SPLITTING FUSION SYSTEMS OVER 2-GROUPS

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Abstract. We find conditions which imply that a saturated fusion system over a product of 2-groups splits as a product of fusion systems over the factors.

A *saturated fusion system* $\mathcal{F}$ over a p-group S is a category whose objects are the subgroups of S, where each morphism in $\mathcal{F}$ is a monomorphism of groups, and which satisfies certain axioms listed in Definition 1.1 below. As one of the main motivating examples, let G be a finite group, fix $S \in \mathrm{Syl}_p(G)$, and let $\mathcal{F}_S(G)$ be the category whose objects are the subgroups of S and where $\mathrm{Hom}_{\mathcal{F}_S(G)}(P,Q)$ is the set of homomorphisms induced by conjugation in G. Then $\mathcal{F}_S(G)$ is a saturated fusion system over S. A saturated fusion system $\mathcal{F}$ which is not isomorphic to $\mathcal{F}_S(G)$ for any finite group G is called *exotic*.

In an ongoing project with Kasper Andersen and Joana Ventura, we are attempting, with the help of computer computations, to list all saturated fusion systems over 2-groups of small order, where we try to make "small" be as large as possible. Since it is clearly impossible to do this explicitly, we want to restrict attention to some appropriate smaller class of fusion systems. Define a saturated fusion system to be *reduced* if it has no proper normal subsystems of p-power index or of index prime to p (Definition 1.6) and no nontrivial normal p-subgroups (Definition 1.2); and to be *indecomposable* if it is not isomorphic to a product of fusion systems over strictly smaller p-groups (Definition 4.1). In [AOV, Theorems A & C], we show that exotic fusion systems can be "detected" in an explicit way on reduced, indecomposable fusion systems. This means, for example, that if we list all reduced indecomposable fusion systems over 2-groups of order $\leq 2^n$ for some fixed n, and show they are all realized by finite groups satisfying a certain "tameness" condition on their automorphism groups ([AOV, Definition 2.10]), then there are no exotic fusion systems over 2-groups of order $\leq 2^n$.

In earlier work with Ventura [OV], and in the ongoing work with her and with Andersen, we derive some fairly strong conditions on a 2-group S which must be satisfied if there are any reduced fusion systems over S. When we use a computer to eliminate those groups of given order which do not satisfy these conditions, we are left with the groups which are known to be Sylow 2-subgroups of simple groups, a very few others, and a long list of groups which split as products of smaller groups also satisfying the conditions. This is what motivated us to study reduced fusion systems over products of (nontrivial) 2-groups, and look for conditions which guarantee that any such fusion system factors as a product.

Our main results are the following two theorems. As usual, for a p-group Q, $\Omega_1(Q)$ is the subgroup generated by all elements of order p.

Theorem A. *Fix 2-groups S_1 and S_2, and set $S = S_1 \times S_2$. Assume, for $i = 1, 2$,*

(a) *S_i is nontrivial and indecomposable, and $\Omega_1(Z(S_i)) \leq [S_i, S_i]$; and*

(b) *S_{3-i} contains no subgroup isomorphic to $S_i \times S_i$.*

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Then for every saturated fusion system $\mathcal{F}$ over S which has no proper normal subsystems of p-power index or of index prime to p, $\mathcal{F} \cong \mathcal{F}_1 \times \mathcal{F}_2$ for some pair of saturated fusion systems $\mathcal{F}_i$ over S_i.

The isomorphism $\mathcal{F} \cong \mathcal{F}_1 \times \mathcal{F}_2$ in the above theorem is induced by an isomorphism $S \cong S_1 \times S_2$, but that isomorphism need *not* send S_i to itself. Condition (a) in Theorem A holds whenever S_i is nonabelian and $Z(S_i)$ is cyclic. But it also holds for many 2-groups whose center is not cyclic; for example, for the groups $UT_3(\mathbb{F}_{2^n})$ (strictly upper triangular matrices) when $n \geq 2$.

The second theorem puts stronger conditions on S_1, while weakening those on S_2. Let D_{2^n} and SD_{2^n} denote the dihedral and semidihedral groups, respectively, of order 2^n.

Theorem B. *Fix 2-groups S_1 and S_2, and set $S = S_1 \times S_2$. Assume*

(a) *$S_1 \cong D_{2^n}$ ($n \geq 3$), SD_{2^n} ($n \geq 4$), or $C_{2^n} \wr C_2$ ($n \geq 2$); and*

(b) *S_2 contains no proper subgroup isomorphic to $S_1 \times S_1$.*

Then for every saturated fusion system $\mathcal{F}$ over S which has no proper normal subsystems of p-power index or of index prime to p, $\mathcal{F} \cong \mathcal{F}_1 \times \mathcal{F}_2$ for some pair of saturated fusion systems $\mathcal{F}_i$ over S_i.

Theorems A and B are shown as Corollary 4.7 and Theorem 4.9, respectively. They are both consequences of Proposition 4.4, which is a more general (and more technical) splitting result.

Examples are given in Section 4.4 to show why certain conditions in the above theorems are needed. For example, set $p = 2$, let G be the subgroup of index two in $\Sigma_6 \times PGL_2(9)$ which contains neither factor, and let $\mathcal{F}$ be the fusion system of G (over the Sylow subgroup $\cong D_8 \times D_{16}$). Then $\mathcal{F}$ satisfies all of the hypotheses in Theorems A and B except the condition that $\mathcal{F}$ have no normal subsystems of 2-power index, but does not factor as a product of smaller fusion systems.

The example of the fusion system of A_{14} over the 2-group $D_8 \times (D_8 \wr C_2)$ helps show why the condition $S_2 \ngtr S_1 \times S_1$ is needed in Theorems A and B (and more examples are given in Section 4.4). However, the following theorem (proven as Theorem 4.10) shows some cases where we can avoid this problem. Note that the factors in the following statement can be, but do not have to be, isomorphic to each other.

Theorem C. *Assume $S = S_1 \times S_2 \times \cdots \times S_m$, where each factor S_i is isomorphic to one of the groups D_{2^n} ($n \geq 3$), SD_{2^n} ($n \geq 4$), or $C_{2^n} \wr C_2$ ($n \geq 2$). Let $\mathcal{F}$ be a saturated fusion system over S which has no proper normal subsystems of p-power index or of index prime to p. Then $\mathcal{F}$ is isomorphic to a product of saturated fusion systems over D.*

Theorems B and C also hold (with only minor modifications to the proofs) when S_1 (or D) is a quaternion 2-group. However, in that case, it is very easy to see directly that $Z(S_1)$ (in Theorem B) or $Z(S)$ (in Theorem C) is normal in any fusion system satifying the hypotheses of the theorems, and hence that no such fusion system can be reduced.

Section 1 contains the background material which is needed on fusion systems. Sections 2 and 3 contain technical results about actions and representations of groups, and automorphisms of p-groups, respectively. The results about splittings of fusion systems are shown in Section 4.

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version of the Krull-Schmidt theorem, which allowed us to greatly simplify the proof of Proposition 3.2.

1. Background results about fusion systems

A *fusion system* over a finite p-group S is a category $\mathcal{F}$, where $\mathrm{Ob}(\mathcal{F})$ is the set of all subgroups of S, and where for all $P, Q \le S$,

$$\mathrm{Hom}_S(P,Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P,Q) \subseteq \mathrm{Inj}(P,Q) ,$$

and each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ is the composite of an $\mathcal{F}$-isomorphism followed by an inclusion. Here, $\mathrm{Inj}(P,Q)$ denotes the set of injective homomorphisms from P to Q.

If $\mathcal{F}$ is a fusion system over S, we say two subgroups $P, Q \le S$ are *$\mathcal{F}$-conjugate* if they are isomorphic as objects of the category $\mathcal{F}$. Two elements $g, h \in S$ are $\mathcal{F}$-conjugate if there is $\varphi \in \mathrm{Iso}_{\mathcal{F}}(\langle g\rangle, \langle h\rangle)$ such that $\varphi(g) = h$. For $P \le S$ and $g \in S$, we write

$$P^{\mathcal{F}} = \{Q \le S \mid Q \text{ is } \mathcal{F}\text{-conjugate to } P\} \quad \text{and} \quad g^{\mathcal{F}} = \{h \in S \mid h \text{ is } \mathcal{F}\text{-conjugate to } g\}.$$

Definition 1.1 ([Pg], see [BLO, Definition 1.2])**.** *Let $\mathcal{F}$ be a fusion system over a finite p-group S.*

- *A subgroup $P \le S$ is* fully centralized in $\mathcal{F}$ *if $|C_S(P)| \ge |C_S(Q)|$ for each $Q \in P^{\mathcal{F}}$.*
- *A subgroup $P \le S$ is* fully normalized in $\mathcal{F}$ *if $|N_S(P)| \ge |N_S(Q)|$ for each $Q \in P^{\mathcal{F}}$.*
- *$\mathcal{F}$ is a* saturated fusion system *if the following two conditions hold:*
  (I) (Sylow axiom) *If $P \le S$ is fully normalized in $\mathcal{F}$, then P is fully centralized in $\mathcal{F}$, and $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$.*

  (II) (Extension axiom) *For each $P, Q \le S$ such that Q is fully centralized in $\mathcal{F}$, and each $\varphi \in \mathrm{Iso}_{\mathcal{F}}(P,Q)$, if we set*
  $$N_\varphi = \{g \in N_S(P) \mid \varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(Q)\},$$
  then there is $\bar\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ such that $\bar\varphi|_P = \varphi$.

If G is a finite group, $S \in \mathrm{Syl}_p(G)$, and $\mathcal{F}_S(G)$ is the fusion system defined in the introduction, then a subgroup $P \le S$ is fully normalized (fully centralized) in $\mathcal{F}_S(G)$ exactly when $N_S(P) \in \mathrm{Syl}_p(N_G(P))$ ($C_S(P) \in \mathrm{Syl}_p(C_G(P))$). For a proof that $\mathcal{F}_S(G)$ is a saturated fusion system, see, for example, [BLO, Proposition 1.3].

We next recall some of the other definitions associated with a fusion system.

Definition 1.2. *Fix a prime p, a p-group S, and a fusion system $\mathcal{F}$ over S. Let $P \le S$ be any subgroup.*

- *P is* $\mathcal{F}$-centric *if $C_S(Q) = Z(Q)$ for all $Q \in P^{\mathcal{F}}$.*
- *P is* strongly closed *in $\mathcal{F}$ if for each $g \in P$, $g^{\mathcal{F}} \subseteq P$. Equivalently, for each $Q \le P$ and each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,S)$, $\varphi(Q) \le P$.*
- *P is* normal *in $\mathcal{F}$ ($P \trianglelefteq \mathcal{F}$) if every morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$ in $\mathcal{F}$ extends to a morphism $\bar\varphi \in \mathrm{Hom}_{\mathcal{F}}(PQ,PR)$ such that $\bar\varphi(P) = P$.*
- $\mathfrak{foc}(\mathcal{F}) = \big\langle x^{-1}y \,\big|\, x,y \in S, y \in x^{\mathcal{F}} \big\rangle$.
- $\mathfrak{hyp}(\mathcal{F}) = \big\langle x^{-1}\alpha(x) \,\big|\, x \in P \le S, \alpha \in \mathrm{Aut}_{\mathcal{F}}(P) \text{ has order prime to } p \big\rangle$.

- *Let $\mathcal{H}$ be a set of subgroups of S closed under $\mathcal{F}$-conjugacy. The fusion system $\mathcal{F}$ is $\mathcal{H}$-*saturated *if all subgroups in $\mathcal{H}$ satisfy axioms* (I) *and* (II) *in Definition 1.1. Also, $\mathcal{F}$ is $\mathcal{H}$-*generated *if each morphism in $\mathcal{F}$ is a composite of restrictions of $\mathcal{F}$-morphisms between subgroups in $\mathcal{H}$.*

The following theorem will be applied several times in Section 4, when showing that certain fusion systems are saturated.

Theorem 1.3 ([5a1, Theorem A]). *Fix a fusion system $\mathcal{F}$ over a p-group S, and a set $\mathcal{H}$ of subgroups of S closed under $\mathcal{F}$-conjugacy such that $\mathcal{F}$ is $\mathcal{H}$-saturated and $\mathcal{H}$-generated. Assume, for each $P \le S$ which is $\mathcal{F}$-centric and not in $\mathcal{H}$, that there is some $Q \in P^{\mathcal{F}}$ such that $\mathrm{Out}_S(Q) \cap O_p(\mathrm{Out}_{\mathcal{F}}(Q)) \ne 1$. Then $\mathcal{F}$ is a saturated fusion system.*

We also need to work with fusion subsystems, and weakly normal fusion subsystems, of a saturated fusion system. When $\mathcal{F}$ is a fusion system over S and $S_0 \le S$, $\mathcal{F}|_{S_0}$ denotes the full subcategory whose objects are the subgroups of S_0 (a fusion system over S_0).

Definition 1.4. *Fix a prime p, and a fusion system $\mathcal{F}$ over a p-group S.*

- *A (saturated) fusion subsystem of $\mathcal{F}$ is a subcategory $\mathcal{F}_0 \subseteq \mathcal{F}$ which is itself a (saturated) fusion system over a subgroup $S_0 \le S$.*
- *For any $\alpha \in \mathrm{Aut}(S)$, ${}^{\alpha}\mathcal{F}$ denotes the fusion system over S defined by*

$$\mathrm{Hom}_{{}^{\alpha}\mathcal{F}}(P,Q) = \alpha \cdot \mathrm{Hom}_{\mathcal{F}}(\alpha^{-1}(P), \alpha^{-1}(Q)) \cdot \alpha^{-1}$$

  for all $P, Q \le S$.
- *If $\mathcal{E}$ is another fusion system over S, then $\mathcal{E} \cong \mathcal{F}$ if $\mathcal{E} = {}^{\alpha}\mathcal{F}$ for some $\alpha \in \mathrm{Aut}(S)$.*
- *If $S_0 \le S$, and $X_1, \ldots, X_m$ are subcategories and/or sets of morphisms in $\mathcal{F}|_{S_0}$, then $\langle X_1, \ldots, X_m \rangle$ denotes the smallest fusion subsystem over S_0 in $\mathcal{F}$ (not necessarily saturated) which contains the X_i. Thus $\langle X_1, \ldots, X_m \rangle$ is a subcategory of $\mathcal{F}$ whose objects are the subgroups of S_0, and which contains those morphisms which are composites of restrictions of inner automorphisms of S_0, and restrictions of morphisms in the X_i and their inverses.*
- *A fusion subsystem $\mathcal{F}_0 \subseteq \mathcal{F}$ over $S_0 \trianglelefteq S$ is $\mathcal{F}$-*invariant *if S_0 is strongly closed in $\mathcal{F}$, ${}^{\alpha}\mathcal{F}_0 = \mathcal{F}_0$ for each $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S_0)$, and $\mathcal{F}|_{S_0} = \langle \mathrm{Aut}_{\mathcal{F}}(S_0), \mathcal{F}_0 \rangle$.*
- *A fusion subsystem $\mathcal{F}_0 \subseteq \mathcal{F}$ over $S_0 \trianglelefteq S$ is* weakly normal *in $\mathcal{F}$ ($\mathcal{F}_0 \trianglelefteq \mathcal{F}$) if $\mathcal{F}_0$ and $\mathcal{F}$ are both saturated and $\mathcal{F}_0$ is $\mathcal{F}$-invariant.*

Thus an "$\mathcal{F}$-invariant" fusion subsystem is one which satisfies all conditions for being weakly normal in $\mathcal{F}$ except for being saturated. This is equivalent to what some authors define as a "normal" fusion subsystem, but for the sake of consistency with the terminology in [AKO], we limit "(weakly) normal" fusion systems to those which are saturated. (See [AKO, Section I.6] for a definition of normal fusion subsystems.) Our main reason for defining $\mathcal{F}$-invariant subsystems here is so that we can apply the following lemma in Section 4, without knowing that the fusion subsystem is saturated.

Lemma 1.5. *Let $\mathcal{F}$ be a saturated fusion system over the finite p-group S, and let $\mathcal{F}_0 \subseteq \mathcal{F}$ be an $\mathcal{F}$-invariant fusion subsystem over S. Then for any $P \le S$, P is fully normalized in $\mathcal{F}_0$ (fully centralized in $\mathcal{F}_0$, $\mathcal{F}_0$-centric) if and only if P is fully normalized in $\mathcal{F}$ (fully centralized in $\mathcal{F}$, $\mathcal{F}$-centric).*

Proof. Assume $P, Q \le S_0$ are $\mathcal{F}$-conjugate. Since $\mathcal{F}|_{S_0} = \langle \mathcal{F}_0, \mathrm{Aut}_{\mathcal{F}}(S_0)\rangle$, any $\varphi \in \mathrm{Iso}_{\mathcal{F}}(P, Q)$ is a composite of morphisms in $\mathcal{F}_0$ and restrictions of $\mathcal{F}$-automorphisms of S_0. Since ${}^{\psi}\mathcal{F}_0 = \mathcal{F}_0$ for all $\psi \in \mathrm{Aut}_{\mathcal{F}}(S_0)$, these morphisms can be rearranged so that the morphisms in $\mathcal{F}_0$ all come first, followed by a composite of restrictions of morphisms in $\mathrm{Aut}_{\mathcal{F}}(S_0)$. Thus there is $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S_0)$ such that $\alpha(Q) \in P^{\mathcal{F}_0}$.

Since $P^{\mathcal{F}} \supseteq P^{\mathcal{F}_0}$, P is fully normalized in $\mathcal{F}_0$ if it is fully normalized in $\mathcal{F}$. Conversely, assume P is fully normalized in $\mathcal{F}_0$, let $Q \in P^{\mathcal{F}}$ be such that Q is fully normalized in $\mathcal{F}$, and choose $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ such that $\alpha(Q) \in P^{\mathcal{F}_0}$. Then $|N_S(P)| \ge |N_S(\alpha(Q))| = |N_S(Q)| \ge |N_S(P)|$, so these are all equal, and P is also fully normalized in $\mathcal{F}$.

The argument for fully centralized subgroups is similar. The centric case follows since P is $\mathcal{F}$-centric if and only if it is fully centralized in $\mathcal{F}$ and contains $C_S(P)$. □

The next definitions and results are taken from [5a2]. As usual, when G is a finite group, $O^p(G)$ and $O^{p'}(G)$ denote the smallest normal subgroups of p-power index and of index prime to p, respectively.

Definition 1.6. *Let $\mathcal{F}_0 \subseteq \mathcal{F}$ be saturated fusion systems over p-groups $S_0 \le S$.*

(a) *$\mathcal{F}_0$ is of* p-power index *in $\mathcal{F}$ if $S_0 \ge \mathfrak{hyp}(\mathcal{F})$, and $\mathrm{Aut}_{\mathcal{F}_0}(P) \ge O^p(\mathrm{Aut}_{\mathcal{F}}(P))$ for all $P \le S_0$.*

(b) *$\mathcal{F}_0$ is of* index prime to p *in $\mathcal{F}$ if $S_0 = S$, and $\mathrm{Aut}_{\mathcal{F}_0}(P) \ge O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))$ for all subgroups $P \le S$.*

Note that, despite the terminology, a fusion subsystem of p-power index (index prime to p) is analogous to a subgroup of a finite group G which contains a *normal* subgroup of p-power index (index prime to p); i.e., a subgroup which contains $O^p(G)$ ($O^{p'}(G)$). However, there are many examples of groups G with $S \in \mathrm{Syl}_p(G)$, where $O^{p'}(G) = G$ but $\mathcal{F}_S(G)$ does have proper subsystems of index prime to p (for example, take $p = 2$ and $G = A_5$).

Theorem 1.7. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S.*

(a) *There is a unique saturated fusion subsystem $O^p(\mathcal{F})$ over $\mathfrak{hyp}(\mathcal{F})$ of p-power index in $\mathcal{F}$, $O^p(\mathcal{F})$ is contained in all other saturated fusions subsystems of p-power index in $\mathcal{F}$, and $O^p(\mathcal{F}) \trianglelefteq \mathcal{F}$. Also, $O^p(\mathcal{F}) = \mathcal{F}$ if and only if $\mathfrak{foc}(\mathcal{F}) = S$.*

(b) *There is a unique minimal saturated fusion subsystem $O^{p'}(\mathcal{F})$ over S of index prime to p in $\mathcal{F}$, and $O^{p'}(\mathcal{F}) \trianglelefteq \mathcal{F}$. If $\mathcal{F}_0 \trianglelefteq \mathcal{F}$ is any weakly normal saturated fusion subsystem over S, then $\mathcal{F}_0 \supseteq O^{p'}(\mathcal{F})$.*

Proof. The first statement in (a) is shown in [5a2, Theorem 4.3], aside from the weak normality of $O^p(\mathcal{F})$ which is shown in [AOV, Proposition 1.16(a)]. From this, it follows immediately that $O^p(\mathcal{F}) = \mathcal{F}$ if and only if $\mathfrak{hyp}(\mathcal{F}) = S$. To see that this is equivalent to the condition $\mathfrak{foc}(\mathcal{F}) = S$, see, e.g., [AOV, Theorem 1.13(a)].

The first statement in (b) is shown in [5a2, Theorem 5.4], the weak normality of $O^{p'}(\mathcal{F})$ in [AOV, Proposition 1.16(b)], and the last statement in [AOV, Lemma 1.17]. □

We now look at essential subgroups of a fusion system, beginning with the definition.

Definition 1.8. *Fix a prime p.*

(a) *A subgroup H of a finite group G is* strongly p-embedded *if $H < G$, $p \big| |H|$, and for all $g \in G \smallsetminus H$, $p \nmid |H \cap gHg^{-1}|$.*

(b) *If $\mathcal{F}$ is a fusion system over a p-group S, then a subgroup $P \le S$ is $\mathcal{F}$-*essential *if P is $\mathcal{F}$-centric and fully normalized in $\mathcal{F}$, and $\mathrm{Out}_{\mathcal{F}}(P)$ contains a strongly p-embedded subgroup.*

The following lemma lists some of the well known properties of strongly p-embedded subgroups.

Lemma 1.9. *Fix a finite group G and a prime p. For each p-subgroup $P \le G$, set $\Gamma_{P,1}(G) = \bigl\langle N_G(Q) \bigm| 1 \ne Q \le P \bigr\rangle \le H$. Then the following hold.*

(a) *Each strongly p-embedded subgroup $H < G$ contains at least one Sylow p-subgroup of G.*

(b) *For each $S \in \mathrm{Syl}_p(G)$, either $\Gamma_{S,1}(G) = G$ and G contains no strongly p-embedded subgroups; or else $\Gamma_{S,1}(G) < G$, $\Gamma_{S,1}(G)$ is strongly p-embedded, and each strongly p-embedded subgroup of G which contains S also contains $\Gamma_{S,1}(G)$.*

(c) *If G contains a strongly p-embedded subgroup, then $O_p(G) = 1$.*

Proof. For any $H < G$ with $p\bigm||H|$, H is strongly p-embedded in G if and only if $\Gamma_{P,1}(G) \le H$ for some (all) $P \in \mathrm{Syl}_p(H)$. This is shown, for example, [A, (46.4)] or [GLS2, Proposition 17.11]. In particular, if H is strongly p-embedded and $P \in \mathrm{Syl}_p(H)$, then $N_G(P) \le H$, so $p \nmid [N_G(P){:}P]$, which implies $P \in \mathrm{Syl}_p(G)$. This proves (a) and (b). Finally, if $O_p(G) \ne 1$, then $\Gamma_{S,1}(G) \ge N_G(O_p(G)) = G$ for each $S \in \mathrm{Syl}_p(G)$, so G has no strongly p-embedded subgroup by (b). □

The properties of essential subgroups which we will need are listed in the following proposition.

Proposition 1.10. *The following hold for any saturated fusion system $\mathcal{F}$ over a p-group S.*

(a) $\mathcal{F} = \bigl\langle \mathrm{Aut}_{\mathcal{F}}(P) \bigm| P = S \text{ or } P \text{ is } \mathcal{F}\text{-essential} \bigr\rangle$.

