AUTOMORPHISMS OF FUSION SYSTEMS OF SPORADIC SIMPLE GROUPS

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ABSTRACT. We prove here that with a very small number of exceptions, when G is a sporadic simple group and p is a prime such that the Sylow p-subgroups of G are nonabelian, then $\mathrm{Out}(G)$ is isomorphic to the outer automorphism groups of the fusion and linking systems of G. In particular, the p-fusion system of G is tame in the sense of [AOV1], and is tamely realized by G itself except when $G \cong M_{11}$ and $p = 2$. From the point of view of homotopy theory, these results also imply that $\mathrm{Out}(G) \cong \mathrm{Out}(BG^\wedge_p)$ in many (but not all) cases.

This paper is centered around the comparison of certain outer automorphism groups associated to a sporadic simple group: outer automorphisms of the group itself, those of its fusion at different primes, and those of its classifying space completed at different primes. In most, but not all cases (under conditions made precise in Theorem A), these automorphism groups are all isomorphic. This comparison is important when studying extensions of fusion systems, and through that plays a role in Aschbacher's program (see, e.g., [A5]) for reproving certain parts of the classification theorem from the point of view of fusion systems.

When G is a finite group, p is a prime, and $S \in \mathrm{Syl}_p(G)$, the *p-fusion system* of G is the category $\mathcal{F}_S(G)$ whose objects are the subgroups of G, and which has morphism sets

$$\mathrm{Mor}_{\mathcal{F}_S(G)}(P,Q) = \{\varphi \in \mathrm{Hom}(P,Q) \,|\, \varphi = c_x, \text{ some } x \in G \text{ with } xPx^{-1} \le Q\}.$$

A p-subgroup $P \le G$ is called *p-centric* in G if $Z(P) \in \mathrm{Syl}_p(C_G(P))$; equivalently, if $C_G(P) = Z(P) \times C'_G(P)$ for some (unique) subgroup $C'_G(P)$ of order prime to p. The *centric linking system* of G at p is the category $\mathcal{L}^c_S(G)$ whose objects are the subgroups of S which are p-centric in G, and where

$$\mathrm{Mor}_{\mathcal{L}^c_S(G)}(P,Q) = T_G(P,Q)/C'_G(P) \quad \text{where} \quad T_G(P,Q) = \{x \in G \,|\, xPx^{-1} \le Q\}.$$

Note that there is a natural functor $\pi\colon \mathcal{L}^c_S(G) \longrightarrow \mathcal{F}_S(G)$ which is the inclusion on objects, and which sends the class of $x \in T_G(P,Q)$ to $c_x \in \mathrm{Hom}(P,Q)$. Outer automorphism groups of these systems were defined in [BLO] and later papers (see below). We say that $\mathcal{F} = \mathcal{F}_S(G)$ is *tamely realized* by G if the natural homomorphism $\kappa_G\colon \mathrm{Out}(G) \longrightarrow \mathrm{Out}(\mathcal{L}^c_S(G))$ is surjective and splits. The fusion system $\mathcal{F}$ is *tame* if it is tamely realized by some finite group.

In terms of homotopy theory, it was shown in [BLO, Theorem B] that for a finite group G and $S \in \mathrm{Syl}_p(G)$, there is a natural isomorphism $\mathrm{Out}(\mathcal{L}^c_S(G)) \cong \mathrm{Out}(BG^\wedge_p)$. Here, $BG^\wedge_p$ is the p-completion, in the sense of Bousfield-Kan, of the classifying space of G, and $\mathrm{Out}(X)$ means the group of homotopy classes of self equivalences of the space X. Thus $\mathcal{F}_S(G)$ is tamely realized by G if the natural map from $\mathrm{Out}(G)$ to $\mathrm{Out}(BG^\wedge_p)$ is split surjective.

When $p = 2$, our main result is easily stated: if G is a sporadic simple group, then the 2-fusion system of G is simple except when $G \cong J_1$, and is tamely realized by G except when $G \cong M_{11}$. The 2-fusion system of M_{11} is tamely realized by $PSL_3(3)$.

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For p odd, information about fusion systems of the sporadic groups at odd primes is summarized in Table 0.1. In that table, for a given group G and prime p and $S \in \mathrm{Syl}_p(G)$,

- a dash "—" means that S is abelian or trivial;
- "constr." means that $\mathcal{F}_S(G)$ is constrained; and
- an almost simple group L in brackets means that $\mathcal{F}_S(G)$ is almost simple but not simple, and is shown in [A4, 16.10] to be isomorphic to the fusion system of L.

For all other pairs (G, p), $\mathcal{F}$ is simple by [A4, 16.10], and we indicate what is known about the nature of $\kappa_G \colon \mathrm{Out}(G) \longrightarrow \mathrm{Out}(\mathcal{L}_S^c(G))$. In addition,

- a dagger (†) marks the pairs (G, p) for which S is extraspecial of order p^3.

G	$\lvert\mathrm{Out}(G)\rvert$	$p=3$	$p=5$	$p=7$	$p \geq 11$
M_{12}	2	κ isom.†	—	—	—
M_{24}	1	$[M_{12}{:}2]^\dagger$	—	—	—
J_2	2	constr.†	—	—	—
J_3	2	constr.	—	—	—
J_4	1	$[{}^2F_4(2)]^\dagger$	—	—	11: constr.†
Co_3	1	κ isom.	constr.†	—	—
Co_2	1	κ isom.	constr.†	—	—
Co_1	1	κ isom.	$[SO_5(5)]$	—	—
HS	2	—	constr.†	—	—
McL	2	κ isom.	constr.†	—	—
Suz	2	κ isom.	—	—	—
He	2	$\mathrm{Out}(\mathcal{L}) = 1^\dagger$	—	κ isom.†	—
Ly	1	κ isom.	κ isom.	—	—
Ru	1	$[{}^2F_4(2)]^\dagger$	$[L_3(5){:}2]^\dagger$	—	—
$O'N$	2	—	—	κ isom.†	—
Fi_{22}	2	κ isom.	—	—	—
Fi_{23}	1	κ isom.	—	—	—
Fi_{24}'	2	κ isom.	—	κ isom.†	—
F_5	2	κ isom.	κ isom.	—	—
F_3	1	κ isom.	κ isom.†	—	—
F_2	1	κ isom.	κ isom.	—	—
F_1	1	κ isom.	κ isom.	κ isom.	13: κ isom.†

TABLE 0.1. Summary of results for odd p

Here, a fusion system $\mathcal{F} = \mathcal{F}_S(G)$ is *constrained* if it contains a normal p-subgroup $Q \trianglelefteq \mathcal{F}$ such that $C_S(Q) \leq Q$. The fusion system $\mathcal{F}$ is *simple* if it has no proper nontrivial normal fusion subsystems. It is *almost simple* if it contains a proper normal subsystem $\mathcal{F}_0 \trianglelefteq \mathcal{F}$ which is simple, and such that $C_{\mathcal{F}}(\mathcal{F}_0) = 1$. We refer to [AKO, Definitions I.4.1 & I.6.1] for the definitions of normal p-subgroups and normal fusion subsystems, and to [A4, § 6] for the definition of the centralizer of a normal subsystem.

Thus when G is a sporadic simple group and p is an odd prime such that the p-fusion system $\mathcal{F}$ of G is simple, we show in all cases that $\mathcal{F}$ is tamely realized by G, and in fact that $\mathrm{Out}(G) \cong \mathrm{Out}(\mathcal{L}_S^c(G))$ except when $G \cong He$ and $p = 3$ (Theorem A).

Before going further, we need to define more precisely the automorphism groups which we are working with. All of the definitions given here apply to abstract fusion and linking systems (see, e.g., [AKO, § III.4.3]), but for simplicity, we always assume that $\mathcal{F} = \mathcal{F}_S(G)$ and $\mathcal{L} = \mathcal{L}_S^c(G)$ for some finite group G with $S \in \mathrm{Syl}_p(G)$.

Automorphisms of $\mathcal{F} = \mathcal{F}_S(G)$ are straightforward. An automorphism $\alpha \in \mathrm{Aut}(S)$ is *fusion preserving* if it induces an automorphism of the category $\mathcal{F}$ (i.e., a functor from $\mathcal{F}$ to itself which is bijective on objects and on morphisms). Set

$$\begin{aligned}\mathrm{Aut}(\mathcal{F}) &= \mathrm{Aut}(S,\mathcal{F}) = \bigl\{\alpha \in \mathrm{Aut}(S) \bigm| \alpha \text{ is fusion preserving}\bigr\}\\ \mathrm{Out}(\mathcal{F}) &= \mathrm{Out}(S,\mathcal{F}) = \mathrm{Aut}(S,\mathcal{F})/\mathrm{Aut}_{\mathcal{F}}(S).\end{aligned}$$

Here, by definition, $\mathrm{Aut}_{\mathcal{F}}(S) = \mathrm{Aut}_G(S)$: the automorphisms induced by conjugation in $N_G(S)$. These groups have been denoted $\mathrm{Aut}(S,\mathcal{F})$ and $\mathrm{Out}(S,\mathcal{F})$ in earlier papers, to emphasize that they are groups of automorphisms of S, but it seems more appropriate here to regard them as automorphisms of the *fusion system* $\mathcal{F}$ (as opposed to the category $\mathcal{F}$).

Now assume $\mathcal{L} = \mathcal{L}_S^c(G)$. For each $P \in \mathrm{Ob}(\mathcal{L})$, set $\iota_P = [1] \in \mathrm{Mor}_{\mathcal{L}}(P,S)$ (the "inclusion" of P in S in the category $\mathcal{L}$), and set $[\![P]\!] = \bigl\{[g] \bigm| g \in P\bigr\} \le \mathrm{Aut}_{\mathcal{L}}(P)$. Define

$$\begin{aligned}\mathrm{Aut}(\mathcal{L}) &= \mathrm{Aut}_{\mathrm{typ}}^I(\mathcal{L}) = \bigl\{\beta \in \mathrm{Aut}_{\mathrm{cat}}(\mathcal{L}) \bigm| \beta(\iota_P) = \iota_{\beta(P)},\ \beta([\![P]\!]) = [\![\beta(P)]\!],\ \forall\, P \in \mathcal{F}^c\bigr\}\\ \mathrm{Out}(\mathcal{L}) &= \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}) = \mathrm{Aut}(\mathcal{L})\big/\langle c_x \,|\, x \in N_G(S)\rangle.\end{aligned}$$

Here, $\mathrm{Aut}_{\mathrm{cat}}(\mathcal{L})$ is the group of automorphisms of $\mathcal{L}$ as a category, and $c_x \in \mathrm{Aut}(\mathcal{L})$ for $x \in N_G(S)$ sends P to ${}^xP$ and $[g]$ to $[{}^xg]$. There are natural homomorphisms

$$\mathrm{Out}(G) \xrightarrow{\ \kappa_G\ } \underset{\cong \mathrm{Out}(BG^\wedge_p)}{\mathrm{Out}(\mathcal{L})} \xrightarrow{\ \mu_G\ } \mathrm{Out}(\mathcal{F}) \qquad \text{and} \qquad \bar{\kappa}_G = \mu_G \circ \kappa_G.$$

Here, κ_G is defined by sending the class of $\alpha \in \mathrm{Aut}(G)$, chosen so that $\alpha(S) = S$, to the class of $\widehat{\alpha} \in \mathrm{Aut}(\mathcal{L})$, where $\widehat{\alpha}(P) = \alpha(P)$ and $\widehat{\alpha}([g]) = [\alpha(g)]$. For $\beta \in \mathrm{Aut}(\mathcal{L})$, μ_G sends the class of $\beta \in \mathrm{Aut}(\mathcal{L})$ to the class of

$$\widehat{\beta} = \Bigl(S \xrightarrow[\cong]{\ g\mapsto[g]\ } [\![S]\!] \xrightarrow[\cong]{\ \beta|_{[\![S]\!]}\ } [\![S]\!] \xrightarrow[\cong]{\ [g]\mapsto g\ } S\Bigr) \in \mathrm{Aut}(\mathcal{F}) \le \mathrm{Aut}(S).$$

Then $\bar{\kappa}_G \colon \mathrm{Out}(G) \longrightarrow \mathrm{Out}(\mathcal{F})$ is induced by restriction to S. See [AKO, § III.4.3] or [AOV1, § 1.3] for more details on these definitions.

By recent work of Chermak, Oliver, and Glauberman and Lynd, the nature of μ_G is now fairly well known in all cases.

