\gamma^2(g) \in C_A(\gamma) = 1$, $[t, g] = 1$, and $t\gamma(g)t^{-1} = \gamma^2(g)$. Since $C_A(\gamma) = 1$, γ acts on the coset Ar fixing exactly one element h, and $c_t(h) = h$ since c_t normalizes $\langle \gamma \rangle$ in $\mathrm{Aut}_{\mathcal{F}}(A_+)$. Define $\varphi \in \mathrm{Aut}(S)$ by setting $\varphi(g) = a^{-1}b^{-1}$, $\varphi(\gamma(g)) = a$, $\varphi(\gamma^2(g)) = b$, $\varphi(h) = r$, and $\varphi(t) = t$. Upon replacing $\mathcal{F}$ by ${}^{\varphi}\mathcal{F}$ (and γ by ${}^{\varphi}\gamma$), we can assume that $\gamma = \sigma$, and hence that $\mathrm{Out}_{\mathcal{F}}(A_+) = \big\langle [\sigma], [c_t] \big\rangle$.

By Lemma 3.8, for $i = 1, 2$, $\mathrm{Aut}_{\mathcal{F}}(P_i) \cong \Sigma_3$ or $\Sigma_3 \times C_3$. In the latter case, by the extension lemma, there is an element of order 3 which extends to an element in $\mathrm{Aut}_{\mathcal{F}}(N_S(P_i))$, which is impossible since $\mathrm{Out}_{\mathcal{F}}(S) = 1$ and no essential subgroup contains $N_S(P_i)$. Thus $\mathrm{Aut}_{\mathcal{F}}(P_i) = \langle \mathrm{Aut}_S(P_i), \delta \rangle$ for some $\delta \in \mathrm{Aut}^*_{\mathcal{F}}(P_i)$ of order 3. By (4.2), $[\delta, P_i] = U_i$, so $\delta(U_i) = U_i$, and $\delta|_{C_S(U_i)} = \mathrm{Id}$ (and Lemma A.9). So $\delta \in \tau_i^{\pm 1}\mathrm{Aut}_{U_i}(P_i)$, and hence

$$i = 1, 2,\ n \ge 3 \quad\Longrightarrow\quad \mathrm{Aut}_{\mathcal{F}}(P_i) = \langle \mathrm{Aut}_S(P_i), \tau_i \rangle. \tag{4.3}$$

Case 1: Assume $\tau_i \in \mathrm{Aut}_{\mathcal{F}}(P_i)$ for $i = 1, 2$. Thus $\mathcal{F}$ is the fusion system over S generated by $\mathrm{Inn}(S)$, $\sigma \in \mathrm{Aut}(A_+)$, and the $\tau_i \in \mathrm{Aut}(U_iC_S(U_i))$.

Let q be a prime power such that $v_2(q^2 - 1) = n + 1$, and set $G = G_2(q)$ and $\overline{G} = G_2(\overline{\mathbb{F}}_q) > G$. For any $x \in I(G)$, $C_{\overline{G}}(x) \cong SL_2(\overline{\mathbb{F}}_q) \times_{C_2} SL_2(\overline{\mathbb{F}}_q)$: this follows, for example, from the description of centralizers in [**Ca2**, Theorem 3.5.3]. Hence $C_G(x)$ contains a subgroup $SL_2(q) \times_{C_2} SL_2(q)$ with index 2 (see, e.g., [**Kl**, Theorem A]), so the Sylow 2-subgroups of G contain $Q_{2^{n+1}} \times_{C_2} Q_{2^{n+1}} \cong Q$ with index 2, and are contained in $\widehat{S} = Q_{2^{n+2}} \times_{C_2} Q_{2^{n+2}}$. Fix generators $c_1, d_1, c_2, d_2 \in \widehat{S}$, where $\langle c_i, d_i \rangle \cong Q_{2^{n+2}}$, $|c_i| = 2^{n+1}$, and $z = c_1^{2^n} = c_2^{2^n} = d_1^2 = d_2^2 \in Z(\widehat{S})$, and define $\chi\colon S \longrightarrow \widehat{S}$ by setting

$$\chi(a) = c_1c_2, \qquad \chi(b) = c_1^{-1}c_2, \qquad \chi(r) = d_1d_2, \qquad \chi(t) = c_2^{2^{n-1}}d_1\,.$$

This defines an isomorphism from S onto some $T \in \mathrm{Syl}_2(G)$, and χ preserves fusion in P_1 and P_2 since $G > SL_2(q) \times_{C_2} SL_2(q)$. Since $\mathcal{F}_T(G)$ is reduced by Proposition 1.12, $\mathcal{F} \cong \mathcal{F}_T(G)$.

Case 2: Now assume $\tau_i \notin \mathrm{Aut}_{\mathcal{F}}(P_i)$ for $i = 1$ or 2. Thus $n = 2$ by (4.3), and $P_1 = P_2 = Q \cong 2^{1+4}_+$. Each automorphism of Q either normalizes the Δ_i or exchanges them, and an automorphism of odd order must normalize them. Hence $\mathrm{Aut}^*_{\mathcal{F}}(Q) \le \big\langle \tau_1, \tau_2, \mathrm{Inn}(Q) \big\rangle$. Since $S = \mathfrak{foc}(\mathcal{F}) = \big\langle \mathfrak{foc}(\mathcal{F}, A_+), \mathfrak{foc}(\mathcal{F}, Q) \big\rangle$ and $\mathfrak{foc}(\mathcal{F}, A_+) = A$, we must have $\mathfrak{foc}(\mathcal{F}, Q) = [\mathrm{Aut}^*_{\mathcal{F}}(Q), Q] = Q$. So $\mathrm{Out}_{\mathcal{F}}(Q)$ must be one of the groups $\langle [\tau_1], [\tau_2], [c_t] \rangle$, $\langle [\tau_1\tau_2], [c_t] \rangle$, or $\langle [\tau_1\tau_2^{-1}], [c_t] \rangle$, and we are assuming $\tau_i \notin \mathrm{Aut}_{\mathcal{F}}(Q)$. If $\mathrm{Out}_{\mathcal{F}}(Q) = \langle [\tau_1\tau_2], [c_t] \rangle$, then the subgroup $\langle a^2, b^2, r \rangle$ is normalized by $\mathrm{Aut}_{\mathcal{F}}(Q)$ and by $\mathrm{Aut}_{\mathcal{F}}(A_+)$, and hence by Lemma 1.15 is normal in $\mathcal{F}$. This is impossible since $\mathcal{F}$ is reduced, so $\mathrm{Out}_{\mathcal{F}}(Q) = \langle [\tau_1\tau_2^{-1}], [c_a] \rangle$.

By [**A2**, Lemma 5.3(2)], M_{12} contains as involution centralizer a split extension of 2^{1+4}_+ by Σ_3 where each of the Q_8 factors is normal. (Note that $S = Q \rtimes \langle ar \rangle$.) The fusion system of M_{12} is reduced by Proposition 1.12. Hence $\mathcal{F}$ is the fusion system of M_{12} in this case. □

It remains to look at fusion systems over a Sylow subgroup of $\mathrm{Aut}(M_{12})$.

PROPOSITION 4.3. *Set $S = S_{2,-1}$ with the presentation in Notation 4.1. Let $\mathcal{F}$ be the fusion system over S generated by* $\mathrm{Inn}(S)$, $\sigma \in \mathrm{Aut}(A_+)$, *and* $\tau_1\tau_2^{-1} \in \mathrm{Aut}(Q)$. *Then $\mathcal{F}$ is isomorphic to the fusion system of M_{12}. Define $\beta \in \mathrm{Aut}(S)$ by setting $\beta(a) = ab^2$, $\beta(b) = a^2b^{-1}$, $\beta(r) = r$, and $\beta(t) = rt$.*

(a) *The class of β generates* $\mathrm{Out}(S, \mathcal{F}) \cong C_2$.

(b) *Set $\widehat{S} = S\langle u\rangle$, where $ugu^{-1} = \beta(g)$ for $g \in S$, and $u^2 = 1$. Then $\widehat{S}$ is of type* $\mathrm{Aut}(M_{12})$.

(c) *There are no reduced fusion systems over $\widehat{S}$.*

PROOF. Set $G = M_{12}$. By the proof of Proposition 4.2(c), we can identify S as a Sylow 2-subgroup of G with $\mathcal{F} = \mathcal{F}_S(G)$.

(a) By direct computation, ${}^{\beta}\sigma = \sigma$, and ${}^{\beta}\tau_i \equiv \tau_{3-i}$ (mod $\mathrm{Inn}(Q)$) for $i = 1, 2$. Hence $\beta \in \mathrm{Aut}(S, \mathcal{F})$. Also, $\beta(\Delta_1) = \Delta_2$ and $\beta(\Delta_2) = \Delta_1$.

Assume $\varphi \in \mathrm{Aut}(S)$ is fusion preserving. Then $\varphi(A) = A$ since $A < S$ is the unique abelian subgroup of index 4 (Lemma A.4(b)). By (4.1) in the proof of Proposition 4.2, either φ normalizes the subgroups $\Delta_i \cong Q_8$, or it exchanges them.

Assume $\varphi(\Delta_i) = \Delta_i$ for $i = 1, 2$. After composing with inner automorphisms, we can assume that $\varphi(ab) = ab$ and $\varphi(ab^{-1}) = ab^{-1}$. Also, $\varphi(r) = r$ (the unique involution in $A_+ = A\langle r\rangle$ fixed by σ), and φ sends the σ-orbit of ab to itself. Since $\varphi|_{\langle a^2,b^2\rangle} = \mathrm{Id}$, this proves that $\varphi|_{A_+} = \mathrm{Id}$. Finally, since $\varphi(\Delta_1) = \Delta_1$, $\varphi(t) = a^ib^jt$ for some i, j, $4|(i+j)$ since $\varphi(a^2t) = a^{i+2}b^jt \in \Delta_1$, and $2|i$ since $[r, \varphi(t)] = 1$. Upon replacing φ by $c_{b^i} \circ \varphi$, we can arrange that $\varphi = \mathrm{Id}$.

Thus $\varphi \in \mathrm{Aut}(S, \mathcal{F})$ and $\varphi(\Delta_i) = \Delta_i$ ($i = 1, 2$) imply that $\varphi \in \mathrm{Inn}(S)$. So $\mathrm{Out}(S, \mathcal{F})$ has order 2 and is generated by the class of β.

(b) Set $H = N_G(A)$. Then $H = AK$, where $K = \langle r, t, s\rangle \cong D_{12}$, $|s| = 3$, $[s, r] = 1$, ${}^ts = s^{-1}$, and ${}^sg = \sigma(g)$ for $g \in A$ (see, e.g., [**Gd2**, (3.8)]).

By [**A2**, Lemma 5.9(1,3)], $|\mathrm{Out}(G)| = 2$, and no $\alpha \in \mathrm{Aut}(G)\smallsetminus\mathrm{Inn}(G)$ centralizes S. Hence by (a), $\mathrm{Aut}(G) \cong G\langle v\rangle$ for some v such that $c_v|_S = \beta$ and (since $\beta^2 = \mathrm{Id}_S$) $v^2 \in Z(S)$. Then ${}^vA = A$, so ${}^vH = H$. By Lemma A.7, applied with H, A, K, and $\langle r, t\rangle$ in the role of G, Q, H, and H_0, ${}^v\langle s\rangle = {}^g\langle s\rangle$ for some $g \in C_A(r, t) = \langle a^2b^2\rangle$. So upon replacing v by a^2b^2v if necessary, we can assume that ${}^v\langle s\rangle = \langle s\rangle$, hence that $[v^2, H] = 1$, and so $v^2 = 1$. Thus $S\langle v\rangle \in \mathrm{Syl}_2(\mathrm{Aut}(G))$ is isomorphic to $\widehat{S}$.

(c) By Lemma A.4(b) and the commutator relations in $\widehat{S}$ (and since $|\widehat{S}/[\widehat{S}, \widehat{S}]| = 8$), $Q/Z(\widehat{S})$ is the only abelian subgroup of rank 4 in $\widehat{S}/Z(\widehat{S})$, and so Q is the only extraspecial subgroup of order 2^5 in $\widehat{S}$. Also, $Z(\widehat{S}) = \langle a^2b^2\rangle$, $Z_2(\widehat{S}) = \langle a^2, b^2\rangle$, and $Z_3(\widehat{S}) = \langle a^2, b^2, ab, r\rangle$. By Lemma A.2(b), each normal subgroup of order 8 in $\widehat{S}$ contains $Z_2(\widehat{S})$ and is contained in $Z_3(\widehat{S})$, hence is abelian, and thus $\mathscr{X}(\widehat{S}) = \varnothing$. Also, $\mathscr{Y}(\widehat{S}) = \varnothing$ since $|\widehat{S}| = 2^7$ and $\widehat{S} \not\cong D_8 \wr C_2$.

Assume there is $T < \widehat{S}$ such that $T \cong UT_3(4)$. Then $Z(T) \trianglelefteq \widehat{S}$ implies $Z(T) = Z_2(S) = \langle a^2, b^2\rangle$ (Lemma A.2(b) again), so $T = C_{\widehat{S}}(\langle a^2, b^2\rangle) = \langle a, b, r, u\rangle$. Since $I(UT_3(4)) = A_1^{\#} \cup A_2^{\#}$ where $A_i \cong C_2^4$ and $A_1 \cap A_2 = Z(T)$ (Lemma C.6(a)), a subgroup of index 2 in $UT_3(4)$ has at most 19 involutions, and has exactly 19 only if it contains A_1 or A_2. Since $|I(\langle a, b, r\rangle)| = |Z(T)^{\#}| + |rA| = 19$, and since $r(\langle a, b, r\rangle) = 3$, this proves that $T \not\cong UT_3(4)$, and hence that $\widehat{S} \notin \mathcal{U}$.

Let $\mathcal{F}$ be a reduced fusion system over $\widehat{S}$. Since $\mathscr{X}(\widehat{S}) = \varnothing = \mathscr{Y}(\widehat{S})$ and $\widehat{S} \notin \mathcal{U}$, Theorem 3.1 implies that each $\mathcal{F}$-essential subgroup has index 2 in $\widehat{S}$.

If $\gamma \in \mathrm{Aut}_{\mathcal{F}}(\widehat{S})$ has odd order, then $\gamma(Q) = Q$, and $[\gamma|_Q] \in \mathrm{Out}(Q)$ normalizes $\mathrm{Out}_{\widehat{S}}(Q) \cong C_2^2$. Since $\mathrm{Out}(Q) \cong \Sigma_3 \wr C_2$, this implies that $[\gamma|_Q] = 1$, and hence that $\gamma|_Q = \mathrm{Id}$. Since $C_{\widehat{S}}(Q) \leq Q$, γ induces the identity on $\widehat{S}/Q$, and hence $\gamma = \mathrm{Id}_{\widehat{S}}$ by Lemma A.9. Thus $\mathrm{Out}_{\mathcal{F}}(\widehat{S}) = 1$.

Assume $P < \widehat{S}$ is $\mathcal{F}$-essential of index 2; thus $P \geq \mathrm{Fr}(\widehat{S}) = \langle ab, a^2, r \rangle$. If $P > Q$, then $\mathfrak{foc}(\mathcal{F}, P) \leq Q$. If $P > A$ and $P \ngeq Q$, then either $P = A\langle r, u \rangle \cong A \rtimes C_2^2$ or $P = A\langle tu \rangle \cong A \rtimes C_4$, and in either case, $\mathfrak{foc}(\mathcal{F}, P) \leq A$. This leaves the two subgroups

$$Q_1 = \langle at, ab, a^2, r, u \rangle \qquad [Q_1, Q_1] = \langle a^2, b^2, r \rangle \qquad Q_1/[Q_1, Q_1] \cong C_4 \times C_2$$

$$Q_2 = \langle at, ab, a^2, r, tu \rangle \qquad [Q_1, Q_1] = \langle a^2, b^2, abr \rangle \qquad Q_2/[Q_2, Q_2] \cong C_4 \times C_2.$$

By Corollary A.10(a), $\mathrm{Aut}(Q_i)$ is a 2-group for $i = 1, 2$, so neither can be $\mathcal{F}$-essential.

Thus $\mathfrak{foc}(\mathcal{F}) \leq AQ = S < \widehat{S}$, so by Proposition 1.14(b), $\mathcal{F}$ is not reduced. □

CHAPTER 5

Dihedral and semidihedral wreath products

We next study reduced fusion systems over 2-groups in $\mathcal{V}$. These are the groups where the set of subgroups $\mathscr{Y}(S)$ plays a central role. The one exception to this is the case $S = UT_4(2)$ (where $\mathscr{Y}(S) = \varnothing$) which we handle first.

Note, in the statement of the following proposition, that $PSU_4(2) \cong PSp_4(3)$ (cf. [**Ta**, Corollary 10.19]).

PROPOSITION 5.1. *Assume $S = UT_4(2)$. Let $\mathcal{F}$ be a fusion system over S such that $O_2(\mathcal{F}) = 1$. Then $\mathcal{F}$ is isomorphic either to the fusion system of $GL_4(2)$, or to that of $PSp_4(q)$ for each $q \equiv \pm 3 \pmod 8$.*

PROOF. By Lemma C.4(a,b), there are unique subgroups $Q, A < S$ such that $Q \cong 2_+^{1+4}$ and $A \cong C_2^4$. Set $Z = \langle z \rangle = Z(S)$. Let $\Delta_1, \Delta_2 \trianglelefteq Q$ be the unique subgroups isomorphic to Q_8; thus $[\Delta_1, \Delta_2] = 1$ and $\Delta_1 \cap \Delta_2 = Z$.

Define automorphisms

$$\gamma \in \mathrm{Aut}(S), \qquad \varphi \in Z(\mathrm{Aut}(S)), \qquad \text{and} \qquad \tau_1, \tau_2 \in \mathrm{Aut}(Q)$$

as follows. Fix some $t \in A \smallsetminus Q$; then ${}^t\Delta_1 = \Delta_2$ by Lemma C.4(b). Choose any $\tau_1 \in \mathrm{Aut}(Q)$ which acts with order 3 on Δ_1 and via the identity on Δ_2, and set $\tau_2 = c_t \tau_1 c_t^{-1} \in \mathrm{Aut}(Q)$. Then c_t commutes with $\tau_1\tau_2 = \tau_2\tau_1$, so there is $\gamma \in \mathrm{Aut}(S)$ of order 3 such that $\gamma|_Q = \tau_1\tau_2$ and $\gamma(t) = t$. Finally, set $\varphi(g) = g$ if $g \in Q$ and $\varphi(g) = zg$ if $g \in S \smallsetminus Q$. Then $\varphi \in Z(\mathrm{Aut}(S))$ since $\alpha(Q) = Q$ for each $\alpha \in \mathrm{Aut}(S)$.

Let $B_1, B_2, B_3 < S$ be the three subgroups of index 2 which contain A. Since $S \cong C_2 \wr C_2^2$ by Lemma C.4(a), $B_i \cong C_2^2 \wr C_2$ for each i. Choose $v_i \in I(B_i \smallsetminus A)$, and set $V_i = C_{B_i}(v_i) = \langle v_i \rangle \times C_A(v_i) \cong C_2^3$. Note that $V_i = \langle I(B_i \smallsetminus A) \rangle$, so V_i is independent of the choice of v_i, and is characteristic in B_i since A is. By Lemma C.4(d), γ permutes the B_i transitively, and hence also permutes the V_i transitively.

We claim that

$$\tau_1\tau_2^{-1}(V_i) = V_i \qquad \text{for } i = 1, 2, 3. \tag{5.1}$$

To see this, let $\mathscr{V}$ be the set of all $V < Q$ such that $V \cong C_2^3$. Each involution in Q has the form g_1g_2 where $g_i \in \Delta_i \smallsetminus Z$. Hence for each $V \in \mathscr{V}$, there is $\chi \in \mathrm{Iso}(\Delta_1, \Delta_2)$ such that $V = V_\chi \stackrel{\text{def}}{=} \langle z, g\chi(g) \,|\, g \in \Delta_1 \rangle$. Also, $V_\chi = V_{\chi'}$ if $\chi' \in \chi{\cdot}\mathrm{Inn}(\Delta_1)$. Hence $|\mathscr{V}| = |\mathrm{Out}(Q_8)| = 6$, each of the automorphisms τ_1 and τ_2 permutes $\mathscr{V}$ freely, and $\langle \tau_1, \tau_2 \rangle \cong C_3 \times C_3$ permutes it in two orbits of length 3. Since γ permutes the set $\{V_1, V_2, V_3\} \subseteq \mathscr{V}$ freely, it must be one of the orbits, is permuted freely by τ_1, τ_2, and $\tau_1\tau_2 = \gamma|_Q$, and hence is pointwise fixed by $\tau_1\tau_2^{-1}$. This proves (5.1).

We next claim that

$$\text{there are exactly two bases } \mathcal{B}_1, \mathcal{B}_2 \subseteq A \text{ which are permuted freely by } \mathrm{Aut}_S(A) \cong C_2^2, \text{ and } \varphi(\mathcal{B}_1) = \mathcal{B}_2. \tag{5.2}$$

By Lemma C.4(a), there is at least one such basis $\mathcal{B}_1$, and $\mathcal{B}_1 \subseteq A \smallsetminus Q$ since it is normalized by S and generates A. Thus $\varphi(b) = bz$ for each $b \in \mathcal{B}_1$. If $\varphi(\mathcal{B}_1) = \mathcal{B}_1$, then $\mathcal{B}_1$ is the union of two cosets of Z, which is impossible since $\langle \mathcal{B}_1 \rangle = A$. Thus $\mathcal{B}_1$ and $\varphi(\mathcal{B}_1)$ are distinct orbits of $\mathrm{Aut}_S(A)$, so $\mathcal{B}_1 \cup \varphi(\mathcal{B}_1) = A \smallsetminus Q$, and these are the only such bases.

Let $\mathcal{F}$ be a saturated fusion system over S such that $O_2(\mathcal{F}) = 1$. By points (b.1)–(b.3) in Theorem 3.1, if $R \in \mathbf{E}_{\mathcal{F}}^{(\mathrm{II})}$, then either $\mathfrak{foc}(\mathcal{F}, R) \in \mathscr{X}(S) \cup \mathscr{Y}(S)$ or $S \in \mathcal{U}$. By Lemma C.4(e), $\mathscr{X}(S) = \varnothing$. Also, $\mathscr{Y}(S) = \varnothing$ since $|S| < |D_8 \wr C_2| = 2^7$, and $S \notin \mathcal{U}$ ($S \not\cong UT_3(4)$) since $A < S$ is the unique subgroup isomorphic to C_2^4. Thus $\mathbf{E}_{\mathcal{F}}^{(\mathrm{II})} = \varnothing$.

By Theorem 3.1(a), if $R \in \mathbf{E}_{\mathcal{F}}^{(\mathrm{I})}$, then $R \cong C_2^4$ or 2_-^{1+4}. Since $[S{:}R] \geq 4$, this implies $\mathbf{E}_{\mathcal{F}}^{(\mathrm{I})} \subseteq \{A\}$. By Lemma C.4(c), the only subgroups of index 2 in S whose automorphism groups are not 2-groups are Q and the B_i. Thus $\mathbf{E}_{\mathcal{F}} \subseteq \{A, Q, B_1, B_2, B_3\}$.

For $i = 1, 2, 3$, $\mathrm{Out}_{\mathcal{F}}(B_i) \cong C_{\mathrm{Aut}_{\mathcal{F}}(A)}(c_{v_i})/\langle c_{v_i} \rangle$ by Lemma 1.5(a). Also, since $\mathrm{Out}_S(B_i) \in \mathrm{Syl}_2(\mathrm{Out}_{\mathcal{F}}(B_i))$ and $|\mathrm{Out}_S(B_i)| = 2$, $\mathrm{Out}_{\mathcal{F}}(B_i)$ has a strongly embedded subgroup if and only if $O_2(\mathrm{Out}_{\mathcal{F}}(B_i)) = 1$. Hence

$$\text{for } i = 1, 2, 3, \;\; B_i \in \mathbf{E}_{\mathcal{F}} \iff O_2(C_{\mathrm{Aut}_{\mathcal{F}}(A)}(c_{v_i})) = \langle c_{v_i} \rangle . \tag{5.3}$$

If $Q \notin \mathbf{E}_{\mathcal{F}}$, then all $\mathcal{F}$-essential subgroups and S contain A as a characteristic subgroup, so $A \trianglelefteq \mathcal{F}$ by Lemma 1.15, and $O_2(\mathcal{F}) \neq 1$. Thus $Q \in \mathbf{E}_{\mathcal{F}}$. The images of $\langle \tau_1\tau_2 \rangle$ and $\langle \tau_1\tau_2^{-1} \rangle$ in $\mathrm{Out}(Q) \cong \Sigma_3 \wr C_2$ are the only subgroups of order 3 normalized by $\mathrm{Out}_S(Q)$. Since $\langle [\tau_1\tau_2], \mathrm{Out}_S(Q) \rangle \cong C_6$ while $\langle [\tau_1\tau_2^{-1}], \mathrm{Out}_S(Q) \rangle \cong \Sigma_3$, $\mathrm{Out}_{\mathcal{F}}(Q)$ must be equal to $\langle [\tau_1\tau_2^{-1}], \mathrm{Out}_S(Q) \rangle \cong \Sigma_3$ or to $\langle [\tau_1], [\tau_2], \mathrm{Out}_S(Q) \rangle \cong C_3 \times \Sigma_3$.

If $\tau_1\tau_2 \in \mathrm{Aut}_{\mathcal{F}}(Q)$, then since it normalizes $\mathrm{Aut}_S(Q)$, it extends to some $\gamma' \in \mathrm{Aut}_{\mathcal{F}}(S)$ by the extension axiom. Also, $\gamma' \in \{\gamma, \varphi\gamma\}$, since $\alpha \in \mathrm{Aut}(S)$ and $\alpha|_Q = \mathrm{Id}_Q$ imply $\alpha \in \langle \varphi \rangle$. If $\gamma' = \varphi\gamma$, then since $\varphi \in Z(\mathrm{Aut}(S))$, $(\gamma')^3 = \varphi \in \mathrm{Aut}_{\mathcal{F}}(S)$, which contradicts the Sylow axiom. Thus $\gamma' = \gamma$. Since Q is characteristic in S, restriction induces an isomorphism $\mathrm{Out}_{\mathcal{F}}(S) \cong C_{\mathrm{Out}_{\mathcal{F}}(Q)}(\mathrm{Out}_S(Q))/\mathrm{Out}_S(Q)$ by Lemma 1.5(a), so $\mathcal{F}$ has either

Type (1): $\mathrm{Out}_{\mathcal{F}}(Q) = \big\langle [\tau_1\tau_2^{-1}], \mathrm{Out}_S(Q) \big\rangle \cong \Sigma_3$ and $\mathrm{Out}_{\mathcal{F}}(S) = 1$; or

Type (2): $\mathrm{Out}_{\mathcal{F}}(Q) = \big\langle [\tau_1], [\tau_2], \mathrm{Out}_S(Q) \big\rangle \cong C_3 \times \Sigma_3$ and $\mathrm{Out}_{\mathcal{F}}(S) = \big\langle [\gamma] \big\rangle \cong C_3$.

If $\mathcal{F}$ has type (1), then $N_{\mathrm{Aut}_{\mathcal{F}}(A)}(\mathrm{Aut}_S(A))/\mathrm{Aut}_S(A) \cong \mathrm{Out}_{\mathcal{F}}(S) = 1$ by Lemma 1.5(a). Thus $\mathrm{Aut}_{\mathcal{F}}(A)$ does not contain a subgroup isomorphic to A_5, so $A \notin \mathbf{E}_{\mathcal{F}}$ by Lemma 3.3(b,c). If $\mathbf{E}_{\mathcal{F}} = \{Q\}$, then $Q \trianglelefteq \mathcal{F}$. If $\mathbf{E}_{\mathcal{F}} = \{Q, B_i\}$ for some $i = 1, 2, 3$, then since $V_i \trianglelefteq S$ is characteristic in B_i (as shown above), each automorphism of B_i sends V_i to itself. Also, $\tau_1\tau_2^{-1}(V_i) = V_i$ by (5.1), so $V_i \trianglelefteq \mathcal{F}$ in this case.

Since $O_2(\mathcal{F}) = 1$, this shows that at least two of the B_j must be in $\mathbf{E}_{\mathcal{F}}$. Upon replacing $\mathcal{F}$ by ${}^{\gamma^i}\mathcal{F}$ for appropriate i, we can arrange that $B_1, B_2 \in \mathbf{E}_{\mathcal{F}}$. Then $O_2(C_{\mathrm{Aut}_{\mathcal{F}}(A)}(c_{v_i})) = \langle c_{v_i} \rangle$ for $i = 1, 2$ by (5.3), so by Proposition D.1(e.1), $\mathrm{Out}_{\mathcal{F}}(A) \cong \Sigma_3 \times \Sigma_3$. Then $c_{v_3} = c_{v_1v_2} \in \mathrm{Aut}(A)$ inverts $O_3(\mathrm{Aut}_{\mathcal{F}}(A)) \cong C_3 \times C_3$, so $C_{\mathrm{Aut}_{\mathcal{F}}(A)}(c_{v_3}) = \mathrm{Aut}_S(A)$, and $B_3 \notin \mathbf{E}_{\mathcal{F}}$ by (5.3) again.

Now, $O_3(\mathrm{Aut}_{\mathcal{F}}(A)) \cong C_3 \times C_3$ is determined by a choice of two complementary subgroups $W_1, W_2 < A$ of rank 2, and since $O_3(\mathrm{Aut}_{\mathcal{F}}(A))$ is normalized by $\mathrm{Aut}_S(A)$, each c_{v_i} ($i = 1, 2, 3$) either normalizes the W_i or exchanges them. Since c_{v_3} inverts $O_3(\mathrm{Aut}_{\mathcal{F}}(A))$, there is $i \in \{1, 2\}$ such that $\{W_1, W_2\}$ is the pair $\{\langle \mathcal{B}_i' \rangle, \langle \mathcal{B}_i'' \rangle\}$, where

$\mathcal{B}_i', \mathcal{B}_i'' \subseteq \mathcal{B}_i$ are the two $\langle c_{g_3}\rangle$-orbits. Thus $\mathrm{Aut}_{\mathcal{F}}(A)$ is determined by $\mathcal{B}_i$, and upon replacing $\mathcal{F}$ by ${}^{\varphi}\mathcal{F}$, if necessary, we can assume that $i = 1$.

By Lemma 1.5(b), $\mathrm{Aut}_{\mathcal{F}}(B_i)$ is determined by $\mathrm{Aut}_{\mathcal{F}}(A)$ in all cases. So $\mathcal{F}$ is uniquely determined by the choice of $\mathrm{Aut}_{\mathcal{F}}(A)$, which we just saw is determined by the choice of basis $\mathcal{B}_1$.