(b) *Let $\mathcal{H}$ be any set of $\mathcal{F}$-essential subgroups with the property that if P, Q are $\mathcal{F}$-essential, $P \in \mathcal{H}$, and P is $\mathcal{F}$-conjugate to a subgroup of Q, then $Q \in \mathcal{H}$. Then*

$$\langle \mathrm{Aut}_{\mathcal{F}}(S),\, \mathrm{Aut}_{\mathcal{F}}(P) \,|\, P \in \mathcal{H} \rangle = \langle \mathrm{Aut}_{\mathcal{F}}(S),\, O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P)) \,|\, P \in \mathcal{H} \rangle \;.$$

(c) *Assume $P < S$ is $\mathcal{F}$-centric and fully normalized in $\mathcal{F}$. Let $H \le \mathrm{Aut}_{\mathcal{F}}(P)$ be the subgroup generated by those automorphisms which extend to morphisms in $\mathcal{F}$ between strictly larger subgroups of S. Then either $H < \mathrm{Aut}_{\mathcal{F}}(P)$, $H/\mathrm{Inn}(P)$ is strongly p-embedded in $\mathrm{Out}_{\mathcal{F}}(P)$, and P is $\mathcal{F}$-essential; or $H = \mathrm{Aut}_{\mathcal{F}}(P)$ and P is not $\mathcal{F}$-essential.*

(d) *If $P \le S$ is such that $[N_S(P), P] \le \mathrm{Fr}(P)$, then P is not $\mathcal{F}$-essential.*

Proof. Point (a) is shown, for example, in [OV, Corollary 2.6] and point (d) in [OV, Proposition 3.2 & Lemma 3.4].

Point (c) is mostly shown in [OV, Proposition 2.5]. Fix $P < S$ which is $\mathcal{F}$-centric and fully normalized in $\mathcal{F}$, and let $H \le \mathrm{Aut}_{\mathcal{F}}(P)$ be defined as in (c). By (a) and the extension axiom for a saturated fusion system, H is generated by the subgroups $N_{\mathrm{Aut}_{\mathcal{F}}(P)}(\mathrm{Aut}_Q(P))$ for all $P < Q \le N_S(P)$. Thus

$$\begin{aligned} H/\mathrm{Inn}(P) &= \bigl\langle N_{\mathrm{Out}_{\mathcal{F}}(P)}(\mathrm{Out}_Q(P)) \bigm| P < Q \le N_S(P) \bigr\rangle \\ &= \bigl\langle N_{\mathrm{Out}_{\mathcal{F}}(P)}(R) \bigm| 1 \ne R \le \mathrm{Out}_S(P) \bigr\rangle. \end{aligned}$$

Since P is fully centralized, $\mathrm{Out}_S(P) \in \mathrm{Syl}_p(\mathrm{Out}_{\mathcal{F}}(P))$. Thus by Lemma 1.9(b), either $H/\mathrm{Inn}(P) < \mathrm{Out}_{\mathcal{F}}(P)$ is strongly p-embedded and P is $\mathcal{F}$-essential, or $H = \mathrm{Aut}_{\mathcal{F}}(P)$ and P is not $\mathcal{F}$-essential. This proves (c).

It remains to prove (b). For each $P \in \mathcal{H}$ and $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$, $\mathrm{Aut}_S(P)$ and $\alpha\mathrm{Aut}_S(P)\alpha^{-1}$ are both Sylow p-subgroups of $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))$, and hence there is $\beta \in O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))$ such that $\beta\alpha \in N_{\mathrm{Aut}_{\mathcal{F}}(P)}(\mathrm{Aut}_S(P))$. In other words,

$$\mathrm{Aut}_{\mathcal{F}}(P) = O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))\cdot N_{\mathrm{Aut}_{\mathcal{F}}(P)}(\mathrm{Aut}_S(P))$$

by a Frattini argument. By the extension axiom, each automorphism in the normalizer extends to an element of $\mathrm{Aut}_{\mathcal{F}}(N_S(P))$, and by (a), this is a composite of restrictions of $\mathcal{F}$-automorphisms of S and of strictly larger $\mathcal{F}$-essential subgroups, all of which are in $\mathcal{H}$ by the hypotheses. So by downwards induction on $|P|$, we see that each $\mathrm{Aut}_{\mathcal{F}}(P)$ for $P \in \mathcal{H}$ is in the fusion subsystem generated by $\mathrm{Aut}_{\mathcal{F}}(S)$ and the groups $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(Q))$ for $Q \in \mathcal{H}$. □

We will also need the following lemma about essential subgroups for a fusion system over a product of p-groups.

Lemma 1.11. *Let S_1, S_2 be a pair of p-groups, and set $S = S_1 \times S_2$. Then the following hold for each $P \leq S$.*

(a) *If $P < P_1P_2$, where $P_i \leq S_i$ is the image of P under projection, then there is $g \in N_S(P)\smallsetminus P$ such that $c_g \in O_p(\mathrm{Aut}(P))$. In particular, P cannot be $\mathcal{F}$-essential for any saturated fusion system $\mathcal{F}$ over S.*

(b) *If $p = 2$, and P is $\mathcal{F}$-essential for some saturated fusion system $\mathcal{F}$ over S, then $P \geq S_1$ or $P \geq S_2$.*

Proof. The first statement in (a) is shown in [AOV, Lemma 3.1]. Hence if P is $\mathcal{F}$-centric and $P < P_1P_2$, then $O_p(\mathrm{Out}_{\mathcal{F}}(P)) \neq 1$. Thus $\mathrm{Out}_{\mathcal{F}}(P)$ does not contain a strongly p-embedded subgroup by Lemma 1.9(c). So no such P can be $\mathcal{F}$-essential, and this finishes the proof of (a).

Now assume $p = 2$, $P = P_1P_2$, and $O_2(\mathrm{Out}_{\mathcal{F}}(P)) = 1$. Assume also $P_1 < S_1$ and $P_2 < S_2$. Choose elements $x_i \in N_{S_i}(P_i)\smallsetminus P_i$ such that c_{x_i} has order 2 in $\mathrm{Out}(P_i)$. Then

$$\mathrm{rk}([x_1x_2, P/\mathrm{Fr}(P)]) = \mathrm{rk}([x_1, P/\mathrm{Fr}(P)]) + \mathrm{rk}([x_2, P/\mathrm{Fr}(P)]).$$

Since $O_2(\mathrm{Out}_{\mathcal{F}}(P)) = 1$, $\mathrm{Out}_{\mathcal{F}}(P)$ acts faithfully on $P/\mathrm{Fr}(P)$ by Lemma 2.1, and hence $\mathrm{rk}([x_i, P/\mathrm{Fr}(P)]) > 0$ for $i = 1, 2$. So x_1x_2 is not $\mathrm{Out}_{\mathcal{F}}(P)$-conjugate to x_1 or x_2. If $\mathrm{Out}_{\mathcal{F}}(P)$ did contain a strongly 2-embedded subgroup, then all of its involutions would be conjugate (cf. [Sz2, (6.4.4(i))]), and this is not the case. Thus P is not $\mathcal{F}$-essential, and this proves (b). □

The following transfer homomorphism for abstract fusion systems will be needed.

Proposition 1.12. *Fix a p-group S, and a saturated fusion system $\mathcal{F}$ over S. Then there is an injective homomorphism*

$$\mathrm{trf}_{\mathcal{F}}\colon S/\mathfrak{foc}(S) \longrightarrow S^{\mathrm{ab}} \overset{\mathrm{def}}{=} S/[S,S]$$

which has the following form. There are proper subgroups $P_1, \ldots, P_m < S$, and morphisms $\varphi_i \in \mathrm{Hom}_{\mathcal{F}}(P_i, S)$ $(i = 1, \ldots, m)$, such that for $g \in S$,

$$\mathrm{trf}_{\mathcal{F}}([g]) = \prod_{[\alpha]\in\mathrm{Out}_{\mathcal{F}}(S)} [\alpha(g)] \cdot \prod_{i=1}^{m} \varphi_{i*}\big(\mathrm{trf}_{P_i}^{S}([g])\big).$$

Here, $[g]$ denotes the class of g in $S/\mathfrak{foc}(\mathcal{F})$ or in $S^{\mathrm{ab}} = S/[S,S]$, $\mathrm{trf}_{P_i}^{S}$ is the transfer homomorphism in $(-)^{\mathrm{ab}} = H_1(-)$, and the terms on the right are regarded as lying in the abelianization of S. If $g \in \Omega_1(Z(S))$, then $\mathrm{trf}_{\mathcal{F}}([g]) = \prod_{[\alpha]\in\mathrm{Out}_{\mathcal{F}}(S)}[\alpha(g)]$. If in addition, g is $\mathrm{Out}_{\mathcal{F}}(S)$-invariant, then $\mathrm{trf}_{\mathcal{F}}([g]) = [g]^k$ where $k = |\mathrm{Out}_{\mathcal{F}}(S)|$ is prime to p.

Proof. See [AKO, Section I.8]. □

2. Lemmas on groups and groups acting on groups

We collect here the results which will be needed later on finite groups and their actions on other finite groups. Note that our commutators are always of the form $[a,b]=aba^{-1}b^{-1}$.

The first three results are well known, and listed for future reference.

Lemma 2.1. *Fix a prime p, a p-group P, a subgroup $P_0 \le \mathrm{Fr}(P)$, and a sequence of subgroups*

$$P_0 < P_1 < \cdots < P_k = P$$

all normal in P. Set

$$\mathcal{A} = \bigl\{\alpha \in \mathrm{Aut}(P) \bigm| x^{-1}\alpha(x) \in P_{i-1},\ \text{all } x \in P_i, \text{ all } i=1,\dots,k\bigr\} \le \mathrm{Aut}(P):$$

the group of automorphisms which leave each P_i invariant, and which induce the identity on each quotient group P_i/P_{i-1}. Then $\mathcal{A}$ is a p-group. If the P_i are all characteristic in P, then $\mathcal{A} \trianglelefteq \mathrm{Aut}(P)$, and hence $\mathcal{A} \le O_p(\mathrm{Aut}(P))$.

Proof. See, for example, [G, Theorems 5.1.4 & 5.3.2]. □

Lemma 2.2. *If G is a finite group of order $2n$, where n is odd, then G contains a normal subgroup of order n and index two.*

Proof. This is a special case of Burnside's normal p-complement theorem (cf. [G, Theorem 7.4.3]). The following, more elementary proof was shown to us by Dave Benson. Consider the action of G on itself by left translation. Elements of even order act via odd permutations, and elements of odd order via even permutations. Hence the elements of odd order form a subgroup of index two. □

Lemma 2.3. *Fix a prime p, and a finite group $G=HT$, where $H \trianglelefteq G$, $p \nmid |H|$, and T is a p-group. Then*

(a) *$H=[H,T]{\cdot}C_H(T)$ and $[H,T]=[[H,T],T]$; and*

(b) *if T is abelian and $\mathcal{U}=\{U \le T \,|\, T/U$ is cyclic$\}$, then $H=\bigl\langle C_H(U) \bigm| U \in \mathcal{U}\bigr\rangle$,*

Proof. Let $q_1,\dots,q_k$ be the distinct primes which divide $|H|$. For each $1 \le i \le k$, $|\mathrm{Syl}_{q_i}(H)|$ divides $|H|$ and hence is prime to p. Since the p-group T acts on the set $\mathrm{Syl}_{q_i}(H)$, it must fix at least one element. Thus there is some $S_i \in \mathrm{Syl}_{q_i}(H)$ such that $T \le N_H(S_i)$; i.e., T acts on each S_i.

(a) By [A, 24.4], $S_i=[S_i,T]{\cdot}C_{S_i}(T)$ for each i, and hence $H=\langle S_i\rangle=\langle [H,T],C_H(T)\rangle$. Since $[H,T] \trianglelefteq H$ by the relation ${}^g[h,t]=[gh,t][g,t]^{-1}$ (cf. [A, 8.5.6]), this implies that $H=[H,T]{\cdot}C_H(T)$.

In particular, $[H,T]$ is generated by elements $[ab,t]$ for $a \in [H,T]$, $b \in C_H(T)$, and $t \in T$; and $[ab,t]=[a,t]$ since $[b,t]=1$. Thus $[H,T]=[[H,T],T]$.

(b) Let $\mathrm{Fr}(S_i) \trianglelefteq S_i$ be the Frattini subgroup of S_i: the intersection of all maximal proper subgroups. Then $S_i/\mathrm{Fr}(S_i)$ is an elementary abelian q_i-group (cf. [A, 23.2]), and hence can be regarded as an $\mathbb{F}_{q_i}[T]$-module. Since T is a p-group and $p \ne q_i$, $S_i/\mathrm{Fr}(S_i)$ splits as a product of irreducible modules. For $U \le T$, T/U has a faithful irreducible $\mathbb{F}_{q_i}[T]$-module only if T/U is cyclic (cf. [G, Theorem 3.2.2]). Thus each irreducible factor in $S_i/\mathrm{Fr}(S_i)$ is pointwise fixed by some $U \in \mathcal{U}$, and so $S_i/\mathrm{Fr}(S_i)$ is generated by its subgroups $C_{S_i/\mathrm{Fr}(S_i)}(U)$ for $U \in \mathcal{U}$.

If $g\mathrm{Fr}(S_i) \in C_{S_i/\mathrm{Fr}(S_i)}(U)$ for some $U \le T$, then U acts on the coset $g\mathrm{Fr}(S_i)$, and this action fixes at least one element since the coset has order prime to p. Hence every element of $C_{S_i/\mathrm{Fr}(S_i)}(U)$ lifts to an element of $C_{S_i}(U)$, and so $S_i = \langle \mathrm{Fr}(S_i), C_{S_i}(U) \,|\, U \in \mathcal{U}\rangle$. Since $\mathrm{Fr}(S_i)$ is contained in each maximal proper subgroup of S_i, it follows that $S_i = \langle C_{S_i}(U) \,|\, U \in \mathcal{U}\rangle$. Since H is generated by the S_i, this proves (b). □

Let $I(G)$ denote the set of involutions in a group G. The next result is also very elementary.

Proposition 2.4. *Fix an abelian 2-group A, and a subgroup $G \le \mathrm{Aut}(A)$ of order $2n$ for n odd. Assume, for some $x \in I(G)$, that $[x, G] \ne 1$, and that $[x, A] \cong C_{2^m}$ for some $m \ge 1$. Set $G_1 = \langle I(G)\rangle$ and $G_2 = C_G(G_1)$. Then $G_1 \cong \Sigma_3$, $|G_2|$ is odd, and $G = G_1 \times G_2$. Also, there is a unique decomposition $A = A_1 \times A_2$ such that the G-action on A splits as a product of G_i-actions on A_i, and such that $A_1 \cong C_{2^m} \times C_{2^m}$.*

Proof. If there is any decomposition of A as described above, then $A_1 = [G_1, A]$ and $A_0 = C_A(G_1)$. So there is at most one such decomposition.

Set $H_1 = O^2(G_1)$, so $|H_1|$ is odd and $[G_1{:}H_1] = 2$ by Lemma 2.2. Also, $H_1 \ne 1$ by the assumption that $[x, G] \ne 1$. Set $A_0 = C_A(H_1)$ and $A_1 = [H_1, A]$, so that $A = A_0 \times A_1$ (cf. [A, 24.6]). Since $H_1 \trianglelefteq G$, the action of G sends each A_i to itself. Hence $[x, A] = [x, A_0] \times [x, A_1]$, and one of the two factors must vanish since $[x, A]$ is cyclic. If $[x, A_i] = 1$, then the normal closure G_1 of $\langle x\rangle$ in G acts trivially on A_i. Since $H_1 \ne 1$ acts nontrivially on A_1, it follows that G_1 acts trivially on A_0.

Let $\{x_1, \dots, x_k\} \subseteq I(G)$ be a minimal subset which generates $G_1 = \langle I(G)\rangle$. Then $\mathrm{rk}([G_1, A]) \le \sum_{i=1}^k \mathrm{rk}([x_i, A]) = k$. In particular, if $k = 2$, then $G_1 \cong GL_2(2) \cong \Sigma_3$. If $k \ge 3$, then set $K_1 = \langle x_1, x_2, x_3\rangle$, so that $\mathrm{rk}([K_1, A]) \le 3$, and K_1 is isomorphic to a subgroup of $GL_3(2)$. Then $|K_1|\big|42$ (since $|GL_3(2)| = 168$). If $|K_1| \ne 6$, then it contains a normal subgroup of order 7 by Lemma 2.2; and this is impossible since the normalizer of a Sylow 7-subgroup in $GL_3(2)$ has order 21. Thus $K_1 \cong \Sigma_3$, contradicting the assumption that $\{x_i\}$ was a minimal generating set. We conclude that $k = 2$, $G_1 \cong \Sigma_3$, and $\mathrm{rk}([G_1, A]) = 2$.

Now, $G_0 = C_G(G_1)$ has index at most 6 (since G permutes the three elements in $I(G)$), $G_0 \cap G_1 = 1$, and thus $G = G_0 \times G_1$. We already saw that $\mathrm{rk}(A_1) = 2$. Since $H_1 \cong C_3$ acts nontrivially on A_1, A_1 must be homocyclic, and thus $A_1 \cong C_{2^m} \times C_{2^m}$. Also, $\mathrm{Aut}(A_1)/O_2(\mathrm{Aut}(A_1)) \cong \Sigma_3$ (Lemma 2.1), $|G_0|$ is odd, and $[G_0, G_1] = 1$ — so G_0 acts trivially on A_1. □

Throughout the rest of the section, we look at some conditions which imply that two subgroups of a larger group commute. The first one is very simple.

Lemma 2.5. *Fix a prime p, a finite group G, and a pair of subgroups $G_1, G_2 \le G$. Choose $S_i \in \mathrm{Syl}_p(G_i)$, and choose normal p-subgroups $P_i \trianglelefteq G_i$. Assume $[G_1, S_2] = 1 = [G_2, S_1]$ and $[G_1, G_2] \le P_1P_2$. Then $[G_1, G_2] = 1$.*

Proof. We must show, for each pair of elements $g_i \in G_i$ of order prime to p, that $[g_1, g_2] = 1$. Upon replacing G_i by $\langle P_i, g_i\rangle$, we are reduced to the case where G_i/P_i has order prime to p.

Consider the conjugation action $c_{g_1} \in \mathrm{Aut}(P_1G_2)$. By assumption, $c_{g_1}(P_1) = P_1$, and c_{g_1} induces the identity on P_2 and on P_1G_2/P_1P_2. The subgroups P_1P_2/P_2 and G_2/P_2 generate P_1G_2/P_2, the first is a p-group and the second has order prime to p, and they commute since $[P_1, G_2] = 1$ by assumption. Thus $P_1G_2/P_2 = (P_1P_2/P_2) \times (G_2/P_2)$, $P_1P_2 \cap G_2 = P_2$ and the induced action of c_{g_1} on P_1G_2/P_2 leaves G_2/P_2 invariant. Hence $[g_1, G_2] \le P_1P_2 \cap G_2 = P_2$.

Thus $c_{g_1} \in \mathrm{Aut}(G_2)$ induces the identity on P_2 and on G_2/P_2. By the Schur-Zassenhaus theorem (cf. [A, Theorem 18.1]), c_{g_1} is conjugation by some element of P_2, and hence is the identity since P_2 is a p-group and g_1 has order prime to p. Thus $[g_1, G_2] = 1$. □

The following lemma, which appears as [A, Exercise 8.9], was suggested to us by the referee as a means of simplifying the statement and proof of Lemma 2.7 below.

Lemma 2.6. *Fix a prime p, and a finite group $G = HT$, where $H \trianglelefteq G$, $p \nmid |H|$, H is solvable, and T is an abelian p-group. Set $\mathcal{U} = \{U \le T \,|\, T/U \text{ is cyclic}\}$. Then for each $t \in T$,*

$$[H, t] = \langle [C_H(U), t] \,|\, t \notin U \in \mathcal{U} \rangle \ .$$

Proof. By Lemma 2.3(a), $[[H, t], t] = [H, t]$. So upon replacing H by $[H, t]$ (which is T-invariant since T is abelian), we can assume $H = [H, t]$.

Set $X = \langle [C_H(U), t] \,|\, t \notin U \in \mathcal{U} \rangle$. We must show $X = [H, t] = H$. By Lemma 2.3(b), $H = [H, t]$ is generated by elements $[g, t]$, where $g = g_1 \cdots g_k$, and for each i, $g \in C_H(U_i)$ for some $U_i \in \mathcal{U}$. From the relation $[ab, t] = {}^a[b, t]{\cdot}[a, t]$, we now get

$$[g, t] = {}^{g_1 \cdots g_{k-1}}[g_k, t] \cdots {}^{g_1}[g_2, t]{\cdot}[g_1, t] \ .$$

Since $[g_i, t] \in X$ for each i, this proves that H is the normal closure of X in H.

Since H is solvable, there is a nontrivial characteristic abelian subgroup $N \trianglelefteq H$ (e.g., the next-to-last term in the derived series). We can assume inductively that the lemma holds for H/N; i.e., that $H/N = X^* \overset{\text{def}}{=} \langle [C_{H/N}(U), t] \,|\, t \notin U \in \mathcal{U} \rangle$. For each U and $gN \in C_{H/N}(U)$, U acts on gN with $C_{gN}(U) \neq \varnothing$ since $p \nmid |gN|$. The projection of H onto H/N thus sends $C_H(U)$ onto $C_{H/N}(U)$ for each U, and hence $X^* = XN/N$. Thus $H = XN$.

Since $N = \langle C_N(U) \,|\, U \in \mathcal{U} \rangle$ by Lemma 2.3(b),

$$[N, t] = \langle [C_N(U), t] \,|\, U \in \mathcal{U} \rangle = \langle [C_N(U), t] \,|\, t \notin U \in \mathcal{U} \rangle \le X$$

since N is abelian. Let $N_0 \trianglelefteq H$ be the normal closure of $[N, t]$ in H. Then $N_0 \trianglelefteq \langle H, t \rangle$,

$$N_0 = \langle {}^g[N, t] \,|\, g \in H \rangle = \langle {}^g[N, t] \,|\, g \in X \rangle \le X$$

since $H = XN$ and N is abelian. If $N_0 = N$, then $X = XN_0 = XN = H$, and we are done. Otherwise, upon replacing H by H/N_0, we can assume that $[N, t] = 1$.

Since $[[H, N], t] \le [N, t] = 1$, the 3-subgroup lemma implies that $[H, N] = [[H, t], N] = 1$ (cf. [A, 8.7]). Thus $H = XN$ where $[X, N] = 1$, so $X \trianglelefteq H$. We already saw that H is the normal closure of X in H, and it now follows that $H = X$. □

Lemma 2.6 will now be used to prove:

Lemma 2.7. *Fix an elementary abelian 2-group V and a subgroup $G = HT \le \mathrm{Aut}(V)$, where $H \trianglelefteq G$, $|H|$ is odd, and $T = \langle t_1, t_2 \rangle \cong C_2^2$. Assume $[t_1, V] \cap [t_2, V] = 1$. Set $H_i = [t_i, H]$ and $G_i = \langle H_i, t_i \rangle$ ($i = 1, 2$). Then $[G_1, G_2] = 1$.*

Proof. By construction, $G_i = \langle I(G_i) \rangle$ (it is generated by the H-conjugacy class of t_i). Since $t_i \in N_G(H_i)$, $[G_i{:}H_i] = 2$.

Set $t_3 = t_1 t_2$, and fix $h \in C_H(t_3)$. Set $g = [h, t_1] = [h, t_3 t_2] = [h, t_2]$. Thus g is fixed by t_3 and inverted by t_1 and t_2. If $g \neq 1$, then $[g, V] = [g^{-1}, V]$ is T-invariant; set $W = C_{[g,V]}(t_3)$. Then $[g, V] \neq 1$ since g acts faithfully, and so $W \neq 1$ since $[g, V]$ and $\langle t_3 \rangle$ are both 2-groups. Also, W is invariant under the action of the dihedral group $\langle g, t_1 \rangle$. If t_1 fixes W pointwise, then so does each element in $\langle g, t_1 \rangle$, and in particular $[g, W] = 1$. This is impossible, since $W \le [g, V]$ (and g has odd order). Hence there is $w \in W$ such that $t_1(w) \neq w$,

$1 \neq [t_1, w] = [t_2, w]$, and this contradicts the assumption $[t_1, V] \cap [t_2, V] = 1$. We conclude that $g = [h, t_1] = 1$. Since this holds for all $h \in C_H(t_3)$, $C_H(t_3) = C_H(T)$.

By Lemma 2.6, and since $C_H(t_3) \leq C_H(t_2)$,

$$[H, t_1] = \big\langle [C_H(t_2), t_1], [C_H(t_3), t_1] \big\rangle = [C_H(t_2), t_1] \leq C_H(t_2) \ .$$

Thus t_2 commutes with $[H, t_1]$, and hence with every element H-conjugate to t_1. Thus each involution H-conjugate to t_2 commutes with each involution H-conjugate to t_1. In particular, $[G_1, G_2] = \big[\langle I(G_1)\rangle, \langle I(G_2)\rangle\big] = 1$. □

The next proposition is much more technical in its formulation. It will be applied later to the case where $G = \mathrm{Aut}_{\mathcal{F}}(P)$ for a fusion system $\mathcal{F}$, $P = P_1 \times P_2$, and $G_i \leq G$ are subgroups which leave P_i invariant. The idea is to first get information about the outer automorphism groups commuting (e.g., $G_i/O_2(G_i)$), using Lemma 2.7, and then lift that to information about the actual automorphism groups using Lemma 2.5.