Proposition 0.1 ([O2, Theorem C], [GlL, Theorem 1.1])**.** *For each prime p, and each finite group G with $S \in \mathrm{Syl}_p(G)$, $\mu_G \colon \mathrm{Out}(\mathcal{L}_S^c(G)) \longrightarrow \mathrm{Out}(\mathcal{F}_S(G))$ is surjective, and is an isomorphism if p is odd.*

In fact, [O2] and [GlL] show that the conclusion of Proposition 0.1 holds for all (abstract) fusion systems and associated linking systems.

When G is a sporadic simple group and p is odd, a more direct proof that μ_G is an isomorphism is given in [O1, Propositions 4.1 & 4.4].

The fusion system $\mathcal{F} = \mathcal{F}_S(G)$ is *tamely realized* by G if κ_G is split surjective, and is *tame* if it is tamely realized by some finite group G^* with $S \in \mathrm{Syl}_p(G^*)$ and $\mathcal{F} = \mathcal{F}_S(G^*)$. We refer to [AOV1, Theorems A & B] or [AKO, § III.6.1] for the original motivation for this definition. In practice, it is in many cases easier to study the homomorphism $\bar{\kappa}_G$, which is why we include information about μ_G here. The injectivity of $\bar{\kappa}_G$, when $p = 2$ and G is a sporadic simple group, follows from a theorem of Richard Lyons [Ly2, Theorem 1.1] (see the proof of Proposition 2.2).

Fusion systems of alternating groups were shown to be tame in [AOV1, Proposition 4.8], while those of finite groups of Lie type (including the Tits group) were shown to be tame in [BMO, Theorems C & D]. So the following theorem completes the study of tameness for fusion systems of the known finite nonabelian simple groups.

Theorem A. *Fix a sporadic simple group G, a prime p which divides $|G|$, and $S \in \mathrm{Syl}_p(G)$. Set $\mathcal{F} = \mathcal{F}_S(G)$ and $\mathcal{L} = \mathcal{L}_S^c(G)$. Then $\mathcal{F}$ is tame. Furthermore, κ_G and μ_G are isomorphisms (hence $\mathcal{F}$ is tamely realized by G) if $p = 2$, or if p is odd and S is nonabelian, with the following two exceptions:*

(a) *$G \cong M_{11}$ and $p = 2$, in which case $\mathrm{Out}(G) = 1$ and $|\mathrm{Out}(\mathcal{F})| = |\mathrm{Out}(\mathcal{L})| = 2$; and*

(b) *$G \cong He$ and $p = 3$, in which case $|\mathrm{Out}(G)| = 2$ and $\mathrm{Out}(\mathcal{F}) = \mathrm{Out}(\mathcal{L}) = 1$.*

Proof. By Proposition 0.1, μ_G is surjective in all cases, and is an isomorphism if p is odd. When $p = 2$, μ_G is injective (hence an isomorphism) by Propositions 2.1 (when $|S| \leq 2^9$) and 5.1 (when $|S| \geq 2^{10}$). Thus in all cases, κ_G is an isomorphism if and only if $\bar{\kappa}_G = \mu_G \circ \kappa_G$ is an isomorphism.

When $p = 2$, $\bar{\kappa}_G$ is an isomorphism, with the one exception $G \cong M_{11}$, by Propositions 2.1 (when $|S| \leq 2^9$) and 2.2 (when $|S| \geq 2^{10}$). When p is odd, S is nonabelian, and $\mathcal{F}$ is not simple, then $\bar{\kappa}_G$ is an isomorphism by Proposition 3.1. When p is odd and $\mathcal{F}$ is simple, $\bar{\kappa}_G$ is an isomorphism except when $G \cong He$ and $p = 3$ by Proposition 3.2. The two exceptional cases are handled in Propositions 2.1 and 3.2. □

In the first half of the paper, we compare $\mathrm{Out}(G)$ with $\mathrm{Out}(\mathcal{F})$: first listing some general results in Section 1, and then applying them to determine the nature of $\bar{\kappa}_G$ in Sections 2 (for $p = 2$) and 3 (for p odd). We then compare $\mathrm{Out}(\mathcal{F})$ with $\mathrm{Out}(\mathcal{L})$ (when $p = 2$) in the last half of the paper: general techniques for determining $\mathrm{Ker}(\mu_G)$ are listed in Section 4, and these are applied in Section 5 to finish the proof of the main theorem.

The author plans, in a future paper with Jesper Grodal, to look more closely at the fundamental groups of geometric realizations of the categories $\mathcal{L}_S^c(G)$ when G is a sporadic group. This should give alternative proofs for several of the cases covered by Theorem A.

I would like to thank Michael Aschbacher for explaining to me the potential importance of these results. Kasper Andersen made some computer computations several years ago involving the Rudvalis sporadic group at $p = 2$; while they're not used here, they probably gave me hints as to how to proceed in that case (one of the hardest). I also thank the referee for his many suggestions which helped simplify or clarify several arguments. I would especially like to thank Richard Lyons for the notes [Ly2] he wrote about automorphisms of sporadic groups, without which I might not have known how to begin this project.

Notation: We mostly use Atlas notation [Atl, § 5.2] for groups, extensions, extraspecial groups, etc., as well as for names (**2A**, **2B**, **3A**, ...) of conjugacy classes of elements. An elementary abelian 2-group has type $\mathbf{2A}^n$ if it is **2A**-pure of rank n (similarly for an elementary abelian 3-group of type $\mathbf{3A}^n$); it has type $\mathbf{2A}_i\mathbf{B}_j\ldots$ if it contains i elements of class **2A**, j of class **2B**, etc. Also, A_n and S_n denote the alternating and symmetric groups on n letters, E_{p^k} (for p prime) is an elementary abelian p-group of order p^k, and $UT_n(q)$ (for $n \geq 2$ and q a prime power) is the group of upper triangular matrices in $GL_n(q)$ with 1's on the diagonal. As usual, $G^\# = G \smallsetminus \{1\}$ is the set of nonidentity elements of a group G, and $Z_2(S) \leq S$ (for a p-group S) is the subgroup such that $Z_2(S)/Z(S) = Z(S/Z(S))$. For groups $H \le G$ and elements $g, h \in G$, ${}^{g}h = ghg^{-1}$ and ${}^{g}H = gHg^{-1}$. For each pair of groups $H \le G$,

$$\mathrm{Aut}_G(H) = \{(x \mapsto {}^{g}x) \mid g \in N_G(H)\} \le \mathrm{Aut}(H) \quad \text{and} \quad \mathrm{Out}_G(H) = \mathrm{Aut}_G(H)/\mathrm{Inn}(H).$$

We assume in all cases the known order of $\mathrm{Out}(G)$ for sporadic groups G, without giving references each time.

1. Automorphism groups of fusion systems: generalities

We give here some techniques which will be used to determine the nature of $\bar{\kappa}_G$. We begin with the question of injectivity. Recall that $|\mathrm{Out}(G)| \le 2$ for each sporadic simple group G.

Lemma 1.1. *Fix a prime p. Let G be a finite group, fix $S \in \mathrm{Syl}_p(G)$, and set $\mathcal{F} = \mathcal{F}_S(G)$.*

(a) *For each $\alpha \in \mathrm{Aut}(G)$, the class $[\alpha] \in \mathrm{Out}(G)$ lies in $\mathrm{Ker}(\bar{\kappa}_G)$ if and only if there is $\alpha' \in [\alpha]$ such that $|S| \bigm| |C_G(\alpha')|$.*

(b) *Assume $|\mathrm{Out}(G)| = 2$ and p is odd. If there is no $\alpha \in \mathrm{Aut}(G)$ such that $|\alpha| = 2$ and $|S| \bigm| |C_G(\alpha)|$, then $\bar{\kappa}_G$ is injective.*

(c) *Assume $|\mathrm{Out}(G)| = 2$. If $\mathrm{Out}_{\mathrm{Aut}(G)}(Q) > \mathrm{Out}_G(Q)$ for some $Q \trianglelefteq S$, then $\bar{\kappa}_G$ is injective.*

Proof. **(a)** We can assume α is chosen so that $\alpha(S) = S$. If $[\alpha] \in \mathrm{Ker}(\bar{\kappa}_G)$, then $\alpha|_S \in \mathrm{Aut}_G(S)$: conjugation by some $g \in N_G(S)$. Set $\alpha' = \alpha \circ c_g^{-1} \in \mathrm{Aut}(G)$; then $[\alpha'] = [\alpha]$ in $\mathrm{Out}(G)$, and $C_G(\alpha') \ge S$.

Conversely, assume $|S| \bigm| |C_G(\alpha')|$. Then $C_G(\alpha') \ge {}^{g}S$ for some $g \in G$. Set $\alpha'' = c_g \circ \alpha' \circ c_g$ (composing from right to left), where $c_g \in \mathrm{Inn}(G)$ and $c_g(S) = {}^{g}S$. Then $[\alpha''] = [\alpha'] = [\alpha]$ in $\mathrm{Out}(G)$, $\alpha''|_S = \mathrm{Id}_S$, and hence $\bar{\kappa}_G([\alpha]) = \bar{\kappa}_G([\alpha'']) = 1$.

(b) If $\bar{\kappa}_G$ is not injective, then by (a), there is $\alpha \in \mathrm{Aut}(G) \smallsetminus \mathrm{Inn}(G)$ such that $|S| \bigm| |C_G(\alpha)|$. Since $|\mathrm{Out}(G)| = 2$, $|\alpha| = 2m$ for some $m \ge 1$. Thus $|\alpha^m| = 2$, and $|S| \bigm| |C_G(\alpha^m)|$.

(c) If $Q \trianglelefteq S$ and $\mathrm{Out}_{\mathrm{Aut}(G)}(Q) > \mathrm{Out}_G(Q)$, then there is $\beta \in \mathrm{Aut}(G) \smallsetminus \mathrm{Inn}(G)$ such that $\beta(Q) = Q$ and $\beta|_Q \notin \mathrm{Aut}_G(Q)$. Since $S \in \mathrm{Syl}_p(N_G(Q))$, we can arrange that $\beta(S) = S$ by replacing β by $c_x \circ \beta$ for some appropriate element $x \in N_G(Q)$. We still have $\beta|_Q \notin \mathrm{Aut}_G(Q)$, so $\beta|_S \notin \mathrm{Aut}_G(S)$, and $\bar{\kappa}_G([\beta]) \ne 1$. Thus $\bar{\kappa}_G$ is nontrivial, and is injective if $|\mathrm{Out}(G)| = 2$. □

A finite group H will be called *strictly p-constrained* if $C_H(O_p(H)) \le O_p(H)$; equivalently, if $F^*(H) = O_p(H)$.

Lemma 1.2. *Fix a prime p. Let G be a finite group, fix $S \in \mathrm{Syl}_p(G)$, and set $\mathcal{F} = \mathcal{F}_S(G)$. Let $H < G$ be a subgroup which contains S.*

(a) *If H is strictly p-constrained, then κ_H and μ_H are isomorphisms.*

(b) *Assume $H = N_G(Q)$, where either Q is characteristic in S, or $|Q| = p$, $Q \le Z(S)$, and $\mathrm{Aut}(\mathcal{F})$ sends each G-conjugacy class of elements of order p in $Z(S)$ to itself. Set $\mathcal{F}_H = \mathcal{F}_S(H)$ for short, and set $\mathrm{Aut}^0(\mathcal{F}_H) = \mathrm{Aut}(\mathcal{F}_H) \cap \mathrm{Aut}(\mathcal{F})$ and $\mathrm{Out}^0(\mathcal{F}_H) = \mathrm{Aut}^0(\mathcal{F}_H)/\mathrm{Aut}_H(S)$. Then the inclusion of $\mathrm{Aut}^0(\mathcal{F}_H)$ in $\mathrm{Aut}(\mathcal{F})$ induces a surjection of $\mathrm{Out}^0(\mathcal{F}_H)$ onto $\mathrm{Out}(\mathcal{F})$, and hence $|\mathrm{Out}(\mathcal{F})| \le |\mathrm{Out}^0(\mathcal{F}_H)| \le |\mathrm{Out}(\mathcal{F}_H)|$.*

If in addition, H is strictly p-constrained or $\bar{\kappa}_H$ is onto, and we set $\mathrm{Out}^0(H) = \bar{\kappa}_H^{-1}(\mathrm{Out}^0(\mathcal{F}_H))$, then $|\mathrm{Out}(\mathcal{F})| \le |\mathrm{Out}^0(H)| \le |\mathrm{Out}(H)|$.