If $\mathcal{F}$ has type (2), then by Proposition D.1(e.2) (and since $\mathrm{Out}_S(A)$ permutes a basis for A), $\mathrm{Out}_{\mathcal{F}}(A) \cong A_4$ or A_5. Then $B_i \notin \mathbf{E}_{\mathcal{F}}$ for $i = 1, 2, 3$ by (5.3), so $\mathbf{E}_{\mathcal{F}} \subseteq \{A, Q\}$, with equality since otherwise $Q \trianglelefteq \mathcal{F}$. Hence $\mathrm{Aut}_{\mathcal{F}}(A) \cong A_5$ by Lemma 3.3(b), and A is the orthogonal module since $\mathrm{rk}(C_A(S)) = 1$ (Proposition D.1(d)).

Thus the action of $\mathrm{Aut}_{\mathcal{F}}(A)$ has an orbit $\mathcal{B}^*$ of length 5, where $\mathrm{Aut}_S(A) \cong C_2^2$ acts on $\mathcal{B}^*$ with one orbit of length 4 and one fixed point z. Since the action is irreducible, $\langle \mathcal{B}^* \rangle = A$ and $\prod_{g \in \mathcal{B}^*} g \in C_A(\mathrm{Aut}_{\mathcal{F}}(A)) = 1$. So $\mathcal{B}^* \smallsetminus \{z\}$ is a basis for A, and hence equal to $\mathcal{B}_1$ or $\mathcal{B}_2$ by (5.2). Upon replacing $\mathcal{F}$ by ${}^{\varphi}\mathcal{F}$ if necessary, we can assume that $\mathcal{B}^* = \mathcal{B}_1 \cup \{z\}$, and hence that $\mathrm{Aut}_{\mathcal{F}}(A)$ acts as the group of all even permutations of this set.

Both types: We have now shown that up to isomorphism, there are at most two saturated fusion systems $\mathcal{F}_1$ and $\mathcal{F}_2$ with $O_2(\mathcal{F}) = 1$, one of each type (1) and (2), respectively. Set $G_1 = GL_4(2)$, set $G_2 = PSp_4(q)$ for any prime power $q \equiv \pm 3 \pmod 8$, and choose $S_i \in \mathrm{Syl}_2(G_i)$. We can take $S_1 = UT_4(2) \cong S$. Let $Q < S_1$ be the subgroup of triangular matrices with zero in the entry $(2, 3)$; then $Q \cong 2_+^{1+4}$ and $N_{G_2}(Q)/Q \cong \Sigma_3$, so $\mathcal{F}_{S_1}(G_1) \cong \mathcal{F}_1$ has type (1).

By [**CF**, § 1], $Sp_4(q)$ has Sylow 2-subgroups isomorphic to $Q_8 \wr C_2$, so $S_2 \cong (Q_8 \times_{C_2} Q_8) \overset{t}{\rtimes} C_2 \cong S$. By Proposition 1.12(b), $O_2(\mathcal{F}_{S_2}(G_2)) = 1$. Furthermore, G_2 contains $(Sp_2(q) \times_{C_2} Sp_2(q)) \overset{t}{\rtimes} C_2$, $Sp_2(q) \cong SL_2(q)$ contains a subgroup $Q_8 \rtimes C_3$, and so $\mathcal{F}_{S_2}(G_2) \cong \mathcal{F}_2$ has type (2). □

Before continuing with the other cases, we need a lemma which helps to make more explicit how we apply the results shown in Section 3.2. Recall that for a saturated fusion system $\mathcal{F}$ over S and $Y \trianglelefteq S$, $\mathbf{E}_{\mathcal{F}}(Y)$ denotes the set of all $\mathcal{F}$-essential subgroups $R < S$ such that $\mathfrak{foc}(\mathcal{F}, R) = Y$.

LEMMA 5.2. *Let $\mathcal{F}$ be a saturated fusion system over a 2-group S such that $r(S) \leq 4$ and $\mathscr{Y}(S) \neq \varnothing$. Fix $Y \in \mathscr{Y}(S)$, and assume that $\mathbf{E}_{\mathcal{F}}(Y) \neq \varnothing$. Let $\mathscr{Y}_0$ be the set of all $Y_0 \in \mathscr{Y}_0(S)$ whose normal closure is Y. Let $\Theta_1, \Theta_2 \trianglelefteq Y$ and*

$$\mathscr{U}_{\mathcal{F}}(Y) = \{U \leq \Theta_i \mid i = 1, 2,\ U \cong C_2^2 \text{ or } Q_8\}.$$

be as in Proposition 3.11(a). Then the following hold.

(a) $\mathbf{E}_{\mathcal{F}}(Y) = \mathbf{E}^a_{\mathcal{F}}(Y) \cup \mathbf{E}^c_{\mathcal{F}}(Y)$, *where*

- $\mathbf{E}^a_{\mathcal{F}}(Y) = \bigl\{Y_0\langle g\rangle \bigm| Y_0 \in \mathscr{Y}_0,\ \mathrm{Aut}_{\mathcal{F}}(Y_0) \cong \Sigma_3 \wr C_2,\ g \in N_S(Y_0),$ $g^2 \in Y_0,\ c_g \text{ exchanges } Y_0 \cap \Theta_1 \text{ and } Y_0 \cap \Theta_2\bigr\}$
- $\mathbf{E}^c_{\mathcal{F}}(Y) = \bigl\{UC_S(U) \bigm| U \in \mathscr{U}_{\mathcal{F}}(Y)\bigr\}$.

(b) *For each $R \in \mathbf{E}_{\mathcal{F}}(Y)$, $R \geq Y_0$ for some $Y_0 \in \mathscr{Y}_0$.*

(c) *If $P = Y_0\langle g\rangle \in \mathbf{E}^a_{\mathcal{F}}(Y)$, and $\Gamma \leq \mathrm{Aut}(P)$ is such that $\mathrm{Aut}_S(P) \in \mathrm{Syl}_2(\Gamma)$ and $\bigl\{\gamma|_{Y_0} \bigm| \gamma \in \Gamma\bigr\} = N_{\mathrm{Aut}_{\mathcal{F}}(Y_0)}(\mathrm{Aut}_P(Y_0))$, then $\Gamma = \mathrm{Aut}_{\mathcal{F}}(P)$.*

PROOF. We first claim that

$$P > Y_0 \in \mathscr{Y}_0,\ |P/Y_0| = 2,\ \mathrm{rk}([P, Y_0/\mathrm{Fr}(Y_0)]) = 2 \quad \Longrightarrow \quad Y_0 \text{ char. } P\,. \tag{5.4}$$

In all cases, $\mathrm{Fr}(Y_0)$ is characteristic in P: either $\mathrm{Fr}(Y_0) = 1$, or $\mathrm{Fr}(Y_0) = Z(P)$; or $Y_0 \cong Q_8 \times Q_8$, $|Z(P)| = 2$, $Y_0/Z(P) \cong 2_+^{1+4}$, and hence $\mathrm{Fr}(Y_0) = Z_2(P)$. Since $Y_0/\mathrm{Fr}(Y_0)$ is the unique abelian subgroup of index 2 in $P/\mathrm{Fr}(Y_0)$ by Lemma A.4(a), Y_0 is characteristic in P.

(a) $\mathbf{E}_{\mathcal{F}}(Y) \subseteq \mathbf{E}^{a}_{\mathcal{F}}(Y) \cup \mathbf{E}^{c}_{\mathcal{F}}(Y)$. By Proposition 3.9(b) and since $Y \in \mathscr{Y}(S)$, $\mathbf{E}_{\mathcal{F}}(Y) = \mathbf{E}_{\mathcal{F}}^{(\mathrm{II})}(Y)$, and each $\mathcal{F}$-essential pair (P_1, P_2) of type (II) in $\mathbf{E}_{\mathcal{F}}(Y)$ has the form described in Lemma 3.7(a) or in Lemma 3.8(b). Set $P_{12} = P_1 \cap P_2$.

Case 1: Assume (P_1, P_2) is as in Lemma 3.7(a). By that lemma, $P_{12} \in \mathscr{Y}_0$, $\mathrm{Out}_{\mathcal{F}}(P_{12}) \cong \Sigma_3 \wr C_2$ or Σ_5, and $\mathrm{Out}_{P_1}(P_{12}) \nleq O^2(\mathrm{Out}_{\mathcal{F}}(P_{12}))$. If $\mathrm{Out}_{\mathcal{F}}(P_{12}) \cong \Sigma_5$, then $P_{12}/\mathrm{Fr}(P_{12})$ is the orthogonal module for $\mathrm{Out}_{\mathcal{F}}(P_{12})$. Since $\mathrm{Out}_{P_1}(P_{12}) \nleq O^2(\mathrm{Out}_{\mathcal{F}}(P_{12}))$, $\mathrm{Out}_{P_1}(P_{12})$ is generated by a transposition in Σ_5. Thus

$$\mathrm{Out}_{\mathcal{F}}(P_{12}) \cong \Sigma_5 \quad \Longrightarrow \quad \mathrm{rk}([P_1, P_{12}/\mathrm{Fr}(P_{12})]) = 1\,. \tag{5.5}$$

Set $U_i = P_{12} \cap \Theta_i$ $(i = 1, 2)$. By Proposition 3.11(b.2), $\{U_1, U_2\} \in \mathscr{U}_S(P_{12})$, so $P_{12} = U_1U_2$, $U_1 \cap U_2 \leq \mathrm{Fr}(P_{12})$, and each element of $\mathrm{Out}_S(P_{12}) \cong D_8$ either normalizes the U_i or exchanges them. So we can choose bases $\{b_{i1}, b_{i2}\}$ of U_i $(i = 1, 2)$ such that $\mathcal{B} = \{b_{ij} \,|\, i, j = 1, 2\}$ is a basis of $P_{12}/\mathrm{Fr}(P_{12})$ permuted by $\mathrm{Out}_S(P_{12})$. Then one of the following happens:

- The action of $\mathrm{Out}_{P_1}(P_{12})$ exchanges U_1 and U_2. In particular, a generator of this group acts on $\mathcal{B}$ as a product of two disjoint 2-cycles, so $\mathrm{Out}_{\mathcal{F}}(P_{12}) \ncong \Sigma_5$ by (5.5). Thus $\mathrm{Out}_{\mathcal{F}}(P_{12}) \cong \Sigma_3 \wr C_2$, and $P_1 \in \mathbf{E}^{a}_{\mathcal{F}}(Y)$.
- The action of $\mathrm{Out}_{P_1}(P_{12})$ normalizes each U_i. Since $\mathrm{Out}_{P_1}(P_{12}) \cong C_2$ is noncentral in $\mathrm{Out}_S(P_{12}) \cong D_8$ (since $|N_S(P_1)/P_1| = 2$), it acts on $\mathcal{B}$ as a 2-cycle, and there is exactly one of the groups $U \in \{U_1, U_2\}$ for which $\mathrm{Out}_{P_1}(P_{12})$ acts trivially on $U/\mathrm{Fr}(U) \cong C_2^2$. Then $[P_1, U] \leq \mathrm{Fr}(U)$, so $\mathrm{Aut}_{P_1}(U) \leq \mathrm{Inn}(U)$, and $P_1 \leq UC_S(U)$. If $P_1 < UC_S(U)$, then by Lemma A.1(a), there is $g \in N_S(P_1) \smallsetminus P_1$ with $g \in C_S(U)$. Since $N_S(P_{12})/\mathrm{Fr}(P_{12}) \cong D_8 \wr C_2$ $(P_{12} \in \mathscr{Y}_0)$ and $P_1 < N_S(P_{12})$, there is also $h \in N_{N_S(P_{12})}(P_1) \smallsetminus P_1$ which does not centralize U. Since $|N_S(P_1)/P_1| = 2$, this is impossible, so $P_1 = UC_S(U) \in \mathbf{E}^{c}_{\mathcal{F}}(Y)$.

Case 2: Now assume (P_1, P_2) has the form described in Lemma 3.8(b). By Lemma 3.8(b,d,e), $P_1 = UC_S(U)$ where $U = [\mathrm{Aut}^*_{\mathcal{F}}(P_1), P_1] \cong C_2^2$ or Q_8. Thus $\mathrm{Aut}_{\mathcal{F}}(U) = \mathrm{Aut}(U)$, and $Y = \mathfrak{foc}(\mathcal{F}, P_1)$ is the normal closure of U in S. By the same lemma, $Y = \Delta\Delta^*$ where $\{\Delta, \Delta^*\}$ is an S-conjugacy class, $\Delta, \Delta^* \in \mathcal{DQ}$, $U \leq \Delta$, $[\Delta, \Delta^*] \leq \Delta \cap \Delta^* \leq Z(S)$, and $\Delta \cap \Delta^* = 1$ if $\Delta, \Delta^* \in \mathcal{D}$. Also, Δ and Δ^* are strongly automized in S, since Δ is the normal closure of U in a certain subgroup S_* of index 2 in S.

We must show that $U \in \mathscr{U}_{\mathcal{F}}(Y)$ (i.e., that $U \leq \Theta_i$ for $i = 1$ or 2). Choose $g \in S$ such that ${}^g\Delta = \Delta^*$, and set $U^* = {}^gU$ and $Y_0 = UU^*$. If $Y \in \mathscr{Y}_0$, then $U = \Delta$ and $U^* = \Delta^*$, so $Y_0 = UU^* = Y$. If $Y \notin \mathscr{Y}_0$, then $\Delta \cong D_{2^n}$ for $n \geq 3$ or Q_{2^n} for $n \geq 4$, and $Y_0 \in \mathscr{Y}_0$ by Lemma 2.6(a).

In either case, $Y_0 \in \mathscr{Y}_0$, and $\{U, U^*\} = \{Y_0 \cap \Delta, Y_0 \cap \Delta^*\} \in \mathscr{U}_S(Y_0)$ is an $N_S(Y_0)$-conjugacy class. Let $\alpha \in \mathrm{Aut}^*_{\mathcal{F}}(P_1)$ be of odd order. Since $U = [\mathrm{Aut}^*_{\mathcal{F}}(P_1), P_1]$ and

$U \le Y_0 \le P_1$, α normalizes U and Y_0, and $\alpha|_U \in \mathrm{Aut}_{\mathcal{F}}(U)$ has order 3. So $\{U,U^*\} \in \mathscr{U}_S(Y_0)$ is the unique element compactible with $\mathrm{Out}_{\mathcal{F}}(Y_0)$ (unique by Lemma 2.9(b)), and hence $U \in \mathscr{U}_{\mathcal{F}}(Y)$ by Proposition 3.11(b).

$\mathbf{E}_{\mathcal{F}}(Y) \supseteq \mathbf{E}^{a}_{\mathcal{F}}(Y) \cup \mathbf{E}^{c}_{\mathcal{F}}(Y)$. By Proposition 3.11(c.2), $\mathbf{E}^{c}_{\mathcal{F}}(Y) \subseteq \mathbf{E}_{\mathcal{F}}(Y)$.

Assume $P \in \mathbf{E}^{a}_{\mathcal{F}}(Y)$. Thus $P = Y_0\langle g\rangle$ where $Y_0 \in \mathscr{Y}_0$, c_g exchanges the two subgroups $Y_0 \cap \Theta_i$, and $\mathrm{Out}_{\mathcal{F}}(Y_0) \cong \Sigma_3 \wr C_2$. Then Y_0 is characteristic in P by (5.4). Since Y_0 is fully normalized (Lemma 3.10), and since $\mathrm{Out}_P(Y_0)$ is not $\mathrm{Out}_{\mathcal{F}}(Y_0)$-conjugate to the center of $\mathrm{Out}_S(Y_0) \cong D_8$ (since $\mathrm{Out}_{\mathcal{F}}(Y_0) \cong \Sigma_3 \wr C_2$), P is fully normalized in $N_{\mathcal{F}}(Y_0)$, and hence also in $\mathcal{F}$ by Proposition 1.3(b). So by Lemma 1.5 and since $\mathrm{Out}_P(Y_0)$ is noncentral of order 2 in $\mathrm{Out}_S(Y_0) \cong D_8$,

$$\mathrm{Out}_{\mathcal{F}}(P) \cong N_{\mathrm{Out}_{\mathcal{F}}(Y_0)}(\mathrm{Out}_P(Y_0))/\mathrm{Out}_P(Y_0) \cong \Sigma_3.$$

Thus $P \in \mathbf{E}_{\mathcal{F}}$. Also, $[\mathrm{Aut}_{\mathcal{F}}(P), P] \le Y_0$ by (5.4), so $\mathfrak{foc}(\mathcal{F}, P) \le Y$. Since $\mathfrak{foc}(\mathcal{F}, P) \in \mathscr{Y}(S)$ by Proposition 3.9(a), and since no element of $\mathscr{Y}(S)$ is strictly contained in any other (Lemma 2.4(b)), $P \in \mathbf{E}_{\mathcal{F}}(Y)$ and thus $\mathbf{E}^{a}_{\mathcal{F}}(Y) \subseteq \mathbf{E}_{\mathcal{F}}(Y)$.

(b) Fix $R \in \mathbf{E}_{\mathcal{F}}(Y)$. If $R \in \mathbf{E}^{a}_{\mathcal{F}}(Y)$, then by definition, $R > Y_0$ for some $Y_0 \in \mathcal{Y}_0$.

If $R \in \mathbf{E}^{c}_{\mathcal{F}}(Y)$, then $R = UC_S(U)$ for some $U \in \mathscr{U}_{\mathcal{F}}(Y)$. Also, by definition of $\mathscr{U}_{\mathcal{F}}(Y)$ (Proposition 3.11(b)), $U = Y_0 \cap \Theta_i$ for some $Y_0 \in \mathscr{Y}_0(S)$ and some $i = 1, 2$. Set $U^* = Y_0 \cap \Theta_{3-i}$. By Proposition 3.11(b.2), $\{U, U^*\} \in \mathscr{U}_S(Y_0)$, and in particular, $Y_0 = UU^*$ and $[U, U^*] \le \mathrm{Fr}(U)$ (Definition 2.1(e)). Thus $\mathrm{Aut}_{U^*}(U) \le \mathrm{Inn}(U)$, and so $Y_0 = UU^* \le UC_S(U) = R$.

(c) Assume $P = Y_0\langle g\rangle \in \mathbf{E}^{a}_{\mathcal{F}}(Y)$, where $Y_0 \in \mathscr{Y}_0$. Recall that Y_0 is fully normalized in $\mathcal{F}$ by Lemma 3.10, is $\mathcal{F}$-centric by definition of $\mathscr{Y}_0(S)$, and is characteristic in P by (5.4). Also, P/Y_0 permutes freely a basis for $Z(Y_0)$ if $Y_0 \cong C_2^4$ or $Q_8 \times Q_8$, and $|P/Y_0| = |Z(Y_0)| = 2$ if $Y_0 \cong 2^{1+4}_{\pm}$. So by Lemma 1.5(b), for any $\Gamma \le \mathrm{Aut}(P)$ with the given properties, $\Gamma = \mathrm{Aut}_{\mathcal{F}}(P)$. □

We next consider wreath products $\Delta \wr C_2$ for $\Delta \in \mathcal{DS}$. It is easy to see that $D_8 \wr C_2$ is a Sylow 2-subgroup of Σ_8 and hence of A_{10}. Since SD_{2^n} is a Sylow 2-subgroup of $GL_2(q)$ for appropriate $q \equiv 3 \pmod 4$, $SD_{2^n} \wr C_2$ is a Sylow 2-subgroup of the groups $GL_2(q) \wr C_2 \le GL_4(q)$, and hence of $PSL_5(q)$. We next check that $D_{2^n} \wr C_2$ is a Sylow 2-subgroup of $PSL_4(q)$ for appropriate q.

Lemma 5.3. *Fix a prime power $q \equiv 3 \pmod 4$, and set $n = 1 + v_2(q+1)$. Then the Sylow 2-subgroups of $PSL_4(q)$ are isomorphic to $D_{2^n} \wr C_2$.*

Proof. This is most easily seen via the isomorphism $PSL_4(q) \cong P\Omega_6^+(q)$ (cf. [**Ta**, Corollary 12.21]). By [**CF**, Theorems 2–3] and since $q \equiv \pmod 4$, the general orthogonal group $GO_6^-(q)$ contains $(GO_2^-(q) \wr C_2) \times GO_2^+(q)$ with odd index, where $GO_2^{\pm}(q) \cong D_{2(q\mp1)}$ (see [**Ta**, Theorem 11.4]). Thus the Sylow 2-subgroups of $GO_6^+(q)$ are isomorphic to $(D_{2^n} \wr C_2) \times C_2^2$. The last factor is sent isomorphically to $GO_6^+(q)/\Omega_6^+(q)$ (in particular, $-I \in SO_2^+(q)$ has nontrivial spinor norm since -1 is not a square [**Ta**, p. 163]). Hence $\Omega_6^+(q) \cong P\Omega_6^+(q)$ has Sylow 2-subgroup isomorphic to $D_{2^n} \wr C_2$. □

The following presentation for the groups studied here will be used throughout the rest of chapter.

Notation 5.4. For some $n \ge 3$, $S = \langle a_1, b_1, a_2, b_2, t\rangle$, where for $i = 1, 2$,

$$|a_i| = 2^{n-1}, \quad |b_i| = 2, \quad b_i a_i b_i^{-1} = a_i^{\lambda}, \quad \Delta_i \overset{\text{def}}{=} \langle a_i, b_i\rangle \cong D_{2^n} \text{ or } SD_{2^n},$$

and $\lambda = -1$ or $\lambda = -1 + 2^{n-2}$ (and $\lambda = -1$ if $n = 3$). Also,

$$[\Delta_1, \Delta_2] = 1, \quad ta_1t^{-1} = a_2, \quad tb_1t^{-1} = b_2, \quad \text{and} \quad t^2 = 1.$$

Either $\Delta_1 \cap \Delta_2 = 1$, or $\Delta_1 \cap \Delta_2 = Z(\Delta_1) = Z(\Delta_2)$. Also, set $w_i = a_i^{2^{n-3}}$ and $z_i = w_i^2 \in Z(\Delta_i)$, and set $z = z_1 = z_2$ if $\Delta_1 \cap \Delta_2 \neq 1$.

Proposition 5.5. *Let $\mathcal{F}$ be a reduced fusion system over $S \cong \Delta \wr C_2$, where $\Delta \in \mathcal{DS}$.*

(a) *If $\Delta \cong D_{2^n}$ for $n \geq 3$, then either $\mathcal{F}$ is isomorphic to the fusion system of $PSL_4(q)$ for each q such that $v_2(q+1) = n-1$, or $n = 3$ and $\mathcal{F}$ is isomorphic to the fusion system of A_{10}.*

(b) *If $\Delta \cong SD_{2^n}$ for $n \geq 4$, then $\mathcal{F}$ is isomorphic to the fusion system of $PSL_5(q)$ for each q such that $v_2(q+1) = n-2$.*

Proof. Let S have the presentation in Notation 5.4, where $\Delta_1 \cap \Delta_2 = 1$. Set $Z_* = \langle a_1^{2^{n-3}}, a_2^{2^{n-3}} \rangle$, and set

$$Y_1 = \langle a_1^2, a_2^2, b_1, b_2 \rangle, \quad Y_2 = \langle a_1^2, a_2^2, a_1b_1, a_2b_2 \rangle, \quad Y_3 = \langle a_1a_2^{-1}, a_1a_2, b_1b_2, t \rangle.$$

Thus $S/Z_* \cong D_8 \wr C_2$ (the unique normal subgroup of index 2^7 by Lemma 2.4(a)), $Y_1/Z_* \cong Y_2/Z_* \cong C_2^4$, and $Y_3/Z_* \cong 2_+^{1+4}$. So by Proposition 3.9, $\mathrm{Out}_{\mathcal{F}}(S) = 1$,

$$\mathbf{E}_{\mathcal{F}} = \mathbf{E}_{\mathcal{F}}(Y_1) \cup \mathbf{E}_{\mathcal{F}}(Y_2) \cup \mathbf{E}_{\mathcal{F}}(Y_3), \qquad \mathbf{E}_{\mathcal{F}}(Y_i) \subseteq \mathbf{E}_{\mathcal{F}}^{(\mathrm{II})} \text{ if } Y_i \in \mathscr{Y}(S),$$

and $\mathbf{E}_{\mathcal{F}}(Y_i) \neq \varnothing$ for each $i = 1, 2, 3$.

By Lemma 2.4(b), $\mathscr{Y}(S) \subseteq \{Y_1, Y_2, Y_3\}$. We claim that

$$\mathscr{Y}(S) = \begin{cases} \{Y_1, Y_2\} & \text{if } n = 3 \\ \{Y_1, Y_2, Y_3\} & \text{if } n \geq 4. \end{cases} \tag{5.6}$$

When $S \cong D_8 \wr C_2$, $Y_1, Y_2 \in \mathscr{Y}(S)$ by definition, and $Y_3 \notin \mathscr{Y}(S)$ by Lemma 2.4(d) (and since $[S{:}Y_3] = 4$ and $|S| < 2^8$). This proves (5.6) when $n = 3$. For $n \geq 4$, it follows from Lemma 2.6(a), except when $\Delta \cong SD_{16}$, in which case $Y_2 \cong Q_8 \times Q_8$ lies in $\mathscr{Y}_0(S)$ (hence in $\mathscr{Y}(S)$) by definition.

For each $i = 1, 2, 3$, let $\mathscr{U}_i = \mathscr{U}_{\mathcal{F}}(Y_i)$ and $\mathbf{E}_{\mathcal{F}}(Y_i) = \mathbf{E}_{\mathcal{F}}^{a}(Y_i) \cup \mathbf{E}_{\mathcal{F}}^{c}(Y_i)$ be as in Lemma 5.2. Let $\mathscr{Y}_{0i}$ be the set of subgroups $P \in \mathscr{Y}_0(S)$ whose normal closure is Y_i.

Step 1: We first consider $\mathcal{F}$-essential subgroups associated to Y_3. Set

$$\Theta_{31} = \langle a_1a_2^{-1}, z_2t \rangle \qquad \text{and} \qquad \Theta_{32} = \begin{cases} \langle a_1a_2, z_2b_1b_2t \rangle & \text{if } \Delta_i \in \mathcal{D} \\ \langle z_2a_1a_2, z_2b_1b_2t \rangle & \text{if } \Delta_i \in \mathcal{S}. \end{cases}$$

Then $\Theta_{31} \cong \Theta_{32} \cong Q_{2^n}$, $\Theta_{31}\Theta_{32} = Y_3$, and $\Theta_{31} \cap \Theta_{32} = \langle z_1z_2 \rangle$. Also, $[\Theta_{31}, \Theta_{32}] = 1$ if $\Delta_i \in \mathcal{D}$, while $[\Theta_{31}, \Theta_{32}] = \langle z_1z_2 \rangle = Z(S)$ if $\Delta_i \in \mathcal{S}$.

Case $n = 3$: If $S \cong D_8 \wr C_2$, then $Y_3 \notin \mathscr{Y}(S)$ by (5.6), and $\mathbf{E}_{\mathcal{F}}(Y_3) = \mathbf{E}_{\mathcal{F}}^{(\mathrm{III})}$ by Proposition 3.9(c) (and since $Y_3 \cong 2_+^{1+4}$). Since each automorphism of Y_3 either normalizes the subgroups $\Theta_{3i} \cong Q_8$ or exchanges them, $\mathrm{Out}(Y_3) \cong \Sigma_3 \wr C_2$.

Let $\tau_1, \tau_2 \in \mathrm{Aut}(Y_3)$ be the automorphisms of order 3 defined by setting

$$\tau_1\colon \Big(a_1a_2^{-1} \mapsto z_2t \mapsto a_1a_2t \mapsto a_1a_2^{-1}\Big) \quad \text{and} \quad \tau_1|_{\Theta_{32}} = \mathrm{Id}$$
$$\tau_2\colon \Big(a_1a_2 \mapsto z_2b_1b_2t \mapsto a_1a_2^{-1}b_1b_2t \mapsto a_1a_2\Big) \quad \text{and} \quad \tau_2|_{\Theta_{31}} = \mathrm{Id}$$

Thus $\langle[\tau_1],[\tau_2]\rangle = O_3(\mathrm{Out}(Y_3)) = O^2(\mathrm{Out}(Y_3))$. For $R \in \mathbf{E}_{\mathcal{F}}(Y_3)$, $[S{:}R] = [R{:}Y_3] = 2$, Y_3 is characteristic in R since it is the only subgroup in S of its isomorphism type (Lemma C.5(a)), and hence

$$\mathrm{Out}_{\mathcal{F}}(R) \cong N_{\mathrm{Out}_{\mathcal{F}}(Y_3)}(\mathrm{Out}_R(Y_3))/\mathrm{Out}_R(Y_3) \tag{5.7}$$

by Lemma 1.5(a). Thus $N_{\mathrm{Out}(Y_3)}(\mathrm{Out}_R(Y_3))$ is not a 2-group, where $\mathrm{Out}(Y_3) \cong \Sigma_3 \wr C_2$, so $R \neq Y_3\langle a_2\rangle$, and thus R is one of the groups

$$R_1 = Y_3\langle b_2\rangle \qquad \text{or} \qquad R_2 = Y_3\langle a_2b_2\rangle\,.$$

Furthermore, by (5.7), one of the following holds:

$$\begin{array}{lll} \mathrm{Out}_{\mathcal{F}}(Y_3) = \langle[\tau_1],[\tau_2],\mathrm{Out}_S(Y_3)\rangle \cong (C_3\times C_3) \overset{-1,t}{\rtimes} C_2^2 & \mathbf{E}_{\mathcal{F}}(Y_3) = \{R_1,R_2\} & \\ \mathrm{Out}_{\mathcal{F}}(Y_3) = \langle[\tau_1\tau_2],\mathrm{Out}_S(Y_3)\rangle \cong C_2\times\Sigma_3 & \mathbf{E}_{\mathcal{F}}(Y_3) = \{R_1\} & (5.8) \\ \mathrm{Out}_{\mathcal{F}}(Y_3) = \langle[\tau_1\tau_2^{-1}],\mathrm{Out}_S(Y_3)\rangle \cong C_2\times\Sigma_3 & \mathbf{E}_{\mathcal{F}}(Y_3) = \{R_2\} & \end{array}$$

Let $\psi \in \mathrm{Aut}(S)$ be the automorphism $\psi(a_i) = a_i^{-1}$, $\psi(b_i) = a_ib_i$, $\psi(t) = t$. Upon replacing $\mathcal{F}$ by ${}^{\psi}\mathcal{F}$ if necessary, we can assume $R_1 \in \mathbf{E}_{\mathcal{F}}(Y_3)$.