Proposition 2.8. *Fix a 2-group P and a subgroup $G \leq \mathrm{Aut}(P)$. Let $G_1, G_2 \leq G$ be such that $[G_1, P] \cap [G_2, P] = 1$ and $[O_2(G_i), P] \leq \mathrm{Fr}(P)$. Fix $S_i \in \mathrm{Syl}_2(G_i)$, and set $Q_i = O_2(G_i)$. Assume the following hold for $i = 1, 2$.*

(a) *G_i/Q_i has a strongly 2-embedded subgroup $H_i/Q_i \geq S_i/Q_i$, and $[H_i, G_{3-i}] = 1$.*

(b) *$S_1S_2 \in \mathrm{Syl}_2(G)$, and S_i is strongly closed in S_1S_2 with respect to G.*

Then $[G_1, G_2] = 1$.

Proof. Set $Q = \{g \in G \,|\, [g, P] \leq \mathrm{Fr}(P)\}$: the kernel of the induced action of G on $P/\mathrm{Fr}(P)$. Then $Q_1Q_2 \leq Q$, and Q is a normal 2-subgroup of G by Lemma 2.1. Hence $Q \leq S_1S_2$ since $S_1S_2 \in \mathrm{Syl}_2(G)$, and $G_i \cap Q = Q_i$ ($i = 1, 2$) since it is a normal 2-subgroup of G_i. If $g = g_1g_2 \in Q$ where $g_i \in S_i$, then $[G_i, g] = [G_i, g_i] \leq Q \cap G_i = Q_i$, so $\langle Q_i, g_i\rangle \trianglelefteq G_i$, and $g_i \in Q_i$ since $Q_i = O_2(G_i)$. Thus $Q = Q_1Q_2$.

By Lemma 2.5, it suffices to prove that $[G_1, G_2] \leq Q_1Q_2 = Q$.

Set $G^* = G/Q$, and set $G_i^* = G_iQ/Q \cong G_i/Q_i$ for $i = 1, 2$. By definition of Q, G^* and the G_i^* all act faithfully on $P^* \overset{\mathrm{def}}{=} P/\mathrm{Fr}(P)$. We must prove that $[G_1^*, G_2^*] = 1$. So upon replacing G by G^*, G_i by G_i^*, and P by P^*, we are reduced to proving the proposition when $Q_i = 1$, H_i is strongly 2-embedded in G_i, and P is elementary abelian.

Let $\widehat{G}_i$ be the normal closure of S_i in G, and set $N = \widehat{G}_1 \cap \widehat{G}_2 \geq [\widehat{G}_1, \widehat{G}_2]$. By assumption, $S_1S_2 \in \mathrm{Syl}_2(G)$, S_1 and S_2 are strongly closed in S_1S_2 with respect to G, and $[S_1, S_2] = 1$. Hence N has odd order by a theorem of Goldschmidt [Gd, Corollary A2].

For each $i = 1, 2$, fix some $x_i \in I(S_i)$, and set $\mathcal{G}_i = \{g \in G_i \,\big|\, |g| \text{ odd}, {}^{x_i}g = g^{-1}\}$. For each $g_1 \in \mathcal{G}_1$ and $g_2 \in \mathcal{G}_2$, $\langle x_i, g_i\rangle$ is dihedral and its involutions are G_i-conjugate to x_i ($i = 1, 2$), so $[\langle x_1, g_1\rangle, \langle x_2, g_2\rangle] \leq [\widehat{G}_1, \widehat{G}_2] \leq N$. Hence $\langle g_1, g_2\rangle$ has odd order since $N \trianglelefteq G$ has odd order. So $[g_1, g_2] = 1$ by Lemma 2.7, applied with $H = \langle g_1, g_2\rangle$, $T = \langle x_1, x_2\rangle$, and $V = P$. (Since $[x_i, g_{3-i}] \in [H_i, G_{3-i}] = 1$ by assumption, T normalizes H.) Thus $[\langle \mathcal{G}_1\rangle, \langle \mathcal{G}_2\rangle] = 1$, and hence $[\langle H_1, \mathcal{G}_1\rangle, \langle H_2, \mathcal{G}_2\rangle] = 1$ since $[H_i, G_{3-i}] = 1$.

We will be done upon showing that $G_i = \langle H_i, \mathcal{G}_i\rangle$. Fix $g_i \in G_i \smallsetminus H_i$, and set $y_i = g_ix_ig_i^{-1}$. Then $y_i \notin H_i$, and $|x_iy_i|$ is odd since otherwise the involution in $\langle x_iy_i\rangle$ would commute with both x_i and y_i (impossible since H_i is strongly 2-embedded). Thus $y_i = h_ix_ih_i^{-1}$ for some $h_i \in \langle x_iy_i\rangle$, $h_i \in \mathcal{G}_i$, and $g_i^{-1}h_i \in C_{G_i}(x_i) \leq H_i$ since H_i is strongly 2-embedded. Hence $g_i \in \langle H_i, \mathcal{G}_i\rangle$, and thus $G_i = \langle H_i, \mathcal{G}_i\rangle$. □

We finish the section with the following easy result, which describes one consequence of two groups of automorphisms commuting.

Lemma 2.9. *Fix a finite group K, and subgroups $G_1, G_2 \le \mathrm{Aut}(K)$ such that $[G_1, G_2] = 1$ and $[G_1, K] \cap [G_2, K] = 1$. Then $[G_1, K] \le C_K(G_2)$ and $[G_2, K] \le C_K(G_1)$.*

Proof. For each $g_1 \in G_1$, $g_2 \in G_2$, and $x \in K$,

$$[g_1, [g_2, x]] = g_1g_2(x){\cdot}g_1(x)^{-1}{\cdot}[g_2, x]^{-1} = g_1g_2(x){\cdot}x^{-1}{\cdot}[g_1, x]^{-1}{\cdot}[g_2, x]^{-1}$$
$$[g_2, [g_1, x]] = g_2g_1(x){\cdot}g_2(x)^{-1}{\cdot}[g_1, x]^{-1} = g_2g_1(x){\cdot}x^{-1}{\cdot}[g_2, x]^{-1}{\cdot}[g_1, x]^{-1},$$

and $\big[[g_1, x], [g_2, x]\big] = 1$ since $[G_1, K]$ and $[G_2, K]$ are normal in K (cf. [A, 8.5.6]) with trivial intersection. Hence $[g_1, [g_2, x]] = [g_2, [g_1, x]]$ lies in $[G_1, K] \cap [G_2, K] = 1$. □

3. Automorphisms of products of nonabelian p-groups

Throughout this section, p is an arbitrary prime. To shorten notation, for any finite p-group P, we write

$$\overline{\mathrm{Out}}(P) = \mathrm{Aut}(P)/O_p(\mathrm{Aut}(P)) \cong \mathrm{Out}(P)/O_p(\mathrm{Out}(P)).$$

We want to compare the group of automorphisms of a product of p-groups with the groups of automorphisms of its factors. For example, Proposition 3.2 implies as a special case that if $S = S_1 \times S_2$, where S_1 is indecomposable, S_2 has no direct factor isomorphic to S_1, and $\Omega_1(Z(S_1)) \le [S_1, S_1]$, then $\overline{\mathrm{Out}}(S) \cong \overline{\mathrm{Out}}(S_1) \times \overline{\mathrm{Out}}(S_2)$.

We begin with the following, very general consequence of the Krull-Schmidt theorem.

Proposition 3.1. *Fix a finite group $G = G_1 \times \cdots \times G_m \times H$, where for each $i = 1, \ldots, m$, G_i is indecomposable and nonabelian, and is not isomorphic to any direct factor of H. Then for each $\alpha \in \mathrm{Aut}(G)$, there is a unique permutation $\sigma \in \Sigma_m$ such that for each i, $\alpha(G_iZ(G)) = G_{\sigma(i)}Z(G)$ and $\alpha([G_i, G_i]) = [G_{\sigma(i)}, G_{\sigma(i)}]$.*

Proof. We adopt here the terminology of Suzuki in [Sz1, Definition 1.6.17]: an endomorphism of a group G is *normal* if it commutes with all inner automorphism of G. By [Sz1, 1.6.18(ii)], for any normal automorphism ν of G, there is $\zeta \in \mathrm{Hom}(G, Z(G))$ such that $\nu(g) = \zeta(g)^{-1}g$ for each $g \in G$. In particular, $\nu|_{[G,G]} = \mathrm{Id}$.

For each $\alpha \in \mathrm{Aut}(G)$, the Krull-Schmidt theorem, in the form stated in [Sz1, Theorem 2.4.8] and applied to the decompositions $G = G_1\times\cdots\times G_m\times H = \alpha(G_1)\times\cdots\times\alpha(G_m)\times\alpha(H)$, says that there is a normal automorphism ν of G, and a permutation $\sigma \in \Sigma_m$, such that $\nu(\alpha(G_i)) = G_{\sigma(i)}$ for each i and $\nu(\alpha(H)) = H$. Also, $\alpha([G_i, G_i]) = [G_{\sigma(i)}, G_{\sigma(i)}]$ for each i, since $[G_iZ(G), G_iZ(G)] = [G_i, G_i]$. □

We next look at automorphisms of a product of p-groups which partly fix one factor.

Proposition 3.2. *Fix a pair of p-groups S_1 and S_2, set $S = S_1\times S_2$, and let $\mathrm{pr}_i \in \mathrm{Hom}(S, S_i)$ be the projection. Set*

$$\mathrm{Aut}^0(S) = \{\alpha \in \mathrm{Aut}(S) \,|\, \alpha(\Omega_1(Z(S_1))) = \Omega_1(Z(S_1))\}\ .$$

Then the following hold.

(a) *For each $\alpha \in \mathrm{Aut}^0(S)$, $\mathrm{pr}_i(\alpha(S_i)) = S_i$ and $\alpha(S_iZ(S_{3-i})) = S_iZ(S_{3-i})$ for $i = 1, 2$.*

(b) *There is an isomorphism*

$$\mathrm{Aut}^0(S)/O_p(\mathrm{Aut}^0(S)) \xrightarrow{\;\cong\;} \overline{\mathrm{Out}}(S_1) \times \overline{\mathrm{Out}}(S_2) \ ,$$

which sends the class of $\alpha \in \mathrm{Aut}^0(S)$ *to the class of* $(\mathrm{pr}_1 \circ\alpha|_{S_1}, \mathrm{pr}_2 \circ\alpha|_{S_2})$.

(c) *Assume* $S_1 \neq 1$, S_1 *is indecomposable, and* $\Omega_1(Z(S_1)) \leq [S_1, S_1]$. *Let* $n \geq 1$ *be the largest integer such that* S_2 *contains a direct factor isomorphic to* $(S_1)^{n-1}$. *Then* $[\mathrm{Aut}(S){:}\mathrm{Aut}^0(S)] = n$.

Proof. **(a)** Set $Z_i = Z(S_i)$ for short. Fix $\alpha \in \mathrm{Aut}^0(S)$. If $\mathrm{pr}_1 \circ\alpha|_{S_1}$ were not injective, then some $1 \neq x \in \Omega_1(Z_1)$ would be in the kernel, which contradicts the assumption $\alpha(\Omega_1(Z_1)) = \Omega_1(Z_1)$. If $\mathrm{pr}_2 \circ\alpha|_{S_2}$ were not injective, then some $1 \neq y \in \Omega_1(Z_2)$ would be in the kernel, so $\alpha(y) \in S_1 \cap \Omega_1(Z(S)) = \Omega_1(Z_1)$, which is impossible since $\alpha(\Omega_1(Z_1)) = \Omega_1(Z_1)$. Thus $\mathrm{pr}_1 \circ\alpha|_{S_1}$ and $\mathrm{pr}_2 \circ\alpha|_{S_2}$ are both automorphisms. Furthermore, for $i = 1, 2$,

$$\alpha(S_iZ_{3-i}) = \alpha(C_S(S_{3-i})) = C_S(\alpha(S_{3-i})) \leq C_{S_i}(\mathrm{pr}_i(\alpha(S_{3-i})))\cdot C_{S_{3-i}}(S_{3-i}) \leq S_i\cdot Z_{3-i} \ ,$$

and hence $\alpha(S_iZ_{3-i}) = S_iZ_{3-i}$.

(b) Define $\mathrm{Aut}^1(S) \trianglelefteq \mathrm{Aut}^0(S)$ by setting

$$\mathrm{Aut}^1(S) = \bigl\{\alpha \in \mathrm{Aut}^0(S) \bigm| \alpha \text{ induces the identity on } \Omega_1(Z_1), \text{ on } \Omega_1(Z(S))/\Omega_1(Z_1), \text{ and on } S/Z(S)\bigr\} \ .$$

Since each element of $\mathrm{Aut}^0(S)$ leaves the subgroups $\Omega_1(Z_1)$, $\Omega_1(Z(S))$, and $Z(S)$ invariant, $\mathrm{Aut}^1(S) \trianglelefteq \mathrm{Aut}^0(S)$. For each $\alpha \in \mathrm{Aut}^1(S)$, $\alpha|_{\Omega_1(Z(S))}$ has p-power order by Lemma 2.1, so $\alpha|_{Z(S)}$ has p-power order by [G, Theorem 5.2.4], and α has p-power order by Lemma 2.1 again. Thus $\mathrm{Aut}^1(S)$ is a p-group, and hence is contained in $O_p(\mathrm{Aut}^0(S))$.

Consider the following maps:

$$\mathrm{Aut}^0(S) \xrightarrow{\;\chi\;} \mathrm{Aut}(S_1) \times \mathrm{Aut}(S_2) \xrightarrow{\;\psi\;} \mathrm{Aut}^0(S) \xrightarrow{\;\rho\;} \mathrm{Aut}^0(S)/O_p(\mathrm{Aut}^0(S)) \ ,$$

where $\chi(\alpha) = (\mathrm{pr}_1 \circ\alpha|_{S_1}, \mathrm{pr}_2 \circ\alpha|_{S_2})$ (as a map of sets), $\psi(\alpha_1, \alpha_2) = \alpha_1 \times \alpha_2$, and ρ is the projection. Here, for $\alpha_i \in \mathrm{Aut}(S_i)$, $\alpha_1 \times \alpha_2$ is the automorphism which sends (s_1, s_2) to $(\alpha_1(s_1), \alpha_2(s_2))$, and thus $\chi \circ \psi$ is the identity on $\mathrm{Aut}(S_1) \times \mathrm{Aut}(S_2)$. Fix $\alpha \in \mathrm{Aut}^0(S)$, and set $\widehat{\alpha} = \alpha \circ ((\psi \circ \chi)(\alpha))^{-1}$. Then $\widehat{\alpha}|_{\Omega_1(Z_1)} = \mathrm{Id}$, since $\alpha(\Omega_1(Z_1)) = \Omega_1(Z_1)$. Also, $\widehat{\alpha} \equiv \mathrm{Id}$ (mod $Z(S)$), since for $g \in S_i$, $\alpha(g) \in S_iZ_{3-i}$ and hence $\mathrm{pr}_i(\alpha(g)) \equiv \alpha(g)$ (mod Z_{3-i}). By a similar argument, $\widehat{\alpha}|_{S_2} \equiv \mathrm{Id}$ (mod Z_1). This proves that $\alpha \equiv \psi(\chi(\alpha))$ (mod $\mathrm{Aut}^1(S)$) for each $\alpha \in \mathrm{Aut}^0(S)$, and hence (since $\mathrm{Aut}^1(S) \leq O_p(\mathrm{Aut}^0(S))$) that $\rho \circ \psi \circ \chi = \rho$.

Thus $\rho \circ \psi$ is surjective, and $\mathrm{Ker}(\rho \circ \psi)$ is a p-group since ψ is injective. So ψ induces an isomorphism $\widehat{\psi}$ from $\overline{\mathrm{Out}}(S_1) \times \overline{\mathrm{Out}}(S_2)$ onto $\mathrm{Aut}^0(S)/O_p(\mathrm{Aut}^0(S))$, and $\widehat{\psi}^{-1}$ sends the class of each $\alpha \in \mathrm{Aut}^0(S)$ to the class of $\chi(\alpha)$.

(c) Write $S_2 = T_2 \times \cdots \times T_n \times U$, where $T_i \cong S_1$ for each i, and U contains no direct factors isomorphic to S_1. Set $T_1 = S_1$, so that $S = T_1 \times T_2 \times \cdots \times T_n \times U$. Fix $\alpha \in \mathrm{Aut}(S)$. By Proposition 3.1, there is $\sigma \in \Sigma_n$ such that $\alpha([T_i, T_i]) = [T_{\sigma(i)}, T_{\sigma(i)}]$ for each i. Since $\alpha(\Omega_1(Z(S))) = \Omega_1(Z(S))$, and since $\Omega_1(Z(S)) \cap [T_i, T_i] = \Omega_1(Z(T_i))$ by assumption, $\alpha(\Omega_1(Z(T_i))) = \Omega_1(Z(T_{\sigma(i)}))$ for each $i = 1, \dots, n$.

Since $\mathrm{Aut}^0(S)$ is the subgroup of automorphisms which send $\Omega_1(Z(T_1))$ to itself, each of its (left) cosets is the set of automorphisms which send $\Omega_1(Z(T_1))$ to $\Omega_1(Z(T_i))$ for some fixed $1 \leq i \leq n$. Thus $[\mathrm{Aut}(S){:}\mathrm{Aut}^0(S)] = n$, and this proves (a). □

With a little more work, one can show that in the situation of Proposition 3.2(c), for any fixed choice of isomorphism $S \cong (S_1)^n \times T$, the composite

$$\mathrm{Aut}(S_1) \wr \Sigma_n \times \mathrm{Aut}(T) \xrightarrow{\text{incl}} \mathrm{Aut}(S) \twoheadrightarrow \overline{\mathrm{Out}}(S) \qquad (1)$$

of the natural inclusion followed by the projection is surjective, and its kernel is a p-group. Thus $\overline{\mathrm{Out}}(S) \cong \overline{\mathrm{Out}}(S_1) \wr \Sigma_n \times \overline{\mathrm{Out}}(T)$ if $\mathrm{Aut}(S_1)$ is not a p-group.

The following is a more technical consequence of Proposition 3.2. It will be needed in the next section, when identifying potential essential subgroups.

Lemma 3.3. *Fix a finite p-group S, and a subgroup $P \leq S$. Assume that $Z(P) = Z(S)$, or more generally, that $\mathrm{Aut}_S(P)$ acts trivially on $Z(P)$. Assume also, for some p-group T and some saturated fusion system $\mathcal{F}$ over $S \times T$, that PT is $\mathcal{F}$-essential. Then there is a subgroup $\Gamma \leq \mathrm{Out}(P)$ such that $\mathrm{Out}_S(P) \in \mathrm{Syl}_p(\Gamma)$ and Γ has a strongly p-embedded subgroup.*

Proof. Let $\mathrm{pr}_1 \colon S \times T \longrightarrow S$ and $\mathrm{pr}_2 \colon S \times T \longrightarrow T$ be the projections. Set

$$\Gamma^* = O^{p'}(\mathrm{Out}_{\mathcal{F}}(PT)) \qquad \text{and} \qquad \widetilde{\Gamma}^* = O^{p'}(\mathrm{Aut}_{\mathcal{F}}(PT)) ,$$

so that $\widetilde{\Gamma}^*/\mathrm{Inn}(PT) = \Gamma^*$. Since PT is $\mathcal{F}$-essential, there is a strongly p-embedded subgroup $H < \mathrm{Out}_{\mathcal{F}}(PT)$ which contains $\mathrm{Out}_{ST}(PT)$ (Lemma 1.9(a)). Also, PT is fully normalized in $\mathcal{F}$, and hence $\mathrm{Out}_{ST}(PT) \in \mathrm{Syl}_p(\mathrm{Out}_{\mathcal{F}}(PT))$.

Set $H^* = H \cap \Gamma^*$. Then $H^* < \Gamma^*$ (H does not contain all Sylow p-subgroups of $\mathrm{Out}_{\mathcal{F}}(PT)$ while Γ^* does), and $H^* \geq \mathrm{Out}_{ST}(PT) \neq 1$. If $g \in \Gamma^* \smallsetminus H^*$, then $g \notin H$ implies $H^* \cap gH^*g^{-1} \leq H \cap gHg^{-1}$ has order prime to p. Thus H^* is strongly p-embedded in Γ^*.

Since $\mathrm{Out}_{ST}(PT)$ acts trivially on $Z(PT)$ (and PT is fully normalized in $\mathcal{F}$ since it is $\mathcal{F}$-essential), each Sylow p-subgroup of Γ^* acts trivially. Thus Γ^* acts trivially on $Z(PT)$, since it is generated by its Sylow p-subgroups. In particular, $\widetilde{\Gamma}^*$ is contained in the group $\mathrm{Aut}^0(PT)$ of automorphisms which send $\Omega_1(Z(P))$ to itself. Define

$$\chi_1 \colon \mathrm{Aut}^0(PT) \longrightarrow \overline{\mathrm{Out}}(P), \qquad \text{and} \qquad \chi_2 \colon \mathrm{Aut}^0(PT) \longrightarrow \overline{\mathrm{Out}}(T)$$

by setting $\chi_1(\alpha) = [\mathrm{pr}_1 \circ \alpha|_P]$ and $\chi_2(\alpha) = [\mathrm{pr}_2 \circ \alpha|_T]$. Set

$$\chi = (\chi_1, \chi_2) \colon \mathrm{Aut}^0(PT) \longrightarrow \overline{\mathrm{Out}}(P) \times \overline{\mathrm{Out}}(T).$$

By Proposition 3.2(b), χ is a surjective homomorphism, and $\mathrm{Ker}(\chi)$ is a p-group.

The Sylow subgroup $\mathrm{Aut}_{ST}(PT)$ is contained in $\mathrm{Ker}(\chi_2)$, and hence $\widetilde{\Gamma}^* \leq \mathrm{Ker}(\chi_2)$ since $\widetilde{\Gamma}^*$ is generated by the Sylow p-subgroups of $\mathrm{Aut}^0(PT)$. Since Γ^* has a strongly p-embedded subgroup, $O_p(\Gamma^*) = 1$ (Lemma 1.9(c)), and hence χ_1 induces an injection of Γ^* into $\overline{\mathrm{Out}}(P)$.

By Proposition 3.2(a), each element of $\widetilde{\Gamma}^*$ sends $PZ(T)$ to itself. We already saw that each element of $\widetilde{\Gamma}^*$ acts trivially on $Z(PT)$, and in particular on $Z(T)$. Hence there is a homomorphism ψ from $\Gamma^* \leq \mathrm{Out}(PT)$ to $\mathrm{Out}(P)$ which sends the class of $\alpha \in \widetilde{\Gamma}^*$ to the class of the automorphism of $PZ(T)/Z(T)$ induced by $\alpha|_{PZ(T)}$. This is equal (mod $O_p(\mathrm{Aut}(P))$) to χ_1, and hence ψ is injective. Set $\Gamma = \psi(\Gamma^*) \leq \mathrm{Out}(P)$. Then Γ contains a strongly p-embedded subgroup since Γ^* does (and $\Gamma \cong \Gamma^*$), and $\psi(\mathrm{Out}_{ST}(PT)) = \mathrm{Out}_S(P) \in \mathrm{Syl}_p(\Gamma)$. □

4. Fusion systems over products of nonabelian 2-groups

We want to prove, under certain hypotheses, that a saturated fusion system over a product of 2-groups splits as a product of fusion systems. Before doing this, we first define what we mean by the product of two fusion systems.

Definition 4.1. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be fusion systems over p-groups S_1 and S_2, respectively, and set $S = S_1 \times S_2$. Then $\mathcal{F}_1 \times \mathcal{F}_2$ is the fusion system over S where for $P, Q \leq S$, if $P_i, Q_i \leq S_i$ denote the images of P and Q under projection, then*

$$\mathrm{Hom}_{\mathcal{F}_1\times\mathcal{F}_2}(P,Q) = \bigl\{(\varphi_1,\varphi_2)|_P \bigm| \varphi_i \in \mathrm{Hom}_{\mathcal{F}_i}(P_i,Q_i),\ (\varphi_1,\varphi_2)(P) \leq Q\bigr\} .$$

It is not hard to see, in the above situation, that $\mathcal{F}_1 \times \mathcal{F}_2$ is the smallest fusion system over $S_1 \times S_2$ which contains each morphism set $\mathrm{Hom}_{\mathcal{F}_1}(P_1,Q_1) \times \mathrm{Hom}_{\mathcal{F}_2}(P_2,Q_2)$ (for $P_i, Q_i \leq S_i$), when regarded as a set of homomorphisms from $P_1 \times P_2$ to $Q_1 \times Q_2$. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are both saturated, then $\mathcal{F}_1 \times \mathcal{F}_2$ is also saturated (cf. [BLO, Lemma 1.5]).