Proof. **(a)** See, e.g., [BMO, Proposition 1.6(a)].

(b) We first claim that

$$\mathrm{Aut}(\mathcal{F}) = \mathrm{Aut}_G(S)\cdot N_{\mathrm{Aut}(\mathcal{F})}(Q) \le \mathrm{Aut}_G(S)\cdot\mathrm{Aut}^0(\mathcal{F}_H) \tag{1}$$

as subgroups of $\mathrm{Aut}(S)$. If Q is characteristic in S, then the equality is clear. If $|Q| = p$, $Q \le Z(S)$, and each $\alpha \in \mathrm{Aut}(\mathcal{F})$ sends Q to a subgroup which is G-conjugate to Q, then the equality follows from the Frattini argument (and since each subgroup of $Z(S)$ which is G-conjugate to Q is $N_G(S)$-conjugate to Q). If $\alpha \in \mathrm{Aut}(S)$ normalizes Q and preserves fusion in G, then it preserves fusion in $H = N_G(Q)$. Thus $N_{\mathrm{Aut}(\mathcal{F})}(Q) \le \mathrm{Aut}^0(\mathcal{F}_H)$, proving the second relation in (1).

Now, $\mathrm{Aut}_H(S) \le \mathrm{Aut}_G(S) \cap \mathrm{Aut}^0(\mathcal{F}_H)$. Together with (1), this implies that the natural homomorphism

$$\mathrm{Out}^0(\mathcal{F}_H) = \mathrm{Aut}^0(\mathcal{F}_H)/\mathrm{Aut}_H(S) \longrightarrow \mathrm{Aut}(\mathcal{F})/\mathrm{Aut}_G(S) = \mathrm{Out}(\mathcal{F})$$

is well defined and surjective. The last statement now follows from (a). □

The next lemma will be useful when determining $\mathrm{Out}(H)$ for the subgroups H which appear when applying Lemma 1.2(b).

Lemma 1.3. *Let H be a finite group, and let $Q \trianglelefteq H$ be a characteristic subgroup such that $C_H(Q) \le Q$. Set $H^* = \mathrm{Out}_H(Q) \cong H/Q$.*

(a) *There is an exact sequence*

$$1 \longrightarrow H^1(H^*; Z(Q)) \longrightarrow \mathrm{Out}(H) \xrightarrow{\ R\ } N_{\mathrm{Out}(Q)}(H^*)/H^*,$$

where R sends the class of $\alpha \in \mathrm{Aut}(H)$ to the class of $\alpha|_Q$.

(b) *Assume $R \le Z(Q)$ and $R \trianglelefteq H$. Let $\alpha \in \mathrm{Aut}(H)$ be such that $\alpha|_R = \mathrm{Id}_R$ and $[\alpha, H] \le R$. Then there is $\psi \in \mathrm{Hom}_H(Q/R, R)$ such that $\alpha(g) = g\psi(gR)$ for each $g \in Q$, and hence $\alpha|_Q = \mathrm{Id}_Q$ if $\mathrm{Hom}_H(Q/R, R) = 1$. If $\alpha|_Q = \mathrm{Id}_Q$, $[\alpha, H] \le R$, and $H^1(H^*; R) = 0$, then $\alpha \in \mathrm{Aut}_R(H)$.*

(c) *Fix a prime p, assume Q is an extraspecial or elementary abelian p-group, and set $\overline{Q} = Q/\mathrm{Fr}(Q)$. Set $H_0^* = O^{p'}(H^*)$, and $X = N_{\mathrm{Out}(Q)}(H^*)/H^*$.*

  (c.i) *If $\overline{Q}$ is absolutely irreducible as an $\mathbb{F}_pH^*$-module, then there is $Y \trianglelefteq X$ such that $Y \cong (\mathbb{Z}/p)^\times/Z(H^*)$ and X/Y is isomorphic to a subgroup of $\mathrm{Out}(H^*)$.*

  (c.ii) *If $\overline{Q}$ is absolutely irreducible as an $\mathbb{F}_pH_0^*$-module, then there is $Y \trianglelefteq X$ such that $Y \cong (\mathbb{Z}/p)^\times/Z(H^*)$ and*

$$|X/Y| \le |\mathrm{Out}(H_0^*)|/|\mathrm{Out}_{H^*}(H_0^*)|.$$

Here, $Z(H^*)$ acts on $\overline{Q}$ via multiplication by scalars, and we regard it as a subgroup of $(\mathbb{Z}/p)^\times$ in that way.

Proof. **(a)** The exact sequence is a special case of [OV, Lemma 1.2].

(b) By assumption, there is a function $\psi\colon Q/R \longrightarrow R$ such that $\alpha(g) = g\psi(gR)$ for each $g \in Q$, and ψ is a homomorphism since $R \le Z(Q)$. For each $h \in H$, $\alpha(h) = rh$ for some $r \in R$. So for $g \in Q$, since $[r, Q] = 1$, we get $\psi({}^h gR) = ({}^h g)^{-1}\alpha({}^h g) = ({}^h g)^{-1}{}^{rh}(\alpha(g)) = {}^h(g^{-1}\alpha(g)) = {}^h\psi(gR)$. Thus $\psi \in \mathrm{Hom}_H(Q/R, R)$.

If $\alpha|_Q = \mathrm{Id}$ and $[\alpha, H] \le R$, then there is $\chi\colon H^* \longrightarrow R$ such that $\alpha(g) = \chi(gQ)g$ for each $g \in H$. Then $\chi(ghQ) = \chi(gQ)\cdot{}^g\chi(hQ)$ for all $g, h \in H$, so χ is a 1-cocycle. If $H^1(H^*; R) = 0$, then there is $r \in R$ such that $\chi(gQ) = r({}^g r)^{-1}$ for each $g \in H$, and α is conjugation by r.

(c) If $\overline{Q}$ is absolutely irreducible as an $\mathbb{F}_pH^*$-module, then $C_{\mathrm{Out}(Q)}(H^*)\cong(\mathbb{Z}/p)^\times$ consists of multiplication by scalars (see [A, 25.8]), so its image Y in $X=N_{\mathrm{Out}(Q)}(H^*)/H^*$ is isomorphic to $(\mathbb{Z}/p)^\times/Z(H^*)$. Also, $X/Y\cong\mathrm{Out}_{\mathrm{Out}(Q)}(H^*)$: a subgroup of $\mathrm{Out}(H^*)$. This proves (c.i).

If $\overline{Q}$ is absolutely irreducible as an $\mathbb{F}_pH_0^*$-module, let Y be the image of $C_{\mathrm{Out}(Q)}(H_0^*)$ in $X=N_{\mathrm{Out}(Q)}(H^*)/H^*$. Then $Y\cong(\mathbb{Z}/p)^\times/Z(H^*)$ (by [A, 25.8] again), and

$$\begin{aligned}|X/Y|&=|N_{\mathrm{Out}(Q)}(H^*)/C_{\mathrm{Out}(Q)}(H_0^*){\cdot}H^*|\le|N_{\mathrm{Out}(Q)}(H_0^*)|/|C_{\mathrm{Out}(Q)}(H_0^*){\cdot}H^*|\\&=|\mathrm{Aut}_{\mathrm{Out}(Q)}(H_0^*)|/|\mathrm{Aut}_{H^*}(H_0^*)|\\&\le|\mathrm{Aut}(H_0^*)|/|\mathrm{Aut}_{H^*}(H_0^*)|=|\mathrm{Out}(H_0^*)|/|\mathrm{Out}_{H^*}(H_0^*)|.\end{aligned}$$

This proves (c.ii). □

The next lemma provides some simple tools for showing that certain representations are absolutely irreducible.

Lemma 1.4. *Fix a prime p, a finite group G, and an irreducible $\mathbb{F}_pG$-module V.*

(a) *The module V is absolutely irreducible if and only if $\mathrm{End}_{\mathbb{F}_pG}(V)\cong\mathbb{F}_p$.*

(b) *If $\dim_{\mathbb{F}_p}(C_V(H))=1$ for some $H\le G$, then V is absolutely irreducible.*

(c) *Assume $H\le G$ is a subgroup such that $V|_H$ splits as a direct sum of absolutely irreducible pairwise nonisomorphic $\mathbb{F}_pH$-submodules. Then V is absolutely irreducible.*

Proof. **(a)** See, e.g., [A, 25.8].

(b) Set $\mathrm{End}_{\mathbb{F}_pG}(V)=K$: a finite extension of $\mathbb{F}_p$. Then V can be considered as a KG-module, so $[K{:}\mathbb{F}_p]$ divides $\dim_{\mathbb{F}_p}(C_V(H))$ for each $H\le G$. Since there is H with $\dim_{\mathbb{F}_p}(C_V(H))=1$, this implies $K=\mathbb{F}_p$, and so V is absolutely irreducible by (a).

(c) The hypothesis implies that the ring $\mathrm{End}_{\mathbb{F}_pH}(V)$ is isomorphic to a direct product of copies of $\mathbb{F}_p$, one for each irreducible summand of $V|_H$. Since $\mathrm{End}_{\mathbb{F}_pG}(V)$ is a subring of $\mathrm{End}_{\mathbb{F}_pH}(V)$, and is a field since V is irreducible, it must be isomorphic to $\mathbb{F}_p$. So V is absolutely irreducible by (a). □

Lemma 1.5. *Let G be a finite group, and let V be a finite $\mathbb{F}_pG$-module.*

(a) *If $C_V(O_{p'}(G))=0$, then $H^1(G;V)=0$.*

(b) *If $|V|=p$, and $G_0=C_G(V)$, then $H^1(G;V)\cong\mathrm{Hom}_{G/G_0}\big(G_0/[G_0,G_0],V\big)$.*

Proof. **(a)** Set $H=O_{p'}(G)$ for short. Assume $W\ge V$ is an $\mathbb{F}_pG$-module such that $[G,W]\le V$. Then $[H,W]=[H,V]=V$ since $C_V(H)=0$, and so $W=C_W(H)\oplus[H,W]=C_W(H)\oplus V$. Thus $H^1(G;V)\cong\mathrm{Ext}^1_{\mathbb{F}_pG}(\mathbb{F}_p,V)=0$.

Alternatively, with the help of the obvious spectral sequence, one can show that $H^i(G;V)=0$ for all $i\ge0$.

(b) This is clear when G acts trivially on V. It follows in the general case since for $G_0\trianglelefteq G$ of index prime to p and any $\mathbb{F}_pG$-module V, $H^1(G;V)$ is the group of elements fixed by the action of G/G_0 on $H^1(G_0;V)$. □

We end with a much more specialized lemma, which is needed when working with the Thompson group F_3.

Lemma 1.6. *Set $H=A_9$. Assume V is an 8-dimensional $\mathbb{F}_2H$-module such that for each 3-cycle $g\in H$, $C_V(g)=0$. Then V is absolutely irreducible, $\dim(C_V(T))=1$ for $T\in\mathrm{Syl}_2(H)$, and $N_{\mathrm{Aut}(V)}(H)/H=1$.*

Proof. Consider the following elements in A_9:

$$a_1 = (1\,2\,3), \qquad a_2 = (4\,5\,6), \qquad a_3 = (7\,8\,9),$$
$$b_1 = (1\,2)(4\,5), \qquad b_2 = (1\,2)(7\,8), \qquad b_3 = (1\,2)(4\,7)(5\,8)(6\,9).$$

Set $A = \langle a_1, a_2, a_3 \rangle \cong E_{27}$ and $B = \langle b_1, b_2, b_3 \rangle \cong D_8$. Set $\overline{V} = \bar{\mathbb{F}}_2 \otimes_{\mathbb{F}_2} V$. As an $\bar{\mathbb{F}}_2A$-module, $\overline{V}$ splits as a sum of 1-dimensional submodules, each of which has character $A \longrightarrow \bar{\mathbb{F}}_2^\times$ for which none of the a_i is in the kernel. There are eight such characters, they are permuted transitively by B, and so each occurs with multiplicity 1 in the decomposition of $\overline{V}$. Thus $\overline{V}$ is AB-irreducible, and hence H-irreducible (and V is absolutely irreducible). Also, $\dim_{\bar{\mathbb{F}}_2}(C_{\overline{V}}(B)) = 1$, so $\dim(C_V(B)) = 1$, and $\dim(C_V(T)) = 1$ since $C_V(T) \neq 0$.