Case $n \geq 4$: By Lemma 2.6(a), $\mathscr{Y}_{03}$ is the set of subgroups S-conjugate to one of the groups

$$Y_{03}^{(1)} = \langle z_1, w_1w_2, b_1b_2, t\rangle = \langle w_1w_2^{-1}, z_2t\rangle{\cdot}\langle w_1w_2, z_2b_1b_2t\rangle \cong 2_+^{1+4}$$

$$\begin{aligned} Y_{03}^{(2)} &= \langle z_1, w_1w_2, a_1a_2b_1b_2, t\rangle \\ &= \begin{cases} \langle w_1w_2^{-1}, z_2t\rangle{\cdot}\langle w_1w_2, z_2a_1a_2b_1b_2t\rangle \cong 2_+^{1+4} & \text{if } \Delta_i \in \mathcal{D} \\ \langle w_1w_2^{-1}, z_2t\rangle{\cdot}\langle w_1w_2, a_1a_2b_1b_2t\rangle \cong 2_-^{1+4} & \text{if } \Delta_i \in \mathcal{S}. \end{cases} \end{aligned}$$

By the uniqueness in Lemma C.3, Θ_{31} and Θ_{32} are the subgroups Θ_i which appear in Proposition 3.11(a). So by definition (Proposition 3.11(b)), $\mathscr{U}_3$ is the set of subgroups of Θ_{31} or of Θ_{32} isomorphic to Q_8, and thus the S-conjugacy class of $\langle w_1w_2^{-1}, z_1t\rangle$.

Now, $\mathrm{Out}_{\mathcal{F}}(Y_{03}^{(i)}) \in \mathscr{A}_S(Y_{03}^{(i)})$ by Proposition 3.11(b.1). Hence $\mathrm{Out}_{\mathcal{F}}(Y_{03}^{(i)}) = \mathrm{Out}(Y_{03}^{(i)}) \cong \Sigma_3 \wr C_2$ or Σ_5, depending on whether $Y_{03}^{(i)} \cong 2_+^{1+4}$ or 2_-^{1+4}. By Lemma 5.2(a), $\mathbf{E}_{\mathcal{F}}(Y_3) = \mathbf{E}_{\mathcal{F}}^a(Y_3) \cup \mathbf{E}_{\mathcal{F}}^c(Y_3)$, where

(i) $\mathbf{E}_{\mathcal{F}}^c(Y_3)$ is the set of all $P = UC_S(U)$ for $U \in \mathscr{U}_3$; and

(ii) $\mathbf{E}_{\mathcal{F}}^a(Y_3)$ is the union of the S-conjugacy classes of

$$R_1 = Y_{03}^{(1)}\langle b_2\rangle \qquad \text{and} \qquad R_2 = Y_{03}^{(2)}\langle a_2b_2\rangle. \tag{5.9}$$

By Lemma 5.2(c), $\mathrm{Aut}_{\mathcal{F}}(R_i)$ $(i=1,2)$ is uniquely determined by $\mathrm{Aut}_{\mathcal{F}}(Y_{03}^{(i)})$. For $R = UC_S(U) \in \mathbf{E}_{\mathcal{F}}^c(Y_3)$ $(U \in \mathscr{U}_3)$, $\mathrm{Aut}_{\mathcal{F}}^*(R)$ is uniquely determined by Proposition 3.11(c.4): $\mathrm{Aut}_{\mathcal{F}}^*(UC_S(U)) = O^2\big(\mathrm{Inn}(UC_S(U))\langle\alpha\rangle\big)$ for some $\alpha \in \mathrm{Aut}_{\mathcal{F}}^*(UC_S(U))$ of order 3 which normalizes $U \cong Q_8$ and acts via the identity on $C_S(U)$.

Step 2: We now examine subgroups in $\mathbf{E}_{\mathcal{F}}(Y_1)\cup\mathbf{E}_{\mathcal{F}}(Y_2)$ and their automorphisms, by first showing how this is influenced by the subgroups in $\mathbf{E}_{\mathcal{F}}(Y_3)$. Set

$$Y_{01} = \langle z_1, z_2, b_1, b_2\rangle \cong C_2^4$$

$$Y_{02} = \begin{cases} \langle z_1, z_2, a_1b_1, a_2b_2\rangle \cong C_2^4 & \text{if } \Delta_i \in \mathcal{D} \\ \langle w_1, w_2, a_1b_1, a_2b_2\rangle \cong Q_8\times Q_8 & \text{if } \Delta_i \in \mathcal{S}. \end{cases}$$

Then $Y_{0i} \in \mathscr{Y}_{0i}$ in all cases, and $\mathscr{Y}_{0i}$ is the S-conjugacy class of Y_{0i} by Lemma 2.6(a) (or by definition when $Y_{0i} = Y_i$).

Consider the subgroup $R_1 = Y_{03}^{(1)}\langle b_2\rangle = Y_{01}\langle w_1w_2, t\rangle$ ($R_1 = Y_3\langle b_2\rangle$ if $n = 3$). Since $Y_{03}^{(1)} \cong 2_+^{1+4}$ and c_{b_2} exchanges the two quaternion factors, $R_1 \cong UT_4(2)$ by Lemma C.4(b), and $Y_{01} < R_1$ is the unique subgroup isomorphic to C_2^4 (Lemma C.4(a)). We just showed (in (5.8) and (5.9)) that $R_1 \in \mathbf{E}_{\mathcal{F}}(Y_3)$. Hence $\mathrm{Aut}_{\mathcal{F}}(R_1)$ contains an automorphism of order 3, and it permutes cyclically the three subgroups of index 2 which contain Y_{01} (Lemma C.4(d)) and hence acts nontrivally on $\mathrm{Aut}_{R_1}(Y_{01}) \cong C_2^2$. So $\mathrm{Aut}_{\mathcal{F}}(Y_{01}) \not\cong \Sigma_3 \wr C_2$, and hence $\mathrm{Aut}_{\mathcal{F}}(Y_{01}) \cong \Sigma_5$ by Proposition 3.11(b.1).

To identify $\mathrm{Aut}_{\mathcal{F}}(Y_{02})$, we consider three different cases:

Type (1): Assume $\Delta_1, \Delta_2 \in \mathcal{D}$ and $R_2 \in \mathbf{E}_{\mathcal{F}}(Y_3)$. Then $\mathrm{Aut}_{\mathcal{F}}(Y_{02}) \cong \Sigma_5$ by an argument similar to the above, applied with R_2 in place of R_1.

Type (2): Assume $\Delta_1, \Delta_2 \in \mathcal{D}$ and $R_2 \notin \mathbf{E}_{\mathcal{F}}(Y_3)$. By Step 1, $n = 3$ and $S \cong D_8 \wr C_2$. Then $\mathrm{Out}_{\mathcal{F}}(Y_{02}) \cong \Sigma_3 \wr C_2$, since otherwise $\mathrm{Out}_{\mathcal{F}}(Y_{02}) \cong \Sigma_5$ by Proposition 3.11(b.1), which by the extension axiom (and since $R_2 = Y_{02}\langle w_1w_2, t\rangle$) would imply $\mathrm{Out}_{\mathcal{F}}(R_2) \geq \Sigma_3$.

Type (3): Assume $\Delta_i \in \mathcal{S}$. Then $Y_{02} \cong Q_8 \times Q_8$, so $\mathrm{Out}_{\mathcal{F}}(Y_{02}) \cong \Sigma_3 \wr C_2$ by Proposition 3.11(b.1).

Thus in each case, $\mathrm{Out}_{\mathcal{F}}(P)$ is determined up to isomorphism for $P \in \mathscr{Y}_{01} \cup \mathscr{Y}_{02}$.

For $i = 1, 2$, let $\{\Theta_{i1}, \Theta_{i2}\}$ be as in Proposition 3.11(a). Thus $Y_i = \Theta_{i1} \times \Theta_{i2}$ where $\{\Theta_{i1}, \Theta_{i2}\}$ is an S-conjugacy class (hence both are normal in $\Delta_1\Delta_2$), and $\Theta_{ij} \in \mathcal{DQ}$. By the Krull-Schmidt theorem (Theorem A.8(a)) (and after exchanging indices if necessary), $\Theta_{11} \leq \langle a_m^2, b_m\rangle \times \langle z_{3-m}\rangle$, and after reindexing if necessary, we can assume $m = 1$. Then $b_1z_2^j \in \Theta_{11}$ for some $j = 0, 1$, and ${}^{a_1}(b_1z_2^j) = a_1^2b_1z_2^j \in \Theta_{11}$ since $\Theta_{11} \trianglelefteq \Delta_1\Delta_2$. Thus $\Theta_{11} = \langle a_1^2, b_1z_2^j\rangle$, and hence $\Theta_{12} = {}^t\Theta_{11} = \langle a_2^2, b_2z_1^j\rangle$. By a similar argument, $\Theta_{21} = \langle a_1^2, a_1b_1z_2^k\rangle$ for some $k = 0, 1$, and $\Theta_{22} = {}^t\Theta_{21}$.

Define $\varphi_{jk} \in \mathrm{Aut}(S)$ by setting $\varphi_{jk}(t) = t$, $\varphi_{jk}(b_i) = b_iz_{3-i}^j$, and $\varphi_{jk}(a_ib_i) = a_ib_iz_{3-i}^k$. Then upon replacing $\mathcal{F}$ by ${}^{\varphi_{jk}}\mathcal{F}$, we have $\Theta_{11} = \langle a_1^2, b_1\rangle$ and $\Theta_{21} = \langle a_1^2, a_1b_1\rangle$. Also, by Proposition 3.11(b), $\mathscr{U}_1$ and $\mathscr{U}_2$ are the S-conjugacy classes of

$$U_1 = \Theta_{11} \cap Y_{01} = \langle z_1, b_1\rangle \quad \text{and} \quad U_2 = \Theta_{21} \cap Y_{02} = \begin{cases} \langle z_2, a_2b_2\rangle & \text{if } \Delta_2 \in \mathcal{D} \\ \langle w_2, a_2b_2\rangle & \text{if } \Delta_2 \in \mathcal{S}. \end{cases}$$

By Lemma 3.11(b.2), for $i = 1, 2$, $\mathrm{Out}_{\mathcal{F}}(Y_{0i})$ is the unique subgroup of $\mathrm{Out}(Y_{0i})$ of its isomorphism type (as determined above) which is compatible with the pair $\{Y_{0i} \cap \Theta_{i1}, Y_{0i} \cap \Theta_{i2}\} \in \mathscr{U}_S(Y_{0i})$ (compatible in the sense of Definition 2.2(b)). In particular, each automorphism of order 3 of $Y_{0i} \cap \Theta_{ij}$ ($j = 1, 2$) extends to an element of $\mathrm{Aut}_{\mathcal{F}}(Y_{0i})$. We refer to Lemma 2.9(c) and its proof for more details on how the above pair determines $\mathrm{Out}_{\mathcal{F}}(Y_{0i})$.

By Lemma 5.2(a), for $i = 1, 2$, $\mathbf{E}_{\mathcal{F}}(Y_i) = \mathbf{E}^a_{\mathcal{F}}(Y_i) \cup \mathbf{E}^c_{\mathcal{F}}(Y_i)$, where $\mathbf{E}^a_{\mathcal{F}}(Y_i) = \varnothing$ if $\mathrm{Aut}_{\mathcal{F}}(Y_{0i}) \cong \Sigma_5$. Thus $\mathbf{E}^a_{\mathcal{F}}(Y_1) = \varnothing$. By Lemma 5.2(c), the $\mathcal{F}$-automorphisms of $P \in \mathbf{E}^a_{\mathcal{F}}(Y_2)$ are uniquely determined by the above information.

Step 3: It remains to determine $\mathrm{Aut}_{\mathcal{F}}(R)$ when $R = UC_S(U) \in \mathbf{E}^c_{\mathcal{F}}(Y_i)$ for $U \in \mathscr{U}_i$ ($i = 1, 2$). By Proposition 3.11(c.3–4), $[\mathrm{Aut}^*_{\mathcal{F}}(R), R] = U$ in this situation, and $\mathrm{Aut}^*_{\mathcal{F}}(R) = O^2\big(\mathrm{Inn}(R)\langle\alpha\rangle\big)$ for some α of order 3 such that $\alpha(U) = U$ and α induces the identity on R/U. It remains to determine α more precisely.

Consider the group $P = U_1U_2$. Each element of $N_S(P)$ either normalizes or exchanges the two subgroups $U_i = P \cap \Delta_i$, and they are not S-conjugate since $\langle U_1^S\rangle = Y_1$ while $\langle U_2^S\rangle = Y_2$. Hence $N_S(P) = N_{\Delta_1\Delta_2}(P) = N_{\Delta_1}(U_1)N_{\Delta_2}(U_2)$, $N_{\Delta_1}(U_1) \cong D_8$, and $N_{\Delta_2}(U_2) \cong D_8$, Q_{16}, or SD_{16}. Thus $\mathrm{Out}_S(P) \cong N_S(P)/P \cong C_2^2$.

We claim that P is fully normalized in $\mathcal{F}$. If not, then by Lemma 1.16(a), there is $T \in \mathbf{E}_{\mathcal{F}}$ such that $T \geq N_S(P)$. Let j be such that $T \in \mathbf{E}_{\mathcal{F}}(Y_j)$ $(j = 1, 2, 3)$. If $T = UC_S(U)$ for some $U \in \mathscr{U}_j$, then $U \leq N_S(P)$ by Lemma A.6(c) (applied with T in the role of S), which is impossible. (Recall that if $U \cong C_2^2$, then it is a direct factor in T.) If $T \in \mathbf{E}^a_{\mathcal{F}}(Y_j)$ for $j = 1, 2$, then $N_S(P) \leq T \cap \Delta_1\Delta_2 \in \mathscr{Y}_{0j}$, which is also impossible. Finally, $N_S(P) \nleq R_1, R_2 \in \mathbf{E}^a_{\mathcal{F}}(Y_3)$ as defined in Step 1. Hence there is no such $T \in \mathbf{E}_{\mathcal{F}}$, and P is fully normalized.

Let $x_1 \in \mathrm{Out}_{\Delta_1}(P)$ and $x_2 \in \mathrm{Out}_{\Delta_2}(P)$ be the generators. For each $i = 1, 2$, by Proposition 3.11(b.4) and since $U_i \in \mathscr{U}_i$, there is $\alpha_i \in \mathrm{Aut}_{\mathcal{F}}(U_iC_S(U_i))$ of order 3 which normalizes U_i and induces the identity on $U_iC_S(U_i)/U_i$. Thus $\alpha_i(P) = P$. Set $\overline{\alpha}_i \overset{\text{def}}{=} \alpha_i|_P \in \mathrm{Aut}_{\mathcal{F}}(P)$. Since $U_i\Delta_{3-i} \leq U_iC_S(U_i)$, $[\overline{\alpha}_i] \in \mathrm{Out}_{\mathcal{F}}(P)$ normalizes (hence centralizes) $\mathrm{Out}_{\Delta_{3-i}}(P) = \langle x_{3-i}\rangle$. The hypotheses of Proposition D.1(e.1) thus hold, applied to the action of $\mathrm{Out}_{\mathcal{F}}(P)$ on $P/\mathrm{Fr}(P) \cong C_2^4$, and hence $\mathrm{Out}_{\mathcal{F}}(P) \cong \Sigma_3 \times \Sigma_3$. If $\Delta_2 \in \mathcal{D}$, so $P = U_1U_2 \cong C_2^4$, then U_1 and U_2 are the irreducible summands of the action of $\langle\overline{\alpha}_1, \overline{\alpha}_2\rangle \cong C_3 \times C_3$. If $\Delta_2 \in \mathcal{S}$, so $P \cong C_2^2 \times Q_8$, then $\overline{\alpha}_1$ normalizes U_2 by a similar argument applied to $P/\mathrm{Fr}(P)$, and $\overline{\alpha}_2$ normalizes U_1 since $\overline{\alpha}_2|_{Z(P)} = \mathrm{Id}$. Thus in both cases, $\overline{\alpha}_1$ is the identity on U_2 and $\overline{\alpha}_2$ is the identity on U_1. This, together with $\mathrm{Aut}_{\mathcal{F}}(Y_{01})$ and $\mathrm{Aut}_{\mathcal{F}}(Y_{02})$, determine uniquely the automorphism groups $\mathrm{Aut}^*_{\mathcal{F}}(U_iC_S(U_i))$ for $i = 1, 2$, and hence determine $\mathrm{Aut}^*_{\mathcal{F}}(R)$ for each $R \in \mathbf{E}^c_{\mathcal{F}}(Y_1) \cup \mathbf{E}^c_{\mathcal{F}}(Y_2)$.

For example, $\mathrm{Aut}_{\mathcal{F}}(Y_{01}) \cong \Sigma_5$ is the group compatible with the pair $\{U_1, {}^tU_1\} \in \mathscr{U}_S(Y_{01})$. As shown explicitly in the proof of Lemma 2.9(c), it is the group of automorphisms of $Y_{01} = \langle z_1, b_1, z_2, b_2\rangle$ which permute the set

$$X = \{b_1z_2, b_1z_1z_2, z_1z_2, b_2z_1, b_2z_1z_2\}.$$

Any element of $\mathrm{Aut}_{\mathcal{F}}(Y_{01})$ which permutes cyclically the first three elements in X acts on $U_1 = \langle z_1, b_1\rangle$ with order 3, and any element which permutes cyclically the last three acts on ${}^tU_1 = \langle z_2, b_2\rangle$ with order 3. If $R = U_1C_S(U_1) \in \mathbf{E}^c_{\mathcal{F}}(Y_1)$, then $\mathrm{Aut}^*_{\mathcal{F}}(R) = O^2\big(\mathrm{Inn}(R)\langle\alpha\rangle\big)$ where $\alpha|_{Y_{01}}$ permutes cyclically the first three elements in X, fixes the last two, and is the identity on U_2. Thus α acts on $R = U_1 \times \langle b_2z_1, a_2b_2\rangle$ with order 3 on the first factor and as the identity on the second factor.

We have now shown that up to isomorphism, there is at most one reduced fusion system of each of the three types listed above.

Step 4: It remains to find explicit fusion systems of each type. Assume $\Delta_i \in \mathcal{D}$, let q be a prime power such that $v_2(q+1) = n-1$, and identify S with a Sylow 2-subgroup of $G_1 = PSL_4(q)$ (Lemma 5.3). Since $SL_2(q)$ contains subgroups isomorphic to $Q_8 \rtimes C_3$, G_1 contains subgroups $Y_3 \cong 2^{1+4}_+$ with $9\big||\mathrm{Out}_{\mathcal{F}}(Y_3)|$. So $R_1, R_2 \in \mathbf{E}_{\mathcal{F}}(Y_3)$ by (5.8) (when $n = 3$), and $\mathcal{F}_S(G_1)$ has type (1). Also, $\mathcal{F}_S(G_1)$ is reduced by Proposition 1.12.

Next assume $S \cong D_8 \wr C_2$, identify S with a Sylow 2-subgroup of $\Sigma_8 < A_{10}$, and set $G_2 = A_{10}$. There are two G_2-conjugacy classes of subgroups isomorphic to

C_2^4, represented by

$$V_1 = A_{10} \cap \big\langle (1\,2), (3\,4), (5\,6), (7\,8), (9\,10) \big\rangle$$
$$V_2 = \big\langle (1\,2)(3\,4), (1\,3)(2\,4), (5\,6)(7\,8), (5\,7)(6\,8) \big\rangle.$$

Since $\mathrm{Aut}_{G_2}(V_1) \cong \Sigma_5$ and $\mathrm{Aut}_{G_2}(V_2) \cong \Sigma_3 \wr C_2$, $\mathcal{F}_S(G_2)$ has type (2). Again, $\mathcal{F}_S(G_2)$ is reduced by Proposition 1.12.

Now assume $\Delta_i \in \mathcal{S}$, let q be such that $v_2(q+1) = n-2$, and identify S with a Sylow 2-subgroup of $GL_2(q) \wr C_2 < GL_4(q)$, and hence of $G_3 = PSL_5(q)$. Thus $\mathcal{F}_S(G_3)$ has type (3), and $\mathcal{F}_S(G_3)$ is reduced by Proposition 1.12 again. □

It remains to consider the central wreath products.

Proposition 5.6. *Let $\mathcal{F}$ be a reduced fusion system over $S \cong (\Delta \times_{C_2} \Delta) \overset{t}{\rtimes} C_2$, where $\Delta \in \mathcal{DS}$, and $|\Delta| = 2^n$ for $n \geq 4$. Then $\Delta \cong D_{2^n}$, and $\mathcal{F}$ is isomorphic to the fusion system of $PSp_4(q)$ for each odd prime power q such that $v_2(q^2-1) = n$.*

Proof. Let S have the presentation and subgroups of Notation 5.4, where $n \geq 4$, $z = z_1 = z_2$, and $\Delta_1 \cap \Delta_2 = \langle z \rangle = Z(S)$. Set $Z_* = \langle a_1^{2^{n-3}}, a_2^{2^{n-3}} \rangle$, and set

$$Y_1 = \langle a_1^2, a_2^2, b_1, b_2 \rangle, \quad Y_2 = \langle a_1^2, a_2^2, a_1b_1, a_2b_2 \rangle, \quad Y_3 = \langle a_1a_2^{-1}, a_1a_2, b_1b_2, t \rangle.$$

Thus $S/Z_* \cong D_8 \wr C_2$ (the unique normal subgroup of index 2^7 by Lemma 2.4(a)), $Y_1/Z_* \cong Y_2/Z_* \cong C_2^4$, and $Y_3/Z_* \cong 2_+^{1+4}$. So by Lemma 2.4(b),

$$\mathscr{Y}(S) \subseteq \{Y_1, Y_2, Y_3\}.$$

For $j = 1, 2$, set

$$\Theta_{1j} = \langle a_j^2, w_{3-j}b_j \rangle \qquad \text{and} \qquad \Theta_{2j} = \begin{cases} \langle a_j^2, w_{3-j}a_jb_j \rangle & \text{if } \Delta \in \mathcal{D} \\ \langle a_j^2, a_jb_j \rangle & \text{if } \Delta \in \mathcal{S}. \end{cases}$$

Then for $i = 1, 2$, $\Theta_{i1} \cong \Theta_{i2} \cong Q_{2^{n-1}}$, $[\Theta_{i1}, \Theta_{i2}] = 1$ since $[w_1b_2, w_2b_1] = 1$, $\Theta_{i1} \cap \Theta_{i2} = \langle z \rangle$, and thus $Y_i = \Theta_{i1}\Theta_{i2} \cong Q_{2^{n-1}} \times_{C_2} Q_{2^{n-1}}$. Also, set

$$\Theta_{31} = \langle a_1a_2^{-1}, t \rangle \cong D_{2^{n-1}} \qquad \text{and} \qquad \Theta_{32} = \langle a_1a_2, b_1b_2t \rangle \cong D_{2^{n-1}},$$

so that $Y_3 = \Theta_{31}\Theta_{32} \cong D_{2^{n-1}} \times D_{2^{n-1}}$.

When $n \geq 5$ or $i = 3$, $Y_i \in \mathscr{Y}(S)$ by Lemma 2.6(a). When $n = 4$, $Y_1, Y_2 \in \mathscr{Y}_0(S)$ by definition (and since $S/\langle z \rangle \cong D_8 \wr C_2$), and hence $Y_1, Y_2 \in \mathscr{Y}(S)$ since they are normal. Thus $\mathscr{Y}(S) = \{Y_1, Y_2, Y_3\}$. So by Proposition 3.9(a,b), $\mathrm{Out}_{\mathcal{F}}(S) = 1$,

$$\mathbf{E}_{\mathcal{F}} = \mathbf{E}_{\mathcal{F}}^{(\mathrm{II})} = \mathbf{E}_{\mathcal{F}}(Y_1) \cup \mathbf{E}_{\mathcal{F}}(Y_2) \cup \mathbf{E}_{\mathcal{F}}(Y_3), \quad \text{and} \quad \mathbf{E}_{\mathcal{F}}(Y_i) \neq \varnothing \ \forall\, i = 1, 2, 3. \tag{5.10}$$

For each $i = 1, 2, 3$, set $\mathscr{U}_i = \mathscr{U}_{\mathcal{F}}(Y_i)$ as defined in Proposition 3.11(b). For $i = 1, 2$, if $g \in Y_i = \Theta_{i1}\Theta_{i2}$ and $g^2 = z$, then $g \in \Theta_{i1} \cup \Theta_{i2}$. So for each $U < Y_i$ with $U \cong Q_8$, $U \leq \Theta_{ij}$ for some j. Since all such subgroups of Y_i are S-conjugate to each other, this proves that

$$\text{for } i = 1, 2, \ \mathscr{U}_i = \{U < Y_i \,|\, U \cong Q_8\}. \tag{5.11}$$

Let $\mathbf{E}_{\mathcal{F}}(Y_i) = \mathbf{E}_{\mathcal{F}}^a(Y_i) \cup \mathbf{E}_{\mathcal{F}}^c(Y_i)$ be the decomposition of Proposition 5.2(a).

Case 1: Assume $\Delta_1, \Delta_2 \in \mathcal{S}$. Set $U_1 = \langle w_1, w_2b_1 \rangle \in \mathscr{U}_1$, $U_2 = \langle w_2, a_2b_2 \rangle \in \mathscr{U}_2$, and $P = U_1U_2$. The given generators for U_1 commute with those for U_2 except that $[w_2b_1, a_2b_2] = z$, so $P \cong 2_-^{1+4}$ by Lemma C.2(a). For $i = 1, 2$, $\mathrm{Aut}_P(U_i) \leq \mathrm{Inn}(U_i)$ since $[U_1, U_2] = Z(S)$, and hence $P \leq U_iC_S(U_i)$. By Proposition 3.11(c.4), there is $\alpha_i \in \mathrm{Aut}_{\mathcal{F}}^*(U_iC_S(U_i))$ such that $|\alpha_i| = 3$, $\alpha_i(U_i) = U_i$, and α_i induces the identity on $U_iC_S(U_i)/U_i$. Thus $\alpha_i(P) = P$ and $|\alpha_i|_P| = 3$. So by Lemma C.2(b) (with α_i in the role of γ_i), $\mathrm{Out}_{\mathcal{F}}(P) \cong \Sigma_5$ or A_5.

Let $v_1 \in \langle a_1 \rangle$ be such that $v_1^2 = w_1$. Then v_1 normalizes U_1 and centralizes U_2. Set $\eta = c_{v_1} \in \mathrm{Aut}_S(P)$. Then $\eta|_{U_1} \notin \mathrm{Inn}(U_1)$ while $\eta|_{U_2} = \mathrm{Id}$. So by Lemma C.2(b) again, $\mathrm{Out}_{\mathcal{F}}(P) \cong \Sigma_5$.

Set $\Delta_i^* = \langle w_{3-i}a_i, w_{3-i}b_i \rangle \cong Q_{2^n}$ $(i = 1, 2)$. Then $[\Delta_1^*, \Delta_2^*] = \langle z \rangle$, so each element of S normalizes or exchanges the Δ_i^*, and each element of $N_S(P)$ either normalizes or exchanges the two subgroups $U_i = P \cap \Delta_i^*$. Also, U_1 and U_2 are not S-conjugate, since $\langle U_1^S \rangle = Y_1$ while $\langle U_2^S \rangle = Y_2$. Hence

$$\begin{aligned} N_S(P) &= N_{\Delta_1^*\Delta_2^*}(P) = N_{\Delta_1^*}(U_1)N_{\Delta_2^*}(U_2) \\ &= P\langle v_1, v_2 \rangle \qquad\qquad \text{where } v_i \in \langle a_i \rangle \text{ and } v_i^2 = w_i \text{ for } i = 1, 2\,. \end{aligned}$$

Thus $\mathrm{Out}_S(P) \cong N_S(P)/P \cong C_2^2$, so $\mathrm{Out}_S(P) \notin \mathrm{Syl}_2(\mathrm{Out}_{\mathcal{F}}(P))$, and P is not fully normalized in $\mathcal{F}$.

By Lemma 1.16(a) and since P is not fully normalized, there is an $\mathcal{F}$-essential subgroup $R \geq N_S(P)$. Let $i = 1, 2, 3$ be such that $R \in \mathbf{E}_{\mathcal{F}}(Y_i)$. If $R \in \mathbf{E}_{\mathcal{F}}^a(Y_i)$, then there is $T < R$ with $|R/T| = 2$ and $T \in \mathscr{Y}_0(S)$; and since $|N_S(P)| = 2^7$, $R = N_S(P)$ and $T \cong Q_8 \times Q_8$, which is impossible. If $R = UC_S(U) \in \mathbf{E}_{\mathcal{F}}^c(Y_i)$ for some $U \in \mathscr{U}_i$, then $U \leq N_S(P)$ by Lemma A.6(c) (applied with R and $N_S(P)$ in the role of S and Q), so $N_S(P)$ contains a direct factor C_2^2 (if $i = 3$) or a central factor Q_8, which is also impossible. We thus have a contradiction, and there is no reduced fusion system over S.

Case 2: Now assume $\Delta_1, \Delta_2 \in \mathcal{D}$. Let $\mathscr{Y}_{0i}$ $(i = 1, 2, 3)$ be the set of subgroups $P \in \mathscr{Y}_0(S)$ whose normal closure is Y_i. By Lemma 2.6(a) (or by definition if $Y_i \cong 2_+^{1+4}$), for $i = 1, 2$, $\mathscr{Y}_{0i}$ is the set of all $U_1U_2 \cong 2_+^{1+4}$, where $U_1, U_2 \in \mathscr{U}_i$, and $U_j \leq \Theta_{ij}$ for $j = 1, 2$ by (5.11). By Proposition 3.11(b.1), $\mathrm{Out}_{\mathcal{F}}(U_1U_2) = \mathrm{Out}(U_1U_2) \cong \Sigma_3 \wr C_2$, and by Lemma 5.2(c), this determines $\mathrm{Out}_{\mathcal{F}}(R)$ for each $R \in \mathbf{E}_{\mathcal{F}}^a(Y_i)$. Also, $\mathbf{E}_{\mathcal{F}}^c(Y_i)$ is the set of subgroups of the form $R = UC_S(U) \cong Q_8 \times_{C_2} Q_{2^n}$ for $U \in \mathscr{U}_i$. By Proposition 3.11(c.4), $\mathrm{Aut}_{\mathcal{F}}^*(R) = O^2\bigl(\mathrm{Inn}(R)\langle\alpha\rangle\bigr)$ for some α such that $|\alpha| = 3$, $\alpha(U) = U$, and $\alpha|_{C_S(U)} = \mathrm{Id}$. Hence $\mathrm{Aut}_{\mathcal{F}}^*(R)$ is also determined uniquely.

It remains to examine the subgroups in $\mathbf{E}_{\mathcal{F}}(Y_3)$. By Proposition 3.11(a), and since $Y_3 \cong D_{2^{n-1}} \times D_{2^{n-1}}$, there is a product decomposition $Y_3 = \Theta_{31}^* \times \Theta_{32}^*$ such that $\mathscr{U}_3$ is the set of subgroups of the $\Theta_{3i}^* \cong D_{2^{n-1}}$ which are isomorphic to C_2^2. By Proposition 3.11(b), the subgroups in $\mathscr{U}_3$ are all S-conjugate. By the Krull-Schmidt theorem (Theorem A.8(a)), $\Theta_{3i}^* \leq \Theta_{3i}\langle z \rangle$ (after changing indices if necessary). Hence $\mathscr{U}_3$ is the S-conjugacy class of $\langle w_1w_2^{-1}, t \rangle$ or of $\langle w_1w_2^{-1}, tz \rangle$. Define $\varphi \in \mathrm{Aut}(S)$ by setting $\varphi|_{\Delta_1\Delta_2} = \mathrm{Id}$ and $\varphi(t) = zt$. Upon replacing $\mathcal{F}$ by ${}^{\varphi}\mathcal{F}$ if necessary, we can arrange that $\mathscr{U}_3$ is the S-conjugacy class of $\langle w_1w_2^{-1}, t \rangle$ (and also that $\Theta_{3i}^* = \Theta_{3i}$).