In Section 4.1, we prove Proposition 4.4, which gives some very general conditions where this happens. Then, in the next two subsections, we prove Theorems A, B, and C as special cases of Proposition 4.4. Afterwards, in Section 4.4, we give some examples to show why some of the hypotheses in those two theorems (and in the proposition) are needed.

Throughout this section, whenever $\mathcal{F}$ is a fusion system over a p-group S, we write

$$\mathcal{F}^c = \bigl\{P \leq S \bigm| P \text{ is } \mathcal{F}\text{-centric}\bigr\} .$$

4.1. **A general splitting proposition.**

By [AOV, Proposition 3.3], if $\mathcal{F}$ is a saturated fusion system over a p-group $S = S_1 \times S_2$, where S_1 and S_2 are strongly closed in $\mathcal{F}$ and $O^{p'}(\mathcal{F}) = \mathcal{F}$, then $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ for some pair of saturated fusion systems $\mathcal{F}_i$ over S_i. This played an important role when studying automorphisms of products of fusion systems in [AOV, §3], but unfortunately, it does not seem to be very helpful for our purposes here.

Our main result in this subsection, Proposition 4.4, can be regarded as a stronger version of [AOV, Proposition 3.3], in that we only assume S_1 and S_2 are strongly closed in certain fusion subsystems of $\mathcal{F}$, but not in $\mathcal{F}$ itself. However, this is shown at the expense of adding additional hypotheses: that $p = 2$, and that $O^2(\mathcal{F}) = \mathcal{F}$.

The following definition, which will be used only within this subsection, will be needed when handling "fusion subsystems" which satisfy all of the conditions in the definition except the one which says that they contain the fusion system of the underlying p-group. The term "restrictive (sub)category" is taken from [5a2], although it is used slightly differently here.

Definition 4.2. *Let $\mathcal{F}$ be a fusion system over a p-group S. A* restrictive subcategory *of $\mathcal{F}$ is a subcategory $\mathcal{E} \subseteq \mathcal{F}$ with the same objects, and with the property that for each $P_0 \leq P \leq S$ and $Q_0 \leq Q \leq S$, and each $\varphi \in \mathrm{Hom}_{\mathcal{E}}(P,Q)$ such that $\varphi(P_0) \leq Q_0$, $\varphi|_{P_0} \in \mathrm{Hom}_{\mathcal{E}}(P_0,Q_0)$, and $\varphi^{-1}|_{Q_0} \in \mathrm{Hom}_{\mathcal{E}}(Q_0,P_0)$ if $\varphi(P_0) = Q_0$.*

Thus when $\mathcal{F}$ is a fusion system over S, a restrictive subcategory $\mathcal{E} \subseteq \mathcal{F}$ is a fusion subsystem if and only if $\mathrm{Hom}_{\mathcal{E}}(S) \geq \mathrm{Inn}(S)$. Just as for fusion systems, when $\mathcal{E}$ is a restrictive subcategory over S, then for $T \leq S$, $\mathcal{E}|_T$ denotes the full subcategory whose objects are the subgroups of T. Also, when $\mathcal{H}$ is a set of subgroups of S, we say $\mathcal{E}$ is $\mathcal{H}$-generated if each morphism in $\mathcal{E}$ is a composite of restrictions of morphisms between subgroups in $\mathcal{H}$.

Lemma 4.3. *Fix a pair of p-groups S_1 and S_2, and set $S = S_1 \times S_2$. Let $\mathcal{F}$ be a saturated fusion system over S. Set*

$$\mathcal{F}_1^\bullet = \bigl\langle \mathrm{Aut}_{\mathcal{F}}(P) \bigm| S_2 \le P \le S,\ P\ \mathcal{F}\text{-essential or } P = S \bigr\rangle\ ,$$

and assume S_1 is strongly closed in $\mathcal{F}_1^\bullet$. Set

$$\mathcal{T} = \{P \le S \,|\, P \ge S_2\} \qquad \text{and} \qquad \mathcal{T}^c = \mathcal{T} \cap \mathcal{F}^c\ .$$

Whenever $P \in \mathcal{T}$, we set $P_1 = P \cap S_1$ (so $P = P_1S_2$). Then the following hold.

(a) *If $P, Q \le S$ are such that $P \in \mathcal{T}$, then $\mathrm{Hom}_{\mathcal{F}_1^\bullet}(P,Q) = \mathrm{Hom}_{\mathcal{F}}(P,Q)$. If $P \in \mathcal{T}^c$ and $Q \in P^{\mathcal{F}}$, then $Q \in \mathcal{T}^c$.*

(b) *There is a $\mathcal{T}^c$-generated, restrictive subcategory $\mathcal{E}_1^\bullet$ of $\mathcal{F}_1^\bullet$ such that for $P, Q \in \mathcal{T}^c$,*

$$\mathrm{Hom}_{\mathcal{E}_1^\bullet}(P,Q) = \bigl\{\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q) \bigm| \varphi(g) \in gS_1 \text{ for all } g \in P\bigr\}\ .$$

(c) *Set $\mathcal{F}_1 = \mathcal{F}_1^\bullet|_{S_1}$ and $\mathcal{E}_1 = \mathcal{E}_1^\bullet|_{S_1}$. Then $\mathcal{F}_1$ and $\mathcal{E}_1$ are both saturated fusion systems over S_1, and $\mathcal{E}_1$ is a weakly normal fusion subsystem of index prime to p in $\mathcal{F}_1$. Also, $\mathcal{F}_1^\bullet = \langle \mathcal{E}_1^\bullet, \mathrm{Aut}_{\mathcal{F}}(S)\rangle$.*

(d) *For all $P \in \mathcal{T}^c$,*
  - (d1) *$\alpha \in \mathrm{Aut}_{\mathcal{E}_1^\bullet}(P)$ and $\alpha|_{P_1} = \mathrm{Id}_{P_1}$ imply $\alpha = \mathrm{Id}_P$*
  - (d2) *$[\mathrm{Aut}_{\mathcal{E}_1^\bullet}(P), P] = [\mathrm{Aut}_{\mathcal{E}_1}(P_1), P_1]$, and*
  - (d3) *restriction to P_1 induces a bijection $\mathrm{Hom}_{\mathcal{E}_1^\bullet}(P,S) \xrightarrow[\cong]{R} \mathrm{Hom}_{\mathcal{E}_1}(P_1,S_1)$.*

(e) *For all $P \in \mathcal{T}$,*

$$\begin{gathered} P_1 \text{ fully norm. in } \mathcal{F}_1 \iff P_1 \text{ fully norm. in } \mathcal{F} \implies P \text{ fully norm. in } \mathcal{F} \\ P_1 \text{ fully centr. in } \mathcal{F}_1 \iff P_1 \text{ fully centr. in } \mathcal{F} \implies P \text{ fully centr. in } \mathcal{F} \\ P_1 \text{ is } \mathcal{F}_1\text{-centric} \iff P \text{ is } \mathcal{F}\text{-centric} \end{gathered}$$

Proof. **(a)** Let $\mathrm{pr}_2 \in \mathrm{Hom}(S,S_2)$ be the projection. By Proposition 1.10(a), for all $P \in \mathcal{T}$ and all $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$, we can write $\varphi = \psi_m \circ \cdots \circ \psi_1$ where each ψ_i is the restriction of an automorphism of some $Q_i \le S$ which is $\mathcal{F}$-essential or equal to S. Let $j \le m$ be such that $Q_i \ge S_2$ for all $i = 1,\dots,j$ (hence $Q_i \in \mathcal{T}^c$), and either $j = m$ or $Q_{j+1} \not\ge S_2$. Set $\varphi^* = \psi_j \circ \cdots \circ \psi_1$. Then $\varphi^* \in \mathrm{Mor}(\mathcal{F}_1^\bullet)$, and $\varphi^*(P \cap S_1) = \varphi^*(P) \cap S_1$ since S_1 is strongly closed. It follows that $\varphi^*(S_2) \cap S_1 = \varphi^*(S_2) \cap \varphi^*(P \cap S_1) = 1$, so $\mathrm{pr}_2 \circ \varphi^*|_{S_2}$ is injective, and hence $\mathrm{pr}_2(\varphi^*(S_2)) = S_2$. If $j < m$, then $\mathrm{pr}_2(Q_{j+1}) \ge \mathrm{pr}_2(\varphi^*(S_2)) = S_2$, $Q_{j+1} = R_1R_2$ for some $R_i \le S_i$ by Lemma 1.11(a), and hence $R_2 = S_2$ and $Q_{j+1} \ge S_2$. This contradicts our assumption on j, and we conclude that $j = m$ and $\varphi^* = \varphi$. In particular, $\varphi \in \mathrm{Hom}_{\mathcal{F}_1^\bullet}(P,S)$.

Now assume $P \in \mathcal{T}^c$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$. We must show $\varphi(P) \ge S_2$; i.e., $\varphi(P) \in \mathcal{T}^c$. Since φ is in $\mathcal{F}_1^\bullet$, it suffices to show this when $\varphi = \alpha|_P$ for some $\alpha \in \mathrm{Aut}_{\mathcal{F}}(Q)$, where Q is $\mathcal{F}$-essential or $Q = S$. Set $Q_1 = Q \cap S_1$, so $Q = Q_1S_2$. Then $\alpha(Q_1) = Q_1$ since S_1 is strongly closed in $\mathcal{F}_1^\bullet$, and $\alpha(Z(Q_1)S_2) = Z(Q_1)S_2$ since $Z(Q_1)S_2 = C_Q(Q_1)$. Since $P \le Q$ is $\mathcal{F}$-centric and contains S_2, $P \ge Z(Q_1)S_2$, and so $\alpha(P) \ge Z(Q_1)S_2 \ge S_2$.

(b) The formula for $\mathrm{Hom}_{\mathcal{E}_1^\bullet}(P,Q)$ clearly defines a category with objects in $\mathcal{T}^c$, which is contained in $\mathcal{F}_1^\bullet$ by (a). Since $Q \ge P \in \mathcal{T}^c$ implies $Q \in \mathcal{T}^c$, restriction to arbitrary subgroups of S now defines a restrictive subcategory of $\mathcal{F}_1^\bullet$, with morphisms between subgroups in $\mathcal{T}^c$ as given.

(d) Fix $P = P_1S_2 \in \mathcal{T}^c$. By definition,

$$\mathrm{Aut}_{\mathcal{E}_1^\bullet}(P) = \mathrm{Ker}\bigl[f\colon \mathrm{Aut}_{\mathcal{F}}(P) \longrightarrow \mathrm{Aut}(P/P_1)\bigr] \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(P). \tag{1}$$

(Recall that $\alpha(P_1) = P_1$ for all $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$ by (a).)

Choose $\varphi \in \mathrm{Iso}_{\mathcal{F}}(P, R)$, where R is fully normalized in $\mathcal{F}$. Then $\varphi \in \mathrm{Mor}(\mathcal{F}_1^\bullet)$, $R \geq S_2$, and $\varphi(P_1) = R_1 \stackrel{\text{def}}{=} R \cap S_1$ by (a). Let

$$c_\varphi\colon \mathrm{Aut}_{\mathcal{F}}(P) \xrightarrow{\;\cong\;} \mathrm{Aut}_{\mathcal{F}}(R)$$

be conjugation by φ ($c_\varphi(\alpha) = \varphi\alpha\varphi^{-1}$). Then $c_\varphi(\mathrm{Aut}_{\mathcal{E}_1^\bullet}(P)) = \mathrm{Aut}_{\mathcal{E}_1^\bullet}(R)$ by (1), and so (d1) and (d2) hold for P if they hold for R. It thus suffices to prove them when P is fully normalized in $\mathcal{F}$.

Set

$$\Gamma = \bigl\{\alpha \in \mathrm{Aut}_{\mathcal{F}}(P) \bigm| \alpha \text{ induces the identity on } P_1 \text{ and on } P/P_1\bigr\} .$$

We must show $\Gamma = 1$. By (a), $\alpha(P_1) = P_1$ for all $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$, and hence $\Gamma \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(P)$. Also, Γ is a p-subgroup by Lemma 2.1, and thus $\Gamma \leq \mathrm{Aut}_S(P) \in \mathrm{Syl}_2(\mathrm{Aut}_{\mathcal{F}}(P))$. But each element of $\mathrm{Aut}_S(P)$ sends S_2 to itself, so Γ contains only the identity. This proves (d1).

We next prove (d2). Since $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$, and $\mathrm{Aut}_{\mathcal{E}_1^\bullet}(P) \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(P)$ by (1), $\mathrm{Aut}_{S_1}(P) = \mathrm{Aut}_S(P) \cap \mathrm{Aut}_{\mathcal{E}_1^\bullet}(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{E}_1^\bullet}(P)$. Also, $[\mathrm{Aut}_{S_1}(P), P] = [\mathrm{Aut}_{S_1}(P_1), P_1] \leq [\mathrm{Aut}_{\mathcal{E}_1}(P_1), P_1]$, so it remains to check that $[H, P] \leq [\mathrm{Aut}_{\mathcal{E}_1}(P_1), P_1]$ for each $H \leq \mathrm{Aut}_{\mathcal{E}_1^\bullet}(P)$ of order prime to p. For each such H, $[H, P] \leq P_1$ by definition of $\mathcal{E}_1^\bullet$, $[H, P] = [H, [H, P]]$ by [G, Theorem 5.3.6] or [A, 24.5], and so $[H, P] \leq [H, P_1] \leq [\mathrm{Aut}_{\mathcal{E}_1}(P_1), P_1]$.

The restriction map R in (d3) is well defined since S_1 is strongly closed in $\mathcal{F}_1^\bullet \supseteq \mathcal{E}_1^\bullet$. If $\chi_1, \chi_2 \in \mathrm{Hom}_{\mathcal{E}_1^\bullet}(P, S)$ are such that $R(\chi_1) = R(\chi_2) = \varphi$, then $\mathrm{Im}(\chi_1) = \varphi(P_1)S_2 = \mathrm{Im}(\chi_2)$, and $\chi_2^{-1} \circ \chi_1 \in \mathrm{Aut}_{\mathcal{E}_1^\bullet}(P)$ is the identity on P_1, hence the identity on P by (d1). Thus $\chi_1 = \chi_2$, and so R is injective.

To see that R is surjective, fix $\psi \in \mathrm{Hom}_{\mathcal{E}_1}(P_1, S_1)$. By definition of $\mathcal{E}_1^\bullet$ (or by (b)), we can write $\psi = \psi_k \circ \cdots \circ \psi_1$, where each ψ_i is the restriction of a $\mathcal{E}_1^\bullet$-morphism between subgroups in $\mathcal{T}^c$. If $\psi_1 = \widehat{\psi}_1|_{P_1}$ where $\widehat{\psi}_1 \in \mathrm{Hom}_{\mathcal{E}_1^\bullet}(R, T)$ for $R, T \in \mathcal{T}^c$, then $R \geq P_1S_2 = P$, $Q \stackrel{\text{def}}{=} \widehat{\psi}_1(P) \in \mathcal{T}^c$ by (a), and $\psi_1(P_1) = Q_1$ since S_1 is strongly closed in $\mathcal{F}_1^\bullet$. Thus ψ_1 extends to a morphism on P with image in $\mathcal{T}^c$; and by induction on k, the same holds for ψ.

(e) Fix $P_1 \leq S_1$. Assume $Q \in {P_1}^{\mathcal{F}}$ is fully centralized in $\mathcal{F}$, and fix $\varphi \in \mathrm{Iso}_{\mathcal{F}}(P_1, Q)$. Since Q is fully centralized, φ extends to some $\bar\varphi \in \mathrm{Hom}_{\mathcal{F}}(P_1S_2, S)$ by the extension axiom. By (a), $\bar\varphi \in \mathrm{Hom}_{\mathcal{F}_1^\bullet}(P_1S_2, S)$. Thus $Q = \bar\varphi(P_1) \leq S_1$ since S_1 is strongly closed in $\mathcal{F}_1^\bullet$, and Q is $\mathcal{F}_1$-conjugate to P_1. Since $C_{S_1}(P_1) = C_{S_1}(Q)$ if and only if $C_S(P_1) = C_S(Q)$ (and similarly for normalizers), P_1 is fully centralized (fully normalized) in $\mathcal{F}_1$ if and only if it is fully centralized (fully normalized) in $\mathcal{F}$.

Now fix $P \in \mathcal{T}$, and set $P_1 = P \cap S_1$ as usual. Assume P_1 is fully normalized in $\mathcal{F}$. Fix $Q \in P^{\mathcal{F}}$ which is fully normalized in $\mathcal{F}$, and choose $\varphi \in \mathrm{Iso}_{\mathcal{F}}(P, Q)$. Then $Q_1 \stackrel{\text{def}}{=} Q \cap S_1 = \varphi(P_1)$ since $\varphi \in \mathrm{Mor}(\mathcal{F}_1^\bullet)$ by (a) (and since S_1 is strongly closed in $\mathcal{F}_1^\bullet$). So $|N_S(P)| = |N_S(P_1)| \geq |N_S(Q_1)| \geq |N_S(Q)|$, and thus P is also fully normalized in $\mathcal{F}$.

To prove the corresponding result for fully centralized subgroups, first note that any $R \leq S$ is fully centralized in $\mathcal{F}$ if and only if $|C_S(R){\cdot}R|$ is maximal in the $\mathcal{F}$-conjugacy class of R. So assume $P_1 \leq S_1$ is fully centralized in $\mathcal{F}$, and choose $Q \leq S$ and $\varphi \in \mathrm{Iso}_{\mathcal{F}}(P, Q)$ as in the last paragraph. Thus $Q_1 = \varphi(P_1)$ by (a). Also, $Q \leq C_S(Q_1){\cdot}Q_1$ since $P \leq C_S(P_1){\cdot}P_1$, so $C_S(Q){\cdot}Q \leq C_S(Q_1){\cdot}Q_1$. Hence

$$|C_S(P){\cdot}P| = |C_S(P_1){\cdot}P_1| \geq |C_S(Q_1){\cdot}Q_1| \geq |C_S(Q){\cdot}Q| ,$$

(the equality since $P \geq S_2$ and the first inequality since P_1 is fully centralized), and so P is fully centralized in $\mathcal{F}$ since Q is.

A subgroup $R \leq S$ is $\mathcal{F}$-centric if and only if R is fully centralized in $\mathcal{F}$ and $R \geq C_S(R)$; and similarly for $\mathcal{F}_1$-centric subgroups. Thus $P_1 \in \mathcal{F}_1^c$ implies $P \in \mathcal{F}^c$ by the corresponding result for fully centralized subgroups. It remains to prove the converse.

Assume $P \in \mathcal{F}^c$; equivalently, $P \in \mathcal{T}^c$. Choose $Q_1 \leq S_1$ which is $\mathcal{F}_1$-conjugate to P_1 and fully centralized in $\mathcal{F}_1$ (hence also in $\mathcal{F}$). Then any $\varphi \in \mathrm{Iso}_{\mathcal{F}_1}(P_1, Q_1)$ extends to some $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ by the extension axiom. Since $P \in \mathcal{T}^c$, $\bar{\varphi}(P) \geq S_2$ by (a), so $\bar{\varphi}(P) = Q_1S_2$, and $C_S(Q_1S_2) = C_{S_1}(Q_1)Z(S_2) \leq Q_1S_2$ since P is $\mathcal{F}$-centric. Hence $C_{S_1}(Q_1) \leq Q_1$, and so $Q_1, P_1 \in \mathcal{F}_1^c$ since Q_1 is fully centralized.

(c) If $P \leq S_1$ is fully normalized in $\mathcal{F}_1$, then it is fully normalized in $\mathcal{F}$ by (e), hence fully centralized in $\mathcal{F}$, and hence fully centralized in $\mathcal{F}_1$ by (e) again. Also, $\mathrm{Aut}_{S_1}(P) = \mathrm{Aut}_S(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}_1}(P) = \mathrm{Aut}_{\mathcal{F}}(P)$, and this proves the Sylow axiom for $\mathcal{F}_1$.

Assume $\varphi \in \mathrm{Iso}_{\mathcal{F}_1}(P, Q)$, where Q is fully centralized in $\mathcal{F}_1$ and hence in $\mathcal{F}$. Let $N_\varphi \leq N_{S_1}(P)$ be the subgroup of those g such that $\varphi c_g \varphi^{-1} \in \mathrm{Aut}_{S_1}(Q)$. Then φ extends to some $\widehat{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi S_2, S)$ by the extension axiom for $\mathcal{F}$, $\widehat{\varphi} \in \mathrm{Hom}_{\mathcal{F}_1^\bullet}(N_\varphi S_2, S)$ by (a), and so $\widehat{\varphi}(N_\varphi) \leq S_1$ since S_1 is strongly closed in $\mathcal{F}_1^\bullet$. Then $\bar{\varphi} = \widehat{\varphi}|_{N_\varphi} \in \mathrm{Hom}_{\mathcal{F}_1}(N_\varphi, S_1)$, and this proves the extension axiom for $\mathcal{F}_1$. Thus $\mathcal{F}_1$ is saturated.

Recall $\mathcal{E}_1 = \mathcal{E}_1^\bullet|_{S_1}$, where $\mathcal{E}_1^\bullet$ is as described in (b). Hence $\mathcal{E}_1^\bullet$ is invariant under conjugation by elements in $\mathrm{Aut}_{\mathcal{F}}(S)$, and $\mathcal{E}_1$ is invariant under conjugation by elements in $\mathrm{Aut}_{\mathcal{F}_1}(S_1)$.

We next show that

$$\mathcal{F}_1^\bullet = \langle \mathcal{E}_1^\bullet, \mathrm{Aut}_{\mathcal{F}}(S) \rangle \tag{2}$$

Assume $P \geq S_2$ and is $\mathcal{F}$-essential. By (1), $\mathrm{Aut}_{\mathcal{E}_1^\bullet}(P) \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(P)$. So $\mathrm{Aut}_{\mathcal{E}_1^\bullet}(P) \cdot \mathrm{Inn}(P)$ is normal in $\mathrm{Aut}_{\mathcal{F}}(P)$ and contains the Sylow 2-subgroup $\mathrm{Aut}_S(P)$ (since $\mathrm{Aut}_{P \cap S_1}(P) \leq \mathrm{Aut}_{\mathcal{E}_1^\bullet}(P)$). It follows that $\mathrm{Aut}_{\mathcal{E}_1^\bullet}(P) \cdot \mathrm{Inn}(P) \geq O^{2'}(\mathrm{Aut}_{\mathcal{F}}(P))$. Since

$$\mathcal{F}_1^\bullet = \big\langle \mathrm{Aut}_{\mathcal{F}}(S), O^{2'}(\mathrm{Aut}_{\mathcal{F}}(P)) \,\big|\, P \geq S_2 \ \mathcal{F}\text{-essential} \big\rangle$$

by Proposition 1.10(b), this finishes the proof of (2).

Upon restricting to S_1, (2) implies that $\mathcal{F}_1 = \langle \mathcal{E}_1, \mathrm{Aut}_{\mathcal{F}_1}(S_1) \rangle$. We already saw that $\mathcal{E}_1$ is invariant under conjugation by elements of $\mathrm{Aut}_{\mathcal{F}_1}(S_1)$, so $\mathcal{E}_1$ is $\mathcal{F}_1$-invariant.

Since $P_1 \leq S_1$ is $\mathcal{E}_1$-centric if and only if $P = P_1S_2 \in \mathcal{T}^c$ by (e) (and Lemma 1.5), and since $\mathcal{E}_1^\bullet$ is $\mathcal{T}^c$-generated by construction, $\mathcal{E}_1$ is $\mathcal{E}_1^c$-generated. Hence by Theorem 1.3, to show $\mathcal{E}_1$ is saturated, it suffices to check the axioms on $\mathcal{E}_1$-centric subgroups (equivalently, $\mathcal{F}_1$-centric by Lemma 1.5). Also by Lemma 1.5, a subgroup of S_1 is fully normalized (fully centralized) in $\mathcal{E}_1$ if and only if it is in $\mathcal{F}_1$. In particular, if P is fully normalized in $\mathcal{E}_1$, then it is fully centralized, and $\mathrm{Aut}_{S_1}(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{E}_1}(P))$ since $\mathrm{Aut}_{S_1}(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}_1}(P))$ and $\mathrm{Aut}_{\mathcal{E}_1}(P) \leq \mathrm{Aut}_{\mathcal{F}_1}(P)$. The Sylow axiom thus holds.