In particular, $C_{\mathrm{Aut}(V)}(H) \cong \mathbb{F}_2^\times = 1$, and hence $N_{\mathrm{Aut}(V)}(H)/H$ embeds into $\mathrm{Out}(H)$. So if $N_{\mathrm{Aut}(V)}(H)/H \neq 1$, then the action of H extends to one of $\widehat{H} \cong S_9$. In that case, if we set $x = (1\,2) \in \widehat{H}$, then $C_V(x)$ has rank 4 since x inverts a_1 and $C_V(a_1) = 0$. But the group $C_{\widehat{H}}(x)/x \cong S_7$ acts faithfully on $C_V(x)$, and this is impossible since $GL_4(2) \cong A_8$ contains no S_7-subgroup. (This argument is due to Richard Lyons [Ly2].) □

2. Automorphisms of 2-fusion systems of sporadic groups

The main result in this section is that when G is a sporadic simple group and $p = 2$, $\mathrm{Out}(\mathcal{F}) \cong \mathrm{Out}(G)$ in all cases except when $G \cong M_{11}$. The first proposition consists mostly of the cases where this was shown in earlier papers.

Proposition 2.1. *Let G be a sporadic simple group whose Sylow 2-subgroups have order at most 2^9. Then the 2-fusion system of G is tame. More precisely, κ_G and μ_G are isomorphisms except when $G \cong M_{11}$, in which case the 2-fusion system of G is tamely realized by $PSL_3(3)$.*

Proof. Fix G as above, choose $S \in \mathrm{Syl}_2(G)$, and set $\mathcal{F} = \mathcal{F}_S(G)$. There are eleven cases to consider.

If $G \cong M_{11}$, then $\mathrm{Out}(G) = 1$. Also, $\mathcal{F}$ is the unique simple fusion system over SD_{16}, so by [AOV1, Proposition 4.4], $|\mathrm{Out}(\mathcal{F})| = 2$, and κ_{G^*} is an isomorphism for $G^* = PSU_3(13)$ (and μ_{G^*} is an isomorphism by the proof of that proposition). Note that we could also take $G^* = PSL_3(3)$.

If $G \cong J_1$, then $\mathrm{Out}(G) = 1$. Set $H = N_G(S)$. Since $S \cong E_8$ is abelian, fusion in G is controlled by $H \cong 2^3{:}7{:}3$, and so $\mathcal{F} = \mathcal{F}_S(H)$ and $\mathcal{L} \cong \mathcal{L}_S^c(H)$. Since H is strictly 2-constrained, $\mathrm{Out}(\mathcal{L}) \cong \mathrm{Out}(\mathcal{F}) \cong \mathrm{Out}(H) = 1$ by Lemma 1.2(a), and so κ_G and μ_G are isomorphisms.

If $G \cong M_{22}$, M_{23}, J_2, J_3, or McL, then $\mathcal{F}$ is tame, and κ_G is an isomorphism, by [AOV1, Proposition 4.5]. Also, μ_G was shown to be an injective in the proof of that proposition, and hence is an isomorphism by Proposition 0.1.

If $G \cong M_{12}$, Ly, HS, or $O'N$, then $\mathcal{F}$ is tame, and κ_G and μ_G are isomorphisms, by [AOV3, Lemmas 4.2 & 5.2 and Proposition 6.3]. □

It remains to consider the larger cases.

Proposition 2.2. *Let G be a sporadic simple group whose Sylow 2-subgroups have order at least 2^{10}. Then $\bar{\kappa}_G$ is an isomorphism.*

Proof. Fix G as above, choose $S \in \mathrm{Syl}_2(G)$, and set $\mathcal{F} = \mathcal{F}_S(G)$. There are fifteen groups to consider, listed in Table 2.2.

We first check that $\bar{\kappa}_G$ is injective in all cases. This follows from a theorem of Richard Lyons [Ly2, Theorem 1.1], which says that if $\mathrm{Out}(G) \neq 1$, then there is a 2-subgroup of G whose centralizer in $\mathrm{Aut}(G) = G.2$ is contained in G [Ly2, Theorem 1.1]. Since that paper has not been published, we give a different argument here: one which is based on Lemma 1.1(c), together with some well known (but hard-to-find-referenced) descriptions of certain subgroups of G and of $\mathrm{Aut}(G)$.

The groups G under consideration for which $|\mathrm{Out}(G)| = 2$ are listed in Table 2.1. In each case, $N_G(R)$ has odd index in G (hence R can be assumed to be normal in S), and $\mathrm{Out}_{\mathrm{Aut}(G)}(R) > \mathrm{Out}_G(R)$. So $\bar{\kappa}_G$ is injective by Lemma 1.1(c).

G	Suz	He	Fi_{22}	Fi'_{24}	F_5
R	2_-^{1+6}	2^{4+4}	2^{5+8}	2_+^{1+12}	2_+^{1+8}
$\mathrm{Out}_G(R)$	$\Omega_6^-(2)$	$S_3 \times S_3$	$S_3 \times A_6$	$3{\cdot}U_4(3).2$	$(A_5 \times A_5).2$
$\mathrm{Out}_{G.2}(R)$	$SO_6^-(2)$	$3^2{:}D_8$	$S_3 \times S_6$	$3{\cdot}U_4(3).2^2$	$(A_5 \times A_5).2^2$
Reference	[GL, p.56]	[W7, § 5]	[A3, 37.8.2]	[W8, Th.E]	[NW, Th.2]

TABLE 2.1

It remains to prove that $|\mathrm{Out}(\mathcal{F})| \leq |\mathrm{Out}(G)|$. Except when $G \cong Ru$, we do this with the help of Lemma 1.2(b) applied with H as in Table 2.2. Set $Q = O_2(H)$, $\bar{Q} = Q/Z(Q)$, and $H^* = \mathrm{Out}_H(Q)$.

When $G \cong Co_3$, and $H = N_G(Z(S)) \cong 2{\cdot}Sp_6(2)$ is quasisimple, $\mathrm{Out}(H) = 1$ since $\mathrm{Out}(H/Z(H)) = 1$ by Steinberg's theorem (see [GLS, Theorem 2.5.1]). Also, $\kappa_{H/Z(H)}$ is surjective by [BMO, Theorem A], so κ_H and $\bar{\kappa}_H$ are surjective by [AOV1, Proposition 2.18]. Hence $|\mathrm{Out}(\mathcal{F})| \leq |\mathrm{Out}(H)| = 1$ by Lemma 1.2(b).

If $G \cong Co_1$, Fi_{22}, Fi_{23}, or Fi'_{24}, then Q is elementary abelian, $H^* \cong M_k$ for $k = 24$, 22, 23, or 24, respectively, and Q is an absolutely irreducible $\mathbb{F}_2H^*$-module by [A3, 22.5]. Also, $Q = J(S)$ (i.e., Q is the unique abelian subgroup of its rank) in each case: by [A2, Lemma 46.12.1] when $G \cong Co_1$, and by [A3, Exercise 11.1, 32.3, or 34.5] when G is one of the Fischer groups. By [MSt, Lemma 4.1] (or by [A3, 22.7–8] when G is a Fischer group), $H^1(H^*; Q)$ has order 2 when $G \cong Fi'_{24}$ (and Q is the Todd module for H^*), and has order 1 when G is one of the other Fischer groups (Q is again the Todd module) or Co_1 (Q is the dual Todd module). So

$$|\mathrm{Out}(\mathcal{F})| \leq |\mathrm{Out}(H)| \leq |H^1(H^*; Q)|{\cdot}|\mathrm{Out}(H^*)| = |\mathrm{Out}(G)|:$$

the first inequality by Lemma 1.2(b), the second by Lemmas 1.3(a) and 1.3(c.i), and the equality by a case-by-case check (see Table 2.2).

In each of the remaining cases covered by Table 2.2, $H = N_G(Z(S))$ and is strictly 2-constrained, and Q is extraspecial. We apply Lemma 1.3(a) to get an upper bound for $|\mathrm{Out}(H)|$. This upper bound is listed in the fourth column of Table 2.2 in the form $m = a{\cdot}b$, where $|H^1(H^*; Z(Q))| \leq a$ and $|N_{\mathrm{Out}(Q)}(H^*)/H^*| \leq b$. By Lemma 1.5(b), $H^1(H^*; Z(Q)) \cong \mathrm{Hom}(H^*, C_2) = 1$ except when $G \cong J_4$ or F_5, in which cases it has order 2. This explains the first factor in the fourth column. The second factor will be established case-by-case, as will be the difference between $|\mathrm{Out}(\mathcal{F})|$ and $|\mathrm{Out}(H)|$ when there is one (noted by an asterisk).

If $G \cong M_{24}$ or He, then $H \cong 2_+^{1+6}.L_3(2)$, and $\bar{Q}$ splits as a sum of two nonisomorphic absolutely irreducible $\mathbb{F}_2H^*$-modules which differ by an outer automorphism of H^*. Hence

G	$\|S\|$	H	$\|\mathrm{Out}(H)\|$	$\|\mathrm{Out}(\mathcal{F})\|$	$\|\mathrm{Out}(G)\|$	Reference
M_{24}	2^{10}	$2_+^{1+6}.L_3(2)$	$2=1{\cdot}2$	1^*	1	[A2, Lm. 39.1.1]
J_4	2^{21}	$2_+^{1+12}.3M_{22}{:}2$	$2=2{\cdot}1$	1^*	1	[KW, § 1.2]
Co_3	2^{10}	$2{\cdot}Sp_6(2)$	1	1	1	[Fi, Lm. 4.4]
Co_2	2^{18}	$2_+^{1+8}.Sp_6(2)$	$1=1{\cdot}1$	1	1	[W1, pp.113–14]
Co_1	2^{21}	$2^{11}.M_{24}$	$1=1{\cdot}1$	1	1	[A2, Lm. 46.12]
Suz	2^{13}	$2_-^{1+6}.U_4(2)$	$2=1{\cdot}2$	2	2	[W2, § 2.4]
He	2^{10}	$2_+^{1+6}{:}L_3(2)$	$2=1{\cdot}2$	2	2	[He, p. 253]
Ru	2^{14}	$2^{3+8}.L_3(2)$ $2.2^{4+6}.S_5$		1^*	1	[A1, 12.12] [AS, Th. J.1.1]
Fi_{22}	2^{17}	$2^{10}.M_{22}$	$2=1{\cdot}2$	2	2	[A3, 25.7]
Fi_{23}	2^{18}	$2^{11}.M_{23}$	$1=1{\cdot}1$	1	1	[A3, 25.7]
Fi_{24}'	2^{21}	$2^{11}.M_{24}$	$2=2{\cdot}1$	2	2	[A3, 34.8, 34.9]
F_5	2^{14}	$2_+^{1+8}.(A_5\times A_5).2$	$4=2{\cdot}2$	2^*	2	[NW, § 3.1]
F_3	2^{15}	$2_+^{1+8}.A_9$	$1=1{\cdot}1$	1	1	[W11, Thm. 2.2]
F_2	2^{41}	$2_+^{1+22}.Co_2$	$1=1{\cdot}1$	1	1	[MS, Thm. 2]
F_1	2^{46}	$2_+^{1+24}.Co_1$	$1=1{\cdot}1$	1	1	[MS, Thm. 1]

TABLE 2.2

$N_{\mathrm{Out}(Q)}(H^*)\cong L_3(2){:}2$, and $|\mathrm{Out}(H)|\le|N_{\mathrm{Out}(Q)}(H^*)/H^*|=2$. These two irreducible submodules in $\overline{Q}$ lift to rank 4 subgroups of Q, of which exactly one is radical (with automizer $SL_4(2)$) when $G\cong M_{24}$ (see [A2, Lemma 40.5.2]). Since an outer automorphism of H exchanges these two subgroups, it does not preserve fusion in G when $G\cong M_{24}$, hence is not in $\mathrm{Out}^0(H)$ in the notation of Lemma 1.2(b). So $|\mathrm{Out}(\mathcal{F})|\le|\mathrm{Out}^0(H)|=1$ in this case.

If $G\cong J_4$, then $H\cong 2_+^{1+12}.3M_{22}{:}2$. The group $3M_{22}$ has a 6-dimensional absolutely irreducible representation over $\mathbb{F}_4$, which extends to an irreducible 12-dimensional representation of $3M_{22}{:}2$ realized over $\mathbb{F}_2$. (See [KW, p. 487]: $3M_{22}<SU_6(2)<SO_{12}^+(2)$.) Hence $|\mathrm{Out}(H)|\le 2$ by Lemmas 1.3(a,c) and 1.5(b), generated by the class of $\beta\in\mathrm{Aut}(H)$ of order 2 which is the identity on $O^2(H)$ and on $H/Z(H)$.