Consider the subgroups

$$\begin{aligned} Y_{01} &= \langle w_1, b_1, w_2, b_2 \rangle \in \mathscr{Y}_{01} & Y_{03}^{(1)} &= \langle w_1w_2^{-1}, t \rangle \times \langle w_1w_2, b_1b_2t \rangle \in \mathscr{Y}_{03} \\ Y_{02} &= \langle w_1, a_1b_1, w_2, a_2b_2 \rangle \in \mathscr{Y}_{02} & Y_{03}^{(2)} &= \langle w_1w_2^{-1}, t \rangle \times \langle w_1w_2, a_1a_2b_1b_2t \rangle \in \mathscr{Y}_{03}\,. \end{aligned}$$

Set $R_i = Y_{0i}\langle t \rangle \in \mathbf{E}_{\mathcal{F}}^a(Y_i)$ for $i = 1, 2$. Then

$$\mathrm{Out}_{\mathcal{F}}(R_i) \cong N_{\mathrm{Out}_{\mathcal{F}}(Y_{0i})}\bigl(\mathrm{Out}_{R_i}(Y_{0i})\bigr)/\mathrm{Out}_{R_i}(Y_{0i}) \cong \Sigma_3$$

by Lemma 1.5(a), and the subgroup of order 3 acts nontrivially on the group $\mathrm{Aut}_{R_i}(Y_{03}^{(i)}) \cong C_2^2$. Also, $R_1 = Y_{03}^{(1)} \rtimes \langle b_1, w_1\rangle$ and $R_2 = Y_{03}^{(2)} \rtimes \langle a_1b_1, w_1\rangle$, so there is $\alpha_i \in \mathrm{Aut}_{\mathcal{F}}(R_i)$ of order 3 which acts nontrivially on $R_i/Y_{03}^{(i)}$. Hence $\mathrm{Aut}_{\mathcal{F}}(Y_{03}^{(i)}) \not\cong \Sigma_3 \wr C_2$. By Proposition 3.11(b.2), $\mathrm{Aut}_{\mathcal{F}}(Y_{03}^{(i)}) \cong \Sigma_5$, and is the unique subgroup in $\mathscr{A}_S^-(Y_{03}^{(i)})$ associated to $\{Y_{03}^{(i)} \cap \Theta_{31}, Y_{03}^{(i)} \cap \Theta_{32}\} \in \mathscr{U}_S(Y_0^{(i)})$. Since each subgroup in $\mathscr{Y}_{03}$ is S-conjugate to $Y_{03}^{(1)}$ or $Y_{03}^{(2)}$ (Lemma 2.6(a)), we have now determined $\mathrm{Aut}_{\mathcal{F}}(P)$ for each $P \in \mathscr{Y}_{03}$. Also, $\mathbf{E}_{\mathcal{F}}^a(Y_3) = \varnothing$ since $\mathrm{Aut}_{\mathcal{F}}(Y_{03}^{(i)}) \not\cong \Sigma_3 \wr C_2$.

By Lemma 5.2(a), $\mathbf{E}_{\mathcal{F}}(Y_3) = \mathbf{E}_{\mathcal{F}}^c(Y_3)$: the set of all $R = UC_S(U)$ for $U \in \mathscr{U}_3$. By Proposition 3.11(c.4), $\mathrm{Aut}_{\mathcal{F}}^*(R) = O^2\big(\mathrm{Inn}(R)\langle\alpha\rangle\big)$ for some α which induces the identity on U and on R/U. When $U = \langle w_1w_2^{-1}, t\rangle$, this shows that α normalizes $Y_{03}^{(1)}$ and $Y_{03}^{(2)}$, hence is uniquely determined on those subgroups, and is uniquely determined on $R = U \times \langle a_1a_2, b_1b_2t\rangle = Y_{03}^{(1)}Y_{03}^{(2)}$.

This proves that $\mathcal{F}$ is completely determined by our choice of $\mathscr{U}_3$. So up to isomorphism, there is at most one unique reduced fusion system over S.

Let q be any prime power such that $n = v_2(q^2-1)$, and set $G = PSp_4(q)$. By [**CF**, § 1], the Sylow 2-subgroups of $Sp_4(q)$ are isomorphic to $Q_{2^n} \wr C_2$, and hence those of G are isomorphic to $S \cong (Q_{2^n} \times_{C_2} Q_{2^n}) \overset{t}{\rtimes} C_2$. The 2-fusion system of G is reduced by Proposition 1.12, and hence is isomorphic to $\mathcal{F}$ as just described. □

CHAPTER 6

Fusion systems over extensions of $UT_3(4)$

Recall that $\mathcal{U}$ is the class of all 2-groups S such that there is $T \trianglelefteq S$ with $T \cong UT_3(4)$ for which $T/Z(T)$ is centric in $S/Z(T)$. We now look at reduced fusion systems over 2-groups in $\mathcal{U}$, using the following notation for their elements and subgroups. For the most part, this is the same notation as that used in [**OV**, § 4–5] (and also in Appendix C).

Notation 6.1. Set $S_0 = UT_3(4)$, the group of strictly upper triangular 3×3 matrices over $\mathbb{F}_4$. Let e_{ij}^a denote the elementary matrix with nonzero entry a in position (i, j). Set

$$A_1 = \left\{ \left(\begin{smallmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) = e_{12}^a e_{13}^b \,\middle|\, a, b \in \mathbb{F}_4 \right\} \quad \text{and} \quad A_2 = \left\{ \left(\begin{smallmatrix} 1 & 0 & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{smallmatrix}\right) = e_{23}^a e_{13}^b \,\middle|\, a, b \in \mathbb{F}_4 \right\}.$$

Let $a \mapsto \bar{a} = a^2$ be the (nontrivial) automorphism of $\mathbb{F}_4$, and write $\mathbb{F}_4 = \{0, 1, \omega, \bar{\omega}\}$. Note the relation

$$[e_{12}^a, e_{23}^b] = e_{13}^{ab} \qquad \text{for all } a, b \in \mathbb{F}_4. \tag{6.1}$$

Let $\tau \in \mathrm{Aut}(S_0)$ be the graph automorphism defined by transpose inverse; thus $\tau(e_{ij}^a) = e_{4-j,4-i}^a$. Let $\phi \in \mathrm{Aut}(UT_3(4))$ be the field automorphism $\phi(e_{ij}^a) = e_{ij}^{\bar{a}}$, and set $\theta = \phi \circ \tau = \tau \circ \phi$.

Set $S_{\phi,\tau} = S_0 \overset{\phi,\tau}{\rtimes} \langle \boldsymbol{\phi}, \boldsymbol{\tau} \rangle$, the semidirect product where $c_{\boldsymbol{\phi}} = \phi$, $c_{\boldsymbol{\tau}} = \tau$, and $\langle \boldsymbol{\phi}, \boldsymbol{\tau} \rangle \cong C_2^2$. Let $S_{\phi,\tau}^* = S_0 \langle \boldsymbol{\phi}, \boldsymbol{\tau} \rangle$ be the nonsplit extension where

$$S_0 \trianglelefteq S_{\phi,\tau}^*,\ c_{\boldsymbol{\phi}} = \phi,\ c_{\boldsymbol{\tau}} = \tau,\ \boldsymbol{\phi}^2 = e_{13}^1,\ [\boldsymbol{\phi}, \boldsymbol{\tau}] = 1,\ \boldsymbol{\tau}^2 = e_{13}^1\,.$$

Set $\boldsymbol{\theta} = \boldsymbol{\phi}\boldsymbol{\tau}$ in both groups. Let $S_\tau, S_\theta, S_\phi < S_{\phi,\tau}$ and $S_\tau^*, S_\theta^*, S_\phi^* < S_{\phi,\tau}^*$ be the subgroups generated by S_0, and $\boldsymbol{\tau}$, $\boldsymbol{\theta}$, or $\boldsymbol{\phi}$, respectively.

Note that S_ϕ^* is a semidirect product, since $(e_{13}^\omega \boldsymbol{\phi})^2 = e_{13}^1 \boldsymbol{\phi}^2 = 1$ in $S_{\phi,\tau}^*$. Thus the choice that $\boldsymbol{\theta}^2 = 1$ and not $\boldsymbol{\phi}^2 = 1$ was arbitrary, and was made to simplify some of the later formulas.

We first show that each $S \in \mathcal{U}$ is isomorphic to one of the groups listed above.

Lemma 6.2. *Each group $S \in \mathcal{U}$ is isomorphic to one of the groups $UT_3(4)$, S_ϕ, S_θ, S_τ, S_τ^*, $S_{\phi,\tau}$, or $S_{\phi,\tau}^*$.*

Proof. Assume $S \in \mathcal{U}$, and fix $T \trianglelefteq S$ such that

$$T \cong UT_3(4) \quad \text{and} \quad C_{S/Z(T)}(T/Z(T)) = T/Z(T).$$

Since $O_2(\mathrm{Out}(T))$ is the subgroup of all $[\alpha] \in \mathrm{Out}(T)$ such that α induces the identity on $T/Z(T)$ (Lemma C.8), the condition that $T/Z(T)$ be centric in $S/Z(T)$ implies that $\mathrm{Out}_S(T) \cap O_2(\mathrm{Out}(T)) = 1$.

We identify $T = S_0 = UT_3(4)$ via some choice of isomorphism $T \cong S_0$. Set $\Gamma = O_2(\mathrm{Out}(S_0))\langle [\phi], [\tau] \rangle \leq \mathrm{Out}(S_0)$. Then $\Gamma \in \mathrm{Syl}_2(\mathrm{Out}(S_0))$ by Lemma C.8, so $\mathrm{Out}_S(S_0)$ is $\mathrm{Out}(S_0)$-conjugate to a subgroup of Γ. Hence after changing our choice of identification isomorphism $T \cong S_0$, we can assume that $\mathrm{Out}_S(S_0) \leq \Gamma$. Since $\langle \phi, \tau \rangle$ permutes freely a basis for $O_2(\mathrm{Out}(S_0)) \cong C_2^4$ by Lemma C.8, $H^1(\langle \phi, \tau \rangle; O_2(\mathrm{Out}(S_0))) = 0$, and similarly for subgroups of $\langle \phi, \tau \rangle$. Hence $\mathrm{Out}_S(S_0)$ is $\mathrm{Out}(S_0)$-conjugate to a subgroup of $\langle [\tau], [\phi] \rangle$ (cf. [**Br**, Proposition IV.2.3]), since both are complementary to $O_2(\mathrm{Out}(S_0))$ in a certain subgroup of Γ. So upon changing the isomorphism $T \cong S_0$ again, we can arrange that $\mathrm{Out}_S(S_0) \leq \langle [\tau], [\phi] \rangle$.

The result now follows from the cohomology computations:

$$H^2(\langle \phi \rangle; Z(S_0)) = H^2(\langle \theta \rangle; Z(S_0)) = 0$$
$$H^2(\langle \tau \rangle; Z(S_0)) \cong Z(S_0)$$
$$H^2(\langle \phi, \tau \rangle; Z(S_0)) \cong H^2(\langle \tau \rangle; \langle e_{13}^1 \rangle) \cong \langle e_{13}^1 \rangle.$$

These all follow from the formula $H^2(\langle \gamma \rangle; M) = C_M(\gamma) / \langle x\gamma(x) \rangle$ when $|\langle \gamma \rangle| = 2$ (cf. [**Br**, pp. 58–59]), except for the first isomorphism in the third line which follows from Shapiro's lemma (cf. [**Br**, Proposition III.6.2]). The three nonsplit extensions of S_0 by τ are isomorphic via the automorphism $\gamma_0 \in \mathrm{Aut}(S_0)$ (see Lemma C.8) which permutes transitively the set $Z(T)^\# = \{e_{13}^a \,|\, a \in \mathbb{F}_4^\times\}$. So all of them are isomorphic to S_τ^*. □

The following lemma about subgroups of S_0, S_τ, and S_τ^* will also be needed.

Lemma 6.3. *Assume $S = S_\tau$ or S_τ^*.*

(a) *There are exactly three subgroups of S isomorphic to $C_4 \times C_4$: the subgroups*

$$H_i = \langle e_{12}^1 e_{23}^{\omega^i}, e_{12}^{\omega} e_{23}^{\omega^{i+1}} \rangle \qquad \text{for } i = 0, 1, 2.$$

(b) *The only subgroups of S isomorphic to C_2^4 are A_1 and A_2.*

Proof. Set $Z_0 = Z(S_0) = \{e_{13}^a \,|\, a \in \mathbb{F}_4\}$ for short.

(a) If $H \leq S_0$ has order 16 and contains Z_0, then

$$H = Z_0 \langle e_{12}^a e_{23}^b, e_{12}^c e_{23}^d \rangle$$

for some $a, b, c, d \in \mathbb{F}_4$. Since $[e_{12}^x, e_{23}^y] = e_{13}^{xy} \in Z(S_0)$ by (6.1), H is abelian if and only if $ad = bc$. Thus the three subgroups H_i ($i = 0, 1, 2$) together with $A_1 \cong C_2^4$ and $A_2 \cong C_2^4$ are the only abelian subgroups of order 16 in S_0, and the H_i are the only ones isomorphic to $C_4 \times C_4$.

Conversely, for each $i = 0, 1, 2$, $\Omega_1(H_i) \subseteq H_i \cap (A_1 \cup A_2) = Z_0$ by Lemma C.6(a), so $H_i \cong C_4 \times C_4$ since it is abelian of order 16.

Assume $H \leq S$ is such that $H \nleq S_0$ and $H \cong C_4 \times C_4$. Then $\Omega_1(H) \leq \mathrm{Fr}(S) < S_0$, $\Omega_1(H) \subseteq (A_1 \cup A_2)$ since all elements of $S_0 \smallsetminus (A_1 \cup A_2)$ have order 4, and $\Omega_1(H) = Z_0$ since no element of $A_i \smallsetminus Z_0$ commutes with any element of $S_0\boldsymbol{\tau}$. Thus $H > Z_0$.

Let $g \in S_0$ and $h \in S_0 \smallsetminus Z_0$ be such that $H = \langle g\boldsymbol{\tau}, h \rangle$. Then $(g\boldsymbol{\tau})^2 \in Z_0$ implies that $g\tau(g) \in Z_0$, and $[h, g\boldsymbol{\tau}] = 1$ implies that $h\tau(h)^{-1} \in Z_0$. Since $C_{S_0/Z_0}(\tau) = H_0/Z_0$, we have $g, h \in H_0$. Thus $[h, g] = 1$ since H_0 is abelian, so $[h, \boldsymbol{\tau}] = 1$, and $h = \tau(h)$. But this is impossible: $h \equiv e_{12}^a e_{23}^a h_0$ for some $h_0 \in Z_0$ and some $0 \neq a \in \mathbb{F}_4$, and $\tau(h) = e_{23}^a e_{12}^a h_0 = h e_{13}^{a^2}$.

(b) Assume $P < S$ and $P \cong C_2^4$. Thus $|P \cap S_0| \geq 8$. Since $I(S_0) \subseteq A_1 \cup A_2$, there is $x \in (A_i \smallsetminus Z_0) \cap P$ for some $i = 1, 2$. But then $P \leq C_S(x) = A_i$ (recall that $\tau(A_i) = A_{3-i}$), so $P = A_i$. □

We are now ready to list the reduced fusion systems over groups in the class $\mathcal{U}$, beginning with $S_0 = UT_3(4)$ itself.

PROPOSITION 6.4. *Each reduced fusion system over $UT_3(4)$ is isomorphic to the fusion system of $PSL_3(4)$.*

PROOF. Set $S = S_0 = UT_3(4)$ and $Z = Z(S) = \{e_{13}^a \,|\, a \in \mathbb{F}_4\}$. Let $\mathcal{F}$ be a saturated fusion system over S such that $O_2(\mathcal{F}) = 1$. Each $\mathcal{F}$-essential subgroup of S contains $Z = \mathrm{Fr}(S)$, and thus is normal in S. If $P \in \mathbf{E}_{\mathcal{F}}^{(\mathrm{I})}$, then $P \cong C_2^4$ by Proposition 3.4, and hence $P = A_1$ or A_2 by Lemma 6.3(b). If $P \le S$ has index 2, then $[P,P] = Z = \mathrm{Fr}(S)$ by Lemma C.6(b), hence $[g,P] \le \mathrm{Fr}(P)$ for each $g \in S \smallsetminus P$, which by Lemma 1.8 implies P is not essential. Thus $\mathbf{E}_{\mathcal{F}} \subseteq \{A_1, A_2\}$. If $\mathbf{E}_{\mathcal{F}} = \varnothing$, then $S \trianglelefteq \mathcal{F}$, while if $\mathbf{E}_{\mathcal{F}} = \{A_i\}$, then $A_i \trianglelefteq \mathcal{F}$. Since we are assuming $O_2(\mathcal{F}) = 1$, $\mathbf{E}_{\mathcal{F}} = \{A_1, A_2\}$.

For each $i = 1, 2$, $\mathrm{Aut}_S(A_i) \cong C_2^2$, and $C_{A_i}(S) = Z$ has rank 2. Hence by Lemma 3.3(c), $\mathrm{Aut}_{\mathcal{F}}(A_i) \cong SL_2(4)$ or $GL_2(4)$, and is conjugate in $\mathrm{Aut}(A_i)$ to $\mathrm{Aut}_{G_i}(A_i)$, where $G_i = PSL_3(4)$ or $PGL_3(4)$.

Fix $\alpha \in \mathrm{Aut}(A_1)$ such that ${}^{\alpha}\mathrm{Aut}_{\mathcal{F}}(A_1) = \mathrm{Aut}_{G_1}(A_1)$. Upon composing with an appropriate element of $\mathrm{Aut}_{\mathcal{F}}(A_1)$, we can assume that α commutes with conjugation by e_{23}^1. Then $\alpha(Z) = Z$ since $Z = [e_{23}^1, A_1]$. Upon composing by $\phi|_{A_1}$ if necessary, we can assume that $(\alpha|_Z)^3 = \mathrm{Id}$, and then upon composing by an appropriate element in $C_{\mathrm{Aut}(A_1)}(\mathrm{Aut}_{\mathcal{F}}(A_1)) \cong C_3$, we can assume that $\alpha|_Z = \mathrm{Id}$ (and still α commutes with conjugation by e_{23}^1). Since conjugation by e_{23}^1 induces an isomorphism from A_1/Z to Z, α also induces the identity on A_1/Z.

By a similar argument, there is $\beta \in \mathrm{Aut}(A_2)$ such that $\beta|_Z = \mathrm{Id}$, $[\beta, A_2] \le Z$, and ${}^{\beta}\mathrm{Aut}_{\mathcal{F}}(A_2) = \mathrm{Aut}_{G_2}(A_2)$. Let $\varphi \in \mathrm{Aut}(S)$ be such that $\varphi|_{A_1} = \alpha$ and $\varphi|_{A_2} = \beta$. (Note that φ has the form $\varphi(g) = g\chi(g)$ for some $\chi \in \mathrm{Hom}(S, Z(S))$.) By the extension axiom (and since all automorphisms of S of odd order normalize A_1 and A_2), for $i = 1$ or 2, $\mathrm{Aut}_{\mathcal{F}}(S) \cong C_3$ if $\mathrm{Aut}_{\mathcal{F}}(A_i) = SL_2(4)$, and $\mathrm{Aut}_{\mathcal{F}}(S) \cong C_3 \times C_3$ otherwise. Thus $\mathrm{Aut}_{\mathcal{F}}(A_1) \cong \mathrm{Aut}_{\mathcal{F}}(A_2)$, $G_1 = G_2$, and ${}^{\varphi}\mathcal{F} = \mathcal{F}_S(G_1)$.

We have now shown that each saturated fusion system $\mathcal{F}$ over S such that $O_2(\mathcal{F}) = 1$ is isomorphic to $\mathcal{F}_S(PSL_3(4))$ or $\mathcal{F}_S(PGL_3(4))$. So if $\mathcal{F}$ is reduced, then it is the fusion system of $PSL_3(4)$. □

We now look at extensions of $UT_3(4)$. Since reduced fusion systems over S_ϕ and S_θ were described in [**OV**, § 4–5], it remains to examine fusion systems over S_τ, S_τ^*, $S_{\phi,\tau}$, and $S_{\phi,\tau}^*$.

PROPOSITION 6.5. *There are no reduced fusion systems over S_τ nor over S_τ^*.*

PROOF. Assume $S = S_\tau$ or S_τ^* in the notation of 6.1. Let $\mathcal{F}$ be any reduced fusion system over S. If $P \in \mathbf{E}_{\mathcal{F}}^{(\mathrm{I})}$, then by Propositions 3.4 and 3.5, $P \cong C_2^4$ or 2_-^{1+4}. The latter case cannot occur (P would have to be normal, and hence contain $Z(S) = Z(S_0) \cong C_2^2$ by Lemma C.9). So $\mathbf{E}_{\mathcal{F}}^{(\mathrm{I})} \subseteq \{A_1, A_2\}$ by Lemma 6.3(b).

Assume $P \in \mathbf{E}_{\mathcal{F}}^{(\mathrm{II})}$. Since $|S| = 2^7$ and $S \ncong D_8 \wr C_2$, $\mathscr{Y}(S) = \varnothing$. By Lemma C.9, there are no normal dihedral or quaternion subgroups, so $\mathscr{X}(S) = \varnothing$. So by Theorem 3.1(b), P is in an $\mathcal{F}$-essential pair of the type described in Lemma 3.7(b). This would require a subgroup $T \cong C_2^4$ with normalizer of order 2^7, which contradicts Lemma 6.3(b). Thus $\mathbf{E}_{\mathcal{F}}^{(\mathrm{II})} = \varnothing$.

Now assume $P \in \mathbf{E}_{\mathcal{F}}^{(\mathrm{III})}$; i.e., $[S{:}P] = 2$. Then by (6.1) and Lemma 6.3(a),

$$P \geq \mathrm{Fr}(S) = H_0 = \langle e_{12}^1 e_{23}^1, e_{12}^{\omega} e_{23}^{\omega} \rangle \cong C_4 \times C_4.$$

For $g \in S \smallsetminus P$, $[g, P] \leq H_0$ and

$$[g, H_0] \leq [S_0, S_0] {\cdot} [\boldsymbol{\phi}, H_0] = Z(S_0) = \mathrm{Fr}(H_0) \leq \mathrm{Fr}(P).$$

So by Lemma 1.8, H_0 is not charateristic in P. Thus H_0 is not the only subgroup of P isomorphic to $C_4 \times C_4$, $H_j \leq P$ for $j \in \{1, 2\}$ by Lemma 6.3(a), and $P \geq H_0 H_j = S_0$. So $P = S_0$ in this case.

Thus $\mathbf{E}_{\mathcal{F}} \subseteq \{A_1, A_2, S_0\}$. Also, $[\mathrm{Aut}_{\mathcal{F}}(S), S] \leq S_0$ since S_0 is a characteristic subgroup of index 2 in S by Lemma C.9. Hence by Proposition 1.14(b), $\mathfrak{foc}(\mathcal{F}) \leq S_0$, and $\mathcal{F}$ is not reduced. □

We now turn to fusion systems over $S = S_{\phi,\tau}$ or $S_{\phi,\tau}^*$. Set $Z = Z(S) = \langle e_{13}^1 \rangle$. There is an epimorphism $\chi\colon S \longrightarrow D_8 \wr C_2$ with kernel Z, whose inverse is defined in Table 6.1. Here, we follow Notation 5.4 for elements $a_i, b_i, z_i, t \in D_8 \wr C_2$. To see

$i =$	$\chi^{-1}(a_i)$	$\chi^{-1}(b_i)$	$\chi^{-1}(a_i b_i)$	$\chi^{-1}(z_i)$	$\chi^{-1}(t)$
1	$e_{23}^1 \boldsymbol{\tau} Z$	$e_{12}^1 \boldsymbol{\phi} Z$	$\boldsymbol{\theta} Z$	$e_{12}^1 e_{23}^1 Z$	$e_{12}^{\omega} Z$
2	$e_{12}^{\omega} e_{23}^{\overline{\omega}} \boldsymbol{\tau} Z$	$\boldsymbol{\phi} Z$	$e_{12}^{\omega} e_{23}^{\overline{\omega}} \boldsymbol{\theta} Z$	$e_{12}^1 e_{23}^1 e_{13}^{\omega} Z$	

TABLE 6.1

that this is a well defined isomorphism, it suffices to check that the images under χ^{-1} of a_1, b_1, $a_1 b_1$, and z_1 satisfy the relations needed to lie in a dihedral group, that conjugation by e_{12}^{ω} sends each coset in the first row to the corresponding coset in the second, and that $\langle e_{23}^1 \boldsymbol{\tau}, e_{12}^1 \boldsymbol{\phi} \rangle$ commutes with $\langle e_{12}^{\omega} e_{23}^{\overline{\omega}} \boldsymbol{\tau}, \boldsymbol{\phi} \rangle$ (modulo Z).

PROPOSITION 6.6. *Let $\mathcal{F}$ be a reduced fusion system over S, where $S \in \mathcal{U}$ and $|S| = 2^8$. Then $S \cong S_{\phi,\tau}$, and $\mathcal{F}$ is isomorphic to the fusion system of Lyons's group. Also,* $\mathrm{Out}(S, \mathcal{F}) = 1$*, and S_τ is the unique $\mathcal{F}$-essential subgroup with noncyclic center.*

PROOF. By Lemma 6.2(a), $S \cong S_{\phi,\tau}$ or $S_{\phi,\tau}^*$. So assume $S = S_{\phi,\tau}$ or $S_{\phi,\tau}^*$. We use the notation of 6.1 for elements and subgroups of S, and in particular use (6.1) (without always saying so) for commutator relations among the elements e_{ij}^a.

Step 1: Set

$$Y_1 = \chi^{-1}(\langle z_1, b_1, z_2, b_2 \rangle) = Z(S_0) \langle e_{12}^1, e_{23}^1, \boldsymbol{\phi} \rangle$$
$$Y_2 = \chi^{-1}(\langle z_1, a_1 b_1, z_2, a_2 b_2 \rangle) = Z(S_0) \langle e_{12}^1 e_{23}^1, e_{12}^{\omega} e_{23}^{\overline{\omega}}, \boldsymbol{\theta} \rangle$$
$$Y_3 = \chi^{-1}(\langle a_1 a_2, b_1 b_2, z_1, \boldsymbol{\tau} \rangle) = S_0\,.$$

We first show that

$$Y_1 \cong 2_+^{1+4}, \qquad Y_2 \cong 2_-^{1+4}, \qquad \text{and} \qquad Y_1 \langle e_{12}^{\omega} \rangle = A_1 \langle e_{23}^1, \boldsymbol{\phi} \rangle \cong UT_4(2)\,. \qquad (6.2)$$

The third isomorphism follows from Lemma C.4(a), since $\langle e_{23}^1, \boldsymbol{\phi} \rangle \cong C_2^2$, and the $\langle e_{23}^1, \boldsymbol{\phi} \rangle$-orbit of e_{12}^{ω} is a basis of A_1. Hence $Y_1 \cong 2_+^{1+4}$ by Lemma C.4(b), and since $\chi(Y_1) \cong Y_1/Z$ is the unique abelian subgroup of index 2 in $\chi(Y_1 \langle e_{12}^{\omega} \rangle) = \langle z_1, b_1, z_2, b_2, t \rangle$. Alternatively, $Y_1 = U_1 U_2$ where $U_i = \chi^{-1}(\langle z_i, b_i \rangle)$, U_1 and U_2

are both dihedral (if $S = S_{\phi,\tau}$) or quaternion (if $S = S^*_{\phi,\tau}$), and $[U_1, U_2] = 1$ (so $Y_1\langle e^{\omega}_{12}\rangle \cong UT_4(2)$ by Lemma C.4(b)). As for Y_2, the five elements

$$w_1 = e^{\omega}_{13}, \quad w_2 = \boldsymbol{\theta}, \quad w_3 = e^{1}_{12}e^{1}_{23}\boldsymbol{\theta}, \quad w_4 = e^{\omega}_{12}e^{\overline{\omega}}_{23}\boldsymbol{\theta}, \quad w_5 = e^{\overline{\omega}}_{12}e^{\omega}_{23}\boldsymbol{\theta} \tag{6.3}$$

all have order 2, and $[w_i, w_j] = e^1_{13}$ for $i \neq j$. So the associated quadratic form ($gZ \mapsto g^2$) on Y_2/Z has exactly five isotropic points, and by Lemma A.5, it is the nondegenerate nonhyperbolic form and hence $Y_2 \cong 2^{1+4}_-$.

Since $\chi(Y_1) \cong \chi(Y_2) \cong C_2^4$ and $\chi(Y_3) = \chi(S_0) \cong 2^{1+4}_+$, $\mathscr{Y}(S) \subseteq \{Y_1, Y_2, Y_3\}$ by Lemma 2.4(b). By Definition 2.1(c,d) and since $S/Z \cong D_8 \wr C_2$, $Y_1, Y_2 \in \mathscr{Y}_0(S)$, and lie in $\mathscr{Y}(S)$ since they are normal. If $Y_3 = S_0 \in \mathscr{Y}(S)$, then since $S_0 \notin \mathscr{Y}_0(S)$ (Definition 2.1(c) again), it must be the normal closure of some $Y_0 \in \mathscr{Y}_0(S)$ of index 4 in S_0 (Lemma 2.4(b)), $Y_0 \cong C_2^4$, so $Y_0 = A_1$ or A_2 by Lemma 6.3(b), which is not possible since $N_S(A_1) = S_0\langle\boldsymbol{\phi}\rangle \ncong D_8 \wr C_2$. Thus

$$\mathscr{Y}_0(S) = \mathscr{Y}(S) = \{Y_1, Y_2\}\,. \tag{6.4}$$

Note also that

$$\mathrm{Aut}(S) \quad \text{and} \quad \mathrm{Aut}(S_0\langle\boldsymbol{\phi}\rangle) \quad \text{are 2-groups:} \tag{6.5}$$

the first by Corollary 2.5 and since $\mathscr{Y}(S) \neq \varnothing$, and the second by [**OV**, Lemma 5.5]. These also follow from the description of $\mathrm{Aut}(S_0)$ in Lemma C.8 and since S_0 is characteristic in both groups.