Fix $\varphi \in \mathrm{Iso}_{\mathcal{E}_1}(P_1, Q_1)$, where P_1 and Q_1 are $\mathcal{F}_1$-centric, and set $P = P_1S_2$ and $Q = Q_1S_2$. Thus $P, Q \in \mathcal{T}^c$ by (e). By (d3) and (b), there is $\psi \in \mathrm{Iso}_{\mathcal{F}}(P, Q)$ such that $\psi|_{P_1} = \varphi$, and $\psi(g) \in gS_1$ for all $g \in P$. Let N_φ be the group of all $g \in N_{S_1}(P_1)$ such that $\varphi c_g \varphi^{-1} \in \mathrm{Aut}_{S_1}(Q_1)$. Fix $g \in N_\varphi$, and choose $h \in N_{S_1}(Q_1)$ such that $\varphi \circ c_g = c_h \circ \varphi \in \mathrm{Iso}(P_1, Q_1)$. Then $\psi \circ c_g$ and $c_h \circ \psi$ are two morphisms in $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ which are equal after restriction to P_1, and which both induce the identity from P/P_1 to Q/Q_1. So by (d1), applied to $(c_h \circ \psi)^{-1} \circ (\psi \circ c_g) \in \mathrm{Aut}_{\mathcal{F}}(P)$, $\psi \circ c_g = c_h \circ \psi$, and thus $g \in N_\psi$. By the extension axiom applied to $\mathcal{F}$, ψ extends to a morphism $\bar{\psi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi S_2, S)$, $\bar{\psi} \in \mathrm{Mor}(\mathcal{E}_1^\bullet)$ since $N_\varphi S_2 \in \mathcal{T}^c$, and hence $\bar{\psi}|_{N_\varphi} \in \mathrm{Hom}_{\mathcal{E}_1}(N_\varphi, S_1)$ extends φ.

We have now shown that $\mathcal{E}_1$ is $\mathcal{E}_1^c$-saturated. Since $\mathcal{E}_1^\bullet$ is $\mathcal{T}^c$-generated by definition, $\mathcal{E}_1$ is $\mathcal{E}_1^c$-generated by (e). So $\mathcal{E}_1$ is saturated by Theorem 1.3. We already showed that it is $\mathcal{F}_1$-invariant, and hence it is weakly normal. By Theorem 1.7(b), $\mathcal{E}_1$ has index prime to p in $\mathcal{F}_1$. □

The following is our main, general proposition for decomposing a fusion system. Theorems 4.6 and 4.9 will follow as consequences of this.

Proposition 4.4. *Fix a pair of 2-groups S_1 and S_2, and set $S = S_1 \times S_2$. Let $\mathcal{F}$ be a saturated fusion system over S. For $i = 1, 2$, define*

$$\mathcal{F}_i^\bullet = \big\langle \mathrm{Aut}_{\mathcal{F}}(P) \,\big|\, S_{3-i} \le P \le S,\ P\ \mathcal{F}\text{-essential or } P = S \big\rangle$$

as a fusion subsystem of $\mathcal{F}$ over S. Assume

(a) *$O^2(\mathcal{F}) = O^{2'}(\mathcal{F}) = \mathcal{F}$; and*

(b) *S_i is strongly closed in $\mathcal{F}_i^\bullet$ for $i = 1, 2$.*

Then $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ for some pair of saturated fusion systems $\mathcal{F}_i$ over S_i.

Proof. For $i = 1, 2$, let $\mathrm{pr}_i \in \mathrm{Hom}(S, S_i)$ be the projection, and set

$$\begin{aligned}
\mathcal{T}_i &= \big\{P \le S \,\big|\, P \ge S_{3-i}\big\}\\
\mathcal{T}_i^c &= \mathcal{T}_i \cap \mathcal{F}^c\\
\mathcal{U} &= \big\{P = P_1P_2 \,\big|\, P_i \le S_i,\ P_1S_2, S_1P_2 \in \mathcal{F}^c\big\} = \big\{P \cap Q \,\big|\, P \in \mathcal{T}_1^c,\ Q \in \mathcal{T}_2^c\big\}\ .
\end{aligned}$$

In general, for any $P \in \mathcal{U}$ or $P \in \mathcal{T}_i$, we set $P_i = \mathrm{pr}_i(P) \le S_i$ (so $P = P_1P_2$).

Since $\mathrm{Aut}_{\mathcal{F}_1^\bullet}(S) = \mathrm{Aut}_{\mathcal{F}_2^\bullet}(S) = \mathrm{Aut}_{\mathcal{F}}(S)$ by definition of $\mathcal{F}_i^\bullet$, (b) implies that

$$\alpha \in \mathrm{Aut}_{\mathcal{F}}(S) \quad \Longrightarrow \quad \alpha(S_1) = S_1 \ \text{ and } \ \alpha(S_2) = S_2\ . \tag{1}$$

Define restrictive subcategories $\mathcal{E}_i^\bullet$ as in Lemma 4.3: $\mathcal{E}_i^\bullet$ is the $\mathcal{T}_i^c$-generated restrictive subcategory of $\mathcal{F}_i^\bullet$ where for each $P, Q \in \mathcal{T}_i^c$,

$$\mathrm{Hom}_{\mathcal{E}_i^\bullet}(P,Q) = \big\{\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q) \,\big|\, \varphi(g) \in gS_i \text{ for all } g \in P\big\}\ .$$

Set $\mathcal{F}_i = \mathcal{F}_i^\bullet|_{S_i}$ and $\mathcal{E}_i = \mathcal{E}_i^\bullet|_{S_i}$. By Lemma 4.3(c), $\mathcal{E}_i$ and $\mathcal{F}_i$ are both saturated fusion systems over S_i.

Step 1: We first show, for each $i = 1$ or $i = 2$, that

$$P \in \mathcal{U},\ \psi \in \mathrm{Hom}_{\mathcal{E}_i^\bullet}(P,S) \implies \psi(P) = \psi(P_i)P_{3-i},\ \psi(g) \in gS_i\ \forall\, g \in P\ . \tag{2}$$

To simplify the notation, we prove this for $i = 1$. Fix $\psi \in \mathrm{Hom}_{\mathcal{E}_1^\bullet}(P,S)$. Since $\mathcal{E}_1^\bullet$ is $\mathcal{T}_1^c$-generated, $\psi = \psi_m \circ \cdots \circ \psi_1$, where each ψ_j is the restriction of some $\chi_j \in \mathrm{Hom}_{\mathcal{E}_1^\bullet}(Q_j, R_j)$ for $Q_j, R_j \in \mathcal{T}_1^c$. Then $P = P_1P_2 \le Q_1 \in \mathcal{T}_1^c$, so $P_1S_2 \le Q_1$, $P_1S_2 \in \mathcal{T}_1^c$, and $\chi_1(P_1S_2) \in \mathcal{T}_1^c$ by Lemma 4.3(a). Thus $\chi_1(P_1S_2) \ge \chi_1(P_1)S_2$, with equality since they have the same order. Also, $\chi_1(g) \in gS_1$ for each $g \in P_1S_2$ by definition of $\mathcal{E}_1^\bullet$, so $\psi_1(P) = \chi_1(P) = \chi_1(P_1)P_2 = \psi_1(P_1)P_2 \in \mathcal{U}$. Upon continuing this argument with the other ψ_j, we see that $\psi(P) = \psi(P_1)P_2$, that $\psi(g) \in gS_1$ for all $g \in P$, and also that ψ extends to $\widehat{\psi} \in \mathrm{Iso}_{\mathcal{E}_1^\bullet}(P_1S_2, \psi(P_1)S_2)$. Since S_1P_2 and $\psi(P_1)S_2$ are both $\mathcal{F}$-centric, $\psi(P) \in \mathcal{U}$. This proves (2).

We next show, for $i = 1$ or $i = 2$, that

for all $P, Q \in \mathcal{U}$ with $P_{3-i} = Q_{3-i}$, restriction induces bijections

$$\mathrm{Hom}_{\mathcal{E}_i^\bullet}(P_iS_{3-i}, Q_iS_{3-i}) \xrightarrow[\cong]{R_1} \mathrm{Hom}_{\mathcal{E}_i^\bullet}(P,Q) \xrightarrow[\cong]{R_2} \mathrm{Hom}_{\mathcal{E}_i}(P_i,Q_i)\ . \tag{3}$$

The map R_1 is defined by (2), and R_2 is defined since S_i is strongly closed in $\mathcal{F}_i^\bullet$ (hence in $\mathcal{E}_i^\bullet$). We just showed that R_1 is surjective, R_2R_1 is bijective by Lemma 4.3(d3), and this proves (3).

Fix $i = 1,2$, $P \in \mathcal{T}_i^c$, and $\varphi \in \mathrm{Hom}_{\mathcal{E}_i^\bullet}(P,S)$. Then $\varphi|_{P_i} \in \mathrm{Mor}(\mathcal{E}_i)$, so by Proposition 1.10(a) (applied to the saturated fusion system $\mathcal{E}_i$), $\varphi|_{P_i} = \psi_k \circ \cdots \circ \psi_1$ where each ψ_i is the restriction of an $\mathcal{E}_i$-automorphism of a subgroup of S_i which contains its source and target. Hence by (3), $\varphi = \widehat{\psi}_k \circ \cdots \circ \widehat{\psi}_1$ where each $\widehat{\psi}_i$ is the restriction of an $\mathcal{E}_i^\bullet$-automorphism of a subgroup in $\mathcal{T}_i^c$. Thus for $i = 1,2$,

$$\mathcal{E}_i^\bullet = \big\langle \mathrm{Aut}_{\mathcal{E}_i^\bullet}(P) \,\big|\, P \in \mathcal{T}_i^c \big\rangle \ . \tag{4}$$

In particular,

$$\mathfrak{foc}(\mathcal{E}_i^\bullet) = \big\langle [\mathrm{Aut}_{\mathcal{E}_i^\bullet}(P), P] \,\big|\, P \in \mathcal{T}_i^c \big\rangle = \big\langle [\mathrm{Aut}_{\mathcal{E}_i}(P_i), P_i] \,\big|\, P \in \mathcal{T}_i^c \big\rangle = \mathfrak{foc}(\mathcal{E}_i) \ , \tag{5}$$

where the second equality holds by Lemma 4.3(d2), and the third since by definition, $\langle [\mathrm{Aut}_{\mathcal{E}_i}(P_i), P_i] \,|\, P \in \mathcal{T}_i^c \rangle \le \mathfrak{foc}(\mathcal{E}_i) \le \mathfrak{foc}(\mathcal{E}_i^\bullet)$.

Notation: For $P \in \mathcal{U}$, $i = 1,2$, $Q_i \le S_i$, and $\varphi \in \mathrm{Hom}_{\mathcal{E}_i}(P_i, Q_i)$, let

$$\varphi{\uparrow}^P \in \mathrm{Hom}_{\mathcal{E}_i^\bullet}(P, Q_iP_{3-i})$$

be the morphism whose restriction to P_i is equal to φ. This exists and is unique by (3).

Step 2: By Lemma 4.3(c), $\mathcal{F}_i^\bullet = \langle \mathcal{E}_i^\bullet, \mathrm{Aut}_{\mathcal{F}}(S) \rangle$. Hence

$$\mathcal{F} = \big\langle \mathrm{Aut}_{\mathcal{F}}(P) \,\big|\, P = S \text{ or } P \text{ is } \mathcal{F}\text{-essential} \big\rangle = \langle \mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet \rangle = \langle \mathcal{E}_1^\bullet, \mathcal{E}_2^\bullet, \mathrm{Aut}_{\mathcal{F}}(S) \rangle \ , \tag{6}$$

where the first equality follows from Proposition 1.10(a) and the second from Lemma 1.11(b) and the definition of $\mathcal{F}_i^\bullet$. We claim that

$$\begin{aligned} &P \in \mathcal{U},\ \ \varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S),\ \ Q = \varphi(P) \Longrightarrow \\ &\qquad\qquad Q \in \mathcal{U},\ \ \varphi(P_1Z(P_2)) = Q_1Z(Q_2),\ \ \varphi(P_2Z(P_1)) = Q_2Z(Q_1) \ . \end{aligned} \tag{7}$$

By (6), it suffices to prove this when $\varphi = \psi|_P$ for some $\psi \in \mathrm{Aut}_{\mathcal{F}}(S)$, or when $\varphi \in \mathrm{Hom}_{\mathcal{E}_i^\bullet}(P,S)$ for $i = 1,2$. In the first case, $\psi(S_i) = S_i$ for $i = 1,2$ by (1). Hence $\varphi(P) = Q = Q_1Q_2$ where $Q_i = \varphi(P_i) \le S_i$, and all of the claims in (7) follow immediately.

Now assume $\varphi \in \mathrm{Hom}_{\mathcal{E}_1^\bullet}(P,S)$ (the argument for $\mathcal{E}_2^\bullet$ is similar). By (2), $Q = Q_1Q_2 \in \mathcal{U}$ ($Q_i \le S_i$), where $Q_2 = P_2$ and $Q_1 = \varphi(P_1)$, and $\varphi(g) \in gQ_1$ for each $g \in P$. In particular, $\varphi(P_1Z(P_2)) = Q_1Z(Q_2)$. Also, φ sends $C_P(P_1) = Z(P_1)P_2$ onto $C_Q(Q_1) = Z(Q_1)Q_2$, and this finishes the proof of (7).

Again fix $P = P_1P_2 \in \mathcal{U}$, and consider $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$. If $g \in N_{S_1}(P)$ is such that $\alpha c_g \alpha^{-1} \in \mathrm{Aut}_S(P)$, then by the extension axiom, α extends to $\beta \in \mathrm{Hom}_{\mathcal{F}}(\langle P,g \rangle, S)$, where $\beta(g) = h = h_1h_2$ for $h_i \in N_{S_i}(P)$. Set $Q = \mathrm{Im}(\beta) = \langle P,h \rangle$. By (7), $\beta(\langle P_1,g \rangle) \le \langle P_1,h_1 \rangle Z(P_2)$ since $Z(Q_2) \le Z(P_2)$, so $h_2 \in Z(P_2)$, and $\alpha c_g \alpha^{-1} = c_h = c_{h_1} \in \mathrm{Aut}_{S_1}(P)$. After applying a similar argument to $\mathrm{Aut}_{S_2}(P)$, we have shown

$$P \in \mathcal{U} \quad\Longrightarrow\quad \mathrm{Aut}_{S_1}(P),\ \mathrm{Aut}_{S_2}(P) \text{ str. closed in } \mathrm{Aut}_S(P) \text{ w. resp. to } \mathrm{Aut}_{\mathcal{F}}(P) \ . \tag{8}$$

Step 3: We next claim that

$$\begin{aligned} &P \in \mathcal{U},\ Q_1 \le S_1,\ Q_2 \le S_2,\ \alpha \in \mathrm{Iso}_{\mathcal{E}_1}(P_1,Q_1),\ \beta \in \mathrm{Iso}_{\mathcal{E}_2}(P_2,Q_2) \\ &\qquad\qquad\Longrightarrow\quad (\beta{\uparrow}^{Q_1P_2}) \circ (\alpha{\uparrow}^P) = (\alpha{\uparrow}^{P_1Q_2}) \circ (\beta{\uparrow}^P) \ . \end{aligned} \tag{9}$$

Assume first $P_2 = S_2$, so that $Q_2 = S_2$ and $P \in \mathcal{T}_1^c$. Set $\widehat{\beta} = \beta{\uparrow}^S \in \mathrm{Aut}_{\mathcal{E}_2^\bullet}(S)$. Then $\widehat{\beta}(S_1) = S_1$ by (1), and hence $\widehat{\beta}|_{S_1} = \mathrm{Id}_{S_1}$ by (2). Thus the composite

$$\psi \overset{\text{def}}{=} \bigl((\widehat{\beta}|_{Q_1P_2}) \circ (\alpha{\uparrow}^P)\bigr)^{-1} \circ \bigl((\alpha{\uparrow}^{P_1Q_2}) \circ (\widehat{\beta}|_P)\bigr) \in \mathrm{Aut}_{\mathcal{F}}(P)$$

induces the identity on P_1 and on P/P_1. Hence $\psi \in \mathrm{Aut}_{\mathcal{E}_1^\bullet}(P)$ by definition of $\mathcal{E}_1^\bullet$, and so $\psi = \mathrm{Id}_P$ by the injectivity in (3). This proves (9) when $P_2 = S_2$, and a similar argument proves it when $P_1 = S_1$.

Now assume $P_i < S_i$ for $i = 1, 2$. We can assume inductively that (9) holds for subgroups of order strictly larger than $|P|$. Thus in the situation of (9), if α is a composite of restrictions of isomorphisms $\widehat{\alpha}_i$ between strictly larger subgroups of S_1, then (9) holds for the $\widehat{\alpha}_i$, and hence holds for α. Similarly, if β is a composite of restrictions of isomorphisms between strictly larger subgroups of S_2, then (9) again holds. So by Proposition 1.10(a), we are now reduced to proving this when $P_1 = Q_1$ is $\mathcal{E}_1$-essential and $P_2 = Q_2$ is $\mathcal{E}_2$-essential.

Set $G_i = \mathrm{Aut}_{\mathcal{E}_i^\bullet}(P)$ and $G = \mathrm{Aut}_{\mathcal{F}}(P)$. Then $[G_i, P] = [\mathrm{Aut}_{\mathcal{E}_i}(P_i), P_i] \le S_i$ by Lemma 4.3(d2), so $[G_1, P] \cap [G_2, P] = 1$. Let $H_i \le G_i$ be the subgroup generated by automorphisms which extend (in $\mathcal{E}_i^\bullet$) to larger subgroups. We just showed, in the last paragraph, that $[H_i, G_{3-i}] = 1$ for $i = 1, 2$. Also,

$$\mathrm{Out}_{\mathcal{E}_i}(P_i) = \mathrm{Aut}_{\mathcal{E}_i}(P_i)/\mathrm{Inn}(P_i) \cong \mathrm{Aut}_{\mathcal{E}_i^\bullet}(P)/\mathrm{Aut}_{P_i}(P) = G_i/\mathrm{Aut}_{P_i}(P)$$

where the isomorphism follows by (3). So by Proposition 1.10(c), applied to $\mathrm{Out}_{\mathcal{E}_i}(P_i)$, $H_i/\mathrm{Aut}_{P_i}(P)$ is strongly 2-embedded in $G_i/\mathrm{Aut}_{P_i}(P)$. By (8), $\mathrm{Aut}_{S_1}(P)$ and $\mathrm{Aut}_{S_2}(P)$ are strongly closed in $\mathrm{Aut}_S(P)$ with respect to $\mathrm{Aut}_{\mathcal{F}}(P)$. So $[G_1, G_2] = 1$ by Proposition 2.8, applied to the actions of $G_1, G_2 \le G$ on P, and this finishes the proof of (9).

Step 4: Set $\mathcal{E} = \langle \mathcal{E}_1^\bullet, \mathcal{E}_2^\bullet \rangle \subseteq \mathcal{F}$, as a fusion system over S. Thus $\mathcal{F} = \langle \mathcal{E}, \mathrm{Aut}_{\mathcal{F}}(S) \rangle$ by (6). For each $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ and each $i = 1, 2$, ${}^\alpha\mathcal{E}_i^\bullet = \mathcal{E}_i^\bullet$ since $\alpha(S_i) = S_i$ by (1), and hence ${}^\alpha\mathcal{E} = \mathcal{E}$. This proves that $\mathcal{E}$ is $\mathcal{F}$-invariant. Throughout the remainder of Step 4, we prove that $\mathcal{E}$ is saturated, and hence a weakly normal subsystem of $\mathcal{F}$.

By (2), for $i = 1, 2$, $P \in Q^{\mathcal{E}_i^\bullet}$ if and only if $P_i \in Q_i{}^{\mathcal{E}_i}$ and $P_{3-i} = Q_{3-i}$. Also, $\mathcal{E}$-conjugacy is the equivalence relation generated by $\mathcal{E}_1^\bullet$- and $\mathcal{E}_2^\bullet$-conjugacy. Hence for all $P, Q \in \mathcal{U}$,

$$P \in Q^{\mathcal{E}} \iff P_i \in Q_i{}^{\mathcal{E}_i} \text{ for } i = 1, 2. \tag{10}$$

This in turn implies that

$$P \in \mathcal{U} \text{ and fully normalized in } \mathcal{E} \implies P_i \text{ is fully normalized in } \mathcal{E}_i \text{ for } i = 1, 2. \tag{11}$$

All elements of $\mathcal{U}$ contain their centralizer by definition. Hence $\mathcal{U} \subseteq \mathcal{F}^c$ and $\mathcal{U} \subseteq \mathcal{E}^c$ since $\mathcal{U}$ is a union of $\mathcal{F}$-conjugacy classes by (7). Together with (10), this shows

$$P \in \mathcal{U} \implies P \in \mathcal{E}^c,\ P \in \mathcal{F}^c, \text{ and } P_i \text{ is } \mathcal{E}_i\text{-centric for } i = 1, 2. \tag{12}$$

We next claim that

$$P, Q \in \mathcal{U},\ \varphi \in \mathrm{Iso}_{\mathcal{E}}(P, Q) \implies \exists!\ \varphi_1 \in \mathrm{Iso}_{\mathcal{E}_1^\bullet}(P, Q_1P_2),\ \varphi_2 \in \mathrm{Iso}_{\mathcal{E}_2^\bullet}(Q_1P_2, Q) \text{ such that } \varphi = \varphi_2 \circ \varphi_1\ . \tag{13}$$

To see this, write $\varphi = \psi_m \circ \cdots \circ \psi_1$, where each ψ_i is an isomorphism in $\mathcal{E}_1^\bullet$ or in $\mathcal{E}_2^\bullet$ (recall $\mathcal{E} = \langle \mathcal{E}_1^\bullet, \mathcal{E}_2^\bullet \rangle$). By (9), if for some i, $\psi_{i+1} \in \mathrm{Mor}(\mathcal{E}_1^\bullet)$ and $\psi_i \in \mathrm{Mor}(\mathcal{E}_2^\bullet)$, then $\psi_{i+1} \circ \psi_i = \psi'_{i+1} \circ \psi'_i$ for some pair of isomorphisms ψ'_{i+1} in $\mathcal{E}_2^\bullet$ and ψ'_i in $\mathcal{E}_1^\bullet$. We can thus arrange that the morphisms in $\mathcal{E}_1^\bullet$ all come before morphisms in $\mathcal{E}_2^\bullet$. Hence there exist isomorphisms φ_i in $\mathcal{E}_i^\bullet$ such that $\varphi = \varphi_2 \circ \varphi_1$. If φ'_1 and φ'_2 are another such pair of isomorphisms, then $\psi \overset{\text{def}}{=} (\varphi'_2)^{-1} \circ \varphi_2 = \varphi'_1 \circ \varphi_1^{-1}$ is an automorphism of Q_1P_2 in both $\mathcal{E}_1^\bullet$ and $\mathcal{E}_2^\bullet$. Hence by (2),

ψ sends Q_1 and P_2 to themselves and induces the identity on Q_1P_2/Q_1 and on Q_1P_2/P_2; so $\psi = \mathrm{Id}$. Thus $\varphi_i' = \varphi_i$ for $i = 1, 2$, and the decomposition in (13) is unique.

We are now ready to prove that $\mathcal{E}$ is saturated. For each $i = 1, 2$, $\mathcal{E}_i^\bullet$ is $\mathcal{T}_i^c$-generated by definition, and hence is $\mathcal{U}$-generated since $\mathcal{U} \supseteq \mathcal{T}_i^c$. So $\mathcal{E} = \langle \mathcal{E}_1^\bullet, \mathcal{E}_2^\bullet \rangle$ is $\mathcal{U}$-generated.

Each $P \in \mathcal{U}$ is $\mathcal{E}$-centric by (12), and hence is fully centralized in $\mathcal{E}$. If $P = P_1P_2 \in \mathcal{U}$ is fully normalized in $\mathcal{E}$, then each P_i is fully normalized in $\mathcal{E}_i$ by (11), $\mathrm{Aut}_S(P_i) \in \mathrm{Syl}_2(\mathrm{Aut}_{\mathcal{E}_i}(P_i))$ for $i = 1, 2$, and so $\mathrm{Aut}_S(P) \in \mathrm{Syl}_2(\mathrm{Aut}_{\mathcal{E}}(P))$ since by (13) and (3), $|\mathrm{Aut}_{\mathcal{E}}(P)| = |\mathrm{Aut}_{\mathcal{E}_1}(P_1)|\cdot|\mathrm{Aut}_{\mathcal{E}_2}(P_2)|$. The Sylow axiom thus holds for subgroups in $\mathcal{U}$.