By [KW, Table 1], there is a four-group of type **2AAB** in H, containing $Z(Q)=Z(H)$ (generated by an element of class **2A**), whose image in $H/O_{2,3}(H)\cong M_{22}{:}2$ is generated by an outer involution of class **2B** in $\mathrm{Aut}(M_{22})$. Thus there are cosets of $Z(Q)$ in $H\smallsetminus O^2(H)$ which contain **2A**- and **2B**-elements. Hence $\beta|_S$ is not G-fusion preserving, so $|\mathrm{Out}(\mathcal{F})|\le|\mathrm{Out}^0(H)|=1$ by Lemma 1.2(b).

If $G\cong Co_2$, then $H=N_G(z)\cong 2_+^{1+8}.Sp_6(2)$. By [Sm, Lemma 2.1], the action of H/Q on $\overline{Q}$ is transitive on isotropic points and on nonisotropic points, and hence is irreducible. If $\overline{Q}$ is not absolutely irreducible, then $\mathrm{End}_{\mathbb{F}_p[H/Q]}(\overline{Q})\ge\mathbb{F}_4$ by Lemma 1.4(a), so $H/Q\cong Sp_6(2)$ embeds into $SL_4(4)$, which is impossible since $Sp_6(2)$ contains a subgroup of type 7:6 while $SL_4(4)$ does not.

Alternatively, $\overline{Q}$ is absolutely irreducible by a theorem of Steinberg (see [GLS, Theorem 2.8.2]), which says roughly that each irreducible $\bar{\mathbb{F}}_2Sp_6(\bar{\mathbb{F}}_2)$-module which is "small enough" is still irreducible over the finite subgroup $Sp_6(2)$.

Thus by Lemma 1.3(c.i), $N_{\mathrm{Out}(Q)}(H^*)/H^*$ is isomorphic to a subgroup of $\mathrm{Out}(H^*)$, where $\mathrm{Out}(H^*)=1$ (see [GLS, Theorem 2.5.1]). This confirms the remaining entries for G in Table 2.2.

If $G\cong Suz$, then $H\cong 2_-^{1+6}.\Omega_6^-(2)$, H^* has index 2 in $\mathrm{Out}(Q)\cong SO_6^-(2)$, and $|\mathrm{Out}(H)|\le 2$ by Lemmas 1.3(a) and 1.5(b).

If $G\cong F_5$, then $H=N_G(z)\cong 2_+^{1+8}.(A_5\wr 2)$ for $z\in\mathbf{2B}$. As described in [NW, § 3.1] and in [Ha, Lemma 2.8], $O^2(H^*)$ acts on Q as $\Omega_4^+(4)$ for some $\mathbb{F}_4$-structure on $\overline{Q}$. Also, the **2B**-elements in $Q\smallsetminus Z(Q)$ are exactly those involutions which are isotropic under the $\mathbb{F}_4$-quadratic form on $\overline{Q}\cong\mathbb{F}_4^4$.

Now, H^* has index 2 in its normalizer $SO_4^+(4).2^2$ in $\mathrm{Out}(Q)\cong SO_8^+(2)$, so $|\mathrm{Out}(H)|\le 4$ by Lemmas 1.3(a) and 1.5(b). Let $\beta\in\mathrm{Aut}(H)$ be the nonidentity automorphism which is the identity on $O^2(H)$ and on $H/Z(H)$. To see that $|\mathrm{Out}(\mathcal{F})|\le 2$, we must show that β does not preserve fusion in S.

By [NW, p. 364], if $W=\langle z,g\rangle\cong E_4$ for $z\in Z(H)$ and $g\in H\smallsetminus O^2(H)$, then W contains an odd number of **2A**-elements, and hence g and zg are in different classes (see also [Ha, Lemma 2.9.ii]). Hence β is not fusion preserving since it doesn't preserve G-conjugacy classes. By Lemma 1.2(b), $|\mathrm{Out}(\mathcal{F})|\le|\mathrm{Out}^0(H)|\le 2=|\mathrm{Out}(G)|$.

If $G\cong F_3$, then $H\cong 2_+^{1+8}.A_9$. By [Pa, § 3], the action of A_9 on $\overline{Q}$ is not the permutation representation, but rather that representation twisted by the triality automorphism of $SO_8^+(2)$. By [Pa, 3.7], if $x\in H^*\cong A_9$ is a 3-cycle, then $C_{\overline{Q}}(x)=1$. Hence we are in the situation of Lemma 1.6, and $N_{\mathrm{Out}(Q)}(H^*)/H^*=1$ by that lemma. So $\mathrm{Out}(H)=1$ by Lemmas 1.3(a) and 1.5(b), and $\mathrm{Out}(\mathcal{F})=1$.

If $G\cong F_2$ or F_1, then $H=H_1\cong 2_+^{1+22}.Co_2$ or $2_+^{1+24}.Co_1$, respectively. Set $Q=O_2(H)$ and $\overline{Q}=Q/Z(Q)$. If $G\cong F_1$, then $\overline{Q}\cong\widetilde{\Lambda}$, the mod 2 Leech lattice, and is Co_1-irreducible by [A2, 23.3]. If $G\cong F_2$, then $\overline{Q}\cong v_2^\perp/\langle v_2\rangle$ where $v_2\in\widetilde{\Lambda}$ is the image of a 2-vector. The orbit lengths for the action of Co_2 on $\widetilde{\Lambda}/\langle v_2\rangle$ are listed in [W1, Table I], and from this one sees that $v_2^\perp/\langle v_2\rangle$ is the only proper nontrivial Co_2-linear subspace (the only union of orbits of order 2^k for $0<k<23$), and hence that $\overline{Q}$ is Co_2-irreducible. The absolute irreducibility of $\overline{Q}$ (in both cases) now follows from Lemma 1.4(b), applied with $H=Co_2$ or $U_6(2){:}2$, respectively.

Since $\mathrm{Out}(Co_1)\cong\mathrm{Out}(Co_2)=1$, $N_{\mathrm{Out}(Q)}(H^*)/H^*=1$ by Lemma 1.3(c.i), and so $\mathrm{Out}(H)=1$ in both cases.

In in the remaining case, we need to work with two of the 2-local subgroups of G.

Assume $G\cong Ru$. We refer to [A1, 12.12] and [AS, Theorem J.1.1] for the following properties. There are two conjugacy classes of involutions in G, of which the **2A**-elements are 2-central. There are subgroups $H_1,H_3<G$ containing S such that

$$H_1\cong 2.2^{4+6}.S_5 \qquad H_3\cong 2^{3+8}.L_3(2).$$

Set $Q_i=O_2(H_i)$ and $V_i=Z(Q_i)$; $V_1\cong C_2$ and $V_3\cong E_8$, and both are **2A**-pure and normal in S. Also, Q_1/V_1 and Q_3 are special of types 2^{4+6} and 2^{3+8}, respectively, and $Z(Q_3)$ and $Q_3/Z(Q_3)$ are the natural module and the Steinberg module, respectively, for $H_3/Q_3\cong SL_3(2)$.

Let $V_5 < Q_1$ be such that $V_5/V_1 = Z(Q_1/V_1)$. Then V_5 is of type $\mathbf{2A}^5$, and $C_{Q_1}(V_5) \cong Q_8 \times E_{16}$. Also, $H_1/Q_1 \cong S_5$, and V_5/V_1 and $Q_1/C_{Q_1}(V_5)$ are both natural modules for $O^2(H_1/Q_1) \cong SL_2(4)$. Also, $V_3/V_1 = C_{V_5/V_1}((S/Q_1) \cap O^2(H_1/Q_1))$: thus a 1-dimensional subspace of V_5/V_1 as an $\mathbb{F}_4$-vector space.

The homomorphism $Q_1/C_{Q_1}(V_5) \longrightarrow \mathrm{Hom}(V_5/V_1, V_1)$ which sends g to $(x \mapsto [g,x])$ is injective and hence an isomorphism. So $Q_3 \cap Q_1 = C_{Q_1}(V_3)$ has index 4 in Q_1, and hence $|Q_3Q_1/Q_3| = 4$.

Fix $\beta \in \mathrm{Aut}(\mathcal{F})$. By Lemma 1.2(a), for $i = 1,3$, $\bar{\kappa}_{H_i}$ is an isomorphism, so β extends to an automorphism $\beta_i \in \mathrm{Aut}(H_i)$. Since V_3 is the natural module for $H_3/Q_3 \cong SL_3(2)$, $\beta_3|_{V_3} = c_x$ for some $x \in H_3$, and $x \in N_{H_3}(S)$ since $\beta_3(S) = S$. Then $x \in S$ since $N_{H_3/Q_3}(S/Q_3) = S/Q_3$, and upon replacing β by $c_x^{-1} \circ \beta$ and β_i by $c_x^{-1} \circ \beta_i$ $(i = 1,3)$, we can arrange that $\beta|_{V_3} = \mathrm{Id}$.

Since $\beta|_{V_3} = \mathrm{Id}$, β_3 also induces the identity on $H_3/Q_3 \cong L_3(2)$ (since this acts faithfully on V_3), and on $Q_3/V_3 \cong 2^8$ (since this is the Steinberg module and hence absolutely irreducible). Since Q_3/V_3 is H_3/Q_3-projective (the Steinberg module), $H^1(H_3/Q_3; Q_3/V_3) = 0$, so by Lemma 1.3(b) (applied with Q_3/V_3 in the role of $R = Q$), the automorphism of H_3/V_3 induced by β_3 is conjugation by some $yV_3 \in Q_3/V_3$. Upon replacing β by $c_y^{-1} \circ \beta$ and similarly for the β_i, we can arrange that $[\beta_3, H_3] \le V_3$.

Now, $Q_3/V_3 \not\cong V_3$ are both irreducible $\mathbb{F}_2[H_3/Q_3]$-modules, so $\mathrm{Hom}_{H_3/Q_3}(Q_3/V_3, V_3) = 0$. By Lemma 1.3(b) again, applied this time with $Q_3 \ge V_3$ in the role of $Q \ge R$, $\beta|_{Q_3} = \mathrm{Id}$.

Now consider $\beta_1 \in \mathrm{Aut}(H_1)$. Since β_1 is the identity on $Q_3 = C_S(V_3) \ge C_S(V_5) = C_{H_1}(V_5)$, $\beta_1 \equiv \mathrm{Id}_{H_1}$ modulo $Z(C_S(V_5)) = V_5$ (since $c_g = c_{\beta_1(g)} \in \mathrm{Aut}(C_S(V_5))$ for each $g \in H_1$). So by Lemma 1.3(b), there is $\psi \in \mathrm{Hom}_{H_1/Q_1}(Q_1/V_5, V_5/V_1)$ such that $\beta(g) \in g\psi(gV_5)$ for each $g \in Q_1$. Also, $\mathrm{Im}(\psi) \le V_3/V_1$ since $[\beta, S] \le V_3$, and hence $\psi = 1$ since V_5/V_1 is irreducible. Thus $[\beta_1, Q_1] \le V_1$.

We saw that $|Q_1Q_3/Q_3| = 4$, so $\mathrm{Aut}_{Q_1}(V_3)$ is the group of all automorphisms which send V_1 to itself and induce the identity on V_3/V_1. Fix a pair of generators $uQ_3, vQ_3 \in Q_1Q_3/Q_3$. Then $\beta(u) \in uV_1$ and $\beta(v) \in vV_1$, and each of the four possible automorphisms of Q_3Q_1 (i.e., those which induce the identity on Q_3 and on Q_1Q_3/V_1) is conjugation by some element of V_3 (unique modulo V_1). So after conjugation by an appropriate element of V_3, we can arrange that $\beta|_{Q_1Q_3} = \mathrm{Id}$ (and still $[\beta_3, H_3] \le V_3$).

Let $V_2 < V_3$ be the unique subgroup of rank 2 which is normal in S, and set $S_0 = C_S(V_2)$. Thus $|S/S_0| = 2$, and $S_0/Q_3 \cong E_4$. Fix $w \in (S_0 \cap Q_1Q_3) \smallsetminus Q_3$ (thus wQ_3 generates the center of $S/Q_3 \cong D_8$). Choose $g \in N_{H_3}(V_2)$ of order 3; thus g acts on V_2 with order 3 and acts trivially on V_3/V_2. So $V_3\langle g\rangle \cong A_4 \times C_2$, and since $|\beta_3(g)| = 3$, we have $\beta_3(g) = rg$ for some $r \in V_2$. Set $w' = {}^g w \in S_0$. Then $S_0 = Q_3\langle w, w'\rangle$, $\beta(w) = w$ since $w \in Q_1Q_3$, and $\beta(w') = \beta(gwg^{-1}) = rgwg^{-1}r^{-1} = {}^r w' = w'$: the last equality since $w' \in S_0 = C_S(V_2)$. Since $S = S_0Q_1$, this proves that $\beta = \mathrm{Id}_S$, and hence that $\mathrm{Out}(\mathcal{F}) = 1$.