Step 2: By Proposition 3.9(a), $\mathrm{Out}_{\mathcal{F}}(S) = 1$,

$$\mathbf{E}_{\mathcal{F}} = \mathbf{E}_{\mathcal{F}}(Y_1) \cup \mathbf{E}_{\mathcal{F}}(Y_2) \cup \mathbf{E}_{\mathcal{F}}(Y_3), \quad \text{and} \quad \mathbf{E}_{\mathcal{F}}(Y_i) \neq \varnothing \text{ for each } i = 1,2,3. \tag{6.6}$$

By Proposition 3.9(c) and (6.4), $\mathbf{E}_{\mathcal{F}}(Y_i) \subseteq \mathbf{E}^{(\mathrm{II})}_{\mathcal{F}}$ for $i = 1, 2$. By Proposition 3.11(b.1),

$$\begin{aligned} Y_1 \cong 2^{1+4}_+ \quad &\Longrightarrow \quad \mathrm{Out}_{\mathcal{F}}(Y_1) = \mathrm{Out}(Y_1) \cong SO^+_4(2) \cong \Sigma_3 \wr C_2 \\ Y_2 \cong 2^{1+4}_- \quad &\Longrightarrow \quad \mathrm{Out}_{\mathcal{F}}(Y_2) = \mathrm{Out}(Y_2) \cong SO^-_4(2) \cong \Sigma_5\,. \end{aligned} \tag{6.7}$$

Next assume $R \in \mathbf{E}_{\mathcal{F}}(Y_3)$. By Proposition 3.9(c) and since $Y_3 \notin \mathscr{Y}(S)$, R has type (II) or (III). If $R \in \mathbf{E}^{(\mathrm{III})}_{\mathcal{F}}(Y_3)$, then by the same lemma, $R > Y_3 = S_0$. Since $S_0\langle\boldsymbol{\phi}\rangle \notin \mathbf{E}_{\mathcal{F}}$ by (6.5), $\mathbf{E}^{(\mathrm{III})}_{\mathcal{F}}(Y_3) \subseteq \{S_0\langle\boldsymbol{\theta}\rangle, S_0\langle\boldsymbol{\tau}\rangle\}$.

Let (R_1, R_2) be an $\mathcal{F}$-essential pair of type (II) in $\mathbf{E}_{\mathcal{F}}(Y_3)$, and set $T = R_1 \cap R_2$. By Theorem 3.1(b), this has the type described in Lemma 3.7(b). Thus $T \cong C_2^4$, $|R_1/T| = 2$, T is the $L_2(4)$-module for $\mathrm{Aut}_{\mathcal{F}}(T) \geq \Sigma_5$, $\mathrm{Out}_{R_1}(T) \nleq O^2(\mathrm{Out}_{\mathcal{F}}(T))$, and $\mathfrak{foc}(\mathcal{F}, R_1) = S_0$ is the normal closure of T. Thus $T = A_1$ or A_2 by Lemma 6.3(b).

Since $N_S(A_1) = S_0\langle\boldsymbol{\phi}\rangle$ and $\mathrm{rk}(C_{A_1}(\langle e^1_{23}, \boldsymbol{\phi}\rangle)) = 1$, and since A_1 is the $L_2(4)$-module for $O^2(\mathrm{Aut}_{\mathcal{F}}(A_1)) \cong A_5$, $\mathrm{Aut}_{\langle e^1_{23}, \boldsymbol{\phi}\rangle}(A_1)$ is not contained in $O^2(\mathrm{Aut}_{\mathcal{F}}(A_1))$. Hence $O^2(\mathrm{Aut}_{\mathcal{F}}(A_1)) \cap \mathrm{Aut}_S(A_1) = \mathrm{Aut}_{S_0}(A_1)$, and similarly for A_2. Thus $R_1 = T\langle x\rangle$ for some $x \in S_0\langle\boldsymbol{\phi}\rangle \smallsetminus S_0$ such that $x^2 \in T$, and this implies that R_1 is S-conjugate to $A_1\langle\boldsymbol{\phi}\rangle$.

Set $H_1 = A_1\langle\boldsymbol{\phi}\rangle$. Set $N_1 = N_S(H_1) = A_1\langle e^1_{23}, \boldsymbol{\phi}\rangle = Y_1\langle e^{\omega}_{12}\rangle$. Then $N_1 \cong UT_4(2)$ by (6.2), and $UT_4(2) \cong (Q_8 \times_{C_2} Q_8) \overset{t}{\rtimes} C_2$ by Lemma C.4(b). So $N_1 \in \mathbf{E}^a_{\mathcal{F}}(Y_1) \subseteq \mathbf{E}_{\mathcal{F}}$ by Lemma 5.2(a), and there is $\mathrm{Id} \neq \gamma \in \mathrm{Aut}_{\mathcal{F}}(N_1)$ of odd order. By Lemma C.4(d), γ permutes transitively the three subgroups of index 2 in N_1 which contain A_1. Thus $H_1 = A_1\langle\boldsymbol{\phi}\rangle$ is $\mathcal{F}$-conjugate to $A_1\langle e_{23}^1\rangle$ with normalizer $S_0\langle\boldsymbol{\phi}\rangle$, so H_1 is not fully normalized, and hence not in $\mathbf{E}_{\mathcal{F}}$. We conclude that:

$$\mathbf{E}_{\mathcal{F}}(Y_3) \subseteq \bigl\{S_0\langle\boldsymbol{\theta}\rangle, S_0\langle\boldsymbol{\tau}\rangle\bigr\}\,. \tag{6.8}$$

Step 3: By Lemma 5.2(b), for $i = 1, 2$ and $R_i \in \mathbf{E}_{\mathcal{F}}(Y_i)$, $R_i \geq Y_i$, and hence $Z(R_i) = Z(Y_i) = Z(S) = \langle e_{13}^1\rangle$ by Lemma A.6(a). So by (6.6) and (6.8), the only (possible) subgroup $R \in \mathbf{E}_{\mathcal{F}}$ with $Z(R) > Z(S)$ is $S_0\langle\boldsymbol{\tau}\rangle$. Since $\mathcal{F}$ is reduced, $Z(S) \nleq O_2(\mathcal{F}) = 1$, so $S_0\langle\boldsymbol{\tau}\rangle \in \mathbf{E}_{\mathcal{F}}$, and there is $\beta \in \mathrm{Aut}_{\mathcal{F}}(S_0\langle\boldsymbol{\tau}\rangle)$ of odd order such that $\beta|_{Z(S_0)}$ has order 3.

For each $g \in S_0$, $(g\boldsymbol{\tau})^2 = g\tau(g)\boldsymbol{\tau}^2$. If $g\tau(g) \in Z(S_0) = \{e_{13}^x \,|\, x \in \mathbb{F}_4\}$, then $g \equiv e_{12}^a e_{23}^a \pmod{Z(S_0)}$ for some $a \in \mathbb{F}_4$, which implies that $g\tau(g) = 1$. Thus the only element of $Z(S_0)$ which is the square of an element in the coset $S_0\boldsymbol{\tau}$ is $\boldsymbol{\tau}^2$. Since β permutes transitively the elements of $Z(S_0)^{\#}$, this implies that $\boldsymbol{\tau}^2 = 1$. In other words, $S = S_{\phi,\tau}$, and there are no reduced fusion systems over $S_{\phi,\tau}^*$.

Step 4: From now on, we assume $S = S_{\phi,\tau}$ (i.e., $\boldsymbol{\tau}^2 = 1$). We can now write $S_\theta = S_0\langle\boldsymbol{\theta}\rangle$, $S_\tau = S_0\langle\boldsymbol{\tau}\rangle$, and $S_\phi = S_0\langle\boldsymbol{\phi}\rangle$.

For $i = 1, 2$, let $\mathscr{U}_i = \mathscr{U}_{\mathcal{F}}(Y_i)$ be as in Proposition 3.11(b). Since $\mathrm{Out}_{\mathcal{F}}(Y_i) = \mathrm{Out}(Y_i)$ by (6.7), these sets are uniquely determined by Proposition 3.11(b.2), and they in turn determine the sets $\mathbf{E}_{\mathcal{F}}(Y_i) = \mathbf{E}_{\mathcal{F}}^a(Y_i) \cup \mathbf{E}_{\mathcal{F}}^c(Y_i)$ (Lemma 5.2(a)). For $P \in \mathbf{E}_{\mathcal{F}}^a(Y_i)$, $\mathrm{Aut}_{\mathcal{F}}(P)$ is uniquely determined by Lemma 5.2(c). For $P \in \mathbf{E}_{\mathcal{F}}^c(Y_i)$, $P = UC_S(U)$ for some $U \in \mathscr{U}_i$, and by Proposition 3.11(c.4), $\mathrm{Aut}_{\mathcal{F}}^*(P) = O^2\bigl(\mathrm{Inn}(P)\langle\sigma\rangle\bigr)$ for some σ of order 3 such that $\sigma(U) = U$ and $\sigma|_{C_S(U)} = \mathrm{Id}$. Thus all $\mathcal{F}$-automorphisms of subgroups in $\mathbf{E}_{\mathcal{F}}(Y_1) \cup \mathbf{E}_{\mathcal{F}}(Y_2)$ are uniquely determined by these conditions.

The elements $w_1, \ldots, w_5 \in Y_2$ of (6.3) are permuted under the action of $\mathrm{Out}_{\mathcal{F}}(Y_2) \cong \Sigma_5$, and $\mathrm{Out}_{S_\theta}(Y_2)$ is generated by the elements $c_{e_{12}^1}$ and $c_{e_{12}^\omega}$, corresponding to the permutations $(2\,3)(4\,5)$ and $(2\,4)(3\,5)$, respectively. Let $\rho \in \mathrm{Out}_{\mathcal{F}}(Y_2)$ be an automorphism of order 3 which induces the 3-cycle $(3\,4\,5)$ (i.e., ρ permutes cyclically the elements w_3, w_4, w_5). Then ρ normalizes $\mathrm{Aut}_{S_\theta}(Y_2)$, and hence by the extension axiom extends to some $\widehat{\rho} \in \mathrm{Aut}_{\mathcal{F}}(S_\theta)$. We showed in Step 3 that there is $\beta \in \mathrm{Aut}_{\mathcal{F}}(S_\tau)$ of order 3 (and thus $\mathbf{E}_{\mathcal{F}}(Y_3) = \{S_\theta, S_\tau\}$ by (6.8)). In particular, $[\widehat{\rho}|_{S_0}, \theta], [\beta|_{S_0}, \tau] \in \mathrm{Inn}(S_0)$.

Recall the description of $\mathrm{Aut}(S_0)$ before Lemma C.8:

$$\mathrm{Aut}(S_0) = O_2(\mathrm{Aut}(S_0))\cdot(\Gamma_0 \times \Gamma_1), \text{ where } \Gamma_0 = \langle\gamma_0, \theta\rangle \cong \Sigma_3,\ \Gamma_1 = \langle\gamma_1, \tau\rangle \cong \Sigma_3.$$

Hence $\widehat{\rho}|_{S_0} \equiv \gamma_1^{\pm 1}$ and $\beta|_{S_0} \equiv \gamma_0^{\pm 1} \pmod{O_2(\mathrm{Aut}(S_0))}$,

$$\mathrm{Out}_{\mathcal{F}}(S_0) = \langle \mathrm{Out}_S(S_0), [\widehat{\rho}|_{S_0}], [\beta|_{S_0}]\rangle \cong \Sigma_3 \times \Sigma_3$$

(it cannot be larger by the Sylow axiom), and $O_2(\mathrm{Out}(S_0))\cdot\mathrm{Out}_{\mathcal{F}}(S_0) = \mathrm{Out}(S_0)$. So by Lemma A.7, applied with $G = \mathrm{Out}(S_0)$, $Q = O_2(G)$, $H_0 = \mathrm{Out}_S(S_0)$, and $H = \langle \mathrm{Out}_S(S_0), [\gamma_0], [\gamma_1]\rangle$, there is $\varphi_0 \in O_2(\mathrm{Aut}(S_0))$ such that

$$[\varphi_0] \in C_{O_2(\mathrm{Out}(S_0))}(\mathrm{Out}_S(S_0)) \quad \text{and} \quad {}^{\varphi_0}\mathrm{Aut}_{\mathcal{F}}(S_0) = \langle \mathrm{Aut}_S(S_0), \gamma_0, \gamma_1\rangle.$$

By Lemma C.8, $\mathrm{Out}_S(S_0) \cong C_2^2$ permutes freely a basis for $O_2(\mathrm{Out}(S_0)) \cong C_2^4$, so $\mathrm{rk}(C_{O_2(\mathrm{Out}(S_0))}(\mathrm{Out}_S(S_0))) = 1$. Define $\psi \in \mathrm{Aut}(S)$ by setting $\psi(g) = g$ for $g \in Y_1Y_2$ and $\psi(g) = e_{13}^1 g$ for $g \in S \smallsetminus Y_1Y_2$. Since $|S/Y_1Y_2| = 2$ and $e_{13}^1 \in Z(S)$, this does define an automorphism of S. Also, $[\psi|_{S_0}]$ centralizes $\mathrm{Out}_S(S_0)$ since it extends to S, $\psi|_{S_0} \in O_2(\mathrm{Aut}(S_0))$ since it is the identity modulo $Z(S_0)$, and

$\psi|_{S_0} \notin \mathrm{Inn}(S_0)$. (Recall that $[g, S_0] = Z(S_0)$ for $g \in S_0 \smallsetminus Z(S_0)$.) We have now shown

$$C_{O_2(\mathrm{Out}(S_0))}(\mathrm{Out}_S(S_0)) = \langle [\psi|_{S_0}] \rangle \neq 1 \quad \text{where} \quad \psi \in \mathrm{Aut}(S),\ \psi|_{Y_1Y_2} = \mathrm{Id}. \tag{6.9}$$

Thus $\varphi_0 \in \mathrm{Inn}(S_0)\langle \psi|_{S_0} \rangle$, and hence φ_0 extends to some $\varphi \in \mathrm{Aut}(S)$. Upon replacing $\mathcal{F}$ by ${}^{\varphi}\mathcal{F}$, we can assume that $\mathrm{Aut}_{\mathcal{F}}(S_0) = \langle \mathrm{Aut}_S(S_0), \gamma_0, \gamma_1 \rangle$.

Define $\dot\gamma_0 \in \mathrm{Aut}(S_\tau)$ and $\dot\gamma_1 \in \mathrm{Aut}(S_\theta)$ by setting $\dot\gamma_i|_{S_0} = \gamma_i$, $\dot\gamma_0(\boldsymbol{\tau}) = \boldsymbol{\tau}$, and $\dot\gamma_1(\boldsymbol{\theta}) = \boldsymbol{\theta}$. Since $H^1(\langle \boldsymbol{\theta} \rangle; Z(S_0)) = 0$ (since θ exchanges the elements in the basis $\{e_{13}^{\omega}, e_{13}^{\bar\omega}\} \subseteq Z(S_0)$), there is a unique extension of γ_1 to $\dot\gamma_1 \in \mathrm{Out}(S_\theta)$ (unique modulo $c_{e_{13}^1}$), and $\mathrm{Aut}_{\mathcal{F}}(S_\theta) = \langle \mathrm{Aut}_S(S_\theta), \dot\gamma_1 \rangle$.

The choice of extension of γ_0 to S_τ is not unique. Set

$$G = \bigl\{\alpha \in \mathrm{Aut}(S_\tau) \bigm| \alpha|_{S_0} \in \langle \gamma_0, \theta \rangle \bigr\} \quad \text{and} \quad V = \bigl\{\alpha \in G \bigm| \alpha|_{S_0} = \mathrm{Id}\bigr\}.$$

Then $V \cong \mathrm{Hom}(S_\tau/S_0, Z(S_0)) \cong C_2^2$, and hence $G \cong \Sigma_4$. Also, $V \cap \mathrm{Aut}_{\mathcal{F}}(S_\tau) = 1$ by the Sylow axiom. For any $\gamma^* \in \mathrm{Aut}_{\mathcal{F}}(S_\tau)$ such that $\gamma^*|_{S_0} = \gamma_0$, we have $G \cap \mathrm{Aut}_{\mathcal{F}}(S_\tau) = \langle \gamma^*, c_{\boldsymbol{\theta}} \rangle$. By Lemma A.7 (or by a direct check since $G \cong \Sigma_4$), there is $\alpha \in C_V(c_{\boldsymbol{\theta}})$ such that ${}^{\alpha}(\dot\gamma_0) = \gamma^*$. Then either $\alpha = \mathrm{Id}$, or $\alpha(\boldsymbol{\tau}) = e_{13}^1\boldsymbol{\tau}$. In either case, α extends to an automorphism $\widehat{\alpha}$ of S, and upon replacing $\mathcal{F}$ by ${}^{\widehat{\alpha}}\mathcal{F}$, we can arrange that $\mathrm{Aut}_{\mathcal{F}}(S_\tau) = \langle \mathrm{Aut}_S(S_\tau), \dot\gamma_0 \rangle$. Also, $\mathrm{Out}_{\mathcal{F}}(S) = 1$ by (6.5).

Step 5: We have now shown that up to isomorphism, there is at most one reduced fusion system $\mathcal{F}$ over $S = S_{\phi,\tau}$. By [**Ly**, Proposition 2.5], Lyons' group Ly contains a subgroup isomorphic to 3McL:2 with odd index. Also, S_ϕ is isomorphic to a Sylow 2-subgroup of McL, and this group has an outer automorphism whose restriction to S_0 is τ (see, e.g., [**AOV1**, Table 4.1] and the proof of Proposition 4.5 there). So by Lemma 6.2(a), Aut(McL) and hence Ly have Sylow 2-subgroups isomorphic to $S_{\phi,\tau}$ or $S_{\phi,\tau}^*$. Since $\mathrm{Aut}(S_{\phi,\tau})$ and $\mathrm{Aut}(S_{\phi,\tau}^*)$ are both 2-groups by Corollary 2.5, the fusion system of Ly is reduced by Proposition 1.12. Hence the Sylow 2-subgroups of Ly are isomorphic to $S_{\phi,\tau}$, and its fusion system is isomorphic to $\mathcal{F}$.

It remains to prove that $\mathrm{Out}(S, \mathcal{F}) = 1$. Fix $\varphi \in \mathrm{Aut}(S, \mathcal{F})$, and set $\varphi_0 = \varphi|_{S_0}$. Upon replacing φ by some other element of $\varphi \circ \mathrm{Inn}(S)$, we can assume that $\varphi_0 \in O_2(\mathrm{Aut}(S_0))\langle \gamma_0, \gamma_1 \rangle$. Since $\mathrm{Aut}(S)$ is a 2-group by (6.5), this implies that $\varphi_0 \in O_2(\mathrm{Aut}(S_0))$. Also, $[\varphi_0] \in \mathrm{Out}(S_0)$ centralizes $\mathrm{Out}_S(S_0) = \langle \tau, \phi \rangle$, since $\varphi_0 = \varphi|_{S_0}$ where φ normalizes each of the subgroups S_θ, S_τ, and S_ϕ (since they are pairwise nonisomorphic). So $\varphi_0 \in \mathrm{Inn}(S_0)\langle \psi|_{S_0} \rangle$ by (6.9). Since φ is fusion preserving, φ_0 normalizes $\mathrm{Aut}_{\mathcal{F}}(S_0) = \langle \mathrm{Aut}_S(S_0), \gamma_0, \gamma_1 \rangle$, and hence $[\varphi_0, \mathrm{Aut}_{\mathcal{F}}(S_0)] \in \mathrm{Inn}(S_0)$ (recall that $\varphi_0 \in O_2(\mathrm{Aut}(S_0))$). Since $[\psi|_{S_0}, \gamma_0] \notin \mathrm{Inn}(S_0)$ (since γ_0 does not normalize $Y_1Y_2 \cap S_0$), this proves that $\varphi_0 \in \mathrm{Inn}(S_0)$. So without changing $[\varphi] \in \mathrm{Out}(S)$, we can assume that $\varphi_0 = \mathrm{Id}$.

Let $g, h \in Z(S_0)$ be such that $\varphi(\boldsymbol{\tau}) = g\boldsymbol{\tau}$ and $\varphi(\boldsymbol{\phi}) = h\boldsymbol{\phi}$. The relations $(h\boldsymbol{\phi})^2 = 1 = [g\boldsymbol{\tau}, h\boldsymbol{\phi}]$ imply that $g, h \in \langle e_{13}^1 \rangle$. If $g = e_{13}^1$, then $[\dot\gamma_0, \varphi|_{S_\tau}]$ sends $\boldsymbol{\tau}$ to $e_{13}^{\omega}\boldsymbol{\tau}$, so $\varphi|_{S_\tau}$ does not normalize $\mathrm{Aut}_{\mathcal{F}}(S_\tau)$. Thus $g = 1$, so $\varphi \in \mathrm{Aut}_{Z(S_0)}(S)$, and hence $\mathrm{Out}(S, \mathcal{F}) = 1$. □

APPENDIX A

Background results about groups

We collect here several general results about finite groups, especially p-groups, and their automorphisms.

LEMMA A.1. (a) *If $P < S$ are p-groups for some prime p, then $P < N_S(P)$.*

(b) *If $P < S$ are p-groups, and P is characteristic in $N_S(P)$, then $P \trianglelefteq S$.*

PROOF. Part (a) is shown, for example, in [**Sz1**, Theorem 2.1.6]. To prove (b), assume P is characteristic in $N_S(P)$. Then each $g \in N_S(N_S(P))$ normalizes P, so $N_S(N_S(P)) = N_S(P)$, and hence $S = N_S(P)$ by (a). □

Recall that $Z_i(G)$ denotes the i-th term in the upper central series for G: $Z_0(G) = G$, and $Z_i(G)/Z_{i-1}(G) = Z(G/Z_{i-1}(G))$ for $i \geq 1$.

LEMMA A.2. *Let S be a p-group, and let $Q \trianglelefteq S$ be a normal subgroup.*

(a) *If $Q \neq 1$, then $Q \cap Z(S) \neq 1$.*

(b) *Assume $|Z_k(S)| = p^k$ for some $k \geq 1$. Then either $Q = Z_i(S)$ for some $i \leq k$, or $Q > Z_k(S)$. If $|Q| = p^{k+1}$, then $Z_k(S) < Q \leq Z_{k+1}(S)$.*

PROOF. Point (a) holds since $Q \cap Z(S) = C_Q(S)$, and $|C_Q(S)| \equiv |Q| \equiv 0 \pmod{p}$. Hence $Z(S) \leq Q$ if $|Z(S)| = p$. If $|Z(S)| = p$ and $|Q| = p^2$, then $Q/Z(S) \leq Z(S/Z(S))$ by (a) again, so $Q \leq Z_2(S)$. This proves (b) when $k = 1$, and the general case follows by induction on k. □

LEMMA A.3. *Fix a p-group S. Assume $Q < S$ is such that $[S{:}Q] = p$ and $Z(Q) > Z(S)$. Then $Q = C_S(x)$ for some $x \in Z_2(S) \smallsetminus Z(S)$. If $\Omega_1(Z(Q)) \nleq Z(S)$, then x can be chosen with order p.*

PROOF. Set $\overline{S} = S/Z(S)$, and set $\overline{P} = PZ(S)/Z(S)$ for each $P \leq S$. For each $x \in Z(Q) \smallsetminus Z(S)$, $Q \leq C_S(x) < S$, and $Q = C_S(x)$ since $[S{:}Q] = p$. Also, $\overline{Z(Q)} \trianglelefteq \overline{S}$ since $Q \trianglelefteq S$, so $\overline{Z(Q)} \cap Z(\overline{S}) \neq 1$ by Lemma A.2(a), and x can be chosen so that $x \in Z_2(S)$. If $\Omega_1(Z(Q)) \nleq Z(S)$, then a similar argument, applied to $\overline{\Omega_1(Z(Q))}$, shows that x can be chosen in $\Omega_1(Z(Q)) \cap Z_2(S)$. □

The next lemma involves abelian subgroups of index 2 or 4 in a 2-group.

LEMMA A.4. *Let S be a nonabelian 2-group, and let $A \trianglelefteq S$ be a normal abelian subgroup.*

(a) *If $[S{:}A] = 2$, and $|[g, A]| \geq 4$ for $g \in S \smallsetminus A$, then A is the unique abelian subgroup of index 2 in S.*

(b) *If $[S{:}A] = 4$, and $|[g, A]| \geq 4$ for each $g \in S \smallsetminus A$, then either A is characteristic in S, or $S/[S, S]$ surjects onto $(S/A) \times (S/A)$. If in addition, $|[g, A]| \geq 8$ for* *some $g \in S \smallsetminus A$, or $|S/[S,S]| \le 8$, then A is the unique abelian subgroup of index 4 in S.*

Proof. Note, for each $g \in G$, that $|[g,A]| = |A/C_A(g)|$, since $C_A(g)$ is the kernel of the homomorphism $A \xrightarrow{a \mapsto [b,a]} A$, while $[g,A]$ is its image.

(a) If $[S{:}A] = 2$, and $B < S$ is another abelian subgroup of index 2, then for $g \in B \smallsetminus A$, $|[g,A]| = |A/C_A(g)| \le |A/(B \cap A)| = 2$.

(b) Assume $[S{:}A] = 4$, and $|[g,A]| \ge 4$ for each $g \in S \smallsetminus A$. If $B \trianglelefteq S$ is another abelian subgroup of index 4, then $AB = S$, since otherwise A and B are both abelian of index 2 in AB, contradicting (a). Also, for each $g \in S$, where $g = ab$ for $a \in A$ and $b \in B$, $|[g,A]| = |[b,A]| = |A/C_A(b)| \le |A/(A \cap B)| = 4$. If B is normal, then $S/[S,S]$ has as quotient $S/(A \cap B) \cong (S/A) \times (S/B)$, so $|S/[S,S]| \ge 16$, and $S/B \cong S/A$ if $B = \varphi(A)$ for some $\varphi \in \mathrm{Aut}(S)$.

If $B \ntrianglelefteq S$, set $S_0 = N_S(B)$. Then $[S{:}S_0] = [S_0{:}B] = 2$ since $B < S_0 < S$ by Lemma A.1, $B \ne {}^xB < S_0$ for $x \in S \smallsetminus S_0$, and thus $S_0 = B{\cdot}{}^xB$. Set $B_0 = B \cap {}^xB$. Then $B_0 \le Z(S_0)$, so $S_0 \cap B_0A$ is abelian of index 2 in B_0A, $|[g,A]| = 2$ for $g \in B_0 \smallsetminus A$ by (a), and this contradicts the original assumption. So B must be normal. □

The next lemma, on quadratic forms over $\mathbb{F}_2$, is very elementary and presumably well known, but we have been unable to find references.

Lemma A.5. *Let V be an $\mathbb{F}_2$-vector space of dimension $2n$ (some $n \ge 1$), and let $\mathfrak{q}\colon V \longrightarrow \mathbb{F}_2$ be a quadratic form. Then*

$$\begin{cases} |\mathfrak{q}^{-1}(0)| = 2^{2n-1} \pm 2^{n-1} & \text{if } \mathfrak{q} \text{ is nondegenerate} \\ |\mathfrak{q}^{-1}(0)| \equiv 0 \pmod{2^n} & \text{if } \mathfrak{q} \text{ is degenerate.} \end{cases} \tag{A.1}$$

If $\dim(V) = 4$ and $\mathfrak{q}$ is nondegenerate, then either

- *$\mathfrak{q}$ is hyperbolic, $\mathfrak{q}^{-1}(1) = V_1^\# \cup V_2^\#$ for some complementary pair of 2-dimensional subspaces $V_1, V_2 < V$, and $\mathrm{Aut}(V,\mathfrak{q}) \cong SO_4^+(2) \cong \Sigma_3 \wr C_2$; or*
- *$\mathfrak{q}^{-1}(0) \smallsetminus \{0\}$ is a set of 5 points permuted transitively by $\mathrm{Aut}(V,\mathfrak{q}) \cong SO_4^-(2) \cong \Sigma_5$.*

Proof. Point (A.1) is easily checked when $n = 1$. So fix $n \ge 2$, and assume (A.1) holds for $n-1$. If $\mathfrak{q}$ is degenerate ($V^\perp \ne 0$), then either there is $v \in V^\perp$ such that $\mathfrak{q}(v) = 1$, in which case $|\mathfrak{q}^{-1}(0)| = 2^{2n-1}$ since $\mathfrak{q}(x+v) = \mathfrak{q}(x) + 1$ for each $x \in V$; or $V = V_1 \perp V_2$ where $\mathrm{rk}(V_1) = 2$ and $\mathfrak{q}|_{V_1} = 0$. In this last case, set $\mathfrak{q}_2 = \mathfrak{q}|_{V_2}$; then $|\mathfrak{q}^{-1}(0)| = 4|\mathfrak{q}_2^{-1}(0)|$ where $|\mathfrak{q}_2^{-1}(0)| \equiv 0 \pmod{2^{n-2}}$ by the induction hypothesis. (See [**Ta**, Theorem 11.5] for a formula which applies to nondegenerate forms over arbitrary finite fields.)

Now assume $\mathfrak{q}$ is degenerate. Then $V = V_1 \perp V_2$, where $\mathrm{rk}(V_1) = 2$, and $\mathfrak{q}_i = \mathfrak{q}|_{V_i}$ is nondegenerate for $i = 1,2$. Hence $\mathfrak{q}_1^{-1}(0) = 2 + \eta$ and $\mathfrak{q}_2^{-1}(0) = 2^{2n-3} + \varepsilon 2^{n-2}$ for some $\eta, \varepsilon \in \{\pm 1\}$, and so

$$\begin{aligned} |\mathfrak{q}^{-1}(0)| &= |\mathfrak{q}_1^{-1}(0)|{\cdot}|\mathfrak{q}_2^{-1}(0)| + |\mathfrak{q}_1^{-1}(1)|{\cdot}|\mathfrak{q}_2^{-1}(1)| \\ &= (2+\eta)(2^{2n-3} + \varepsilon 2^{n-2}) + (2-\eta)(2^{2n-3} - \varepsilon 2^{n-2}) \\ &= 2^{2n-1} + \varepsilon\eta 2^{n-1}. \end{aligned}$$

When $\dim(V) = 4$, each nondegenerate form is equivalent to $\mathfrak{q}_1$ (the hyperbolic form) or $\mathfrak{q}_2$, where

$$\mathfrak{q}_1(x_1, x_2, x_3, x_4) = x_1x_2 + x_3x_4$$
$$\mathfrak{q}_2(x_1, x_2, x_3, x_4) = x_1x_2 + x_3^2 + x_3x_4 + x_4^2$$

(see, e.g., [**A1**, § 21] or [**Sz1**, Proposition 3.5.10]). The properties listed above are easily checked. □

The next two lemmas are is more specialized.

Lemma A.6. *Let S be a 2-group, and let $Q \le S$ be such that $r(Q/Z(Q)) = r(S)$.*

(a) *In all cases, $C_S(Q) \le Q$. In particular, $Z(S) \le Z(N_S(Q)) \le Z(Q)$, and $Z(S) = Z(Q)$ if $|Z(Q)| = 2$.*

(b) *Assume that Q is special of type 2^{2+4} (i.e., $Z(Q) = [Q,Q] \cong C_2^2$ and $Q/Z(Q) \cong C_2^4$), and that $Z(N_S(Q)) < Z(Q)$. Assume also that all involutions in Q are central, or more generally, that the number of classes $gZ(Q) \in (Q/Z(Q))^\#$ such that $g^2 = 1$ is even. Then $Z(Q) = Z_2(S)$.*

(c) *If $U \trianglelefteq S$ is such that $[S,U] \le \mathrm{Fr}(U) \le Z(S)$, then $U \le Q$.*

Proof. **(a)** Since $r(S) \ge r(QC_S(Q)/Z(Q)) = r(Q/Z(Q)) + r(C_S(Q)/Z(Q))$ for any pair $Q \le S$, the assumption $r(S) = r(Q/Z(Q))$ implies that $C_S(Q) = Z(Q) \le Q$.