Next fix $P, Q \in \mathcal{U}$ and $\varphi \in \mathrm{Iso}_{\mathcal{E}}(P, Q)$. Let $\varphi = \varphi_2 \circ \varphi_1$ be the decomposition of (13) ($\varphi_i \in \mathrm{Mor}(\mathcal{E}_i^\bullet)$), and set $\chi_i = \varphi_i|_{P_i} \in \mathrm{Hom}_{\mathcal{E}_i}(P_i, Q_i)$. As usual, let $N_\varphi \le N_S(P)$ be the subgroup of all $g \in N_S(P)$ such that $\varphi c_g \varphi^{-1} \le \mathrm{Aut}_S(Q)$, and similarly for $N_{\chi_i} \le N_{S_i}(P_i)$. Set $N_i = N_{\chi_i}$ for short, and $N = N_1N_2$.

Fix $g \in N_\varphi$, and choose $h \in N_S(Q)$ such that $\varphi c_g \varphi^{-1} = c_h$. Write $g = g_1g_2$ and $h = h_1h_2$, where $g_i, h_i \in S_i$. Thus $c_h \circ \varphi = \varphi \circ c_g$, and hence

$$(c_{h_2} \circ \varphi_2) \circ (c_{h_1} \circ \varphi_1) = c_{h_2} \circ c_{h_1} \circ \varphi_2 \circ \varphi_1 = \varphi_2 \circ \varphi_1 \circ c_{g_2} \circ c_{g_1} = (\varphi_2 \circ c_{g_2}) \circ (\varphi_1 \circ c_{g_1}) \ ,$$

where the first and third equalities follow from (9). By the uniqueness in (13), $c_{h_i} \circ \varphi_i = \varphi_i \circ c_{g_i}$ for $i = 1, 2$, so $g_i \in N_i$, and $g \in N$. Thus $N_\varphi \le N$. Since $\mathcal{E}_i$ is saturated and Q_i is $\mathcal{E}_i$-centric by (12) (hence fully centralized), χ_i extends to a morphism $\bar{\chi}_i \in \mathrm{Hom}_{\mathcal{E}_i}(N_i, S_i)$. Thus by (3), φ extends to

$$\bar{\chi}_2{\uparrow}^{S_1N_2} \circ \bar{\chi}_1{\uparrow}^{N} \in \mathrm{Hom}_{\mathcal{E}}(N, S) \ .$$

Since $N_\varphi \le N$, this proves the extension axiom for $\mathcal{E}$ on subgroups in $\mathcal{U}$.

We have now shown that $\mathcal{E}$ is $\mathcal{U}$-saturated and $\mathcal{U}$-generated. Assume $P \le S$ is $\mathcal{E}$-centric but not in $\mathcal{U}$, and set $P_i = \mathrm{pr}_i(P)$. Then P is $\mathcal{F}$-centric by Lemma 1.5 and since $\mathcal{E}$ is $\mathcal{F}$-invariant, so P_1S_2 and S_1P_2 are $\mathcal{F}$-centric, and thus $P_1P_2 \in \mathcal{U}$. Since $P \notin \mathcal{U}$, this implies $P < P_1P_2$, and hence $\mathrm{Out}_S(P) \cap O_2(\mathrm{Out}(P)) \ne 1$ by Lemma 1.11. Thus $\mathcal{E}$ is saturated by Theorem 1.3.

Step 5: Now, $\mathcal{E}$ is weakly normal in $\mathcal{F}$, since it is saturated and $\mathcal{F}$-invariant by Step 4. Also, $\mathcal{E}$ and $\mathcal{F}$ are both fusion systems over S, so $\mathcal{E}$ has odd index in $\mathcal{F}$ by Theorem 1.7(b). Thus $\mathcal{E} = \mathcal{F}$, since $O^{2'}(\mathcal{F}) = \mathcal{F}$.

Since $\mathcal{E} = \langle \mathcal{E}_1^\bullet, \mathcal{E}_2^\bullet \rangle$, $\mathfrak{foc}(\mathcal{E}) = \big\langle \mathfrak{foc}(\mathcal{E}_1^\bullet), \mathfrak{foc}(\mathcal{E}_2^\bullet) \big\rangle$. By (5), $\mathfrak{foc}(\mathcal{E}_i^\bullet) = \mathfrak{foc}(\mathcal{E}_i) \le S_i$. Since $O^2(\mathcal{F}) = \mathcal{F}$, $\mathfrak{foc}(\mathcal{E}) = S$ by Theorem 1.7(a), and thus $\mathfrak{foc}(\mathcal{E}_i^\bullet) = S_i$ for $i = 1, 2$.

Fix $P \in \mathcal{T}_1^c$ and $Q \in \mathcal{T}_2^c$, and set $R = P \cap Q = P_1Q_2 \in \mathcal{U}$. Then $[\mathrm{Aut}_{\mathcal{E}_1^\bullet}(R), \mathrm{Aut}_{\mathcal{E}_2^\bullet}(R)] = 1$ by (9) and (3), and $[\mathrm{Aut}_{\mathcal{E}_i^\bullet}(R), R] \le R \cap \mathfrak{foc}(\mathcal{E}_i^\bullet) = R_i$ $(i = 1, 2)$. So by Lemma 2.9 (applied with $K = R$ and $G_i = \mathrm{Aut}_{\mathcal{E}_i^\bullet}(R)$), $\mathrm{Aut}_{\mathcal{E}_1^\bullet}(R)$ acts trivially on $[\mathrm{Aut}_{\mathcal{E}_2^\bullet}(R), R]$, and in particular on $[\mathrm{Aut}_{\mathcal{E}_2}(Q_2), Q_2]$. For fixed P, the groups $[\mathrm{Aut}_{\mathcal{E}_2}(Q_2), Q_2]$ (for all $Q \in \mathcal{T}_2^c$) generate $\mathfrak{foc}(\mathcal{E}_2) = \mathfrak{foc}(\mathcal{E}_2^\bullet) = S_2$ by (5), and hence $[\mathrm{Aut}_{\mathcal{E}_1^\bullet}(P), S_2] = 1$. Since by (4), $\mathcal{E}_1^\bullet$ is generated by such automorphisms, this proves that S_2 is strongly closed in $\mathcal{E}_1^\bullet$. A similar argument proves that S_1 is strongly closed in $\mathcal{E}_2^\bullet$.

Since each S_i is strongly closed in $\mathcal{E}_i^\bullet \subseteq \mathcal{F}_i^\bullet$ by assumption, this proves that S_1 and S_2 are strongly closed in $\mathcal{F} = \mathcal{E} = \langle \mathcal{E}_1^\bullet, \mathcal{E}_2^\bullet \rangle$. Each morphism in $\mathcal{E}_i^\bullet$ extends to a morphism which is the identity on S_{3-i}. So by definition of the product of fusion systems (and since $\mathcal{E}_i = \mathcal{E}_i^\bullet|_{S_i}$), $\mathcal{F} = \langle \mathcal{E}_1^\bullet, \mathcal{E}_2^\bullet \rangle = \mathcal{E}_1 \times \mathcal{E}_2$. □

4.2. **A first application of Proposition 4.4.**

Recall that when G is a group and $H \le G$ is a subgroup, K is a *normal complement to H in G* if $K \trianglelefteq G$, $K \cap H = 1$, and $KH = G$. Equivalently, K is a normal complement exactly when the inclusions of H and K into G induce an isomorphism $K \rtimes H \xrightarrow{\cong} G$.

Lemma 4.5. *Fix a pair of p-groups S_1 and S_2, and set $S = S_1 \times S_2$. Let $\mathcal{F}$ be a saturated fusion system over S. Set*

$$\mathcal{F}_1^\bullet = \bigl\langle \mathrm{Aut}_{\mathcal{F}}(P) \bigm| S_2 \le P \le S,\ P\ \mathcal{F}\text{-essential or } P = S \bigr\rangle\ ,$$

and assume $\Omega_1(Z(S_2))$ is strongly closed in $\mathcal{F}_1^\bullet$. Then there is a normal complement S_1^0 to $Z(S_2)$ in $S_1Z(S_2)$ which is strongly closed in $\mathcal{F}_1^\bullet$.

Proof. For $i = 1, 2$, set $Z_i = Z(S_i)$ and $\widehat{S}_i = S_iZ_{3-i}$. If $P = S$ or $P \ge S_2$ is $\mathcal{F}$-essential, then $\mathrm{Aut}_{\mathcal{F}}(P)$ sends $\Omega_1(Z_2)$ to itself, and hence $P \cap \widehat{S}_1$ is $\mathrm{Aut}_{\mathcal{F}}(P)$-invariant by Proposition 3.2(a). Since $\mathcal{F}_1^\bullet$ is generated by such automorphisms, $\widehat{S}_1$ is strongly closed in $\mathcal{F}_1^\bullet$.

Let $\mathrm{pr}_2 \in \mathrm{Hom}(S, S_2)$ be the projection. We claim that

$$\varphi \in \mathrm{Hom}_{\mathcal{F}}(S_2, S) \quad\Longrightarrow\quad \varphi(\Omega_1(Z_2)) = \Omega_1(Z_2) \text{ and } \mathrm{pr}_2(\varphi(S_2)) = S_2. \tag{1}$$

By Proposition 1.10(a), each such φ decomposes as a composite $\varphi = \psi_m \circ \cdots \circ \psi_1$, where each ψ_i is the restriction of an $\mathcal{F}$-automorphism of Q_i, and $Q_i = S$ or Q_i is $\mathcal{F}$-essential. Let $j \le m$ be such that $Q_i \ge S_2$ for all $i = 1, \ldots, j$ (hence $Q_i \in \mathcal{T}^c$), and either $j = m$ or $Q_{j+1} \ngeq S_2$. Set $\varphi^* = \psi_j \circ \cdots \circ \psi_1$. For each $i \le j$, $\psi_i(\Omega_1(Z_2)) = \Omega_1(Z_2)$ by assumption. Hence $\varphi^*(\Omega_1(Z_2)) = \Omega_1(Z_2)$, and $\mathrm{pr}_2 \circ \varphi^*$ is injective since the kernel must contain a central element of order p. Thus $\mathrm{pr}_2(\varphi^*(S_2)) = S_2$. If $j < m$, then $\mathrm{pr}_2(Q_{j+1}) \ge \mathrm{pr}_2(\varphi^*(S_2)) = S_2$, and since $Q_{j+1} = R_1R_2$ for some $R_i \le S_i$ by Lemma 1.11(a), this implies $Q_{j+1} \ge S_2$. That contradicts the original choice of j, and thus $j = m$ and $\varphi^* = \varphi$. This proves (1).

Set $\widehat{\mathcal{F}}_1 = \mathcal{F}|_{\widehat{S}_1}$. In other words, $\mathrm{Hom}_{\widehat{\mathcal{F}}_1}(P, Q) = \mathrm{Hom}_{\mathcal{F}}(P, Q)$ for $P, Q \le \widehat{S}_1$. We will show in Step 1 that $\widehat{\mathcal{F}}_1$ is saturated. Then, in Step 2, we construct S_1^0 as the kernel of a certain homomorphism defined using the transfer for $C_{\widehat{\mathcal{F}}_1}(\Omega_1(Z_2))$.

Step 1: We first claim that

$$P \le S \text{ is fully centralized in } \mathcal{F} \text{ and } \mathcal{F}\text{-conjugate to } Q \le \widehat{S}_1 \Longrightarrow P \le \widehat{S}_1. \tag{2}$$

To see this, fix such P and Q, and choose $\varphi \in \mathrm{Iso}_{\mathcal{F}}(Q, P)$. Since $S_2 \le C_S(Q)$, φ extends to some $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(QS_2, S)$, and $P = \varphi(Q) \le C_S(\bar{\varphi}(S_2))$. By (1), $\mathrm{pr}_2(\bar{\varphi}(S_2)) = S_2$, and hence $C_S(\bar{\varphi}(S_2)) \le \widehat{S}_1$. Thus $P \le \widehat{S}_1$.

Fix $P \le \widehat{S}_1$, and choose $Q \in P^{\mathcal{F}}$ which is fully normalized in $\mathcal{F}$. Then $Q \le \widehat{S}_1$ (hence $Q \in P^{\widehat{\mathcal{F}}_1}$) by (2). For each $R \le \widehat{S}_1$, $|N_S(R)| = |N_{\widehat{S}_1}(R)|{\cdot}[S_2{:}Z_2]$ and $|C_S(R)| = |C_{\widehat{S}_1}(R)|{\cdot}[S_2{:}Z_2]$. Thus Q is fully normalized and fully centralized in $\widehat{\mathcal{F}}_1$ since it is fully normalized and fully centralized in $\mathcal{F}$. Hence

$$\begin{aligned} P \le \widehat{S}_1 \text{ is fully normalized in } \widehat{\mathcal{F}}_1 \iff{}& |N_{\widehat{S}_1}(P)| = |N_{\widehat{S}_1}(Q)| \\ \iff{}& |N_S(P)| = |N_S(Q)| \iff P \text{ is fully normalized in } \mathcal{F}\ . \end{aligned}$$

By a similar argument, P is fully centralized in $\widehat{\mathcal{F}}_1$ if and only if it is fully centralized in $\mathcal{F}$. So if P is fully normalized in $\widehat{\mathcal{F}}_1$, then it is fully centralized in $\widehat{\mathcal{F}}_1$ by the Sylow axiom for $\mathcal{F}$. Also, $\mathrm{Aut}_{\widehat{S}_1}(P) = \mathrm{Aut}_S(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\widehat{\mathcal{F}}_1}(P) = \mathrm{Aut}_{\mathcal{F}}(P)$, and this proves the Sylow axiom for $\widehat{\mathcal{F}}_1$.

Assume $\varphi \in \mathrm{Iso}_{\widehat{\mathcal{F}}_1}(P,Q)$, where Q is fully centralized in $\widehat{\mathcal{F}}_1$ and hence in $\mathcal{F}$. Let $N_\varphi \le N_{\widehat{S}_1}(P)$ be the subgroup of those g such that $\varphi c_g \varphi^{-1} \in \mathrm{Aut}_{\widehat{S}_1}(Q)$. Then φ extends to some $\widehat{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi S_2, S)$ by the extension axiom for $\mathcal{F}$, $\mathrm{pr}_2(\widehat{\varphi}(S_2)) = S_2$ by (1), and so $\widehat{\varphi}(N_\varphi) \le C_S(\widehat{\varphi}(S_2)) \le \widehat{S}_1$. Hence $\widehat{\varphi}|_{N_\varphi} \in \mathrm{Hom}_{\widehat{\mathcal{F}}_1}(N_\varphi, \widehat{S}_1)$ since $\widehat{\mathcal{F}}_1 = \mathcal{F}|_{\widehat{S}_1}$, and this proves the extension axiom for $\widehat{\mathcal{F}}_1$. Thus $\widehat{\mathcal{F}}_1$ is saturated.

Step 2: Fix $P,Q \le \widehat{S}_1$ and $\varphi \in \mathrm{Iso}_{\widehat{\mathcal{F}}_1}(P,Q)$. Choose $R \in Q^{\mathcal{F}}$ which is fully centralized in $\mathcal{F}$, and fix $\psi \in \mathrm{Iso}_{\mathcal{F}}(Q,R)$. By the extension axiom for $\mathcal{F}$, there are morphisms $\bar{\psi} \in \mathrm{Hom}_{\mathcal{F}}(QS_2, S)$ and $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(PS_2, S)$ such that $\bar{\psi}|_Q = \psi$ and $\bar{\varphi}|_P = \psi \circ \varphi$. Then $\bar{\psi}(\Omega_1(Z_2)) = \Omega_1(Z_2)$ and $\bar{\varphi}(\Omega_1(Z_2)) = \Omega_1(Z_2)$ by (1), and so $\bar{\psi}^{-1} \circ \bar{\varphi}|_{P\Omega_1(Z_2)} \in \mathrm{Hom}_{\widehat{\mathcal{F}}_1}(P\Omega_1(Z_2), Q\Omega_1(Z_2))$ is an extension of φ which sends $\Omega_1(Z_2)$ to itself. This proves that $\Omega_1(Z_2) \trianglelefteq \widehat{\mathcal{F}}_1$.

Consider the centralizer fusion subsystem $\widehat{\mathcal{E}}_1 = C_{\widehat{\mathcal{F}}_1}(\Omega_1(Z_2))$. This is the fusion system over $\widehat{S}_1 = C_{\widehat{S}_1}(\Omega_1(Z_2))$ where for each $P,Q \le \widehat{S}_1$,

$$\mathrm{Hom}_{\widehat{\mathcal{E}}_1}(P,Q) = \bigl\{\varphi \in \mathrm{Hom}_{\widehat{\mathcal{F}}_1}(P,Q) \bigm| \varphi = \bar{\varphi}|_P \text{ for some } \bar{\varphi} \in \mathrm{Hom}_{\widehat{\mathcal{F}}_1}(P\Omega_1(Z_2), Q\Omega_1(Z_2)) \text{ with } \bar{\varphi}|_{\Omega_1(Z_2)} = \mathrm{Id}\bigr\} .$$

By [BLO, Proposition A.6], this is a saturated fusion system, and by [AOV, Proposition 1.16(c)] (and since $\Omega_1(Z_2) \trianglelefteq \widehat{\mathcal{F}}_1$), it is weakly normal in $\widehat{\mathcal{F}}_1$. It has index prime to p in $\widehat{\mathcal{F}}_1$ since it is a weakly normal fusion subsystem over the same p-group (Theorem 1.7(b)).

Consider the composite homomorphism

$$f\colon \widehat{S}_1 \xrightarrow{\ \mathrm{proj}\ } \widehat{S}_1/\mathfrak{foc}(\widehat{\mathcal{E}}_1) \xrightarrow{\ \mathrm{trf}\ } \widehat{S}_1/[\widehat{S}_1,\widehat{S}_1] \xrightarrow{\ \mathrm{pr}_2\ } Z_2,$$

where trf is the transfer homomorphism of Proposition 1.12, and the first map is the canonical projection. By that proposition, $f(z) = z$ for $z \in \Omega_1(Z_2)$. The actions of $\mathrm{Aut}_{\widehat{\mathcal{F}}_1}(\widehat{S}_1)$ on $\widehat{S}_1/[\widehat{S}_1,\widehat{S}_1]$, $\widehat{S}_1/\mathfrak{foc}(\widehat{\mathcal{E}}_1)$, and Z_2 all factor through the group $\Gamma \overset{\text{def}}{=} \mathrm{Out}_{\widehat{\mathcal{F}}_1}(\widehat{S}_1)$ of order prime to p. Define $\widehat{f} \in \mathrm{Hom}(\widehat{S}_1, Z_2)$ by taking the product over the elements of this group:

$$\widehat{f}(g) = \prod_{[\alpha]\in\Gamma} \alpha(f(\alpha^{-1}(g))) .$$

This is well defined since $f(g)$ depends only on $[g] \in \widehat{S}_1/[\widehat{S}_1,\widehat{S}_1]$. Then $\widehat{f}$ is $\mathrm{Aut}_{\widehat{\mathcal{F}}_1}(\widehat{S}_1)$-linear, and $\widehat{f}(z) = z^{|\Gamma|}$ for $z \in \Omega_1(Z_2)$. Hence $\widehat{f}|_{Z_2}$ is an isomorphism (recall $p \nmid |\Gamma|$).

Set $S_1^0 = \mathrm{Ker}(\widehat{f})$. Then S_1^0 is a normal complement to Z_2 in $\widehat{S}_1$ since $\widehat{f}|_{Z_2}$ is an isomorphism, S_1^0 is $\mathrm{Aut}_{\widehat{\mathcal{F}}_1}(\widehat{S}_1)$-invariant since $\widehat{f}$ is $\mathrm{Aut}_{\widehat{\mathcal{F}}_1}(\widehat{S}_1)$-linear, and it is strongly closed in $\widehat{\mathcal{E}}_1$ since it contains its focal subgroup. Also, S_1^0 is strongly closed in $\widehat{\mathcal{F}}_1$ since $\widehat{\mathcal{F}}_1 = \langle \widehat{\mathcal{E}}_1, \mathrm{Aut}_{\widehat{\mathcal{F}}_1}(\widehat{S}_1)\rangle$ (recall $\widehat{\mathcal{E}}_1$ is weakly normal in $\widehat{\mathcal{F}}_1$.) We already saw that $\widehat{S}_1$ is strongly closed in $\mathcal{F}_1^\bullet$, and thus S_1^0 is strongly closed in $\mathcal{F}_1^\bullet$. □

The following theorem now gives a more explicit set of conditions which imply a splitting of a fusion system.

Theorem 4.6. *Fix a pair of 2-groups S_1 and S_2 such that $\Omega_1(Z(S_1)) \le [S_1,S_1]$, and set $S = S_1 \times S_2$. Let $\mathcal{F}$ be a saturated fusion system over S such that $O^2(\mathcal{F}) = \mathcal{F} = O^{2'}(\mathcal{F})$. Assume, for $i = 1,2$, that whenever $P = S$ or P is an $\mathcal{F}$-essential subgroup which contains S_i, then $\Omega_1(Z(S_i))$ is $\mathrm{Aut}_{\mathcal{F}}(P)$-invariant. Then there are saturated fusion systems $\mathcal{F}_i$ over S_i and $\alpha \in \mathrm{Aut}(S)$ such that ${}^\alpha\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$.*

Proof. Set $Z_i = Z(S_i)$ for short. As in Lemmas 4.3 and 4.5, set

$$\mathcal{F}_i^\bullet = \big\langle \mathrm{Aut}_{\mathcal{F}}(P) \,\big|\, S_{3-i} \le P \le S,\ P\ \mathcal{F}\text{-essential or } P = S \big\rangle .$$

By assumption, $\Omega_1(Z_i)$ is strongly closed in $\mathcal{F}_{3-i}^\bullet$.

Set $\widehat{S}_i = S_i Z_{3-i}$. By Lemma 4.5, for each $i = 1, 2$, there is a normal complement S_i^0 to Z_{3-i} in $\widehat{S}_i$ which is strongly closed in $\mathcal{F}_i^\bullet$. Also, $[S_1^0, S_1^0] = [\widehat{S}_1, \widehat{S}_1] = [S_1, S_1]$, so $\Omega_1(Z_1) \le [S_1, S_1] \le S_1^0$ by assumption.

Now, $S_1^0 \cap S_2^0 \le \widehat{S}_1 \cap \widehat{S}_2 = Z(S)$. Hence if $S_1^0 \cap S_2^0 \ne 1$, there is some $1 \ne z_1 z_2 \in S_1^0 \cap S_2^0$ where $z_i \in \Omega_1(Z_i)$. Since $z_1 \in \Omega_1(Z_1) \le S_1^0$, this implies $z_2 \in S_1^0$, which is impossible since S_1^0 is a normal complement to Z_2 in $\widehat{S}_1$. Thus $S_1^0 \cap S_2^0 = 1$. Hence there is an automorphism $\alpha \in \mathrm{Aut}(S)$ which is the identity modulo $Z(S)$ such that $\alpha(S_i^0) = S_i$.

Since α is the identity modulo $Z(S)$, it sends each $\mathcal{F}$-essential subgroup to itself. Hence ${}^\alpha\mathcal{F}_1^\bullet$ and ${}^\alpha\mathcal{F}_2^\bullet$ are defined in terms of ${}^\alpha\mathcal{F}$ in the same way as the $\mathcal{F}_i^\bullet$ are defined in terms of $\mathcal{F}$. Also, S_i is strongly closed in ${}^\alpha\mathcal{F}_i^\bullet$ for $i = 1, 2$. So by Proposition 4.4, ${}^\alpha\mathcal{F}$ splits as a product of saturated fusion systems over S_1 and S_2. □

The following corollary gives explicit conditions in terms of the 2-groups S_1 and S_2 which imply the existence of a splitting.

Corollary 4.7. *Fix a pair of nontrivial 2-groups S_1 and S_2, and set $S = S_1 \times S_2$. Assume the following conditions hold for each $i = 1, 2$:*

- (i) *S_i is indecomposable and $\Omega_1(Z(S_i)) \le [S_i, S_i]$; and*
- (ii) *S_{3-i} contains no subgroup isomorphic to $S_i \times S_i$.*

Then for every saturated fusion system $\mathcal{F}$ over S such that $O^2(\mathcal{F}) = \mathcal{F} = O^{2'}(\mathcal{F})$, there are saturated fusion systems $\mathcal{F}_i$ over S_i and $\alpha \in \mathrm{Aut}(S)$ such that ${}^\alpha\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$.