This finishes the proof of Proposition 2.2. □

3. Tameness at odd primes

We now turn to fusion systems of sporadic groups at odd primes, and first look at the groups whose p-fusion systems are not simple.

Proposition 3.1. *Let p be an odd prime, and let G be a sporadic simple group whose Sylow p-subgroups are nonabelian and whose p-fusion system is not simple. Then $\bar{\kappa}_G$ is an isomorphism.*

Proof. Fix $S \in \mathrm{Syl}_p(G)$, and set $\mathcal{F} = \mathcal{F}_S(G)$. By [A4, 16.10], if $\mathcal{F}$ is not simple, then either $N_G(S)$ controls fusion in G ("G is p-Goldschmidt" in the terminology of [A4]), in which case $S \trianglelefteq \mathcal{F}$ and $\mathcal{F}$ is constrained, or $\mathcal{F}$ is almost simple and is realized by an almost simple group L given explicitly in [A4, 16.10] and also in Table 0.1. We handle these two cases separately.

Case 1: Assume first that $S \trianglelefteq \mathcal{F}$ and hence $\mathcal{F}$ is constrained. By [A4, Theorem 15.6], there are seven such cases (G, p), also listed in Table 0.1. By the tables in [GLS, Table 5.3], in each case where $\mathrm{Out}(G) \neq 1$, no involution of $\mathrm{Aut}(G)$ centralizes a Sylow p-subgroup. Thus $\bar{\kappa}_G$ is injective in all seven cases by Lemma 1.1(b). Set $H = N_G(H)/O_{p'}(N_G(H))$. Since $N_G(S)$ controls p-fusion in G,

$$\mathrm{Out}(\mathcal{F}) \cong \mathrm{Out}(H) \quad \text{injects into} \quad N_{\mathrm{Out}(S)}(\mathrm{Out}_G(S))/\mathrm{Out}_G(S): \tag{1}$$

the isomorphism by Lemma 1.2(a) and the injection by 1.3(a).

In the six cases described in Table 3.1, S is extraspecial of order p^3 and exponent p. Note

(G, p)	$(J_2, 3)$	$(Co_3, 5)$	$(Co_2, 5)$	$(HS, 5)$	$(McL, 5)$	$(J_4, 11)$
$\lvert\mathrm{Out}(G)\rvert$	2	1	1	2	2	1
$\mathrm{Out}_G(S)$	C_8	$C_{24} \rtimes C_2$	$4{\cdot}\Sigma_4$	$C_8 \rtimes C_2$	$C_3 \rtimes C_8$	$5 \times 2{\cdot}\Sigma_4$

TABLE 3.1

that $\mathrm{Out}(S) \cong GL_2(p)$. Using that $PGL_2(3) \cong \Sigma_4$, $PGL_2(5) \cong \Sigma_5$, and Σ_4 is maximal in $PGL_2(11)$, we see that in all cases, $|\mathrm{Out}(\mathcal{F})| \leq |\mathrm{Out}(G)|$ by (1). So $\bar{\kappa}_G$ and κ_G are isomorphisms since they are injective.

It remains to consider the case $(G, p) = (J_3, 3)$, where $|S| = 3^5$. Set $T = \Omega_1(S)$ and $Z = Z(S)$. By [J, Lemma 5.4], $T \cong C_3^3$, $T > Z \cong C_3^2$, $Z \leq [S, S]$, and there are two classes of elements of order 3: those in Z and those in $T \smallsetminus Z$. Also, S/Z is extraspecial of order 3^3 with center T/Z, and $N_G(S)/S \cong C_8$ acts faithfully on S/T and on Z.

Consider the bilinear map

$$\Phi\colon S/T \times T/Z \xrightarrow{[-,-]} Z$$

where $\Phi(gT, hZ) = [g, h]$. This is nontrivial (otherwise we would have $T \leq Z$), and hence is surjective since $N_G(S)/S \cong C_8$ acts faithfully on Z. Fix $x \in N_G(S)$ and $h \in T$ whose cosets generate the quotient groups $N_G(S)/S$ and T/Z, respectively. Since x acts on $S/T \cong C_3^2$ with order 8, it acts via an element of $GL_2(3) \smallsetminus SL_2(3)$, and hence acts on T/Z by inverting it (recall that S/Z is extraspecial). So if we let $\Phi_h\colon S/T \longrightarrow Z$ be the isomorphism $\Phi_h(gT) = [g, h]$, then $\Phi_h({}^x gT) = [{}^x g, h] = {}^x[g, h^{-1}] = {}^x\Phi_h(gT)^{-1}$. Thus if $\lambda, \lambda^3 \in \mathbb{F}_9$ are the eigenvalues for the action of x on S/T (for some λ of order 8), then $\lambda^{-1}, \lambda^{-3}$ are the eigenvalues for the action of x on Z. So there is no nontrivial homomorphism $S/T \longrightarrow Z$ that commutes with the actions of x.

Let $\alpha \in \mathrm{Aut}(\mathcal{F})$ be such that $\alpha|_Z = \mathrm{Id}$. Since α commutes with Φ, it must either induce the identity on S/T and on T/Z or invert both quotient groups, and the latter is impossible since S/Z is extraspecial. Since α is the identity on Z and on T/Z, $\alpha|_T$ is conjugation by some element of S, and we can assume (modulo $\mathrm{Inn}(S)$) that $\alpha|_T = \mathrm{Id}$. Thus there is $\varphi \in \mathrm{Hom}(S/T, Z)$ such that $\alpha(g) = g\varphi(gT)$ for each $g \in S$, and φ commutes with the action of $xS \in N_G(S)/S$. We just showed that this is only possible when $\varphi = 1$, and conclude that $\alpha = \mathrm{Id}_S$.

Thus $\mathrm{Aut}(\mathcal{F})$ is isomorphic to a subgroup of $\mathrm{Aut}(Z) \cong GL_2(3)$. Since $\mathrm{Aut}_G(S) \cong C_8$ acts faithfully on Z, and the Sylow 2-subgroups of $GL_2(3)$ are semidihedral of order 16, this shows that $|\mathrm{Aut}(\mathcal{F})| \leq 16$ and $|\mathrm{Out}(\mathcal{F})| \leq 2$. Since $\bar{\kappa}_G$ is injective, it is an isomorphism.

Case 2: We now show that κ_G is an isomorphism when $\mathcal{F}$ is almost simple. Let L be as in Table 0.1. If $L \cong {}^2F_4(2)$ and $p = 3$, then $\mathrm{Out}(\mathcal{F}) \cong \mathrm{Out}(L) = 1 = \mathrm{Out}(G)$ since $\bar{\kappa}_L$ is an isomorphism by [BMO, Proposition 6.9].

Otherwise, set $L_0 = O^{p'}(L)$ and $\mathcal{F}_0 = \mathcal{F}_S(L_0)$. By [A4, 16.3 & 16.10], $\mathcal{F}_0$ is simple, and hence $Z(\mathcal{F}_0) = 1$, when $L_0 \cong M_{12}$ and $p = 3$, and when $L_0 \cong \Omega_5(5)$ or $PSL_3(5)$ and $p = 5$. Also, $\bar{\kappa}_{L_0}$ is an isomorphism in these cases by Proposition 3.2 and [BMO, Theorem A], respectively, and $L \cong \mathrm{Aut}(L_0)$ and $|L/L_0| = 2$ (hence $\mathrm{Out}(L) = 1$) by [A4, 16.10]. If $\mathrm{Out}(\mathcal{F}) \neq 1$, then there is $\alpha \in \mathrm{Aut}(\mathcal{F}) \smallsetminus \mathrm{Aut}_{\mathcal{F}}(S)$ such that $\alpha|_{S_0} = \mathrm{Id}$, and by the pullback square in [AOV1, Lemma 2.15], this would lie in the image of a nontrivial element of $\mathrm{Out}(L) = 1$. Thus $\mathrm{Out}(\mathcal{F}) = 1$, $\mathrm{Out}(G) = 1$ by Table 0.1, and so $\bar{\kappa}_G$ and hence κ_G are isomorphisms. □

It remains to handle the cases (G, p) where the p-fusion system of G is simple.

Proposition 3.2. *Let p be an odd prime, and let G be a sporadic simple group whose p-fusion system is simple. Then $\bar{\kappa}_G$ is an isomorphism, except when $p = 3$ and $G \cong He$, in which case $|\mathrm{Out}(G)| = 2$ and $|\mathrm{Out}(\mathcal{F}_S(G))| = 1$ for $S \in \mathrm{Syl}_3(G)$.*

Proof. Fix G and p, choose $S \in \mathrm{Syl}_p(G)$, set $\mathcal{F} = \mathcal{F}_S(G)$, and assume $\mathcal{F}$ is simple (see Table 0.1 or [A4, 16.10]). Set $\mathcal{L} = \mathcal{L}_S^c(G)$.

The centralizers of all involutions in $\mathrm{Aut}(G)$ are listed in, e.g., [GLS, Tables 5.3a–z]. By inspection, for each pair (G, p) in question other than $(He, 3)$ for which $\mathrm{Out}(G) \neq 1$ (see Tables 3.2 and 3.3), there is no $\alpha \in \mathrm{Aut}(G)$ of order 2 for which $|S|$ divides $|C_G(\alpha)|$. So by Lemma 1.1(b), $\bar{\kappa}_G$ is injective in all such cases.

To prove that $\bar{\kappa}_G$ is an isomorphism (with the one exception), it remains to show that $|\mathrm{Out}(\mathcal{F})| \leq |\mathrm{Out}(G)|$.

Assume S is extraspecial of order p^3. Set $H = N_G(S)$ and $H^* = \mathrm{Out}_G(S) \cong H/S$. We list in Table 3.2 all pairs (G, p) which occur, together with a description of H^* and of $N_{\mathrm{Out}(S)}(H^*)$. To determine $|N_{\mathrm{Out}(S)}(H^*)/H^*|$ in each case, just recall that $GL_2(3) \cong 2{\cdot}S_4 \cong Q_8{:}S_3$, that $PGL_2(5) \cong S_5$, and that when $p = 7$ or 13, each subgroup of order prime to p in $PGL_2(p)$ is contained in a subgroup isomorphic to $D_{2(p\pm1)}$ or S_4 (cf. [Sz1, Theorem 3.6.25]).

(G,p)	$(M_{12},3)$	$(He,3)$	$(F_3,5)$	$(He,7)$	$(O'N,7)$	$(Fi'_{24},7)$	$(F_1,13)$
H^*	2^2	D_8	$4{\cdot}S_4$	$3\times S_3$	$3\times D_8$	$6\times S_3$	$3\times 4{\cdot}S_4$
$N_{\mathrm{Out}(S)}(H^*)$	D_8	SD_{16}	$4{\cdot}S_4$	$6\times S_3$	$3\times D_{16}$	$C_6\wr C_2$	$3\times 4{\cdot}S_4$
$\lvert\mathrm{Out}(G)\rvert$	2	2	1	2	2	2	1
Ref.	[GLS, 5.3b]	[He, 3.9]	[W11, § 3]	[He, 3.23]	[GLS, Tbl.5.3s,v]		[W10, § 11]

TABLE 3.2

In all cases, we have

$$|\mathrm{Out}(\mathcal{F})| \leq |\mathrm{Out}(N_G(S))| \leq |N_{\mathrm{Out}(S)}(H^*)/H^*|.$$

The first inequality holds by Lemma 1.2(b). The second holds by Lemma 1.3(a), applied with $H = N_G(S)$, and since $H^1(H^*; Z(S)) = 0$ (Lemma 1.5(b)). By Table 3.2, $|N_{\mathrm{Out}(S)}(H^*)/H^*| = |\mathrm{Out}(G)|$ in all cases. Hence $|\mathrm{Out}(\mathcal{F})| \leq |\mathrm{Out}(G)|$, and so $\bar{\kappa}_G$ is an isomorphism if it is injective.