(b) Assume Q is special of type 2^{2+4}, and $Z(N_S(Q)) < Z(Q)$. Thus $Z(S) = Z(N_S(Q)) = \langle z_0 \rangle$ for some $z_0 \in Z(Q)^\#$. Let $z_1, z_2 \in Z(Q)$ be the other two involutions. Set $V = Q/Z(Q)$, let $\mathfrak{q}\colon V \longrightarrow Z(Q)$ be the quadratic map $\mathfrak{q}(gZ(Q)) = g^2$, and let $\mathfrak{q}_i\colon V \longrightarrow Z(Q)/\langle z_i \rangle \cong \mathbb{F}_2$ $(i = 0, 1, 2)$ be the quadratic form induced by $\mathfrak{q}$. Set $m = |\mathfrak{q}^{-1}(1)|$, and for $i = 0, 1, 2$, set $n_i = |\mathfrak{q}^{-1}(z_i)|$. Note that $n_1 = n_2$, since there is $g \in N_S(Q)$ such that ${}^g z_1 = z_2$.

By assumption, the number k of classes in $(Q/Z(Q))^\#$ which lift to involutions in Q is even, so $m = k+1$ is odd. For each i, by point (A.1) in Lemma A.5, $m+n_i = \mathfrak{q}_i^{-1}(0)$ is even, and thus n_i is odd. Hence $|\mathfrak{q}_0^{-1}(0)| = m + n_0 = 16 - 2n_1 \equiv 2 \pmod 4$, so $\mathfrak{q}_0$ is nondegenerate by (A.1) again. In particular, $Z(Q/\langle z_0 \rangle) = Z(Q)/\langle z_0 \rangle$, and thus $Z_2(S)/\langle z_0 \rangle = Z(S/\langle z_0 \rangle) = Z(Q)/\langle z_0 \rangle$ by (a).

(c) Set $U_0 = \mathrm{Fr}(U) \trianglelefteq S$ for short. Then $QU/U_0 = (U/U_0)(QU_0/U_0)$, where U/U_0 is elementary abelian and $[U/U_0, QU_0/U_0] = 1$, so either $U \le QU_0$ or $r(QU/U_0) > r(QU_0/U_0)$. If $r(QU/U_0) > r(QU_0/U_0)$, then $r(S) > r(QU_0/U_0) = r(Q/(Q \cap U_0)) \ge r(Q/Z(Q))$ since $Q \cap U_0 \le Q \cap Z(S) \le Z(Q)$. Since this contradicts our hypothesis, $U \le QU_0$, so $(U \cap Q)\mathrm{Fr}(U) = U$, and hence $U \le Q$ (cf. [**G**, § 5.1]). □

Lemma A.7 ([**OV**, Proposition 1.8]). *Fix a prime p, a finite group G, and a normal abelian p-subgroup $Q \trianglelefteq G$. Let $H \le G$ be such that $Q \cap H = 1$, and let $H_0 \le H$ be of index prime to p. Consider the set*

$$\mathcal{H} = \bigl\{ H' \le G \bigm| H' \cap Q = 1,\ QH' = QH,\ H_0 \le H' \bigr\}.$$

Then for each $H' \in \mathcal{H}$, there is $g \in C_Q(H_0)$ such that $H' = {}^gH$.

Throughout the rest of the chapter, we recall some general results about automorphisms.

THEOREM A.8 (Krull-Schmidt theorem). *Let G be a finite group, and assume $G = G_1 \times \cdots \times G_k$ for some sequence of indecomposable subgroups $1 \neq G_i \trianglelefteq G$.*

(a) *If $G = G_1^* \times \cdots \times G_\ell^*$ is a second factorization into nontrivial indecomposables, then $k = \ell$, and there are $\sigma \in \Sigma_k$ and $\beta \in \mathrm{Aut}(G)$ such that $\beta \equiv \mathrm{Id}_G$ (mod $Z(G)$) and $\beta(G_i) = G^*_{\sigma(i)}$.*

(b) *For any $\alpha \in \mathrm{Aut}(G)$, there is $\sigma \in \Sigma_k$ such that $\alpha(G_iZ(G)) = G_{\sigma(i)}Z(G)$ for each i.*

PROOF. Point (a) is a special case of the Krull-Schmidt theorem in the form shown in [**Sz1**, Theorem 2.4.8]. Note that by [**Sz1**, 1.6.18], a "normal automorphism" of G is one which is the identity modulo $Z(S)$. Point (b) follows from (a), applied with $G_i^* = \alpha(G_i)$. □

LEMMA A.9. *Fix a prime p, a p-group S, a subgroup $P_0 \leq \mathrm{Fr}(S)$, and a sequence of subgroups*

$$P_0 \trianglelefteq P_1 \trianglelefteq \cdots \trianglelefteq P_k = S$$

all normal in S. Set

$$\mathcal{A} = \bigl\{\alpha \in \mathrm{Aut}(S) \,\big|\, \forall\, 0, \leq i \leq k-1, \quad \alpha(P_i) = P_i \text{ and } [\alpha, P_{i+1}] \leq P_i\bigr\} \leq \mathrm{Aut}(S):$$

the group of automorphisms which induce the identity on each of the quotient groups P_i/P_{i-1}. Then $\mathcal{A}$ is a p-group. If the P_i are all characteristic in S, then $\mathcal{A} \trianglelefteq \mathrm{Aut}(S)$, and hence $\mathcal{A} \leq O_p(\mathrm{Aut}(S))$.

PROOF. See, e.g., [**G**, Theorems 5.1.4 & 5.3.2]. □

As an easy exercise, Lemma A.9 implies the following list of 2-groups whose automorphism groups are 2-groups.

COROLLARY A.10. *For a 2-group S, $\mathrm{Aut}(S)$ is a 2-group if any of the following hold: either*

(a) *S is cyclic, or $S/[S,S] \cong C_{2^m} \times C_{2^n}$ for $m \geq n \geq 2$; or*

(b) *$S \cong D_{2^k}$ with $k \geq 3$, or $S \cong Q_{2^k}$ or SD_{2^k} with $k \geq 4$; or*

(c) *$S \cong D_8 \times D_8$, $D_8 \wr C_2$, or $D_8 \times C_2$.*

PROOF. Each of these follows upon applying Lemma A.9 to an appropriate chain of characteristic subgroups of S. When $S = S_1 \times S_2$ for $S_i \cong D_8$, there is a simpler argument using the Krull-Schmidt theorem (Theorem A.8): each automorphism of S normalizes or exchanges the subgroups $S_iZ(S) \cong D_8 \times C_2$, where $\mathrm{Aut}(D_8 \times C_2)$ is a 2-group. □

LEMMA A.11 ([**O1**, Proposition 2.3]). *Fix an abelian 2-group A, and a subgroup $G \leq \mathrm{Aut}(A)$ with $|G| = 2m$ for some odd m. Assume, for each $x \in I(G)$, that $x \notin Z(G)$ and $[x, A] \cong C_{2^n}$ (some $n \geq 1$). Set $G_1 = O^{2'}(G)$, $G_2 = C_G(G_1)$, and $A_1 = [G_1, A]$. Then $G_1 \cong \Sigma_3$, G_2 has odd order, $G = G_1 \times G_2$, and $A_1 \cong C_{2^n} \times C_{2^n}$.*

APPENDIX B

Subgroups of 2-groups of sectional rank 4

We list here some properties of 2-groups S with $r(S) \leq 4$, starting with the case $r(S) = 2$.

Since $r(P) \leq r(P/Q) + r(Q)$ when $Q \trianglelefteq P$ are p-groups, all noncyclic metacyclic p-groups have sectional rank 2. The converse to this also holds when $p = 2$, as shown in the following lemma. It is not true for odd p: the nonabelian groups of order p^3 and exponent p have sectional rank 2 and are not metacyclic.

LEMMA B.1. *The following hold for any 2-group S with $r(S) = 2$.*

(a) *S is metacyclic.*

(b) *If S contains a subgroup isomorphic to D_8 or Q_8, then $S \in \mathcal{DSQ}$.*

(c) *If $\mathrm{Aut}(S)$ is not a 2-group, then $S \cong C_{2^k} \times C_{2^k}$ for some k, or $S \cong Q_8$.*

PROOF. **(a)** Assume otherwise, and let S be a counterexample of minimal order. Thus $r(S) = 2$, and S is not metacyclic. Also, S is nonabelian, so there is a central involution $z \in Z(S) \cap [S,S]$. Set $Z = \langle z \rangle$.

By the minimality assumption, S/Z is metacyclic. Hence we can choose $a, b \in S$ such that $S = \langle z, a, b \rangle$ where $A \stackrel{\text{def}}{=} \langle a, z \rangle$ is normal; and A is noncyclic since otherwise S would be metacyclic. Also, $[S,S] = [b,A] = \big\langle [b,a] \big\rangle$ (since $[b,z] = 1$), so $[S,S]$ is cyclic. Since $z \in [S,S] \leq A$, and z is not a square in A, this implies that $[a,b] = z$ and $[S,S] = \langle z \rangle$.

Thus S/Z is abelian. So we can assume that a and b were chosen so that $S/Z = \langle aZ \rangle \times \langle bZ \rangle$. Also, $z \notin \langle b \rangle$, since otherwise S would be metacyclic. Set $|a| = 2^j$ and $|b| = 2^k$. Then either $j = k = 1$ and $S \cong D_8$; or one of the subgroups $\langle z, a^2, b \rangle$ or $\langle z, a, b^2 \rangle$ is abelian of rank 3. In either case, this contradicts our original assumption on S.

(b) Since S is metacyclic by (a), there are elements $a, b \in S$ such that $S = \langle a, b \rangle$, $\langle a \rangle \trianglelefteq S$, $|a| = 2^k$, $|S/\langle a \rangle| = 2^\ell$, and $bab^{-1} = a^j$ (j odd). If $P \leq S$ is isomorphic to D_8 or Q_8, then the image of P in $S/\langle a \rangle$ must have order 2, and thus $P \cap \langle a \rangle \cong C_4$. Since the elements in $P \smallsetminus \langle a \rangle$ invert $P \cap \langle a \rangle$, $b^{2^{\ell-1}}$ inverts $P \cap \langle a \rangle$, so $\ell = 1$ ($b^{2^{\ell-1}}$ cannot be a square), and $j = -1$ or $2^{k-1}-1$ since $j^2 \equiv 1 \pmod{2^k}$. Thus $S \in \mathcal{DSQ}$.

(c) See [**Cr**, Proposition 6.7]. □

The following well known result is needed frequently.

PROPOSITION B.2. *Let S be a 2-group such that $S/[S,S] \cong C_2^2$. Then either $S \cong C_2^2$, or $S \in \mathcal{DSQ}$.*

PROOF. See, e.g., [**G**, Theorem 5.4.5]. □

Lemma B.3. *Let S be a 2-group such that $r(S) \le 4$, and let $P,Q \trianglelefteq S$ be normal nonabelian subgroups such that $|Z(P)| = |Z(Q)| = 2$ and $Z(P) \ne Z(Q)$. Then $[P,Q] = P \cap Q = 1$, and $C_S(PQ) \le PQ$ (i.e., PQ is centric).*

Proof. Since $P,Q \trianglelefteq S$, $Z(P),Z(Q) \trianglelefteq S$, and hence $Z(P)Z(Q) \le Z(S)$. If $P \cap Q \ne 1$, then $P \cap Q \ge Z(P)$ by Lemma A.2(a) (and since $P \cap Q \trianglelefteq P$), so $Z(P) \le Q \cap Z(S) \le Z(Q)$, a contradiction. Thus $[P,Q] \le P \cap Q = 1$.

In particular, $r(PQ/Z(PQ)) = 4$, since $r(X/Z(X)) \ge 2$ for any nonabelian 2-group X. So $C_S(PQ) \le PQ$ by Lemma A.6(a). □

The next lemma is a corollary of Lemma B.3.

Lemma B.4. *Let S be a 2-group such that $r(S) \le 4$, and let $P < S$ be a nonabelian subgroup such that $|Z(P)| = 2$. Set $S_0 = C_S(Z(P))$, and assume $P \trianglelefteq S_0$. Then either $S_0 = S$, or $r(P) = 2$ and $[S{:}S_0] = 2$.*

Proof. Assume $S_0 < S$. Choose $g \in N_S(S_0) \smallsetminus S_0$ such that $g^2 \in S_0$, and set $Q = {}^gP$ and $S_1 = S_0\langle g\rangle$. Then $Z(P) \ne Z(Q)$ since $g \notin S_0 = C_S(Z(P))$, so by Lemma B.3, $[P,Q] \le P \cap Q = 1$ and $C_S(PQ) = Z(PQ)$. Hence $4 \ge r(PQ) \ge 2r(P) \ge 2r(P/Z(P)) \ge 4$ implies that $r(P) = r(P/Z(P)) = 2$ and $r(PQ/Z(PQ)) = 4$.

Set $\widehat{Z} = Z(P)Z(Q)$ for short. Since $Z(S) \le C_S(PQ) = \widehat{Z}$, $Z(S) < \widehat{Z} \cong C_2^2$ is the subgroup of order 2 distinct from $Z(P)$ and $Z(Q)$. Hence $Z(PQ/Z(S)) = \widehat{Z}/Z(S)$. By Lemma A.6(a), and since $r(PQ/\widehat{Z}) = 4$, $C_{S/Z(S)}(PQ/Z(S)) = \widehat{Z}/Z(S)$, so $\widehat{Z} = Z_2(S) \trianglelefteq S$. Thus $[S{:}S_0] = [S{:}C_S(\widehat{Z})] = 2$ since $|\mathrm{Aut}_S(\widehat{Z})| = 2$. □

The next two lemmas involve 2-groups of sectional rank 2 or 4 which contain several normal dihedral or quaternion subgroups.

Lemma B.5. *Fix a 2-group S such that $r(S/Z(S)) = 2$, and a subgroup $Z \le Z(S)$ of order 2. Let $\mathscr{P}$ be a set of subgroups of S such that $S = Z(S)\langle\mathscr{P}\rangle$, and such that for each $P \in \mathscr{P}$, $PZ(S) \trianglelefteq S$, and either $P \in \mathcal{DQ}$ and $Z(P) = Z$, or $P \cong C_2^2$ and $P \cap Z(S) = Z$. Assume also that at least one subgroup in $\mathscr{P}$ is nonabelian. Then there is a subgroup $\widehat{P} \trianglelefteq S$ such that $\widehat{P} \in \mathcal{DSQ}$, $\widehat{P}Z(S) = S$, and $\widehat{P} \cap Z(S) = Z$.*

Proof. For each $X \le S$, let $\overline{X} = XZ(S)/Z(S)$ be the image of X in $\overline{S} = S/Z(S)$, and set $\overline{\mathscr{P}} = \{\overline{P} \mid P \in \mathscr{P}\}$. Thus $\overline{P} \trianglelefteq \overline{S}$ for each $P \in \mathscr{P}$, and $\overline{S} = \langle\overline{\mathscr{P}}\rangle$. Since $r(\overline{S}) = 2$ and $\overline{S}$ is generated by involutions (since each $\overline{P}$ is generated by involutions), $\overline{S} \notin \mathcal{SQ}$ and $\overline{S}/[\overline{S},\overline{S}] \cong C_2^2$, so $\overline{S} \in \mathcal{D}$ by Proposition B.2.

If $\overline{S} \cong C_2^2$, then $\overline{S} = \overline{P}$ for any nonabelian subgroup $P \in \mathscr{P}$, and we set $\widehat{P} = P$.

Now assume $\overline{S}$ is nonabelian. We must find $\widehat{P} \trianglelefteq S$ such that $\widehat{P} \in \mathcal{DSQ}$, $\overline{\widehat{P}} = \overline{S}$, and $\widehat{P} \cap Z(S) = Z$. Let $H_1, H_2 < \overline{S}$ be the two noncyclic subgroups of index 2 (recall $\overline{S} \in \mathcal{D}$). Choose $x,y \in S$ whose images $\overline{x},\overline{y} \in \overline{S}$ have order 2, such that each of x and y lies in some $P \in \mathscr{P}$, $\overline{x} \in \overline{S} \smallsetminus H_1$, and $\overline{y} \in \overline{S} \smallsetminus H_2$. (This is possible since each $\overline{P} \in \overline{\mathscr{P}}$ is generated by elements of order 2.) Thus $\langle\overline{x},\overline{y}\rangle = \overline{S}$. Set $\widehat{P} = \langle x,y\rangle$. Then

- $[\widehat{P},\widehat{P}] = [S,S] \ge Z$ since $\widehat{P}Z(S) = S$; and
- $x^2, y^2 \in Z$, since each lies in some $P \in \mathscr{P}$ and $P \cap Z(S) = Z$.

Hence $\widehat{P}/[\widehat{P},\widehat{P}] \cong C_2^2$. So $\widehat{P} \in \mathcal{DSQ}$ by Proposition B.2, and $\widehat{P} \cap Z(S) = Z(\widehat{P}) \geq Z$ with equality since $|Z(\widehat{P})| = 2$. □

Lemma B.6. *Fix a 2-group S with $r(S) \leq 4$. Assume $S = \langle \mathscr{P} \rangle$, where $\mathscr{P}$ is a set of normal subgroups of S such that for each $P \in \mathscr{P}$, either $P \in \mathcal{DQ}$, or $P \cong C_2^2$ and $|P \cap Z(S)| = 2$. Assume also that not all of the $P \in \mathscr{P}$ have the same center (or intersection with $Z(S)$ when $P \cong C_2^2$). Then there are subgroups $S_1, S_2 \trianglelefteq S$ such that $S_1, S_2 \in \mathcal{DSQ}$, $S = S_1S_2$, $S_1 \cap S_2 = 1$, and for some partition $\mathscr{P} = \mathscr{P}_1 \amalg \mathscr{P}_2$ of $\mathscr{P}$, $S_i \leq \langle \mathscr{P}_i \rangle \leq S_iZ(S)$ for $i = 1, 2$.*

Proof. Set $\mathcal{Z} = \{P \cap Z(S) \,|\, P \in \mathscr{P}\}$, and let $Z_1, \cdots, Z_m$ be the distinct subgroups in $\mathcal{Z}$. By assumption, $m \geq 2$, and each Z_i has order 2. For each i, set $\mathscr{P}_i = \{P \in \mathscr{P} \,|\, P \geq Z_i\}$. Set $\Delta_i = \langle \mathscr{P}_i \rangle$.

Assume $i \neq j$, $P \in \mathscr{P}_i$, and $Q \in \mathscr{P}_j$. If P and Q are both nonabelian, then $P \cap Q = 1$ by Lemma B.3. If $P \cong C_2^2$ and Q is nonabelian, then $P \cap Q \trianglelefteq Q$ and $P \cap Z(S) = Z_i \neq Z_j = Z(Q)$ again imply $P \cap Q = 1$. If $P \cong Q \cong C_2^2$, then either $P \cap Q = 1$, or $|P \cap Q| = 2$, $P = Z_i(P \cap Q)$, and $Q = Z_j(P \cap Q)$. Thus $[P, Q] = 1$ in all cases, so $[\Delta_i, \Delta_j] = 1$ for $i \neq j$.

If Δ_i is abelian for some i, then $\Delta_i \leq Z(S)$ since it commutes with the other Δ_j, which is impossible since no $P \in \mathscr{P}$ is contained in $Z(S)$. Thus each Δ_i is nonabelian. If, for some i, all subgroups in $\mathscr{P}_i$ are abelian, then there are $P, Q \in \mathscr{P}_i$ such that $P \cong Q \cong C_2^2$ and $[P, Q] \neq 1$; and $PQ \cong D_8$ since they are both normal in S. So after replacing P and Q by PQ in this situation, we can assume each $\mathscr{P}_i$ contains a nonabelian subgroup.

If $m \geq 3$, then for $P_i \in \mathscr{P}_i$ nonabelian ($i = 1, 2, 3$), $P_3 \leq C_S(P_1P_2)$, and $C_S(P_1P_2) = Z(P_1P_2)$ by Lemma B.3. Since this is impossible, $m = 2$, $S = \Delta_1\Delta_2$, $[\Delta_1, \Delta_2] = 1$, and hence $\Delta_1 \cap \Delta_2 \leq Z(S)$ and $Z(\Delta_i) \leq Z(S)$ $(i = 1, 2)$.

For each $X \leq S$, let $\overline{X} = XZ(S)/Z(S)$ be the image of X in $\overline{S} = S/Z(S)$. Then $\overline{S} = \overline{\Delta}_1 \times \overline{\Delta}_2$, so $r(\overline{\Delta}_i) = 2$ $(i = 1, 2)$ since $r(\overline{S}) \leq 4$ and $\overline{\Delta}_i \cong \Delta_i/Z(\Delta_i)$ is noncyclic. So by Lemma B.5, applied with Δ_i, $\mathscr{P}_i$, and Z_i in the role of S, $\mathscr{P}$, and Z, there is $S_i \trianglelefteq \Delta_i$ such that $S_i \in \mathcal{DSQ}$, $\overline{S}_i = \overline{\Delta}_i$, and $Z(S_i) = S_i \cap \Delta_i = Z_i$. By Lemma B.3, $Z(S) \leq S_1S_2$, so $S = S_1S_2 \cong S_1 \times S_2$. □

Lemma B.7. *Let S be a 2-group such that $r(S) \leq 4$, and let $Q \trianglelefteq S$ be a normal nonabelian subgroup such that $|Z(Q)| = 2$. Then for every $\alpha \in \mathrm{Aut}(S)$ of odd order, $\alpha(Z(Q)) = Z(Q)$.*

Proof. If $\alpha(Z(Q)) \neq Z(Q)$, then Q, $\alpha(Q)$, and $\alpha^2(Q)$ are three normal nonabelian subgroups with distinct centers of order 2. So by Lemma B.3, $\alpha^2(Q) \leq C_S(Q\alpha(Q)) = Z(Q\alpha(Q))$, which is impossible. □

APPENDIX C

Some explicit 2-groups of sectional rank 4

In this chapter, we collect some (mostly) technical results about subgroups and automorphisms of certain 2-groups, such as 2_-^{1+4}, $UT_4(2)$, and $UT_3(4)$. We begin with products of dihedral groups.

LEMMA C.1. *Assume $S \in \mathcal{D} \times \mathcal{D}$: a product of two nonabelian dihedral groups. Then there is a unique abelian subgroup $A < S$ of index 4 and rank 2. Of the three elements in $Z(S)^\#$, exactly two are squares of elements in $S \smallsetminus A$.*

PROOF. Fix dihedral subgroups $D_i = \langle a_i, b_i \rangle$ ($i = 1, 2$) such that $S = D_1 \times D_2$ and $[D_i{:}\langle a_i \rangle] = 2$. If $A < S$ is abelian of index 4, then for each i, the image $A_i \le D_i$ of A under the projection is abelian, so $[D_i{:}A_i] = 2$ and $A = A_1A_2$. Each A_i is cyclic since A has rank 2, so $A = \langle a_1, a_2 \rangle$ is the unique such subgroup.

Let z_i be the generator of $Z(D_i)$. If $g \in S \smallsetminus A$, then either $g \in b_1A$ and $g^2 \in \langle a_2^2 \rangle$, or $g \in b_2A$ and $g^2 \in \langle a_1^2 \rangle$, or $g \in b_1b_2A$ and $g^2 = 1$. Thus z_1 and z_2 can occur as squares of such elements, while z_1z_2 cannot. □

We now look at certain 2-groups, beginning with 2_-^{1+4}.

LEMMA C.2. *Assume $S = \Delta_1\Delta_2$, where $\Delta_1 \cong \Delta_2 \cong Q_8$, $[\Delta_1, \Delta_2] = \Delta_1 \cap \Delta_2 = Z(\Delta_1) = Z(\Delta_2)$, and $|C_{\Delta_1}(\Delta_2)| = 4$. Then the following hold.*

(a) *$S \cong 2_-^{1+4}$. There are exactly five involutions in $S/Z(S) \cong C_2^4$ which lift to involutions in S, and they are permuted transitively by $\mathrm{Out}(S) \cong \Sigma_5$.*

(b) *Let $\Gamma \le \mathrm{Aut}(S)$ be a subgroup which contains $\mathrm{Inn}(S)$. Assume, for each $i = 1, 2$, that there is $\gamma_i \in \Gamma$ of order 3 such that $\gamma_i(\Delta_i) = \Delta_i$ and $\gamma_i|_{\Delta_i} \ne \mathrm{Id}$. Then $[\mathrm{Aut}(S){:}\Gamma] \le 2$. If in addition, there is $\eta \in \Gamma$ such that $\eta(\Delta_i) = \Delta_i$ ($i = 1, 2$), $\eta|_{\Delta_1} \notin \mathrm{Inn}(\Delta_1)$, and $\eta|_{\Delta_2} \in \mathrm{Inn}(\Delta_2)$, then $\Gamma = \mathrm{Aut}(S)$.*

PROOF. Set $Z = Z(\Delta_1) = Z(\Delta_2)$, $V = \overline{S} = S/Z$, and $\overline{X} = XZ/Z$ for $X \le S$. Let $\mathfrak{q}\colon V \longrightarrow Z$ be the quadratic form $\mathfrak{q}(gZ) = g^2$, and let $\mathfrak{b}$ be its associated bilinear form ($\mathfrak{b}(gZ, hZ) = [g, h]$).

(a) If $\mathfrak{b}$ is degenerate, then $\dim(V^\perp) \ge 2$, so $V^\perp = (\overline{\Delta}_1)^\perp = (\overline{\Delta}_2)^\perp$, which is impossible since $|C_{\Delta_1}(\Delta_2)| = 4$ ($\dim(\overline{\Delta}_1 \cap \overline{\Delta}_2^\perp) = 1$). Thus $\mathfrak{b}$ and $\mathfrak{q}$ are nondegenerate, $S \not\cong 2_+^{1+4}$ since that group contains exactly two quaternion subgroups (and they commute), and hence $S \cong 2_-^{1+4} \cong Q_8 \times_{C_2} D_8$.

By Lemma A.5, there are exactly five isotropic points in V (which lift to involutions in S), and $\mathrm{Out}(S) \cong SO(V, \mathfrak{q}) \cong SO_4^-(2) \cong \Sigma_5$ is the group of all permutations of this set.

(b) By assumption, for $i = 1, 2$, $|\gamma_i| = 3$, $\gamma_i(\Delta_i) = \Delta_i$, and $\gamma_i|_{\Delta_i} \ne \mathrm{Id}$. Thus $C_S(\gamma_i) \le C_S(\Delta_i)$, so $C_S(\langle \gamma_1, \gamma_2 \rangle) = Z(S) = Z$. Each subgroup of Σ_5 generated by two elements of order 3 is isomorphic to A_3, A_4, or A_5, and so $\langle [\gamma_1], [\gamma_2] \rangle \cong A_5$ (as a subgroup of $\mathrm{Out}(S) \cong \Sigma_5$) since it doesn't fix any of the five isotropic points in $V = \overline{S}$.

If $\eta(\Delta_i) = \Delta_i$ for $i = 1, 2$ and $\eta|_{\Delta_2} \in \mathrm{Inn}(\Delta_2)$, then the induced automorphism $\overline{\eta} \in \mathrm{Aut}(V, \mathfrak{q})$ is the identity on $\overline{\Delta}_2$ and sends $\overline{\Delta}_1$ to itself. Hence $\mathrm{rk}(C_V(\overline{\eta})) = 3$ (since $C_{\overline{\Delta}_1}(\overline{\eta}) \neq 1$), so $\overline{\eta}$ permutes the isotropic points in V as a 2-cycle. Thus $[\eta] \notin O^2(\mathrm{Out}(S)) \cong A_5$, and $\Gamma = \mathrm{Aut}(S)$. □

The next lemma involves other "near central products" of dihedral and quaternion groups.

Lemma C.3. *Assume $S = \Delta_1\Delta_2$, where $\Delta_1, \Delta_2 \in \mathcal{DSQ}$, $|\Delta_1|, |\Delta_2| \geq 16$, and $[\Delta_1, \Delta_2] \leq \Delta_1 \cap \Delta_2 = Z(S)$. Then there is a unique pair of subgroups $\Theta_1, \Theta_2 \in \mathcal{Q}$ such that $S = \Theta_1\Theta_2$, $|\Theta_i| = |\Delta_i|$ ($i = 1, 2$), and $[\Theta_1, \Theta_2] \leq \Theta_1 \cap \Theta_2 = Z(S)$.*

Proof. Set $B_i = Z_2(\Delta_i) \cong C_4$ for short. Then $Z_2(S) = B_1B_2$, since

$$S/Z(S) = (\Delta_1/Z(S)) \times (\Delta_2/Z(S)) \tag{C.1}$$

by assumption. Also, $B_i \leq \mathrm{Fr}(\Delta_i)$ for $i = 1, 2$ since $|\Delta_i| \geq 16$, so $[\Delta_i, B_{3-i}] = 1$, and $\Delta_i Z_2(S) = \Delta_i B_{3-i} \cong \Delta_i \times_{C_2} C_4$.

Again fix $i = 1, 2$. We claim that

$$\text{there is a unique } \Theta_i^* \leq \Delta_i Z_2(S) \text{ such that } \Theta_i^* \in \mathcal{Q} \text{ and } |\Theta_i^*| = |\Delta_i|. \tag{C.2}$$

To see this, fix $x, y \in \Delta_i$ such that $\Delta_i = \langle x, y\rangle$, $x^2, y^2 \in Z(S)$, and $\langle xy\rangle < \Delta_i$ has index 2. There are $x' \in xB_i$ and $y' \in yB_i$ (unique modulo $Z(S)$) such that $|x'| = 4 = |y'|$. Set $\Theta_i^* = \langle x', y'\rangle$. Then $\Theta_i^*/Z(S) \in \mathcal{D}$ since it is generated by two involutions, and hence $\Theta_i^* \in \mathcal{Q}$. Also, $\Theta_i^* B_i = \Delta_i B_i$, $\Theta_i^* \cap B_i = Z(\Theta_i^*) = Z(S)$, and so $|\Theta_i^*| = |\Delta_i|$. Any other quaternion subgroup of index 2 in $\Delta_i B_i$ must contain elements of order 4 in xB_i and yB_i, hence contains x' and y', and is equal to Θ_i^*.

By the Krull-Schmidt theorem (Theorem A.8(a)), applied to the factorization of $S/Z(S)$ in (C.1), for any $\Theta_1, \Theta_2 \in \mathcal{Q}$ such that $S = \Theta_1\Theta_2$ and $[\Theta_1, \Theta_2] \leq \Theta_1 \cap \Theta_2 = Z(S)$, $\Theta_i \leq \Delta_i Z_2(S)$ (possibly after an exchange of indices). Hence $\Theta_i = \Theta_i^*$ exists and is uniquely determined by (C.2). □

We next look at the group $UT_4(2)$.