Proof. Fix a saturated fusion system $\mathcal{F}$ over S. For $i = 1$ or 2, fix $P \le S$ containing S_i such that $P = S$ or P is $\mathcal{F}$-essential. Thus $P = S_i P_0$ for some $P_0 \le S_{3-i}$, and we must show that $\Omega_1(Z(S_i))$ is $\mathrm{Aut}_{\mathcal{F}}(P)$-invariant. Let $\mathrm{Aut}_{\mathcal{F}}^0(P) \le \mathrm{Aut}_{\mathcal{F}}(P)$ be the subgroup of elements which send $\Omega_1(Z(S_i))$ to itself. Since S_{3-i} contains no subgroup isomorphic to $S_i \times S_i$ by (ii), $\mathrm{Aut}_{\mathcal{F}}^0(P)$ has index at most two in $\mathrm{Aut}_{\mathcal{F}}(P)$ by Proposition 3.2(c). Since $\mathrm{Aut}_{\mathcal{F}}^0(P)$ contains the Sylow 2-subgroup $\mathrm{Aut}_S(P)$ (P is fully normalized since it is $\mathcal{F}$-essential), it is equal to $\mathrm{Aut}_{\mathcal{F}}(P)$. The hypotheses of Theorem 4.6 thus hold, and the result follows. □

4.3. **Dihedral, semidihedral, and wreathed 2-groups.**

We now look at a different set of conditions which imply a splitting of a fusion system. Theorem 4.6 puts some fairly strict conditions on both factors S_1 and S_2. When one of the factors is dihedral or semidihedral, or a wreath product $C_{2^n} \wr C_2$, then the conditions on the other factor can be greatly relaxed.

Lemma 4.8. *Assume $S \cong D_{2^n}$ ($n \ge 3$), SD_{2^n} ($n \ge 4$), or $C_{2^n} \wr C_2$ ($n \ge 2$). Then*

(a) $\mathrm{Aut}(S)$ *is a 2-group, and* $[S, S]$ *is cyclic.*

Define a set $\mathcal{H}$ of subgroups of S as follows:

- *If $S = \langle a, b \rangle \cong D_{2^n}$ ($n \ge 3$), where $|a| = 2^{n-1}$, then $\mathcal{H} = \{T_i \,|\, i \in \mathbb{Z}\}$ where $T_i = \langle a^{2^{n-2}}, a^i b \rangle \cong C_2^2$.*

- *If* $S = \langle a, b\rangle \cong SD_{2^n}$ $(n \geq 4)$*, where* $|a| = 2^{n-1}$ *and* $|b| = 2$*, then* $\mathcal{H} = \{T_{2i}, Q_{2i+1} \,|\, i \in \mathbb{Z}\}$ *where* $T_{2i} = \langle a^{2^{n-2}}, a^{2i}b\rangle \cong C_2^2$ *and* $Q_{2i+1} = \langle a^{2^{n-3}}, a^{2i+1}b\rangle \cong Q_8$*.*
- *If* $S = \langle a, b, t\rangle \cong C_{2^n} \wr C_2$ $(n \geq 2)$*, where* $\langle a, b\rangle \cong C_{2^n} \times C_{2^n}$*,* $|t| = 2$*, and* $tat^{-1} = b$*, then* $\mathcal{H} = \{A, U_{2i} \,|\, i \in \mathbb{Z}\}$*, where* $A = \langle a, b\rangle$ *and* $U_{2i} = \langle ab, a^{2^{n-1}}, a^{2i}t\rangle \cong C_{2^n} \times_{C_2} Q_8$*.*

Then the following hold.

(b) *The set* $\mathcal{H}$ *is the union of exactly two* S*-conjugacy classes.*

(c) *For* $P \in \mathcal{H}$*,*

  (c1) $|N_S(P)/P| = 2$ *and* $[N_S(P), P] = P \cap [S, S]$*;*

  (c2) $N_S(P)$ *is not contained in any subgroup in* $\mathcal{H}$*; and*

  (c3) *either* P *is abelian, or* $[P, P] \cong C_2$ *and* $Z(P) = Z(S)$*.*

(d) *Assume* $\mathcal{F}$ *is a saturated fusion system over* $S \times T$ *for = some 2-group* T*. If* $P < S$ *is such that* PT *is* $\mathcal{F}$*-essential, then* $P \in \mathcal{H}$*.*

(e) *Fix a 2-group* T *and a saturated fusion system* $\mathcal{F}$ *over* $S \times T$*. If* $P \leq S$ *is such that* PT *is* $\mathcal{F}$*-essential, then there is* $\theta_P \in \mathrm{Aut}_{\mathcal{F}}(PT)$ *of order 3 such that*

  (e1) $\mathrm{Out}_{\mathcal{F}}(PT) = \Gamma_P \times H_P$ *where* $\Gamma_P = \langle[\theta_P], \mathrm{Out}_S(PT)\rangle \cong \Sigma_3$ *and* $|H_P|$ *is odd; and*

  (e2) $[N_S(P), P] \leq [\theta_P, PT] \leq PZ(T)$*,* $[\theta_P, T] \leq Z(PT)$*, and* $[\theta_P, PT] \cap Z(T) = 1$*.*

Proof. Points (b) and (c) are easy.

(a) The second statement ($[S, S]$ is cyclic) is easily checked. In all cases (S is dihedral, semidihedral, or wreathed), $S/\mathrm{Fr}(S) \cong C_2^2$. If S is dihedral or semidihedral, let $A \trianglelefteq S$ be the cyclic subgroup of index two; otherwise let A be the unique abelian subgroup of index two. Then A is characteristic in S. Each automorphism of S induces the identity on $S/A \cong C_2$ and on $A/\mathrm{Fr}(S) \cong C_2$, so $\mathrm{Aut}(S)$ is a 2-group by Lemma 2.1.

(d) We prove this case by case. Fix a 2-group T, a saturated fusion system $\mathcal{F}$ over $S \times T$, and a subgroup $P < S$ such that PT is $\mathcal{F}$-essential.

Assume $S = \langle a, b\rangle$ is dihedral or semidihedral, where $|a| = 2^{n-1}$ and $|b| = 2$. If $P \leq S$ is dihedral of order ≥ 8 or quaternion of order ≥ 16, then $Z(P) = Z(S)$, and $\mathrm{Out}(P)$ is a 2-group (Lemma 2.1 or (a)). So PT is not $\mathcal{F}$-essential by Lemma 3.3. If $P \leq S$ is cyclic, then $[N_S(P), P] \leq \mathrm{Fr}(P)$ since it is a proper subgroup, and PT is not $\mathcal{F}$-essential by Proposition 1.10(d). This leaves only the cases where $P \cong C_2^2$ or $P \cong Q_8$, and hence $P \in \mathcal{H}$.

Now assume $S = \langle a, b, t\rangle \cong C_{2^n} \wr C_2$ $(n \geq 2)$, where $A = \langle a, b\rangle \cong C_{2^n} \times C_{2^n}$, $t^2 = 1$, and $tat^{-1} = b$. Set $U_i = \langle ab, a^{2^{n-1}}, a^i t\rangle$ for all $i \in \mathbb{Z}$. Thus $U_i \cong C_{2^n} \times_{C_2} Q_8$ when i is even, U_i contains the cyclic subgroup $\langle a^i t\rangle$ of index two when i is odd, and $aU_i a^{-1} = U_{i+2}$. If $P \nleq A$, and $P \cap A > Z(S) = \langle ab\rangle$, then $Z(P) = Z(S)$, so $\mathrm{Aut}(P)$ is not a 2-group by Lemma 3.3, and $\mathrm{Aut}(P/Z(S))$ is not a 2-group by Lemma 2.1. If $|P/Z(S)| \geq 8$, then $P/Z(S)$ is dihedral, and we just saw that $\mathrm{Aut}(P/Z(S))$ is a 2-group. Thus $|P/Z(S)| = 4$, and $P = U_i$ for some i. When i is odd, $A \cap U_i$ ($\cong C_{2^n} \times C_2$) is characteristic in U_i (the other two subgroups of index two containing $Z(U_i)$ are cyclic), so $\mathrm{Aut}(U_i)$ is a 2-group by Lemma 2.1 again. This leaves the case $P = U_i$ for even i, and thus $P \in \mathcal{H}$.

If $P \nleq A$ and $P \cap A = \langle ab\rangle$, then $N_S(P) = \langle P, a^{2^{n-1}}\rangle$, $[N_S(P), P] = \langle (ab)^{2^{n-1}}\rangle \leq \mathrm{Fr}(P)$, and PT is not $\mathcal{F}$-essential by Proposition 1.10(d). Finally, if $P \leq A$, then $P = A \in \mathcal{H}$ since P is centric in S.

(e) Fix a 2-group T, a saturated fusion system $\mathcal{F}$ over $S \times T$, and a subgroup $P \le S$ such that PT is $\mathcal{F}$-essential. By (d), $P \in \mathcal{H}$.

Since $\mathrm{Out}_{\mathcal{F}}(PT)$ contains a strongly embedded subgroup, $O_2(\mathrm{Out}_{\mathcal{F}}(PT)) = 1$ by Lemma 1.9(c). Since the kernel of the action of $\mathrm{Out}_{\mathcal{F}}(PT)$ on $PT/\mathrm{Fr}(PT)$ is a 2-group by Lemma 2.1, this action must be faithful. Also, $\mathrm{Out}_{ST}(PT) \in \mathrm{Syl}_2(\mathrm{Out}_{\mathcal{F}}(PT))$ since PT is $\mathcal{F}$-essential and hence fully normalized. By (c1,a), $|\mathrm{Out}_{ST}(PT)| = |N_S(P)/P| = 2$ and $[N_S(P), P]$ is cyclic, and hence $[\mathrm{Out}_{ST}(PT), PT/\mathrm{Fr}(PT)]$ is cyclic. So by Proposition 2.4, applied to the action of $\mathrm{Out}_{\mathcal{F}}(PT)$ on $PT/\mathrm{Fr}(PT)$, there is $\theta_P \in \mathrm{Aut}_{\mathcal{F}}(PT)$ such that $\mathrm{Out}_{\mathcal{F}}(PT) = \Gamma_P \times H_P$, where $|H_P|$ is odd, $[\theta_P] \in \Gamma_P$ has order 3, and $\Gamma_P = \langle [\theta_P], \mathrm{Out}_S(PT)\rangle \cong \Sigma_3$. Since $\mathrm{Inn}(PT)$ is a 2-group, we can choose $\theta_P \in \mathrm{Aut}_{\mathcal{F}}(PT)$ to also have order 3. This proves (e1).

Let $\Gamma \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(PT)$ be the normal closure of $\mathrm{Aut}_S(PT)$. The image of Γ in $\mathrm{Out}_{\mathcal{F}}(PT)$ is Γ_P (the normal closure of $\mathrm{Out}_S(PT)$), so Γ has 2-power index in $\langle \theta_P, \mathrm{Aut}_{ST}(PT)\rangle = O^{2'}(\mathrm{Aut}_{\mathcal{F}}(PT))$. Hence $\Gamma \ge O^2\big(O^{2'}(\mathrm{Aut}_{\mathcal{F}}(PT))\big)$, and so $\theta_P \in \Gamma$.

Set $P_0 = [N_S(P), P]$ and $P_\theta = [\theta_P, PT]$ for short. We first show that $P' \overset{\text{def}}{=} [P,P]$ is Γ-invariant. If P is abelian, there is nothing to prove. Otherwise, $|P'| = 2$ and $Z(P) = Z(S)$ by (c3), and hence $P' \le Z(P)$. The group $\mathrm{Aut}_{ST}(PT)$ acts trivially on $Z(PT) = Z(ST)$, so its normal closure Γ also acts trivially. In particular, P' is Γ-invariant.

Thus Γ acts on PT/P', and leaves invariant its center $PZ(T)/P'$. Hence $PZ(T)$ is Γ-invariant, and so Γ acts on $PT/PZ(T) \cong T/Z(T)$. Since $\mathrm{Aut}_S(PT)$ acts trivially on this quotient, so does its normal closure Γ, and hence $[\theta_P, PT] \le [\Gamma, PT] \le PZ(T)$. Also, $Z(P)T = C_{PT}(PZ(T))$ is Γ-invariant, so $[\theta_P, T] \le Z(P)T \cap [\theta_P, PT] \le Z(PT)$.

By Proposition 2.4, now applied to the $\mathrm{Out}_{\mathcal{F}}(PT)$-action on $PZ(T)/P'$, $[\Gamma_P, PZ(T)/P'] = [\theta_P, PZ(T)/P']$, and hence $[\theta_P, PZ(T)/P'] \ge P_0/P'$. Thus $P_0 \le P_\theta{\cdot}P'$. Recall $P_0 = [\mathrm{Aut}_S(PT), PT]$ is cyclic by (a); fix a generator g. Since $P_0 \nleq \mathrm{Fr}(P)$ by Proposition 1.10(d) (and since $P' \le P_0 = [N_S(P), P]$), $P' \le \langle g^2\rangle$. Hence $g = hg^{2k}$ for some k and some $h \in P_\theta$; $h = g^{1-2k}$, and thus $P_0 = \langle h\rangle \le P_\theta$.

Since $P_\theta = [\theta_P, PT] \le PZ(T)$, $P_\theta = [\theta_P, PZ(T)]$ by [G, Theorem 5.3.6]. Hence by Proposition 2.4 again, applied to the action of $\mathrm{Out}_{\mathcal{F}}(PT)$ on $PZ(T)/P'$, $P_\theta/P' = [\theta_P, PZ(T)/P']$ is abelian of rank two. So if $P_\theta \cap Z(T) \ne 1$, then $\Omega_1(P_\theta/P')$ is generated by $\Omega_1(P_0/P')$ and an element of $Z(T)$, both of which are fixed by $\mathrm{Out}_S(PT)$. This is impossible, since $\Gamma_P \cong \Sigma_3$ acts faithfully on $\Omega_1(P_0/P')$, and thus $P_\theta \cap Z(T) = 1$. □

We are now ready to prove Theorem B.

Theorem 4.9. *Fix a pair of 2-groups S_1 and S_2, and set $S = S_1 \times S_2$. Let $\mathcal{F}$ be a saturated fusion system over S such that $O^2(\mathcal{F}) = \mathcal{F} = O^{2'}(\mathcal{F})$. Assume the following conditions hold.*

- (i) *$S_1 \cong D_{2^n}$ ($n \ge 3$), SD_{2^n} ($n \ge 4$), or $C_{2^n} \wr C_2$ ($n \ge 2$).*
- (ii) *Either S_2 contains no proper subgroup isomorphic to $S_1 \times S_1$; or (more generally)*
- (ii′) *if $P = S$ or P is an $\mathcal{F}$-essential subgroup containing S_1, then each element of $\mathrm{Aut}_{\mathcal{F}}(P)$ sends $\Omega_1(Z(S_1))$ to itself.*

Then there are saturated fusion systems $\mathcal{F}_i$ over S_i and $\alpha \in \mathrm{Aut}(S)$ such that ${}^{\alpha}\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$.

Proof. For $i = 1, 2$, set

$$Z_i = Z(S_i), \qquad \widehat{S}_i = S_i Z_{3-i}, \qquad \mathcal{U}_i = \{P \le S_i \mid PS_{3-i} \text{ is } \mathcal{F}\text{-essential}\},$$

and $\mathcal{U}_i^+ = \mathcal{U}_i \cup \{S_i\}$. For each $P \in \mathcal{U}_i^+$, set

$$P^\bullet = PS_{3-i} \qquad \text{and} \qquad \widehat{P} = PZ_{3-i}\,.$$

By Lemma 1.11, the only $\mathcal{F}$-essential subgroups of S are the subgroups $P^\bullet$ for $P \in \mathcal{U}_1 \cup \mathcal{U}_2$. As in Proposition 4.4, define fusion subsystems $\mathcal{F}_1^\bullet$ and $\mathcal{F}_2^\bullet$ over S:

$$\mathcal{F}_i^\bullet = \bigl\langle \mathrm{Aut}_{\mathcal{F}}(P^\bullet) \bigm| P \in \mathcal{U}_i^+ \bigr\rangle \ .$$

We check that (ii) implies (ii′). Assume S_2 contains no subgroup isomorphic to $S_1 \times S_1$. Fix a fully normalized subgroup $P \le S$ containing S_1; thus $P = S_1 \times P_2$ for some $P_2 \le S_2$. Let $\mathrm{Aut}^0_{\mathcal{F}}(P)$ be the subgroup of those automorphisms in $\mathrm{Aut}_{\mathcal{F}}(P)$ which send $\Omega_1(Z_1)$ to itself. Then $\mathrm{Aut}_S(P) \le \mathrm{Aut}^0_{\mathcal{F}}(P)$, so $[\mathrm{Aut}_{\mathcal{F}}(P){:}\mathrm{Aut}^0_{\mathcal{F}}(P)]$ is odd by the Sylow axiom, and is ≤ 2 by Proposition 3.2(c). Thus $\mathrm{Aut}_{\mathcal{F}}(P) = \mathrm{Aut}^0_{\mathcal{F}}(P)$, and (ii′) holds. So from now on, we assume (ii′).

Step 1: By (ii′), $\Omega_1(Z_1)$ is strongly closed in $\mathcal{F}_2^\bullet$. The hypotheses of Lemma 4.5 thus hold, but with the roles of S_1 and S_2 switched. So by that lemma, there is a normal complement S_2^0 to Z_1 in $\widehat{S}_2$ which is strongly closed in $\mathcal{F}_2^\bullet$. Choose $\alpha \in \mathrm{Aut}(S)$ such that $\alpha|_{S_1} = \mathrm{Id}$, $\alpha \equiv \mathrm{Id}_S \pmod{Z_1}$, and $\alpha(S_2^0) = S_2$. Since α is the identity modulo $Z(S)$, it sends each subgroup in $\mathcal{U}_1 \cup \mathcal{U}_2$ to itself, and hence $({}^\alpha\mathcal{F})_i^\bullet = {}^\alpha(\mathcal{F}_i^\bullet)$ for $i = 1, 2$. So upon replacing $\mathcal{F}$ by ${}^\alpha\mathcal{F}$, we can assume $S_2^0 = S_2$; i.e., S_2 is strongly closed in $\mathcal{F}_2^\bullet$.

Thus for $P \in \mathcal{U}_2^+$, $\mathrm{Aut}_{\mathcal{F}}(P^\bullet)$ leaves P invariant, and acts on $P^\bullet/P \cong S_1$ via inner automorphisms since $\mathrm{Aut}(S_1)$ is a 2-group by Lemma 4.8(a). Hence $[\mathrm{Aut}_{\mathcal{F}}(P^\bullet), P^\bullet] \le [S_1, S_1]{\cdot}P$ for each $P \in \mathcal{U}_2^+$, and

$$\mathfrak{foc}(\mathcal{F}_2^\bullet) = \bigl\langle [\mathrm{Aut}_{\mathcal{F}}(P^\bullet), P^\bullet] \bigm| P \in \mathcal{U}_2^+ \bigr\rangle \le [S_1, S_1]{\cdot}S_2 \ . \tag{1}$$

Step 2: By the Sylow axiom, $\mathrm{Aut}_{\mathcal{F}}(S)$ is an extension of the 2-group $\mathrm{Inn}(S)$ by the odd order group $\mathrm{Out}_{\mathcal{F}}(S)$. So by the Schur-Zassenhaus theorem (cf. [G, Theorem 6.2.1]), there is $H \le \mathrm{Aut}_{\mathcal{F}}(S)$ of odd order, unique up to conjugacy by an element of $\mathrm{Inn}(S)$, such that $\mathrm{Aut}_{\mathcal{F}}(S) = H{\cdot}\mathrm{Inn}(S)$.

By Step 1, S_2 is $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant. Since $\mathrm{Aut}(S/S_2)$ is a 2-group by Lemma 4.8(a), $[H, S] \le S_2$. Also, $\widehat{S}_1 = C_S(S_2)$ is $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant, and hence $[H, S_1] \le \widehat{S}_1 \cap S_2 = Z_2$. Since $[\widehat{S}_1, \widehat{S}_1] = [S_1, S_1]$ is H-invariant, H acts trivially on this subgroup. We have now shown that

$$[H, S] \le S_2 \ , \quad [H, S_1] \le Z_2 \ , \quad \text{and} \quad [H, [S_1, S_1]] = 1 \ . \tag{2}$$

Step 3: Throughout this step, we fix a subgroup $P \in \mathcal{U}_1$. By Lemma 4.8(d), $P \in \mathcal{H}$, where $\mathcal{H}$ is an explicitly defined set of subgroups of S_1. By Lemma 4.8(e1),

$$\mathrm{Out}_{\mathcal{F}}(P^\bullet) = \Gamma_P \times H_P \text{ where } |H_P| \text{ odd and } \Gamma_P = \langle [\theta_P], \mathrm{Out}_{S_1}(P^\bullet) \rangle \cong \Sigma_3 \tag{3}$$

for some $\theta_P \in \mathrm{Aut}_{\mathcal{F}}(P^\bullet)$ of order three; and by Lemma 4.8(e2),

$$[N_{S_1}(P), P] \le [\theta_P, P^\bullet] \le PZ_2 = \widehat{P} \qquad \text{and} \qquad [\theta_P, P^\bullet] \cap Z_2 = 1 \ . \tag{4}$$

For each $\beta \in \mathrm{Aut}_{\mathcal{F}}(P^\bullet)$ of odd order whose class $[\beta] \in \mathrm{Out}_{\mathcal{F}}(P^\bullet)$ is in H_P, $[\beta]$ commutes with $\mathrm{Out}_S(P^\bullet) \le \Gamma_P$, so β normalizes $\mathrm{Aut}_S(P^\bullet)$, and extends to $\widehat{\beta} \in \mathrm{Aut}_{\mathcal{F}}(N_S(P^\bullet))$ by the extension axiom. Since neither $N_S(P^\bullet)$ nor any of its conjugates is contained in an $\mathcal{F}$-essential subgroup (Lemma 4.8(c2)), $\widehat{\beta}$ extends to $\bar{\beta} \in \mathrm{Aut}_{\mathcal{F}}(S)$ by Proposition 1.10(a). Upon replacing $\bar{\beta}$ by an appropriate power, we can assume it also has odd order. Then $\bar{\beta}$ is $\mathrm{Inn}(S)$-conjugate to an element of H, and since $[\mathrm{Aut}_{S_1}(S), H] = 1$ (since $[H, S_1] \le Z_2$ by (2)), $\bar{\beta}$ is $\mathrm{Aut}_{S_2}(S)$-conjugate to an element of H. Thus β is $\mathrm{Inn}(P^\bullet)$-conjugate to the restriction of an element of H. Conversely, every element of H restricts to an automorphism of $P^\bullet$ since $[H, S] \le S_2 \le P^\bullet$ by (2). So

$$H_P = \bigl\{ [\eta|_{P^\bullet}] \bigm| \eta \in H \bigr\} \ . \tag{5}$$

By Proposition 2.4, applied to the action of $\mathrm{Out}_{\mathcal{F}}(P^\bullet)$ on $\widehat{P}/[\widehat{P},\widehat{P}]$, H_P acts trivially on $[\theta_P,\widehat{P}/[\widehat{P},\widehat{P}]]$. Hence H acts trivially on this group by (5). Also, H acts trivially on $[\widehat{P},\widehat{P}]\le[S_1,S_1]$ by (2), so it acts trivially on $[\theta_P,\widehat{P}]$ by Lemma 2.1. Since $[\theta_P,P^\bullet]=[\theta_P,[\theta_P,P^\bullet]]=[\theta_P,\widehat{P}]$ by [G, Theorem 5.3.6] (and (4)), this proves that

$$[H,[\theta_P,P^\bullet]]=[H,[\theta_P,\widehat{P}]]=1\ . \tag{6}$$

Step 4: For $P\in\mathcal{U}_1$, set $P_0=[N_{S_1}(P),P]$ and $P_\theta=[\theta_P,P^\bullet]$. We next claim that

for each $P\in\mathcal{U}_1$, there are $x_P\in P$ and $z_P\in Z_2$ such that $P_\theta=\langle P_0,x_Pz_P\rangle$; and such that either P is abelian and $\langle P_0,x_P\rangle=P$, or $\langle P_0,x_P\rangle\cong Q_8$. (7)

When S_1 is dihedral or semidihedral, then $P\cong C_2^2$ or $P\cong Q_8$, and P_0 has index two in P. By (4), P_θ is contained in PZ_2 and has trivial intersection with Z_2 (and strictly contains the cyclic group P_0), so it must have the form in (7). Similarly, when S_1 is wreathed and $P\cong C_{2^n}\times C_{2^n}$, then $P_0\cong C_{2^n}$, P_θ is isomorphic to a subgroup of P by (4) and hence to P since it has an automorphism of order three, and again has the form described in (7).