If $G \cong He$ and $p = 3$, then $H^* = \mathrm{Out}_G(S) \cong D_8$ permutes the four subgroups of index 3 in $S \cong 3^{1+2}_+$ in two orbits of two subgroups each. As described in [Bt, Proposition 10] (see also

[GLS, Table 5.3p, note 4]), the subgroups in one of the orbits are **3A**-pure while those in the other have **3A**- and **3B**-elements, so no fusion preserving automorphism of S exchanges them. So while $|N_{\mathrm{Out}(S)}(H^*)/H^*|=2$, we have $|\mathrm{Out}(\mathcal{F})|\le|\mathrm{Out}^0(H)|=1$ by Lemma 1.2(b). Thus $\bar\kappa_G$ is split surjective (and G tamely realizes $\mathcal{F}_S(G)$), but it is not an isomorphism.

Assume $|S|\ge p^4$. Consider the subgroups $H<G$ described in Table 3.3. In all cases, we can assume $H\ge S$.

G	p	Case	H	$\lvert\mathrm{Out}(H)\rvert$	$\lvert\mathrm{Out}(G)\rvert$	K	$N(-)$	Reference
Co_3	3	4	$3^{1+4}_+.4S_6$	1	1		**3A**	[Fi, 5.12]
Co_2	3	$3b$	$3^{1+4}_+.2^{1+4}_-.S_5$	1	1	2^{1+4}_-	**3A**	[W1, § 3]
Co_1	3	$3a$	$3^{1+4}_+.Sp_4(3).2$	1	1	$Sp_4(3)$	**3C**	[Cu2, p.422]
McL	3	2	$3^{1+4}_+.2S_5$	2	2	$2{\cdot}(5{:}4)$	**3A**	[Fi, Lm.5.5]
Suz	3	1	$3^5.M_{11}$	2	2		$J(S)$	[W2, Thm.]
Ly	3	1	$3^5.(M_{11}\times 2)$	1	1		$J(S)$	[Ly1, Tbl.I]
Fi_{22}	3	4	$3^{1+6}_+.2^{3+4}.3^2.2$	2	2		**3B**	[W5, p.201]
Fi_{23}	3	4	$3^{1+8}_+.2^{1+6}_-.3^{1+2}_+.2S_4$	1	1		**3B**	[W8, § 1.2]
Fi_{24}'	3	2	$3^{1+10}_+.U_5(2){:}2$	2	2	$2{\cdot}(11{:}10)$	**3B**	[W8, Th.B]
F_5	3	4	$3^{1+4}_+.4A_5$	2	2		**3B**	[NW, § 3.2]
F_3	3	5	$3^2.3^{3+4}.GL_2(3)$ $3.[3^8].GL_2(3)$		1		**3B**^{2} **3B**	[A1, 14.1–3] [Pa, §§ 2,4]
F_2	3	$3b$	$3^{1+8}_+.2^{1+6}_-.SO_6^-(2)$	1	1	2^{1+6}_-	**3B**	[W9, § 2]
F_1	3	$3a$	$3^{1+12}_+.2Suz.2$	1	1	$2{\cdot}(13{:}6)$	**3B**	[W10, § 3]
Ly	5	4	$5^{1+4}_+.4S_6$	1	1	$2A_6$	**5A**	[Ly1, Tbl.I]
F_5	5	$3b$	$5^{1+4}_+.2^{1+4}_-{:}5{:}4$	2	2	2^{1+4}_-	**5B**	[NW, § 3.3]
F_2	5	$3b$	$5^{1+4}_+.2^{1+4}_-.A_5.4$	1	1	2^{1+4}_-	**5B**	[W9, § 6]
F_1	5	$3a$	$5^{1+6}_+.4J_2.2$	1	1	$2{\cdot}(7{:}6)$	**5B**	[W10, § 9]
F_1	7	$3a$	$7^{1+4}_+.3\times 2S_7$	1	1	$2{\cdot}(5{:}4)$	**7B**	[W10, § 10]

TABLE 3.3

Case 1: If $G\cong Suz$ or Ly and $p=3$, then $H=N_G(J(S))$, where $J(S)\cong E_{3^5}$ and $H/J(S)\cong M_{11}$ or $M_{11}\times C_2$, respectively, and $|\mathrm{Out}(\mathcal{F})|\le|\mathrm{Out}(H)|$ by Lemma 1.2(b). Set $V=O_3(H)=J(S)$ and $H^*=\mathrm{Aut}_H(V)\cong H/V$. Then V is the Todd module for $O^2(H^*)\cong M_{11}$ (it contains 11 subgroups of type **3A** permuted by H^*), so $H^1(H^*;V)=0$ by [MSt, Lemma 4.1]. Also, V is absolutely $\mathbb{F}_3H^*$-irreducible since $H^*>11{:}5$. So by Lemma 1.3(c.i) and since $\mathrm{Out}(M_{11})=1$, $|N_{\mathrm{Aut}(V)}(H^*)/H^*|\le 2$ if $G\cong Suz$ ($H^*\cong M_{11}$), and is trivial if $G\cong Ly$. Lemma 1.3(a) now implies that $|\mathrm{Out}(H)|\le 2$ or 1 for $G\cong Suz$ or Ly, respectively, and hence that $|\mathrm{Out}(\mathcal{F})|\le|\mathrm{Out}(G)|$.

For each of the remaining pairs (G,p) displayed in Table 3.3, except when $G\cong F_3$ and $p=3$ (Case 5), we set $Q=O_p(H)$, $\bar{Q}=Q/Z(Q)$, $H^*=\mathrm{Out}_H(Q)$, $H_0=O^{p'}(H)$, and $H_0^*=\mathrm{Out}_{H_0}(Q)$. Then H is strictly p-constrained and Q is extraspecial, and hence $Z(S)=Z(Q)$ has order p. Also, $H=N_G(Z(Q))=N_G(Z(S))$ by the above references, so $|\mathrm{Out}(\mathcal{F})|\le|\mathrm{Out}(H)|$ by Lemma 1.2(b), and it remains to show that $|\mathrm{Out}(H)|\le|\mathrm{Out}(G)|$. By Lemma 1.5(b), $H^1(H^*;Z(Q))=0$ in each of these cases, and hence $\mathrm{Out}(H)$ is sent injectively into the quotient group $N_{\mathrm{Out}(Q)}(H^*)/H^*$ by Lemma 1.3(a). So it remains to show that $|N_{\mathrm{Out}(Q)}(H^*)/H^*| \le |\mathrm{Out}(G)|$.

Case 2: If $G \cong McL$ or Fi_{24}' and $p=3$, then $\bar{Q}$ is an absolutely irreducible $\mathbb{F}_pK$-module for $K \le H^*$ as given in Table 3.3, and hence an absolutely irreducible $\mathbb{F}_pH^*$-module. So $|N_{\mathrm{Out}(Q)}(H^*)/H^*| \le 2$ by Lemma 1.3(c): since $|\mathrm{Out}(2S_5)|=2$ in the first case, and since $\mathrm{Out}(U_5(2).2)=1$ and $Z(U_5(2).2)=1$ in the second case.

Case 3: If $G \cong F_1$ and $p=3$, then $\bar{Q}$ splits as a sum of two absolutely irreducible 6-dimensional $\mathbb{F}_3K$-modules. Since $5^2 \bigm| |Suz| \bigm| |H_0^*|$ while $5^2 \nmid |GL_6(3)|$, $\bar{Q}$ is H_0^*-irreducible, hence absolutely H_0^*-irreducible by Lemma 1.4(c). In all other cases under consideration, $\bar{Q}$ is easily checked to be an absolutely irreducible $\mathbb{F}_pK$-module for $K \le H_0^*$ as given in Table 3.3, and hence an absolutely irreducible $\mathbb{F}_pH_0^*$-module.

Thus $|\mathrm{Out}(\mathcal{F})| \le |N_{\mathrm{Out}(Q)}(H^*)/H^*| \le \eta{\cdot}|\mathrm{Out}(H_0^*)|\big/|\mathrm{Out}_{H^*}(H_0^*)|$ by Lemma 1.3(c.ii), where for Y as in the lemma, $\eta=|Y|=2$ when $(G,p)=(F_5,5)$ (and $H^* \not\ge Z(\mathrm{Out}(Q))$), and $\eta=|Y|=1$ otherwise.

In Case 3a, we have $\mathrm{Out}(H_0^*)=\mathrm{Out}_{H^*}(H_0^*)$ in all cases, so $|\mathrm{Out}(\mathcal{F})|=|\mathrm{Out}(H)|=1$.

In Case 3b, we determine $\mathrm{Out}(H_0^*)$ by applying Lemma 1.3(a) again, this time with $O_2(H_0^*)$ in the role of Q. Since $\mathrm{Out}(2_-^{1+4}) \cong S_5$ and $\mathrm{Out}(2_-^{1+6}) \cong SO_6^-(2)$, the lemma implies that $\mathrm{Out}(H_0^*)=\mathrm{Out}_{H^*}(H_0^*)$ in each case, and hence that $|\mathrm{Out}(\mathcal{F})| \le \eta$.

Case 4: We show, one pair (G,p) at a time, that $|N_{\mathrm{Out}(Q)}(H^*)/H^*| \le |\mathrm{Out}(G)|$ in each of these five cases.

If $G \cong Co_3$ and $p=3$, then $Q \cong 3_+^{1+4}$ and $\mathrm{Out}(Q) \cong Sp_4(3){:}2$. Set $Z=Z(\mathrm{Out}(Q)) \cong C_2$. Then $\mathrm{Out}(Q)/Z \cong PSp_4(3){:}2 \cong SO_5(3)$ and $H^*/Z \cong C_2 \times S_6$. Under this identification, the central involution $x \in Z(H^*/Z)$ acts as $-\mathrm{Id}_V \oplus \mathrm{Id}_W$ for some orthogonal decomposition $V \oplus W$ of the natural module $\mathbb{F}_3^5$; and since none of the groups $\Omega_2^\pm(3)$, $\Omega_3(3)$, or $\Omega_4^+(3)$ has order a multiple of 5, $\dim(V)=4$ and $C_{SO_5(3)}(x) \cong GO_4^-(3)$. Since $\Omega_4^-(3) \cong PSL_2(9) \cong A_6$, this shows that $C_{\mathrm{Out}(Q)/Z}(x)=H^*/Z \cong C_2 \times S_6$. So $|N_{\mathrm{Out}(Q)}(H^*)/H^*|=1$.

If $G \cong F_5$ and $p=3$, then $Q \cong 3_+^{1+4}$ and $H^* \cong 4A_5$. By the argument in the last case, $N_{\mathrm{Out}(Q)}(Z(H^*)) \cong 4S_6$, so $|N_{\mathrm{Out}(Q)}(H^*)/H^*|=|N_{S_6}(A_5)/A_5|=2$.

When $G \cong Fi_{22}$ and $p=3$, the subgroup $H \cong 3_+^{1+6}.2^{3+4}.3^2.2$ is described in [W5, p. 201]: H^* can be regarded as a subgroup of $GL_2(3) \wr S_3 < Sp_6(3).2$. More precisely, $2^{3+4} < (Q_8)^3$ (recall $O_2(GL_2(3)) \cong Q_8$) is a subgroup of index 4, one of the factors C_3 normalizes each Q_8 and the other permutes them cyclically, and the C_2 acts by inverting both factors C_3. Then $N_{\mathrm{Out}(Q)}(H^*) \le GL_2(3) \wr S_3$ since it must permute the three $O_2(H^*)$-irreducible subspaces of $\bar{Q}$, so $N_{\mathrm{Out}(Q)}(H^*) \cong 2^{3+4}.(S_3 \times S_3)$, and $|N(H^*)/H^*|=2$.

When $G \cong Fi_{23}$ and $p=3$, the subgroup H is described in [W8, § 1.2]. The subgroup $R^*=O_2(H^*) \cong 2_-^{1+6}$ has a unique faithful irreducible representation over $\mathbb{F}_3$, this is 8-dimensional, and $N_{SL_8(3)}(R^*)/R^*$ is sent injectively into $\mathrm{Out}(R^*) \cong SO_6^-(2) \cong SO_5(3)$. Since $H^*/R^* \cong 3_+^{1+2}{:}2S_4$ is a maximal parabolic subgroup in $SO_5(3)$, we get $N_{\mathrm{Out}(Q)}(H^*)/H^*=1$.