Lemma C.4. *Set $S = UT_4(2)$.*

(a) *There is a unique abelian subgroup $A \leq S$ of order 16, $A \cong C_2^4$, S splits over A, and $\mathrm{Out}_S(A) \cong C_2^2$ permutes freely a basis for A (thus $S \cong C_2 \wr C_2^2$).*

(b) *There is a unique extraspecial subgroup $Q \leq S$ of index 2, $Q \cong 2^{1+4}_+$, and $S \cong (Q_8 \times_{C_2} Q_8) \overset{t}{\rtimes} C_2 \cong (D_8 \times_{C_2} D_8) \overset{t}{\rtimes} C_2$.*

(c) *If $P < S$ has index 2, then either $P = Q$, or $P \geq A$, or $P^{\mathrm{ab}} \cong C_4 \times C_2$ and $\mathrm{Aut}(P)$ is a 2-group. All involutions in S are in $A \cup Q$.*

(d) *If $\mathrm{Id} \neq \alpha \in \mathrm{Aut}(S)$ has odd order, then $|\alpha| = 3$, and α permutes transitively the three subgroups of index 2 which contain A.*

(e) *There is no normal subgroup $P \trianglelefteq S$ with $P \in \mathcal{DQ}$.*

Proof. Fix $V = (\mathbb{F}_2)^4$ with basis $\{b_1, b_2, b_3, b_4\}$. For each $0 \leq i \leq 4$, set $V_i = \langle b_j \,|\, 1 \leq j \leq i\rangle$. Set $G = \mathrm{Aut}(V) \cong GL_4(2)$. We identify S with the group of all $\alpha \in G$ which normalize the chain $0 < V_1 < V_2 < V_3 < V$. Set $Z = Z(S)$.

For $\sigma\in\Sigma_4$, define $\psi_\sigma\in G$ by setting $\psi_\sigma(b_i)=b_{\sigma(i)}$ for each i.

(a) Set $H=\{\alpha\in G\,|\,\alpha(V_2)=V_2\}$ and $A=O_2(H)$. The map $(\alpha\mapsto\alpha-\mathrm{Id})$ defines an isomorphism $A\xrightarrow{\ \cong\ }\mathrm{Hom}(V/V_2,V_2)\cong C_2^4$, and $H=A\rtimes(G_{12}\times G_{34})$. Hence S is H-conjugate to $A\langle\psi_{(1\,2)},\psi_{(3\,4)}\rangle$, and $\langle\psi_{(1\,2)},\psi_{(3\,4)}\rangle\cong C_2^2$ permutes freely the canonical basis for A. Thus $S\cong C_2\wr C_2^2$. So by Lemma A.4(b), and since $|S/[S,S]|=8$, A is the unique abelian subgroup of index 4 in S.

(e) Assume $P\trianglelefteq S$ and $P\in\mathcal{DQ}$. Then $P\nleq A$ since it is nonabelian. If $g\in P\smallsetminus A$, then $P\geq[g,A]=C_A(g)\cong C_2^2$ since P is normal, so $P\geq\langle g,C_A(g)\rangle$ which is abelian of order 8. This is impossible.

(b) Set $K=C_G(Z)=\{\alpha\in G\,|\,\alpha(V_1)=V_1,\ \alpha(V_3)=V_3\}$ and $Q=O_2(K)$. Set

$$W_1=\bigl\{\alpha\in G\,\big|\,[\alpha,V]\leq V_1\bigr\}\qquad\text{and}\qquad W_2=\bigl\{\alpha\in G\,\big|\,\alpha|_{V_3}=\mathrm{Id}\bigr\}.$$

Then $W_1\cong W_2\cong C_2^3$, $W_1W_2=Q$, and $W_1\cap W_2=Z$. Also, $W_1,W_2\trianglelefteq S$, so $[W_1,W_2]\leq W_1\cap W_2=Z$, with equality since Q is nonabelian by (a). Since Q contains no abelian subgroups of order 2^4 by (a) again, it must be extraspecial, and so $Q\cong 2_+^{1+4}\cong Q_8\times_{C_2}Q_8$ since it contains elementary abelian subgroups of rank 3. Since neither of the two quaternion subgroups of Q is normal in S by (e), we get $S\cong(Q_8\times_{C_2}Q_8)\overset{t}{\rtimes}C_2$. Finally, the explicit isomorphisms $D_8\times_{C_2}D_8\cong Q_8\times_{C_2}Q_8$ constructed in [**G**, p. 205] and in [**Sz1**, pp. 139–140] extend to isomorphisms $(D_8\times_{C_2}D_8)\overset{t}{\rtimes}C_2\cong(Q_8\times_{C_2}Q_8)\overset{t}{\rtimes}C_2$ between the semidirect products.

Since $[S,Q]=[S,S]$ has order 2^3, $[S,Q/Z]$ has rank 2. So by Lemma A.4(a), applied with $Q/Z<S/Z$ in the role of $A<S$, Q/Z is the unique abelian subgroup of index 2 in S/Z, and Q is the unique extraspecial subgroup of index 2 in S.

(c) Since S/Q and S/A are elementary abelian, $[S,S]\leq\mathrm{Fr}(S)\leq Q\cap A$. Since $\mathrm{Aut}_S(A)$ permutes freely a basis of A by (a), $|[S,S]|=|[S,A]|=8$, and thus $[S,S]=\mathrm{Fr}(S)=Q\cap A$. So there are 7 subgroups of S of index 2, including Q and the three which contain A.

Let $T_1,T_2,T_3<S$ be the three subgroups of index 2 in $[S,S]$ which contain $Z=Z(Q)$. By (b), $S/Z\cong C_2^2\wr C_2$, where $[S,Q/Z]=[S,S]/Z\cong C_2^2$. Hence $S/T_i\cong D_8\times C_2$ for each $i=1,2,3$. Let $P_i<S$ be the subgroup such that $P_i>T_i$ and $P_i/T_i\cong C_4\times C_2$. Then T_i/Z is nonabelian since Q/Z is the unique abelian subgroup of S/Z of index 2, so $P_i^{\mathrm{ab}}=P_i/T_i\cong C_4\times C_2$, and $\mathrm{Aut}(P_i)$ is a 2-group by Corollary A.10(a). Also, since $Q/T_i\cong C_2^3$ (since Q/Z is elementary abelian), $\Omega_1(P_i/T_i)=Z(S/T_i)\leq Q/T_i$, so $I(P_i)\subseteq Q$, and hence $Q\ngeq A$. Thus P_1,P_2,P_3 are the three subgroups of index 2 which contain neither Q nor A.

In particular, if $g\in I(S)$ and $g\notin Q$, then $g\notin P_1\cup P_2\cup P_3$. Since each element of S is contained in at least three subgroups of index 2, g is contained in all three of the subgroups which contain A, and thus $g\in A$.

(d) If $\mathrm{Id}\neq\alpha\in\mathrm{Aut}(S)$ has odd order, then by Lemma A.9 and since $|A/\mathrm{Fr}(S)|=2$, α has order 3 and acts nontrivially on S/A, and hence permutes transitively the three subgroups of index 2 containing A. □

LEMMA C.5. *Set $S=D_8\wr C_2$.*

(a) *There are exactly two normal subgroups $V_1,V_2\trianglelefteq S$ isomorphic to C_2^4, one (normal) subgroup $Q\trianglelefteq S$ isomorphic to 2_+^{1+4}, and no subgroups isomorphic to 2_-^{1+4}. The images of V_1, V_2, and Q in $S/[S,S]\cong C_2^3$ have order 2 and are linearly independent. Also, $Q\cap V_i\cong C_2^3$ for $i=1,2$.*

(b) *If $P < D_8 \wr C_2$ has index 2, and $\mathrm{Aut}(P)$ is not a 2-group, then $P \cong UT_4(2) \cong (Q_8 \times_{C_2} Q_8) \overset{t}{\rtimes} C_2$. There are two such subgroups in $D_8 \wr C_2$.*

Proof. Set $S = \langle a_1, b_1, a_2, b_2, t\rangle \cong D_8 \wr C_2$, where $|a_i| = 4$, $|b_i| = 2$, $\Delta_i \overset{\text{def}}{=} \langle a_i, b_i\rangle \cong D_8$, $[\Delta_1, \Delta_2] = 1$, $t^2 = 1$, and $ta_it^{-1} = a_{3-i}$, $tb_it^{-1} = b_{3-i}$. Set $Q = \langle a_1a_2, b_1b_2, a_1^2, t\rangle \cong 2_+^{1+4}$. Set $Z = Z(S) = \langle a_1^2a_2^2\rangle$ and $Z_2 = Z_2(S) = \langle a_1^2, a_2^2\rangle$.

(a) If $R \trianglelefteq S$ and $|R| \geq 4$, then $R \geq Z_2$ by Lemma A.2(b). If $R \cong C_2^4$, then $R \leq C_S(Z_2) \cong D_8 \times D_8$, and of the four subgroups in $D_8 \times D_8$ isomorphic to C_2^4, only $V_1 = Z_2\langle b_1, b_2\rangle$ and $V_2 = Z_2\langle a_1b_1, a_2b_2\rangle$ are normal in S.

If $R \leq S$ and $R \cong 2_+^{1+4}$ or 2_-^{1+4}, then $Z(R) = Z$, R/Z and Q/Z are both abelian of index 4 in S/Z, and so $R = Q \cong 2_+^{1+4}$ by Lemma A.4(b). The images of V_1, V_2, and Q in S^{ab} are generated by the classes of b_1, a_1b_1, and t, respectively, and thus are independent. Also, $Q \cap V_1 = Z_2\langle b_1b_2\rangle \cong C_2^3$ and $Q \cap V_2 = Z_2\langle a_1b_1a_2b_2\rangle \cong C_2^3$.

(b) Assume $P < S$ has index 2. If $Z(P) > Z$, then by Lemma A.3, $P = C_S(Z_2) \cong D_8 \times D_8$, and $\mathrm{Aut}(P)$ is a 2-group by Corollary A.10(c). If $Z(P) = Z$ and $\mathrm{Aut}(P)$ is not a 2-group, then $\mathrm{Aut}(P/Z)$ is not a 2-group by Lemma A.9. By Lemma C.4(b,c), and since $S/Z \cong (Q_8 \times_{C_2} Q_8) \overset{t}{\rtimes} C_2$, either P/Z is extraspecial (hence $P \cong D_8 \times D_8$), or $P > Q$.

Of the three subgroups of index 2 which contain Q,

$$P_1 = \langle a_1a_2, b_1, b_2, a_1^2, t\rangle \cong UT_4(2) \quad \text{and} \quad P_2 = \langle a_1a_2, a_1b_1, a_2b_2, a_1^2, t\rangle \cong UT_4(2),$$

while $P_3 = \langle a_1, a_2, b_1b_2, t\rangle$ contains the sequence $\mathrm{Fr}(P) < C_Q(Z_2) < C_{P_3}(Z_2) < P_3$ of characteristic subgroups. (Note that $Z_2 = \Omega_1(Z_2(P_3))$.) So $\mathrm{Aut}(P_3)$ is a 2-group by Lemma A.9. □

Throughout the rest of the chapter, we work with the group $UT_3(4)$. We use the following notation, taken from [**OV**, §§4–5] and from Notation 6.1, for certain subgroups of $UT_3(4)$. Set

$$A_1 = \Big\{ \left(\begin{smallmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) = e_{12}^a e_{13}^b \,\Big|\, a, b \in \mathbb{F}_4 \Big\} \quad \text{and} \quad A_2 = \Big\{ \left(\begin{smallmatrix} 1 & 0 & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{smallmatrix}\right) = e_{23}^a e_{13}^b \,\Big|\, a, b \in \mathbb{F}_4 \Big\}.$$

Thus $A_1 = O_2(\mathfrak{P}_1)$ and $A_2 = O_2(\mathfrak{P}_2)$, where $\mathfrak{P}_1, \mathfrak{P}_2$ are the two maximal parabolic subgroups in $SL_3(4)$ containing $UT_3(4)$. Also,

$$Z(UT_3(4)) = A_1 \cap A_2 = \{e_{13}^a \,|\, a \in \mathbb{F}_4\}.$$

Lemma C.6. *Set $S = UT_3(4)$, and $Z = Z(S) = A_1 \cap A_2$.*

(a) *All involutions in S lie in $A_1 \cup A_2$, and each elementary abelian subgroup of S is contained in A_1 or in A_2. For each $g \in A_i \smallsetminus Z$ ($i = 1, 2$), $C_{A_{3-i}}(g) = [g, A_{3-i}] = Z$.*

(b) *For each $S_0 < S$ of index 2, $[S_0, S_0] = [S, S] = Z$.*

Proof. **(a)** For each $g \in A_1 \smallsetminus Z$ and $h \in A_2 \smallsetminus Z$, $g = e_{12}^a e_{13}^b$ and $h = e_{23}^c e_{13}^d$ for some $a, b, c, d \in \mathbb{F}_4$ with $a, c \neq 0$, and $[g, h] = [e_{12}^a, e_{23}^c] = e_{13}^{ac} \neq 1$. Thus $C_{A_2}(g) = [g, A_2] = Z$, and similarly for h.

If $g \in S \smallsetminus (A_1 \cup A_2)$, then $g = e_{12}^a e_{23}^b e_{13}^c$ for some $a, b, c \in \mathbb{F}_4$ where $a, b \neq 0$, and $g^2 = e_{13}^{ab} \neq 1$. Thus $I(S) = A_1^\# \cup A_2^\#$. Since no element of $A_1 \smallsetminus Z$ commutes with any element of $A_2 \smallsetminus Z$, each elementary abelian subgroup of S is contained in A_1 or in A_2.

(b) Assume $S_0 < S$ has index 2. If $S_0 \cap A_1 < A_1$, then $S = A_1 S_0$. So for any $g \in (S_0 \cap A_1) \smallsetminus Z$, $[S_0, S_0] \geq [g, S_0] = [g, S] = Z$ by (a). If $S_0 \cap A_1 = A_1$, then $S_0 \cap A_2 < A_2$, and a similar argument applies. □

The next lemma gives some criteria for characterizing $UT_3(4)$.

Lemma C.7. *Fix a 2-group S of order 2^6, and set $Z = Z(S)$. Assume that S is special of type 2^{2+4}; i.e., $Z = [S, S] \cong C_2^2$ and $S/Z \cong C_2^4$. Assume also that there are subgroups $B_1, B_2 < S$ such that $B_1 B_2 = S$ and $B_1 \cong B_2 \cong C_2^4$. If either*

(a) $[g, B_1] = Z$ *for each* $g \in B_2 \smallsetminus Z$, *or*

(b) *there is* $\mathrm{Id} \neq \alpha \in \mathrm{Aut}(S)$ *of odd order such that* $\alpha(B_1) = B_1$,

then $S \cong UT_3(4)$.

Proof. Fix $V_i < B_i$ which is complementary to Z; thus $V_i \cong C_2^2$. Let $\chi\colon V_1 \times V_2 \longrightarrow Z$ be the biadditive commutator map $\chi(v, w) = [v, w]$.

(a) For each $v_2 \in V_2^{\#}$, $[v_2, B_1] = Z$ by assumption, so $\chi(-, v_2) \in \mathrm{Hom}(V_1, Z)$ is surjective, and hence an isomorphism. Thus $\chi(v_1, v_2) \neq 1$ for each pair $(v_1, v_2) \in V_1^{\#} \times V_2^{\#}$, and hence $\chi(v_1, -)$ is an isomorphism for each $v_1 \in V_1^{\#}$.

Fix any $e_i \in V_i^{\#}$, and set $e = [e_1, e_2] \in Z^{\#}$. Choose any isomorphism $\rho\colon Z \xrightarrow{\cong} (\mathbb{F}_4, +)$ such that $\varphi(e) = 1$, and set

$$\rho_1\colon V_1 \xrightarrow[\cong]{\chi(-,e_2)} Z \xrightarrow[\cong]{\rho} \mathbb{F}_4 \qquad \text{and} \qquad \rho_2\colon V_2 \xrightarrow[\cong]{\chi(e_1,-)} Z \xrightarrow[\cong]{\rho} \mathbb{F}_4.$$

Set $\mu = \rho \circ \chi \circ (\rho_1^{-1} \times \rho_2^{-1})\colon \mathbb{F}_4 \times \mathbb{F}_4 \longrightarrow \mathbb{F}_4$. By construction, μ is biadditive, $\mu(1, a) = a = \mu(a, 1)$ for each $a \in \mathbb{F}_4$, and $\mu(a, -)$ and $\mu(-, a)$ are isomorphisms for each $a \neq 0$. So if $a \neq 0, 1$, then $\mu(a, a) \notin \{0, a\}$, $\mu(1 + a, 1 + a) = 1 + \mu(a, a) \neq 0$, and hence $\mu(a, a) = 1 + a = a^2$. So $\mu(x, y) = xy$ for all $x, y \in \mathbb{F}_4$.

Now define $\alpha\colon S \longrightarrow UT_3(4)$ by setting $\alpha(v_1) = e_{12}^{\rho_1(v_1)}$, $\alpha(v_2) = e_{23}^{\rho_2(v_2)}$, and $\alpha(z) = e_{13}^{\rho(z)}$ for all $v_i \in V_i$ and $z \in Z$. Then by the relation $[e_{12}^a, e_{23}^b] = e_{13}^{ab}$ in $UT_3(4)$, α is an isomorphism.

(b) Let $\alpha \in \mathrm{Aut}(S)$ be of odd order $k > 1$, and such that $\alpha(B_1) = B_1$. Since the induced action of $\langle\alpha\rangle$ on $(S/B_1) \times (B_1/Z) \times Z$ is faithful by Lemma A.9, $k = |\alpha| = 3$. Since $Z = [S, S] = [S, B_1]$, α induces a nontrivial action on S/B_1 or on B_1/Z (or both).

Assume $S \ncong UT_3(4)$. By the proof of (a), there are elements $v_i \in V_i^{\#}$ such that $\chi(v_1, v_2) = 1$. If $\alpha(v_1) \in v_1 Z$, then α acts trivially on B_1/Z and hence nontrivially on S/B_1, so $S = B_1\langle v_2, \alpha(v_2)\rangle$, $v_1 \in Z(S)$, contradicting the assumption that $Z(S) = Z = [S, S]$. Thus $\alpha(v_1) \notin v_1 Z$, and $\alpha(v_2) \notin v_2 B_1$ by a similar argument.

Let $v_i' \in V_i$ be the unique elements such that $v_1' \in \alpha(v_1) Z$ and $v_2' \in \alpha(v_2) B_1$. Then $V_i = \langle v_i, v_i'\rangle$, $\alpha^2(v_1) \in v_1 v_1' Z$, $\alpha^2(v_2) \in v_2 v_2' B_1$, and $\chi(v_1, v_2) = 1 = \chi(v_1', v_2') = \chi(v_1 v_1', v_2 v_2')$. Thus $[S, S] = \langle \chi(V_1, V_2)\rangle = \langle \chi(v_1, v_2')\rangle$ has rank 1, which contradicts the assumption that $[S, S] = Z$. We conclude that $S \cong UT_3(4)$. □

Fix $\omega \in \mathbb{F}_4 \smallsetminus \{0,1\}$, and let $(a \mapsto \bar{a}) \in \mathrm{Aut}(\mathbb{F}_4)$ be the field automorphism. Thus $\mathbb{F}_4 = \{0,1,\omega,\bar{\omega}\}$. Define $\gamma_0, \gamma_1, \phi, \tau \in \mathrm{Aut}(UT_3(4))$ by setting

$$\gamma_0\left(\left(\begin{smallmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} 1 & \omega a & \bar{\omega} b \\ 0 & 1 & \omega c \\ 0 & 0 & 1 \end{smallmatrix}\right) \qquad \gamma_1\left(\left(\begin{smallmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} 1 & \omega a & b \\ 0 & 1 & \bar{\omega} c \\ 0 & 0 & 1 \end{smallmatrix}\right)$$

$$\phi\left(\left(\begin{smallmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} 1 & \bar{a} & \bar{b} \\ 0 & 1 & \bar{c} \\ 0 & 0 & 1 \end{smallmatrix}\right) \qquad \tau\left(\left(\begin{smallmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} 1 & c & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{smallmatrix}\right)^{-1}.$$

Thus γ_0 and γ_1 are conjugation by $\mathrm{diag}(\omega,1,\bar{\omega})$ and $\mathrm{diag}(1,\bar{\omega},1)$, respectively, while ϕ and τ are restrictions of field and graph automorphisms of $SL_3(4)$. Set

$$\Gamma_0 = \langle \gamma_0, \tau\phi \rangle \qquad \text{and} \qquad \Gamma_1 = \langle \gamma_1, \tau \rangle .$$

As subgroups of $\mathrm{Aut}(UT_3(4))$, $\Gamma_0 \cong \Gamma_1 \cong \Sigma_3$ and $[\Gamma_0, \Gamma_1] = 1$.

Lemma C.8. *Let R be the group of automorphisms of $UT_3(4)$ which induce the identity on $UT_3(4)/Z(UT_3(4))$. There are isomorphisms*

$$\mathrm{Aut}(UT_3(4)) = R{\cdot}(\Gamma_0 \times \Gamma_1) \cong C_2^8 \rtimes (\Sigma_3 \times \Sigma_3),$$
$$\mathrm{Out}(UT_3(4)) = \big(R/\mathrm{Inn}(UT_3(4))\big){\cdot}(\Gamma_0 \times \Gamma_1) \cong C_2^4 \rtimes (\Sigma_3 \times \Sigma_3).$$

Also, the group $\langle \phi, \tau \rangle \cong C_2^2$ permutes freely a basis for $O_2(\mathrm{Out}(UT_3(4))) \cong C_2^4$.

Proof. Set $S = UT_3(4)$ for short. The above descriptions of $\mathrm{Aut}(S)$ and of $\mathrm{Out}(S)$ are proven in [**OV**, Lemma 4.5(a)].

To see that $\langle \phi, \tau \rangle$ permutes freely a basis for $O_2(\mathrm{Out}(S)) \cong C_2^4$, let $R_i \le \mathrm{Aut}(S)$ $(i = 1, 2)$ be the group of automorphisms which induce the identity on A_{3-i} and on $S/Z(S)$. Thus $R/\mathrm{Inn}(S) = (R_1/\mathrm{Aut}_{A_2}(S)) \times (R_2/\mathrm{Aut}_{A_1}(S))$, and τ exchanges these two factors. So it suffices to prove that ϕ acts nontrivially on $R_1/\mathrm{Aut}_{A_2}(S) \cong C_2^2$; which is easily checked. For example, let $\rho \in R_1$ be the automorphism $\rho(e_{12}^1) = e_{12}^1 e_{13}^\omega$ and $\rho(e_{12}^\omega) = e_{12}^\omega$ (and $\rho|_{A_2} = \mathrm{Id}$). Then $\phi\rho\phi^{-1}\rho^{-1}$ sends e_{12}^1 to $e_{12}^1 e_{13}^1$ and e_{12}^ω to $e_{12}^\omega e_{13}^{\bar{\omega}}$, and is not in $\mathrm{Inn}(UT_3(4))$. □

Recall that we define $\mathcal{U}$ to be the family of 2-groups S such that there is $T \trianglelefteq S$ where $T \cong UT_3(4)$ and $C_{S/Z(T)}(T/Z(T)) = T/Z(T)$.

Lemma C.9. *If $S \in \mathcal{U}$, then there is a unique normal subgroup $T \trianglelefteq S$ such that $T \cong UT_3(4)$. For each $Q \trianglelefteq S$ with $|Q| \ge 4$, $Q \ge Z(T)$ and $Q \notin \mathcal{DQ}$.*

Proof. Fix $T \trianglelefteq S$ such that $T \cong UT_3(4)$ and $C_{S/Z(T)}(T/Z(T)) = T/Z(T)$. Set $Z = Z(T)$ for short. For each $Z_0 < Z$ of order 2, $T/Z_0 \cong 2_+^{1+4}$, so $Z(T/Z_0) = Z/Z_0$, and $Z(S/Z_0) = Z/Z_0$ since $C_{S/Z}(T/Z) = T/Z$. If $Q \trianglelefteq S$ and $|Q| \ge 4$, then $Q \cap Z \ge Q \cap Z(S) \ne 1$ (Lemma A.2(a)), so either $Q \ge Z$ or $|Q \cap Z| = 2$. In the latter case, set $Z_0 = Q \cap Z$; then $(Q/Z_0) \cap Z(S/Z_0) = (Q/Z_0) \cap (Z/Z_0) \ne 1$ by Lemma A.2(a) again, and hence $Q \ge Z$.

If $Q \in \mathcal{DQ}$, then $Q \cong D_8$ since it contains a normal subgroup isomorphic to C_2^2. So $Q/Z \le Z(S/Z) \le T/Z$ by Lemma A.2(a), and Q is abelian, a contradiction.

In particular, if $U \trianglelefteq S$ and $U \cong UT_3(4)$, then $Z(U) = Z$ by the first paragraph, applied with $Z(U)$ in the role of Q. By Lemma C.8, and since the automorphisms in R induce the identity on $T/Z(T)$, $S/T \cong \mathrm{Aut}_S(T/Z) \cong C_2^k$ for $k \le 2$, and this group acts on T/Z by permuting freely a basis. Hence T/Z is the unique abelian subgroup of order 2^4 in S/Z by Lemma A.4(a,b), so $U = T$. □

APPENDIX D

Actions on 2-groups of sectional rank at most 4

When studying automorphisms of 2-groups of sectional rank 4, it is natural to begin by looking at subgroups of $GL_4(2)$.

PROPOSITION D.1. *Assume* $V=\mathbb{F}_2^4$, $H<G=\mathrm{Aut}(V)$, *and* $S\in\mathrm{Syl}_2(H)$.

(a) *If* $O_{2'}(H)\neq 1$, *then* H *is contained in one of the following subgroups:*

$$\begin{aligned} &N_G(C_3)\cong\Sigma_3\times\Sigma_3 && N_G(C_3)\cong GL_2(4)\rtimes C_2\cong(C_3\times A_5)\rtimes C_2 \\ &N_G(C_3\times C_3)\cong\Sigma_3\wr C_2 && N_G(C_5)\cong C_{15}\overset{2}{\rtimes}C_4 \quad N_G(C_7)\cong C_7\overset{2}{\rtimes}C_3. \end{aligned} \tag{D.1}$$

(b) *If* $O_{2'}(H)=1$ *and* $O_2(H)\neq 1$, *then* $O_2(H)$ *is centric in* H.

(c) *If* $O_{2'}(H)=1$ *and* $O_2(H)=1$, *then* H *is isomorphic to one of the groups* A_n *for* $5\leq n\leq 8$, Σ_5, Σ_6, *or* $GL_3(2)$. *There are two* G*-conjugacy classes of subgroups isomorphic to* A_5 *or* Σ_5, *three classes of subgroups isomorphic to* $GL_3(2)$, *and a unique class in each of the other cases.*

(d) *If* $H\cong A_5$ *or* Σ_5, *then either* V *is the* $L_2(4)$-module *for* H *(the natural module for* $SL_2(4)\cong A_5$*), or* V *is the* orthogonal module *(the natural module for* $SO_4^-(2)\cong\Sigma_5$*, and the reduced permutation module). In the former case,* H *acts transitively on* $V^\#$, *and* $C_S(V)=[S,V]$ *has rank* 2. *In the latter case,* H *acts on* $V^\#$ *with orbits of length* 5 *and* 10, $\mathrm{rk}(C_S(V))=1$, *and* $\mathrm{rk}([S,V])=3$.

(e) *Assume that* $S\cong C_2^2$, *and that* S *permutes a basis of* V.

  (e.1) *If there are distinct involutions* $x_1,x_2\in S$ *such that* $S\nleq O_2(C_H(x_i))$, *then* $H\cong\Sigma_3\times\Sigma_3$.

  (e.2) *If the three involutions in* S *are all* H*-conjugate, then* $H\cong A_4$ *or* A_5.

(f) *Assume* $S\cong D_8$. *Assume there is a noncentral involution* $x\in S$ *such that* $x\notin O^2(H)$, *and such that* $O_2(C_H(x))=\langle x\rangle$. *Then* $H\cong\Sigma_3\wr C_2$, Σ_5, *or* $(A_5\times C_3)\rtimes C_2\cong\Gamma L_2(4)$.

(g) *Assume* $S\cong D_8$ *and* $O^2(H)=H$. *Then* $H\cong A_6$, A_7, *or* $GL_3(2)$. *The first two are unique up to conjugacy, while* $\mathrm{Aut}(V)$ *contains three conjugacy classes of subgroups isomorphic to* $GL_3(2)$.

PROOF. Since $G\cong A_8$ (cf. [**Ta**, Corollary 6.7]), this is equivalent to looking at subgroups of A_8.

(a) Recall that each minimal normal subgroup of H is isomorphic to a product of simple groups isomorphic to each other (cf. [**G**, Theorem 2.1.5]). So if $O_{2'}(H)\neq 1$, then for some odd prime p, $H\leq N_G(A)$ for some elementary abelian p-subgroup

$A \trianglelefteq H$. Since $|G| = 2^6{\cdot}3^2{\cdot}5{\cdot}7$, either $A \in \mathrm{Syl}_p(G)$ for $p = 3, 5, 7$, or A is in one of the two G-conjugacy classes of subgroups of order 3.

(b) If $O_{2'}(H) = 1$, then the generalized Fitting subgroup $F^*(H)$ is a central product $E(H)O_2(H)$, where $E(H)$ is generated by nonabelian quasisimple subgroups (cf. [**A1**, 31.12] or [**AKO**, Theorem A.13(b)]). If $O_2(H) \neq 1$, then since the centralizer of each involution in A_8 (hence in H) is solvable, $E(H) = 1$, and $F^*(H) = O_2(H)$ is centric in H (cf. [**A1**, 31.13] or [**AKO**, Theorem A.13(c)]).

(c) If $O_{2'}(H) = 1 = O_2(H)$, then $F^*(H) \neq 1$ is a product of nonabelian simple groups. The nonabelian simple subgroups of $G \cong A_8$ are well known. For example, this follows from Burnside's list [**Bu**, § 146] of primitive permutation groups of degree at most 8. Each simple subgroup of G is isomorphic to A_n for $5 \leq n \leq 8$ or to $GL_3(2)$. Also, there are two G-conjugacy classes of subgroups isomorphic to A_5 (one each of degree 5 and 6), and three classes of subgroups isomorphic to $GL_3(2)$ (two of degree 7 and one of degree 8).

Set $H_0 = F^*(H)$. Since no product of two nonabelian simple subgroups is contained in G (the centralizer of each nonabelian simple subgroup is solvable), H_0 is simple, and $H \leq N_G(H_0)$. Since $O_3(H) = 1$, $H \ncong (C_3 \times A_5) \rtimes C_2$.