By Lemma 4.8(d), it remains to consider the case where S_1 is wreathed, and where (in the notation of the lemma) $P=U_{2i}=\langle ab,a^{2^{n-1}},a^{2i}t\rangle\cong C_{2^n}\times_{C_2}Q_8$ for some i. Then P contains a unique subgroup $P_1\cong Q_8$, generated by all elements of order four in $P\smallsetminus Z(P)$, and P_0 has index two in P_1. Set $g=(ab^{-1})^{2^{n-2}}$: a generator of P_0. Since $g\in P_\theta$, $g{\cdot}\theta_P(g){\cdot}\theta_P^2(g)\in[\widehat{P},\widehat{P}]=\langle g^2\rangle$, so $P_\theta=\langle g,\theta_P(g)\rangle$. Set $\theta_P(g)=x_Pz_P$ where $x_P\in P$ and $z_P\in Z_2$. Since g has order 4 and lies in $\widehat{P}\smallsetminus Z(\widehat{P})$, so does $\theta_P(g)$, and hence $x_P\in P\smallsetminus Z(P)$ has order 4 since otherwise $\theta_P(g^2)\in Z_2$ (contradicting (4)). Thus $P_\theta=\langle P_0,x_Pz_P\rangle$ and $\langle P_0,x_P\rangle=P_1\cong Q_8$.

Step 5: Now, $\mathcal{F}=\langle\mathcal{F}_1^\bullet,\mathcal{F}_2^\bullet\rangle$ since by Lemma 1.11(b), each $\mathcal{F}$-essential subgroup has the form $P^\bullet$ for $P\in\mathcal{U}_1\cup\mathcal{U}_2$. So $\mathcal{F}=\langle\mathcal{F}_2^\bullet,\theta_P\,|\,P\in\mathcal{U}_1\rangle$ by (3) and (5). Hence by (1) and (7),

$$\mathfrak{foc}(\mathcal{F})=\big\langle\mathfrak{foc}(\mathcal{F}_2^\bullet),[\theta_P,P^\bullet]\,\big|\,P\in\mathcal{U}_1\big\rangle\le\big\langle[S_1,S_1],S_2,x_Pz_P\,\big|\,P\in\mathcal{U}_1\big\rangle\ .$$

Since $O^2(\mathcal{F})=\mathcal{F}$ by assumption, $\mathfrak{foc}(\mathcal{F})=S$ by Theorem 1.7(a), and so $S_1/[S_1,S_1]$ is generated by the classes $[x_P]$ for $P\in\mathcal{U}_1$. If $P,Q\in\mathcal{U}_1$ are S_1-conjugate, then there is $g\in P$ such that $Q=gPg^{-1}$ and $\theta_Q^{\pm1}=c_g\theta_Pc_g^{-1}$, and hence $Q_\theta=gP_\theta g^{-1}$. Thus $\langle[S_1,S_1],x_P\rangle=\langle[S_1,S_1],x_Q\rangle$. Since $S_1/[S_1,S_1]$ is not cyclic, we conclude that $\mathcal{U}_1$ contains at least two S_1-conjugacy classes. So by Lemma 4.8(b,d), $\mathcal{U}_1=\mathcal{H}$ and contains exactly two conjugacy classes.

Fix representatives P_1,P_2 for the S_1-conjugacy classes in $\mathcal{U}_1=\mathcal{H}$, and set $x_i=x_{P_i}$ and $z_i=z_{P_i}$ for short. Set

$$S_1^0=\big\langle[S_1,S_1],P_\theta\,\big|\,P\in\mathcal{U}_1\big\rangle=\big\langle[S_1,S_1],x_1z_1,x_2z_2\big\rangle\ .$$

Thus $S_1^0Z_2=\langle[S_1,S_1],x_1,x_2\rangle Z_2=S_1Z_2=\widehat{S}_1$. Since least one of the x_i has order two in $S_1/[S_1,S_1]$ by the proof of (7), $S_1/[S_1,S_1]=\langle[x_1]\rangle\times\langle[x_2]\rangle$. Also, $|z_i|$ is at most the order of x_i in $P_{i\theta}/P_{i0}$ (otherwise $P_{i\theta}\cap Z_2\ne1$). Hence $S_1^0\cap Z_2=1$, and so S_1^0 is a normal complement to Z_2 in $\widehat{S}_1$.

Now, $[H,S_1^0]=1$: $[S_1,S_1]\le C_S(H)$ by (2), and $[H,P_\theta]=1$ by (6). Thus S_1^0 is $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant. For $P\in\mathcal{U}_1$, $S_1^0\cap P^\bullet$ is θ_P-invariant since it contains P_θ, it is $\mathrm{Aut}_S(P^\bullet)$-invariant since $[N_{S_1}(P^\bullet),P^\bullet]\le[S_1,S_1]\le S_1^0$ and $[S_2,S_1^0]=1$, and hence it is $\mathrm{Aut}_{\mathcal{F}}(P^\bullet)$-invariant by (3) and (5). This proves that S_1^0 is strongly closed in $\mathcal{F}_1^\bullet$.

Let $\beta\in\mathrm{Aut}(S)$ be the (unique) automorphism such that $\beta|_{S_2}=\mathrm{Id}$, $\beta(S_1^0)=S_1$, and $\beta(g)\equiv g\pmod{Z_2}$ for all $g\in S_1$. By the same reasoning as that used in Step 1, upon replacing $\mathcal{F}$ by ${}^\beta\mathcal{F}$, we can assume $S_1^0=S_1$. So $\mathcal{F}$ satisfies the hypotheses of Proposition 4.4, and splits as a product of saturated fusion systems over the S_i. □

There is one more important case which we want to include, that of a product of three or more 2-groups which are dihedral, semidihedral, or wreathed, and pairwise isomorphic. In fact, this result holds more generally, without assuming the factors are isomorphic.

Theorem 4.10. *Assume $S = S_1 \times \cdots \times S_m$, where each S_i ($1 \le i \le m$) is isomorphic to D_{2^n} ($n \ge 3$), SD_{2^n} ($n \ge 4$), or $C_{2^n} \wr C_2$ ($n \ge 2$). Let $\mathcal{F}$ be a saturated fusion system over S such that $O^2(\mathcal{F}) = \mathcal{F} = O^{2'}(\mathcal{F})$. Then there are saturated fusion systems $\mathcal{F}_i$ over S_i such that ${}^{\alpha}\mathcal{F} = \mathcal{F}_1 \times \cdots \times \mathcal{F}_m$ for some $\alpha \in \mathrm{Aut}(S)$.*

Proof. For each i, let $z_i \in Z(S_i)$ be the element of order two, and set

$$\mathcal{U}_i^\bullet = \bigl\{ P \le S \bigm| P \text{ is } \mathcal{F}\text{-essential}, \quad P \ge S_j \text{ for all } j \ne i \bigr\} .$$

By Lemma 1.11(b), $\bigcup_{i=1}^m \mathcal{U}_i^\bullet$ contains all $\mathcal{F}$-essential subgroups of S.

Fix i and $P \in \mathcal{U}_i^\bullet$. By Lemma 4.8(c1,a), $\mathrm{Out}_S(P) \cong C_2$, and $[N_{S_i}(P), P] \le [S_i, S_i]$ is cyclic. By Lemma 4.8(e), there is an automorphism $\theta_P \in \mathrm{Aut}_{\mathcal{F}}(P)$ of order three, and a factorization $\mathrm{Out}_{\mathcal{F}}(P) = \Gamma_P \times H_P$, such that $\Gamma_P = \langle [\theta_P], \mathrm{Out}_S(P) \rangle \cong \Sigma_3$ and H_P has odd order, and such that

$$z_i \in [N_{S_i}(P), P] \le [\theta_P, P] \le P \cap S_i Z(S) \qquad \text{and} \qquad [\theta_P, S_j] \le Z(P) \text{ for } j \ne i. \tag{1}$$

Also, $[\theta_P, P] \cap \langle z_k \,|\, k \ne i \rangle = 1$ by Lemma 4.8(e) again; and since $z_i \in [\theta_P, P]$,

$$[\theta_P, P] \cap \Omega_1(Z(S)) = [\theta_P, P] \cap \langle z_k \,|\, 1 \le k \le m \rangle = \langle z_i \rangle. \tag{2}$$

For each $\eta \in \mathrm{Aut}_{\mathcal{F}}(P)$ such that $[\eta] \in H_P$, η normalizes $\mathrm{Aut}_S(P)$, hence extends to an $\mathcal{F}$-automorphism of $N_S(P)$ by the extension axiom, and to some $\bar\eta \in \mathrm{Aut}_{\mathcal{F}}(S)$ by Proposition 1.10(a) since no $\mathcal{F}$-essential subgroup contains $N_S(P)$ (Lemma 4.8(c2,d)). Thus

$$P \in \mathcal{U}_i^\bullet \quad \Longrightarrow \quad \mathrm{Aut}_{\mathcal{F}}(P) = \bigl\langle \theta_P, \eta|_P \bigm| \eta \in \mathrm{Aut}_{\mathcal{F}}(S),\ \eta(P) = P \bigr\rangle . \tag{3}$$

Let $I(S)$ be the set of involutions in S. For each $i = 1, \ldots, m$, let $X_i \subseteq I(S)$ be the smallest subset which contains z_i, is invariant under $\mathrm{Inn}(S)$, and is such that for each $P \in \mathcal{U}_i^\bullet$, θ_P sends $X_i \cap P$ to itself. These conditions imply that each element of $O^{2'}(\mathrm{Aut}_{\mathcal{F}}(P)) = \langle \mathrm{Aut}_S(P), \theta_P \rangle$ (for each $P \in \mathcal{U}_i^\bullet$) sends $P \cap X_i$ to itself, so this is independent of the choice of the θ_P. We claim the following hold for all $i \ne j$ in $\{1, \ldots, m\}$:

(4) $\theta_P|_{X_j} = \mathrm{Id}_{X_j}$ for all $P \in \mathcal{U}_i^\bullet$, and

(5) $X_i \cap X_j = \varnothing$.

Assume (4) and (5) hold; we now finish the proof of the theorem. By (3) (and Proposition 1.10(a)), $\mathcal{F}$ is generated by $\mathrm{Aut}_{\mathcal{F}}(S)$ and the θ_P for $P \in \bigcup_{i=1}^m \mathcal{U}_i^\bullet$. Each $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ permutes the subgroups $S_i Z(S)$ by Proposition 3.1, and hence permutes the sets $\mathcal{U}_i^\bullet$, since $\mathcal{U}_i^\bullet$ contains exactly those $\mathcal{F}$-essential subgroups which do not contain $S_i Z(S)$. Hence each $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ permutes the subsets X_i. For $1 \le i \le m$ and $P \in \mathcal{U}_i^\bullet$, $\theta_P(X_j) = X_j$ for all $j \ne i$ by (4), and $\theta_P(P \cap X_i) = P \cap X_i$ by definition of X_i. Hence for each $P, Q \le S$ and each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$, there is $\sigma \in \Sigma_m$ such that $\varphi(P \cap X_i) \subseteq Q \cap X_{\sigma(i)}$ for all $i = 1, \ldots, m$.

If $P \in \mathcal{F}^c$, then $Z(S) \le P$. So for each i, $z_i \in P \cap X_i$ implies $P \cap X_i \ne \varnothing$. Since the X_i are pairwise disjoint by (5), the permutation σ determined by any given $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ is unique. So there is a uniquely defined map

$$\Psi \colon \mathrm{Mor}(\mathcal{F}^c) \longrightarrow \Sigma_m$$

which preserves composition of morphisms and of permutations and sends inclusions to the identity.

Let $\mathcal{F}_0 \subseteq \mathcal{F}$ be the "kernel" subsystem: the fusion system over S generated by those morphisms $\varphi \in \mathrm{Mor}(\mathcal{F}^c)$ such that $\Psi(\varphi) = 1$. Since $\mathcal{F}$ is generated by $\mathrm{Aut}_{\mathcal{F}}(S)$ and the θ_P, and $\Psi(\theta_P) = 1$ for each $P \in \mathcal{U}_i^\bullet$ (for each i), $\mathrm{Im}(\Psi) = \Psi(\mathrm{Aut}_{\mathcal{F}}(S))$. Thus $\mathcal{F}_0$ is $\mathcal{F}$-invariant, and a subgroup of S is $\mathcal{F}_0$-centric if and only if it is $\mathcal{F}$-centric by Lemma 1.5(b). Since $\Psi(\mathrm{Inn}(S)) = 1$, $\mathrm{Im}(\Psi)$ is a subgroup of odd order. The Sylow and extension axioms on centric subgroups hold for $\mathcal{F}_0$ since they hold for $\mathcal{F}$. Since by definition, $\mathcal{F}_0$ is generated by morphisms between centric subgroups, $\mathcal{F}_0$ is saturated by Theorem 1.3. Thus $\mathcal{F}_0$ is a weakly normal fusion subsystem of odd index in $\mathcal{F}$, and $\mathcal{F}_0 = \mathcal{F}$ since $O^{2'}(\mathcal{F}) = \mathcal{F}$ by assumption.

By Proposition 3.1, each element of $\mathrm{Aut}(S)$ permutes the cyclic subgroups $[S_i, S_i]$, and hence permutes the involutions z_i ($i = 1, \dots, m$). Since each element of $\mathrm{Aut}_{\mathcal{F}}(S)$ sends X_1 to itself, $\mathrm{Aut}_{\mathcal{F}}(S)$ fixes z_1. For all $P \in \mathcal{U}_i^\bullet$ for $i \neq 1$, $\theta_P(z_1) = z_1$ by (4). Assumption (ii′) in Theorem 4.9 thus holds, so there are saturated fusion systems $\mathcal{F}_1$ over S_1 and $\widehat{\mathcal{F}}$ over $S_2 \cdots S_m$ such that ${}^{\beta}\mathcal{F} = \mathcal{F}_1 \times \widehat{\mathcal{F}}$ for some $\beta \in \mathrm{Aut}(S)$. The theorem now follows by induction on m.

It remains to prove (4) and (5).

Proof of (4): We first claim, for each $i = 1, \dots, m$, that

$$X_i \subseteq T_i \overset{\text{def}}{=} \bigl\langle [S_i, S_i], [\theta_P, P] \bigm| P \in \mathcal{U}_i^\bullet \bigr\rangle \le S_i Z(S) . \tag{6}$$

By (1), $T_i \le S_i Z(S)$. Hence T_i is $\mathrm{Aut}_{S_j}(S)$-invariant for each $j \neq i$, and is $\mathrm{Aut}_{S_i}(S)$-invariant since it contains $[S_i, S_i]$. Also, for each $P \in \mathcal{U}_i^\bullet$, $T_i \cap P$ is θ_P-invariant since $[\theta_P, P] \le T_i$, and thus $T_i \supseteq X_i$ since $z_i \in T_i$. This proves (6).

Fix subgroups $P_i \in \mathcal{U}_i^\bullet$ and $P_j \in \mathcal{U}_j^\bullet$ for some $i \neq j$, and set

$$R = P_i \cap P_j \cap S_i S_j Z(S) = \bigl\{(x_1, \dots, x_m) \bigm| x_i \in P_i,\ x_j \in P_j,\ x_k \in Z(S_k)\ \forall\, k \neq i, j\bigr\}.$$

Set $G = \mathrm{Aut}_{\mathcal{F}}(R)$, and consider the subgroups

$$\begin{aligned} G_i &= \langle \theta_{P_i}|_R, \mathrm{Aut}_{S_i}(R)\rangle & H_i &= \langle \mathrm{Aut}_{S_i}(R)\rangle & Q_i &= \langle \mathrm{Aut}_{S_i \cap R}(R)\rangle \\ G_j &= \langle \theta_{P_j}|_R, \mathrm{Aut}_{S_j}(R)\rangle & H_j &= \langle \mathrm{Aut}_{S_j}(R)\rangle & Q_j &= \langle \mathrm{Aut}_{S_j \cap R}(R)\rangle . \end{aligned}$$

Note that for $k \in \{i, j\}$, $\theta_{P_k}|_R \in \mathrm{Aut}(R)$ since $[\theta_{P_k}, P_k] \le P_k \cap (S_k Z(S)) \le R$ by (1). Also, $G_k/Q_k = \langle [\theta_{P_k}|_R], \mathrm{Out}_{S_k}(R)\rangle \cong \Sigma_3$ by what was shown above, and thus $Q_k = O_2(G_k)$ and $H_k \in \mathrm{Syl}_2(G_k)$.

By (1), for $k = i$ and $k = j$,

$$[G_k, R] = [\theta_{P_k}, R]{\cdot}[N_{S_k}(P_k), R] \le [\theta_{P_k}, P_k] \le S_k Z(S) .$$

Hence $[G_i, R] \cap [G_j, R] \le S_i Z(S) \cap S_j Z(S) = Z(S)$. If $[G_i, R] \cap [G_j, R] \neq 1$, then

$$\begin{aligned} 1 &\neq [G_i, R] \cap [G_j, R] \cap \Omega_1(Z(S)) \\ &\le \bigl([\theta_{P_i}, P_i] \cap \Omega_1(Z(S))\bigr) \cap \bigl([\theta_{P_j}, P_j] \cap \Omega_1(Z(S))\bigr) = \langle z_i\rangle \cap \langle z_j\rangle = 1 \end{aligned}$$

by (2). Thus $[G_i, R] \cap [G_j, R] = 1$.

For $x_i \in N_{S_i}(R) \smallsetminus Z(R)$ and $x_j \in N_{S_j}(R) \smallsetminus Z(R)$, $[x_i, R]$ and $[x_j, R]$ are cyclic and nontrivial since $[N_{S_i}(P_i), P_i]$ and $[N_{S_j}(P_j), P_j]$ are cyclic by Lemma 4.8(c,a); and hence $[x_i x_j, R] = [x_i, R]{\cdot}[x_j, R]$ is noncyclic. So no element in $H_i \cup H_j$ can be G-conjugate to any element of $H_i H_j \smallsetminus (H_i \cup H_j)$. In particular, for $1 \neq h_i \in H_i$ and $1 \neq h_j \in H_j$, there is no $g \in G$ such that $g h_i g^{-1}, g h_j g^{-1} \in H_k$ for $k = i$ or for $k = j$. Thus if $h \in H_i$ and $g \in G$ are such that $g h g^{-1} \in H_j$, then since $Q_i Q_j = \mathrm{Inn}(R) \trianglelefteq G$, $g Q_i g^{-1} \le Q_j$ and $g Q_j g^{-1} \le Q_i$. Since $H_i H_j = \mathrm{Aut}_S(R) \in \mathrm{Syl}_2(G)$, this is possible only if $Q_i = Q_j = 1$; in which case $H_i H_j \cong C_2^2$, c_g exchanges H_i and H_j, and this is again impossible by the Sylow axiom. We conclude that H_i is strongly closed in $H_i H_j$ with respect to G, and similarly for H_j.

Now, $[\theta_{P_i}|_R, \mathrm{Aut}_{S_j}(R)] = 1$ since $[\theta_{P_i}, S_j] \le Z(P_i)$ by (1), and $[\theta_{P_j}|_R, \mathrm{Aut}_{S_i}(R)] = 1$ since $[\theta_{P_j}, S_i] \le Z(P_j)$. So $[G_i, H_j] = 1 = [G_j, H_i]$. The hypotheses of Proposition 2.8 are thus satisfied, and hence $[G_i, G_j] = 1$. So $[G_i, [G_j, R]] = 1$ by Lemma 2.9.

Now, $[\theta_{P_j}, P_j] \le P_j \cap S_j Z(S) \le R$ by (1), so $[\theta_{P_j}, R] \ge [\theta_{P_j}, [\theta_{P_j}, P_j]] = [\theta_{P_j}, P_j]$ (cf. [G, Theorem 5.3.6]). Thus $[\theta_{P_i}, [\theta_{P_j}, P_j]] = 1$. Also, $[\theta_{P_i}, [S_j, S_j]] = 1$ since by (1), $\theta_{P_i}(x) \in xZ(P_i)$ for each $x \in S_j$. So $\theta_{P_i}|_{X_j} = \mathrm{Id}_{X_j}$ by (6).

Proof of (5): By definition, for each i, X_i is the equivalence class of z_i under the equivalence relation $\underset{i}{\approx}$ generated by setting $g \underset{i}{\approx} h$ if g, h are S-conjugate, or $g, h \in P$ and $g = \theta_P(h)$ for some $P \in \mathcal{U}_i^\bullet$.

Fix $i \ne j$, and assume $x \in X_i \cap X_j$. By (4), $\theta_P(x) = x$ for all $P \in \mathcal{U}_i^\bullet$ since $x \in X_j$, and $\theta_P(x) = x$ for all $P \in \mathcal{U}_j^\bullet$ since $x \in X_i$. Since $x \in Z(S)$, the $\underset{i}{\approx}$- and $\underset{j}{\approx}$-equivalence classes of x each contain only x. Since $z_i \underset{i}{\approx} x \underset{j}{\approx} z_j$, this implies $z_i = x = z_j$, which is impossible. We conclude that $X_i \cap X_j = \varnothing$. □

4.4. **Examples.**

It is easy to see why the condition $O^{2'}(\mathcal{F}) = \mathcal{F}$ is needed in Proposition 4.4 (and it was also needed in the splitting result [AOV, Proposition 3.3]). Let $\mathcal{F}$ be any saturated fusion system over a 2-group S, such that $O^{2'}(\mathcal{F}) \subsetneqq \mathcal{F}$. Then there are fusion subsystems of $\mathcal{F} \times \mathcal{F}$ (over $S \times S$) which contain $O^{2'}(\mathcal{F}) \times O^{2'}(\mathcal{F})$, and which do not split as products of fusion systems over S.

For a more explicit example which satisfies all of the hypotheses of Proposition 4.4 and Theorems 4.6 and 4.9, except the condition $O^{2'}(\mathcal{F}) = \mathcal{F}$, let $\mathcal{F}$ be the 2-fusion system of a subgroup $G \le PGL_3(4) \times PGL_3(4)$ of index 3 which does not contain either factor.

It is a little less obvious why the condition $O^2(\mathcal{F}) = \mathcal{F}$ is needed. As a first example, let $\mathcal{F}$ be the 2-fusion system of the symmetric group Σ_6. This is a fusion system over $S \cong D_8 \times C_2$, and satisfies all of the hypotheses of Proposition 4.4 and of Theorem 4.9 except the condition $O^2(\mathcal{F}) = \mathcal{F}$. (Note that all $\mathcal{F}$-essential subgroups contain the second factor C_2.)

For an example which satisfies all of the other hypotheses of Theorem 4.6, let $G \le \Sigma_6 \times PGL_2(9)$ be the subgroup of index two which contains neither factor. The Sylow 2-subgroups of G are isomorphic to $D_8 \times D_{16}$, but the fusion system of G does not split as a product of fusion systems over D_8 and D_{16}.

The fusion system of the alternating group A_{14} — a fusion system over the 2-group $D_8 \times (D_8 \wr C_2)$ — illustrates why we need to assume S_2 does not contain a subgroup isomorphic to $S_1 \times S_1$ in Theorems 4.6 and 4.9. Fusion systems of larger alternating groups give other examples of this.

To see a larger family of examples, fix any $k \ge 2$ and any odd prime power q, and consider the simple group $G = \Omega_{4k}^-(q)$. Let $\varepsilon \in \{\pm 1\}$ be such that $4|(q-\varepsilon)$, and let 2^n be the largest power of 2 dividing $q^2 - 1$. Then G contains subgroups of odd index

$$G = \Omega_{4k}^-(q) \ge GO_{4k-2}^\varepsilon(q) \ge GO_2^\varepsilon(q) \wr \Sigma_{2k-1} \cong D_{2(q-\varepsilon)} \wr \Sigma_{2k-1} \ge D_{2^n} \wr \Sigma_{2k-1}$$

(where $GO_n^\pm(q)$ is the full orthogonal group). Thus each Sylow 2-subgroup $S \le G$ contains a direct factor D_{2^n}. For $\mathcal{F} = \mathcal{F}_S(G)$, $O^2(\mathcal{F}) = \mathcal{F}$ by the focal subgroup theorem for G (and since $O^2(G) = G$), and $O^{2'}(\mathcal{F}) = \mathcal{F}$ since $\mathrm{Aut}(S)$ is a 2-group.

Similar examples involving semidihedral groups or wreath products $C_{2^n} \wr C_2$ are obtained by considering the groups

$$G = SL_{4k+3}(q) \geq GL_{4k+2}(q) \geq GL_2(q) \wr \Sigma_{2k+1}$$

for odd prime powers q. In particular, for any D as in Theorem 4.10, there is a saturated fusion system $\mathcal{F}$ over $D\times(D\wr C_2)$ which is indecomposable and satisfies $O^2(\mathcal{F}) = \mathcal{F} = O^{2'}(\mathcal{F})$.

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