(d) When $H \cong A_5$ or Σ_5, we already saw that there are only two possibilities (up to isomorphism) for V as an $\mathbb{F}_2[H]$-module. Hence V must be the $L_2(4)$-module or the orthogonal module, as defined above. The other properties are immediate.

(e) Assume $S \cong C_2^2$ permutes a basis for V: either transitively or in two orbits of length 2. If $O_{2'}(H) \neq 1$, then H is contained in one of the normalizers listed in (D.1). By inspection, H must be isomorphic to $C_2 \times \Sigma_3$ or $\Sigma_3 \times \Sigma_3$. (If $H \leq N_G(A) \cong (C_3 \times A_5) \rtimes 2$, where $|A| = 3$, then $H \nleq C_G(A)$ since a Sylow 2-subgroup of $C_G(A)$ permutes no basis, so $H \cong \Sigma_3 \times C_2$ or $\Sigma_3 \times \Sigma_3$.) Hence the situation of (e.2) cannot occur, and $H \cong \Sigma_3 \times \Sigma_3$ in the situation of (e.1).

Now assume $O_{2'}(H) = 1$. If $O_2(H) \neq 1$, then it is centric in H by (b). Since $H \neq S$ (that would satisfy neither of the conditions (e.1) nor (e.2)), $H \cong A_4$. If $O_2(H) = 1$, then $H \cong A_5$ by (c) and since $S \cong C_2^2$.

(f) Assume that $S \cong D_8$, and that there is a noncentral involution $x \in S$ such that $x \notin O^2(H)$ and $O_2(C_H(x)) = \langle x \rangle$.

If $O_{2'}(H) \neq 1$, then by (D.1) (and since $S \cong D_8$), $H \cong (C_3 \times A_4) \rtimes C_2$, $(C_3 \times A_5) \rtimes C_2$, or $\Sigma_3 \wr C_2$. The first of these cannot occur by the assumption that there be $x \in S$ with $O_2(C_H(x)) = \langle x \rangle$.

If $O_{2'}(H) = 1$ and $O_2(H) \neq 1$, then $O_2(H)$ is centric in H by (b), so $H \cong D_8$ or $H \cong \Sigma_4$. In both cases, the centralizer of each involution in H is a 2-group, which contradicts our assumption. If $O_{2'}(H) = O_2(H) = 1$, then by (c) (and since $O^2(H) < H$), $H \cong \Sigma_5$.

(g) Since $S \cong D_8$ and $O^2(H) = H$, all involutions in S are H-conjugate by the focal subgroup theorem, so $O_2(H) = 1$. By inspection of the different cases in (D.1), $O_{2'}(H) = 1$. So by (c), $H \cong A_6$, A_7, or $GL_3(2)$. □

LEMMA D.2. *Let S be a special 2-group of type 2^{2+4}; i.e., $[S, S] = Z(S) \cong C_2^2$ and $S/Z(S) \cong C_2^4$. Assume $\alpha \in \mathrm{Aut}(S)$ is an automorphism of order 3 such that $C_{S/Z(S)}(\alpha) = 1$ and $[\alpha, Z(S)] = 1$. Then either $S \cong Q_8 \times Q_8$ or $UT_3(4)$; or else $S/\langle z \rangle \cong 2_+^{1+4}$ for two of the three elements $z \in Z(S)^\#$, $S/\langle z \rangle \cong Q_8 \times C_2^2$ for the third, and there is a unique subgroup $T < S$ with $T \cong C_2^4$.*

PROOF. Set $Z = Z(S)$ and $V = S/Z$. Let $z_1, z_2, z_3 \in Z^\#$ be the three distinct elements. Then α induces an automorphism α_i of $S/\langle z_i\rangle$ for each $i = 1, 2, 3$, and hence $S/\langle z_i\rangle$ is isomorphic to 2^{1+4}_+ or $Q_8 \times C_2^2$. (It is nonabelian since $[S,S] = Z$, and none of the other nonabelian central extensions, 2^{1+4}_-, $C_2 \times (C_4 \times_{C_2} Q_8)$, nor $C_2^2 \times D_8$, has an automorphism of order 3 of the required type.) There are five α-invariant subspaces in V of rank 2, of which two have nontrivial squares in 2^{1+4}_+, and four have nontrivial squares in $Q_8 \times C_2^2$. For each $i = 1, 2, 3$, let $\theta(z_i)$ be the number of those subgroups the squares of whose elements are z_i. The $\theta(z_i)$ all have the same parity since the sum of any two of them is 2 or 4, and hence $(\theta(z_1), \theta(z_2), \theta(z_3))$ is (up to permutation) one of the triples $(1, 1, 1)$, $(1, 1, 3)$, or $(0, 2, 2)$. In the first case, $S \cong UT_3(4)$ by Lemma C.7(a), while in the second, $S \cong Q_8 \times Q_8$. In the third case, S is a pullback of two copies of 2^{1+4}_+ as described above, and the "exceptional" subspace lifts to a characteristic subgroup isomorphic to C_2^4. □

LEMMA D.3. *Fix a 2-group S, and an elementary abelian subgroup $Z \le Z(S) \cap \mathrm{Fr}(S)$ such that $S/Z \cong 2^{1+4}_+$, $Q_8 \times Q_8$, or $UT_3(4)$. Then $\mathrm{Fr}(S)$ is elementary abelian, and $Z(S)/Z = Z(S/Z)$.*

PROOF. Let $\pi\colon S \longrightarrow S/Z$ be the projection, and set $\widehat{Z} = \pi^{-1}(Z(S/Z)) \ge Z(S)$. When $S/Z \cong 2^{1+4}_+$ or $Q_8 \times Q_8$, the relations $[g^2, g] = 1$ for $g \in S$ suffice to show that $[S, \widehat{Z}] = 1$, and hence that $\widehat{Z} = Z(S)$.

Assume $S/Z \cong UT_3(4)$. We identify these two groups, and use the notation of 6.1 (also used in Appendix C) for elements of $UT_3(4)$. For $g \in S/Z$, let $\widehat{g} \in S$ denote some element in $\pi^{-1}(g)$. If $g, h \in S/Z$ are such that $\langle g, h\rangle \cong C_4 \times C_4$, then $[\widehat{g}, \widehat{h}] \in Z$, so $[\widehat{g}, \widehat{h}^2] = 1$ and $[\widehat{g}, \widehat{g}^2] = 1$. Since $\langle g^2, h^2\rangle = Z(S/Z)$, this shows that $[\widehat{g}, \widehat{Z}] = 1$ for such g. For each $u, v \in \mathbb{F}_4^\times$, $\langle e_{12}^u e_{23}^v, e_{12}^{u\omega} e_{23}^{v\omega}\rangle \cong C_4 \times C_4$ (by Lemma 6.3(a) or by the relation $[e_{12}^x, e_{23}^y] = e_{13}^{xy}$), so $[\widehat{e}_{12}^u \widehat{e}_{23}^v, \widehat{Z}] = 1$. Sine $S/Z = UT_3(4)$ is generated by such elements, $[S, \widehat{Z}] = 1$, and $\widehat{Z} = Z(S)$.

Thus $\widehat{Z} = Z(S) = \mathrm{Fr}(S)$ (so $Z(S/Z) = Z(S)/Z$) in all cases. Hence $[S,S]$ is elementary abelian, since $[g,h]^2 = [g,h^2] = 1$ for $g, h \in S$. Also, $\mathrm{Fr}(S) = [S,S]Z$ since $\mathrm{Fr}(S/Z) = [S/Z, S/Z]$, and hence $\mathrm{Fr}(S)$ is also elementary abelian. □

We say that a finite group G is *strictly 2-constrained* if $O_2(G)$ is centric in G; equivalently, $F^*(G) = O_2(G)$. Let $\mathbf{2Cons_4}$ denote the class of all finite groups which are strictly 2-constrained with sectional 2-rank at most 4. Throughout the rest of the chapter, we list a few results about the structure of such groups. Some of these are taken entirely or in part from [**GH**].

LEMMA D.4 ([**GH**, II.4.1]). *Assume $G \in \mathbf{2Cons_4}$ is such that $G/O_2(G) \cong A_5$. Then $O_2(G)$ is isomorphic to C_2^4 or $D_8 \times_{C_2} Q_8$.*

LEMMA D.5. *Assume $G \in \mathbf{2Cons_4}$ is such that $G/O_2(G) \cong GL_3(2)$. Choose $S \in \mathrm{Syl}_2(G)$, and set $Q = O_2(G) \trianglelefteq S$. Then $Q \cong C_2^3$, C_2^4, or $C_4 \times C_2^3$.*

(a) *If $Q \cong C_2^3$, then $S \cong UT_4(2)$ or S has type M_{12}, and in either case, $r(S) = 4$.*

(b) *If $Q \cong C_2^4$, then either Q is decomposable as an $\mathrm{Aut}_G(Q)$-module, or Q is indecomposable with an invariant subgroup of rank 1 or 3. If Q is decomposable, then $[G,Q] \cong C_2^3$ and $G/[G,Q] \cong SL_2(7)$. If Q is indecomposable, then for each involution $\alpha \in \mathrm{Aut}_G(Q)$, $\mathrm{rk}([\alpha, Q]) = 2$.*

(c) *Assume* $Q \cong C_4 \times C_2^3$, *and set* $V = \Omega_1(Q)$. *Then* V *is decomposable as an* $\mathrm{Aut}_G(Q)$*-module with invariant submodule* $[G,V]$ *of rank* 3, $Q/\mathrm{Fr}(Q)$ *is indecomposable, and* $G/[G,V] \cong C_4 \times_{C_2} SL_2(7)$.

PROOF. Set $\Gamma = \mathrm{Out}_G(Q) \cong GL_3(2)$ for short. The possibilities for Q are listed in [**GH**, Proposition II.3.1].

(a,b) Point (a) is shown in [**GH**, Lemma II.3.4], and the first statement in (b) in [**GH**, Lemma II.3.7]. If $Q \cong C_2^4$ and is Γ-decomposable, then since G does not have a direct factor C_2 (that would imply $r(S) = 5$ by (a)), $G/[G,Q] \cong SL_2(7)$ by [**GH**, Lemma II.3.8].

If $Q \cong C_2^4$ and the conclusion of the last statement in (b) is not true, then $\mathrm{rk}([\alpha,Q]) = 1$ (and hence $\mathrm{rk}(C_Q(\alpha)) = 3$) for each involution $\alpha \in \Gamma$ since the involutions are all Γ-conjugate. Also, $\Gamma \cong GL_3(2)$ is generated by three involutions (e.g., the three elementary matrices e_{12}, e_{23}, and e_{31}). So $\mathrm{rk}(C_Q(\Gamma)) \geq 1$ and $\mathrm{rk}([\Gamma,Q]) \leq 3$, with equality in each case since Γ acts faithfully. Thus Q is decomposable.

(c) Assume $Q \cong C_4 \times C_2^3$, and set $V = \Omega_1(Q)$ and $Z = \mathrm{Fr}(Q)$. By [**GH**, Lemma II.3.12], $G/V \cong C_2 \times GL_3(2)$, and hence Q/Z is Γ-indecomposable by (b).

Fix an involution $\alpha \in \Gamma$, and choose $g \in Q \smallsetminus V$. Then $[\alpha, g] \in V$, and $[\alpha, g] \in C_V(\alpha)$ since $\alpha^2 = \mathrm{Id}$. Since Q/Z is indecomposable, $\mathrm{rk}(C_{Q/Z}(\alpha)) = 2 = \mathrm{rk}(C_{V/Z}(\alpha))$ by (b), so $C_Q(\alpha) \leq V$, $[\alpha, gv] \neq 1$ for each $v \in V$, and hence $[\alpha, g] \notin [\alpha, V]$. Thus $C_V(\alpha) > [\alpha, V]$, which by (b) implies that V is decomposable.

Thus $V = Z \times W$, where $G/Q \cong GL_3(2)$ acts faithfully on W. In particular, $W = [G,V]$, and $Q/[G,V] \cong C_4$. So $G/[G,V]$ is a (central) extension of $Q/[G,V] \cong C_4$ by $G/Q \cong GL_3(2)$, $O^2(G)/[G,V] \cong SL_2(7)$ by (b) (and since $G/V \cong C_2 \times GL_3(2)$), and thus $G/[G,V] \cong C_4 \times_{C_2} SL_2(7)$. □

LEMMA D.6. *If* $G \in \mathbf{2Cons_4}$, *then* $G/O_2(G) \not\cong \Sigma_6$.

PROOF. This is shown in [**GH**, Theorem II.B], but since the proof there is somewhat long and indirect, we give a different argument here. Assume G is strictly 2-constrained with $G/O_2(G) \cong \Sigma_6$; we will show that G has sectional 2-rank at least 5. Set $Q = O_2(G)$. If $\mathrm{rk}(Q/\mathrm{Fr}(Q)) > 4$, then we are done, so upon replacing G by $G/\mathrm{Fr}(Q)$, we can assume that $Q \cong C_2^4$.

Fix a surjection $\psi\colon G \twoheadrightarrow \Sigma_6$ with kernel Q. Since there is a unique conjugacy class of subgroup Σ_6 in $GL_4(2)$ (Proposition D.1(c)), we can identify Q with the group of subsets of even order in $\{1,2,3,4,5,6\}$, modulo the relation of identifying each subset with its complement (and where ψ induces the obvious action of Σ_6). Consider the subgroups

$$\widehat{T} = \big\langle (1\,2), (3\,4), (5\,6) \big\rangle \leq \Sigma_6 \qquad \text{and} \qquad T = \widehat{T} \cap A_6\,.$$

Then $\widehat{T} \cong C_2^3$, $T \cong C_2^2$, and $\widehat{T}$ acts via the identity on $Q_0 = \langle 12, 34 \rangle$ and on Q/Q_0. There is a subgroup $\Gamma \leq \Sigma_6$ such that $\Gamma \cong A_5$ and $T \in \mathrm{Syl}_2(\Gamma)$ (defined via the permutation action of A_5 on its six Sylow 5-subgroups, or on the six pairs of opposite vertices in an icosahedron).

Set $G_0 = \psi^{-1}(\Gamma)$, $\widehat{H} = \psi^{-1}(\widehat{T})$ and $H = \psi^{-1}(T)$. Then $G_0 \cong Q \rtimes \Gamma$, since by [**GH**, Lemma II.2.6], any extension of C_2^4 by A_5 splits. Since $C_Q(T) = Q_0$ has rank 2, Q is the $L_2(4)$-module for Γ (Proposition D.1(d)), so G_0 is isomorphic to a parabolic subgroup in $SL_3(4)$, and $H \cong UT_3(4)$. Let R be the unique subgroup $R < H$ such that $R \neq Q$ and $R \cong C_2^4$ (see Lemma C.6(a)).

Let $x \in \widehat{H}$ be such that $\psi(x) = (1\,2)(3\,4)(5\,6)$. Then $c_x \in \mathrm{Aut}(H)$ induces the identity on Q_0, Q/Q_0, and H/Q. Also, $c_x(R) = R$ since $Q, R < H$ are the only subgroups isomorphic to C_2^4. Since $[x, H] \leq Q$, $[x, R] \leq R \cap Q = Q_0$. Thus c_x induces the identity on R/Q_0 and on Q/Q_0, and hence on $H/Z(H) = QR/Q_0$.

Let $y \in G$ be such that $\psi(y) = (1\,3\,5)(2\,4\,6)$. Since $[x, y] \in \mathrm{Ker}(\psi) = Q$, $[x^2, y] \equiv [x, y]^2 = 1 \pmod{[x, Q] \leq Q_0}$. Since $C_{Q/Q_0}(y) = 1$, this implies that $x^2 \in Q_0$, and hence that $\widehat{H}/Q_0 \cong (Q/Q_0) \times \widehat{T} \cong C_2^5$. □

Two more lemmas of this type are needed. By $(C_3 \times C_3) \rtimes C_4$, we mean the semidirect product where C_4 acts faithfully on $C_3 \times C_3$.

Lemma D.7. *Assume $G \in$ **2Cons$_4$** is such that $G/O_2(G)$ is isomorphic to $\Sigma_3 \wr C_2$ or $(C_3 \times C_3) \rtimes C_4$. Then $O_2(G)$ is isomorphic to one of the groups C_2^4, $Q_8 \times_{C_2} Q_8$, or $Q_8 \times Q_8$.*

Proof. Set $Q = O_2(G)$. Since $\Sigma_3 \wr C_2 \cong (C_3 \times C_3) \rtimes D_8$ contains a subgroup isomorphic to $(C_3 \times C_3) \rtimes C_4$, it suffices to prove this when $G/Q \cong (C_3 \times C_3) \rtimes C_4$. Note that G/Q contains no normal subgroup of order 3. Set $H = O^2(G)$,so that $H/Q \cong C_3 \times C_3$.

Set $Z = \mathrm{Fr}(Q)$ and $V = Q/Z$. By Lemma A.9, H/Q acts faithfully on V, so $\mathrm{rk}(V) = 4$. Let $V_1, V_2 < V$ be the two subgroups of rank 2 in which are normal in H/Z. Thus $V = V_1 \times V_2$. Let

$$\mathfrak{q}\colon V = V_1 \times V_2 \longrightarrow Z \qquad \text{and} \qquad \mathfrak{b}\colon V \times V \longrightarrow Z$$

be the quadratic and bilinear maps where $\mathfrak{q}(xZ) = x^2$ and $\mathfrak{b}(xZ, yZ) = [x, y]$.

Case 1: Assume $Z = \mathrm{Fr}(Q)$ is elementary abelian and $[H, Z] \neq 1$. Thus H/Q acts nontrivially on Z, and since G/Q contains no normal subgroup of order 3, H/Q must act faithfully on Z. Hence $Z/C_Z(H)$ has rank at least 4. Since $r(Q) \leq 4$, we have $\mathrm{rk}(Z) = 4$ and $C_Z(H) = 1$. Let $Z_1, Z_2 < Z$ be the two subgroups of rank 2 normalized by H. For $i = 1, 2$, let $\mathfrak{q}_i\colon V \to Z_i$ and $\mathfrak{b}_i\colon V \times V \to Z_i$ be the composites of $\mathfrak{q}$ and $\mathfrak{b}$ with projection to Z_i.

For each $i, j = 1, 2$, there is $g \in H$ of order 3 which acts nontrivially on both V_i and Z_j. Since $\mathfrak{q}_j|_{V_i}\colon V_i \to Z_j$ commutes with the action of g, it is either a bijection or zero, and in particular is linear. Hence $\mathfrak{q}|_{V_1}$ and $\mathfrak{q}|_{V_2}$ are both linear. They are both nonzero, since otherwise the preimage of V_1 or of V_2 in Q would have rank 6. Thus they are injective. Also, $\mathrm{Im}(\mathfrak{q}|_{V_1}) + \mathrm{Im}(\mathfrak{q}|_{V_2})$ is normalized by the action of G/Q, hence must be equal to Z, and thus $\mathfrak{q}|_{V_1} \oplus \mathfrak{q}|_{V_2}$ is an isomorphism. Since V_1, V_2, Z_1, and Z_2 are the only subgroups of rank 2 in V or Z normalized by H, we can assume (after reindexing if necessary) that $\mathfrak{q}(V_1) = Z_1$ and $\mathfrak{q}(V_2) = Z_2$.

Thus, for $i = 1, 2$, there is $g \in G$ of order 3 which acts trivially on V_i and Z_i and nontrivially on V_{3-i} and Z_{3-i}. Hence for each $v \in V_i$, $\mathfrak{b}_i(v, w) \in Z_i$ is independent of $w \in V_{3-i}^{\#}$, and since $\prod_{w \in V_{3-i}^{\#}} \mathfrak{b}_i(v, w) = 1$ ($\mathfrak{b}_i$ is bilinear), $\mathfrak{b}_i(v, V_{3-i}) = 1$. Thus $\mathfrak{b}_i(V_1, V_2) = 1$ for $i = 1, 2$, so $\mathfrak{b}(V_1, V_2) = 1$, and $\mathfrak{q}$ is linear.

This proves that $Q \cong C_4^4$. Hence G contains a subgroup isomorphic to $T \cong (C_4 \times C_4) \wr C_2$, which is impossible since $r(T) = 5$. So this case is impossible.

Case 2: Next assume $Z = \mathrm{Fr}(Q)$ is elementary abelian and $[H, Z] = 1$. If $Z = 1$, then $Q \cong C_2^4$, so assume $Z \neq 1$. Since $\mathfrak{q}$ commutes with the actions of H/Q on $V = Q/Z$ and on Z, it sends all elements of $V_1^{\#}$ to the same element $z_1 \in Z$, and all elements of $V_2^{\#}$ to the same element $z_2 \in Z$.

Let $g \in H$ be an element of order 3 which acts nontrivially on V_1 and trivially on V_2, and fix $w \in V_2$. Then $\mathfrak{b}(v,w) \in Z$ is constant for $v \in V_1^\#$ (since the three elements are permuted transitively by g), $\prod_{v\in V_1^\#} \mathfrak{b}(v,w) = 1$ since $\mathfrak{b}$ is bilinear, and so $\mathfrak{b}(V_1,w) = 1$.

Thus $\mathfrak{b}(V_1,V_2) = 1$, and $Z = \mathrm{Fr}(Q) = \langle \mathrm{Im}(\mathfrak{q})\rangle\rangle = \langle z_1, z_2\rangle$. So either $z_1 = z_2$ and $Q \cong 2_+^{1+4}$, or $z_1 \neq z_2$ and $Q \cong Q_8 \times Q_8$.

Case 3: Now assume Z is not elementary abelian, and assume G is a minimal example of this type. Set $Z_0 = \mathrm{Fr}(Z)$. By minimality, Z_0 is elementary abelian and central, and $Q/Z_0 \cong 2_+^{1+4}$ or $Q_8 \times Q_8$ by Steps 1 and 2. But then $Z = \mathrm{Fr}(Q)$ is elementary abelian by Lemma D.3, a contradiction. □

LEMMA D.8. *Assume $G \in \mathbf{2Cons_4}$ is such that $G/O_2(G) \cong D_{10}$. Then $O_2(G)$ is isomorphic to C_2^4 or 2_-^{1+4}, or is of type $U_3(4)$.*

PROOF. Set $Q = O_2(G)$ for short. Fix $\sigma, \tau \in \mathrm{Aut}(Q)$ such that $|\sigma| = 5$, $\tau^2 \in \mathrm{Inn}(Q)$, and $\langle[\sigma],[\tau]\rangle = \mathrm{Out}_G(Q) \cong D_{10}$. Then $\mathrm{Out}_G(Q)$ acts faithfully on $Q/\mathrm{Fr}(Q)$ by Lemma A.9, $C_{Q/\mathrm{Fr}(Q)}(\sigma) = 1$, and $\mathrm{rk}(C_{Q/\mathrm{Fr}(Q)}(\tau)) = 2$. So σ acts on $Q/\mathrm{Fr}(Q)$ with three free orbits of involutions each of which contains an element of $C_{Q/\mathrm{Fr}(Q)}(\tau)$ and hence is τ-invariant. Fix $a_1, a_2, a_3, a_4, a_5 \in Q$ whose classes $\bar{a}_i$ generate $Q/\mathrm{Fr}(Q)$ (hence generate Q), whose product is trivial in $Q/\mathrm{Fr}(Q)$, and with indices (modulo 5) chosen such that $\sigma(a_i) = a_{i+1}$ and $\tau(\bar{a}_i) = \bar{a}_{-i}$.

Assume $C_Q(\sigma) \neq 1$. Then by [**GH**, Lemma I.3.9], Q is special of type 2^{k+4} where $1 \leq k \leq 4$. Set $Z = \mathrm{Fr}(Q) = Z(Q) \cong C_2^k$. Then $C_Z(\sigma) \neq 1$, and hence $[\sigma, Z] = 1$ since $\mathrm{rk}(Z) \leq 4$. Thus there are $x, y, z \in Z$ such that $x = a_i^2$, $y = (a_i a_{i+1})^2 = [a_i, a_{i+1}]$, and $z = (a_i a_{i+2})^2 = [a_i, a_{i+2}]$ for all i (indices again taken modulo 5). Then $1 = (a_1a_2a_3a_4a_5)^2 = x^5 \prod_{1\leq i<j\leq 5}[a_i, a_j] = xyz$, so $Z = \mathrm{Fr}(Q) = \langle x, y, z\rangle$ has rank at most 2. By [**GH**, Lemma I.3.9] again, Q is isomorphic to 2_-^{1+4} or is of type $U_3(4)$.

Now assume $C_Q(\sigma) = 1$. Assume G and Q are minimal such that $Q \not\cong C_2^4$, and set $Z = \mathrm{Fr}(Q)$. By minimality, $Z \leq Z(Q)$ and is elementary abelian. Also, $\mathrm{rk}(Z) \geq 4$ since σ acts nontrivially, and $\mathrm{rk}(Z) = 4$ since $r(Q) = 4$. If Q is abelian, then $Q \cong C_4^4$, G has Sylow subgroups isomorphic to $(C_4 \times C_4) \wr C_2$ of sectional rank 5, so $G \notin \mathbf{2Cons_4}$.

Thus Q is nonabelian, and so $[Q,Q] = Z$. For each i, set $x_i = [a_i, a_{i+1}a_{i-1}]$. Then $\sigma(x_i) = x_{i+1}$, and $x_1x_2x_3x_4x_5 \in C_Z(\sigma) = 1$. Also,

$$\begin{aligned} x_i = [a_i, a_{i+1}a_{i-1}] &= [a_i, a_{i-1}]\cdot\sigma([a_i, a_{i-1}]) \\ &= [a_i, a_{i+2}a_{i-2}] = [a_i, a_{i-2}]\cdot\sigma^2([a_i, a_{i-2}]). \end{aligned}$$

Since $C_Z(\sigma) = 1$, this implies that $[a_i, a_{i-1}] = x_{i+1}x_{i-2}$ and $[a_i, a_{i-2}] = x_{i+1}x_{i+2}$ for each i. In particular, $Z = [Q,Q] = \langle x_1, \ldots, x_5\rangle$. Also, $(a_i)^2 \in C_Z(\sigma^{2i}\tau)$, so either $(a_i)^2 = 1$, or one of the following holds:

$$\begin{aligned} (a_i)^2 = x_i &\implies (a_{i+1}a_{i-1})^2 = (a_{i+2}a_{i-2})^2 = x_i \\ (a_i)^2 = x_{i+1}x_{i-1} &\implies (a_{i+1}a_{i-1})^2 = 1 \\ (a_i)^2 = x_{i+2}x_{i-2} &\implies (a_{i+2}a_{i-2})^2 = 1. \end{aligned}$$

If any element of $Q \smallsetminus Z$ has order 2, then Q contains an abelian subgroup of rank 5. If $(a_i)^2 = x_i$, then $Z\langle a_i, a_{i+1}a_{i-1}\rangle \cong Q_8 \times C_2^3$. In either case, $r(Q) \geq 5$, and so $G \notin \mathbf{2Cons_4}$. □

Bibliography

[AOV1] K. Andersen, B. Oliver, & J. Ventura, Reduced, tame, and exotic fusion systems, Proc. London Math. Soc. 105 (2012), 87–152

[AOV2] K. Andersen, B. Oliver, & J. Ventura, Fusion systems and amalgams, Math. Z. 274 (2013), 1119–1154

[A1] M. Aschbacher, Finite group theory, Cambridge Univ. Press (1986)

[A2] M. Aschbacher, A 2-local characterization of M_{12}, Illinois J. Math. 47 (2003), 31–47

[AKO] M. Aschbacher, R. Kessar, & B. Oliver, Fusion systems in algebra and topology, Cambridge Univ. Press (2011)

[Be] H. Bender, Transitive Gruppen gerader Ordnung, in denen jede Involution genau einen Punkt festlässt, J. Algebra 17 (1971), 527–554

[BCGLO1] C. Broto, N. Castellana, J. Grodal, R. Levi, & B. Oliver, Subgroup families controlling p-local finite groups, Proc. London Math. Soc. 91 (2005), 325–354

[BCGLO2] C. Broto, N. Castellana, J. Grodal, R. Levi, & B. Oliver, Extensions of p-local finite groups, Trans. Amer. Math. Soc. 359 (2007), 3791–3858

[BLO2] C. Broto, R. Levi, & B. Oliver, The homotopy theory of fusion systems, J. Amer. Math. Soc. 16 (2003), 779–856

[BMO] C. Broto, J. Møller, & B. Oliver, Equivalences between fusion systems of finite groups of Lie type, Jour. Amer. Math. Soc. 25 (2012), 1–20

[Br] K. Brown, Cohomology of groups, Springer-Verlag (1982)

[Bu] W. Burnside, The Theory of Groups of Finite Order, Cambridge Univ. Press (1897).

[Ca2] R. Carter, Finite groups of Lie type: conjugacy classes and complex characters, Wiley (1985)

[CF] R. Carter & P. Fong, The Sylow 2-subgroups of the finite classical groups, J. Algebra 1 (1964), 139–151

[Cr] D. Craven, The theory of fusion systems, Cambridge Univ. Press (2011)

[Ft] R. Foote, A characterization of finite groups containing a strongly closed 2-subgroup, Comm. Algebra 25 (1997), 593–606

[Gd1] D. Goldschmidt, 2-Fusion in finite groups, Annals of Math. 99 (1974), 70–117

[Gd2] D. Goldschmidt, Automorphisms of trivalent graphs, Annals of Math. 111 (1980), 377–406

[G] D. Gorenstein, Finite groups, Harper & Row (1968)

[GH] D. Gorenstein & K. Harada, Finite groups whose 2-subgroups are generated by at most 4 elements, Memoirs Amer. Math. Soc. 147 (1974)

[Hi] G. Higman, Suzuki 2-groups, Ill. J. Math. 7 (1963), 79–96

[Kl] P. Kleidman, The maximal subgroups of the Chevalley groups $G_2(q)$ with q odd, the Ree groups ${}^2G_2(q)$, and their automorphism groups, J. Algebra 117 (1988), 30–71

[Ly] R. Lyons, Evidence for a new finite simple group, J. Algebra 20 (1972), 540–569

[Mc] S. MacLane, Homology, Springer-Verlag (1975)

[Mz] P. Mazet, Sur les multiplicateurs de Schur des groupes de Mathieu, J. Algebra 77 (1982), 552–576

[O1] B. Oliver, Splitting fusion systems over 2-groups, Proc. Edinburgh Math. Soc. 56 (2013), 263–301

[OV] B. Oliver & J. Ventura, Saturated fusion systems over 2-groups, Trans. Amer. Math. Soc. 361 (2009), 6661–6728.

[Pg] L. Puig, Frobenius categories, J. Algebra 303 (2006), 309–357

[Sz1] M. Suzuki, Group theory I, Springer-Verlag (1982)

[Sz2] M. Suzuki, Group theory II, Springer-Verlag (1986)

[Ta] D. Taylor, The geometry of the classical groups, Heldermann (1992)

[Th] J. Thompson, Nonsolvable finite groups all of whose local subgroups are solvable I, Bull. Amer. Math. Soc. 74 (1968), 383–437