If $G \cong Ly$ and $p=5$, then $\bar{Q}$ is $\mathbb{F}_5[2A_6]$-irreducible since $3^2 \nmid |GL_3(5)|$, and is absolutely irreducible since $2A_6$ is not a subgroup of $SL_2(25)$ (since E_9 is not a subgroup). Thus $|N_{\mathrm{Out}(Q)}(H^*)/H^*| \le |\mathrm{Out}(S_6)|=2$, with equality only if the action of $2A_6$ on $\bar{Q}$ extends to $2A_6.2^2$. This is impossible, since the two classes of 3-elements in $2A_6$ act differently on $\bar{Q}$ (note the action of a Sylow 3-subgroup on $\bar{Q}$), so $N_{\mathrm{Out}(Q)}(H^*)/H^*=1$.

Case 5: When $G \cong F_3$ and $p=3$, we work with two different 3-local subgroups. Set $V_1=Z(S)$ and $V_2=Z_2(S)$, and set $H_i=N_G(V_i)$ and $Q_i=O_3(H_i)$ for $i=1,2$. By [A1, 14.1.2

& 14.1.5] and [Pa, § 4], $V_1 \cong C_3$, $V_2 \cong E_9$, $|Q_1| = |Q_2| = 3^9$, and $H_1/Q_1 \cong H_2/Q_2 \cong GL_2(3)$. Note that $S \le H_1 \cap H_2$, and $|S| = 3^{10}$. Also, the following hold:

(1) Set $V_5 = Z_2(Q_2)$. Then $V_5 = [Q_2, Q_2] \cong E_{3^5}$, $Q_2/V_5 \cong E_{3^4}$, V_2 is the natural module for $G_2/Q_2 \cong GL_2(3)$, and V_5/V_2 is the projective absolutely irreducible $PSL_2(3)$-module of rank 3. Also, $V_5/V_2 = Z(Q_2/V_2)$, and hence Q_2/V_2 is special of type 3^{3+4}. See [A1, 14.2].

(2) By [A1, 14.2.3], the quotient Q_2/V_5 is G_2/Q_2-indecomposable, and is an extension of one copy of the natural $SL_2(3)$-module by another. Let $R_7 < Q_2$ be such that $R_7 > V_5$, and $R_7/V_5 < Q_2/V_5$ is the unique H_2/Q_2-submodule of rank 2 (thus $|R_7| = 3^7$).

(3) We claim that $C_{Q_2}(V_5) = V_5$. Assume otherwise: then $C_{Q_2}(V_5) \ge R_7$ since it is normal in H_2. So $V_5 \le Z(R_7)$, and $|[R_7, R_7]| \le 3$ since $R_7/V_5 \cong E_9$. But $[R_7, R_7] < V_5$ is normal in H_2, so it must be trivial, and R_7 is abelian. This is impossible: V_5 contains elements of all three classes of elements of order 3 [A1, 14.2.2], while the centralizer of a **3A**-element is isomorphic to $(3 \times G_2(3)).2$ whose Sylow 3-subgroups are nonabelian of order 3^7.

(4) Set $V_3 = Z_2(Q_1)$; then $V_3 \cong E_{27}$, and V_3/V_1 is the natural module for G_1/Q_1 [A1, 14.3.1]. Since $V_3 \trianglelefteq S$ and $V_2 = Z_2(S) \cong E_9$, $V_3 > V_2$. Also, $V_3/V_2 \le Z(Q_2/V_2) = V_5/V_2$ since $|V_3/V_2| = 3$. Thus $V_2 < V_3 < V_5$.

By [A1, 14.3.2], $[Q_1, Q_1] > V_3$, and $Q_1/[Q_1, Q_1] \cong E_{3^4}$ is G_1/Q_1-indecomposable and an extension of one copy of the natural $SL_2(3)$-module by another.

(5) Set $W_7 = C_G(V_3) \ge V_5$: a subgroup of S, hence of $Q_1 \cap Q_2$, of order 3^7 [A1, 14.3.4]. We claim that $W_7/V_5 = Z(S/V_5) = C_{Q_2/V_5}(S/Q_2)$, where $S/V_5 \cong C_3 \times (C_3 \wr C_3)$ by [A1, 14.2.5]. To see this, note that for each $g \in Q_2$ such that $gV_5 \in C_{Q_2/V_5}(S/Q_2)$, the map $x \mapsto [x, g]$ is S/Q_2-linear from V_5/V_2 to V_2, so $V_3/V_2 = [S, [S, V_5/V_2]]$ (see (1)) lies in its kernel. Thus $Z(S/V_5) \le W_7/V_5$, and they are equal since they both have order 9.

(6) To summarize, we have defined two sequences of subgroups

$$V_2 < V_5 < R_7 < Q_2 < H_2 \qquad \text{and} \qquad V_1 < V_3 < W_7 < Q_1 < H_1,$$

those in the first sequence normal in H_2 and those in the second normal in H_1, where $V_m \cong E_{3^m}$ and $|R_7| = |W_7| = 3^7$. In addition, $V_1 < V_2 < V_3 < V_5 < W_7 < Q_2$.

Fix $\beta \in \mathrm{Aut}(\mathcal{F})$. By Lemma 1.2(a), κ_{H_2} is an isomorphism, and hence β extends to an automorphism $\beta_2 \in \mathrm{Aut}(H_2)$. Since V_2 is the natural module for $H_2/Q_2 \cong GL_2(3)$, $\beta_2|_{V_2} = c_x$ for some $x \in H_2$, and $x \in N_{H_2}(S)$ since $\beta_2(S) = S$. Upon replacing β by $c_x^{-1} \circ \beta$ and β_2 by $c_x^{-1} \circ \beta_2$, we can arrange that $\beta|_{V_2} = \mathrm{Id}$.

Since $\beta|_{V_2} = \mathrm{Id}$, β_2 also induces the identity on $H_2/Q_2 \cong GL_2(3)$ (since this acts faithfully on V_2), and induces $\varepsilon{\cdot}\mathrm{Id}$ on $V_5/V_2 \cong E_{27}$ for $\varepsilon \in \{\pm 1\}$ since it is absolutely irreducible. By (3), the homomorphism $Q_2/V_5 \longrightarrow \mathrm{Hom}(V_5/V_2, V_2)$ which sends g to $(x \mapsto [g, x])$ is injective. Since β induces the identity on V_2 and $\varepsilon{\cdot}\mathrm{Id}$ on V_5/V_2, it also induces $\varepsilon{\cdot}\mathrm{Id}$ on Q_2/V_5. By (1), $[Q_2/V_2, Q_2/V_2] = V_5/V_2$, so β acts via the identity on V_5/V_2. Thus $\varepsilon = +1$, and β also induces the identity on Q_2/V_5.

Now, $H^1(H_2/Q_2; Q_2/V_5) = 0$ by Lemma 1.5(a) (and since the central involution in $H_2/Q_2 \cong GL_2(3)$ inverts Q_2/V_5). So by Lemma 1.3(b), applied with H_2/V_5 and Q_2/V_5 in the role of H and $Q = R$, $\beta_2 \equiv c_y$ modulo V_5 for some $y \in Q_2$. Upon replacing β_2 by $c_y^{-1} \circ \beta_2$, we can arrange that $[\beta, H_2] \le V_5$.

Next, note that $V_5/V_2 = Z(Q_2/V_2)$ and $\mathrm{Hom}_{H_2/Q_2}(Q_2/V_5, V_5/V_2) = 1$ by (1) and (2), and $H^1(H_2/Q_2; V_5/V_2) = 0$ since V_5/V_2 is H_2/Q_2-projective. So by Lemma 1.3(b), $\beta \equiv c_z$ (mod V_2) for some $z \in V_5$. Upon replacing β_2 by $c_z^{-1} \circ \beta_2$, we can now arrange that $[\beta_2, H_2] \le V_2$.

By Lemma 1.3(b), $\beta|_{Q_2}$ has the form $\beta(u) = u\chi(uV_2)$ for some $\chi \in \mathrm{Hom}_{H_2/Q_2}(Q_2/V_2, V_2)$. Also, χ factors through Q_2/V_5 since $[Q_2, Q_2] = V_5$ by (1). By (2), either $\chi = 1$, or χ is surjective with kernel R_7/V_2. In either case, $\beta|_{R_7} = \mathrm{Id}$. Also, since $W_7/V_5 = C_{Q_2/V_5}(S/Q_2)$ by (5), $\chi(W_7/V_5) \le C_{V_2}(S/Q_2) = V_1$. So $[\beta, W_7] \le V_1$.

By Lemma 1.2(a) again, $\bar{\kappa}_{H_1}$ is an isomorphism, and hence β extends to $\beta_1 \in \mathrm{Aut}(H_1)$. Let $\bar{\beta} \in \mathrm{Aut}(S/V_1)$ and $\bar{\beta}_1 \in \mathrm{Aut}(H_1/V_1)$ be the automorphisms induced by β and β_1. We have just shown that $\bar{\beta}|_{W_7} = \mathrm{Id}$, and that $[\bar{\beta}_1, S/V_1] \le V_2/V_1$. By Lemma 1.3(b) again, $\bar{\beta}|_{Q_1/V_1}$ has the form $\bar{\beta}(g) = g\psi(gW_7)$ for some $\psi \in \mathrm{Hom}_{H_1/Q_1}(Q_1/W_7, V_3/V_1)$ with $\mathrm{Im}(\psi) \le V_2/V_1$. Since Q_1/W_7 and V_3/V_1 are natural modules for $SL_2(3)$ by (5) and (4), ψ must be surjective or trivial. Since ψ is not surjective, $\bar{\beta}|_{Q_1} = \mathrm{Id}$. Also, $H^1(H_1/Q_1; V_3/V_1) = 0$ by Lemma 1.5(a), so $\bar{\beta}_1 \in \mathrm{Aut}_{V_3/V_1}(H_1/V_1)$ by Lemma 1.3(b).

We can thus arrange, upon replacing β_1 by $c_w^{-1} \circ \beta_1$ for some $w \in V_3$, that $\bar{\beta}_1 = \mathrm{Id}$, and hence that $[\beta_1, H_1] \le V_1$. (We can no longer claim that $[\beta_2, H_2] \le V_2$, but this will not be needed.) Set $H_1' = [H_1, H_1]$. By (4), $H_1' \ge Q_1$ and $H_1'/Q_1 \cong SL_2(3)$. Also, $V_1 = Z(H_1')$, so $\beta_1|_{H_1'}$ has the form $\beta_1(g) = g\phi(g)$ for some $\phi \in \mathrm{Hom}(H_1', V_1)$. But H_1' is perfect by (4) again, so $\phi = 1$, and $\beta_1 = \mathrm{Id}$. Thus $\mathrm{Out}(\mathcal{F}) = 1$, and $\bar{\kappa}_G$ is an isomorphism.

This finishes the proof of Proposition 3.2. □

4. Tools for comparing automorphisms of fusion and linking systems

Throughout this section and the next, we assume $p = 2$. Many of the definitions and statements given here are well known to hold for arbitrary primes, but we restrict to this case for simplicity. In particular, a strongly embedded subgroup $H < G$ always means a strongly 2-embedded subgroup; i.e., one such that $2 \big| |H|$ while $2 \nmid |H \cap {}^gH|$ for $g \in G \smallsetminus H$.

Definition 4.1. *Fix a finite group G, choose $S \in \mathrm{Syl}_2(G)$, and set $\mathcal{F} = \mathcal{F}_S(G)$.*

(a) *A subgroup $P \le S$ is* fully normalized in $\mathcal{F}$ *if $N_S(P) \in \mathrm{Syl}_2(N_G(P))$.*

(b) *A 2-subgroup $P \le G$ is* essential *if P is 2-centric in G (i.e., $Z(P) \in \mathrm{Syl}_2(C_G(P))$), and $\mathrm{Out}_G(P)$ has a strongly embedded subgroup. Let $\mathbf{E}_2(G)$ be the set of all essential 2-subgroups of G.*

(c) *A subgroup $P \le S$ is* $\mathcal{F}$-essential *if P is fully normalized in $\mathcal{F}$ and essential in G. Let $\mathbf{E}_{\mathcal{F}}$ be the set of all $\mathcal{F}$-essential subgroups of G.*

(d) $\widehat{\mathcal{Z}}(\mathcal{F}) = \big\{W \le S \,\big|\, W$ elementary abelian, fully normalized in $\mathcal{F}$, $W = \Omega_1(Z(C_S(W)))$, $\mathrm{Aut}_{\mathcal{F}}(W)$ has a strongly embedded subgroup$\big\}$.

Clearly, in the situation of Definition 4.1, $\mathbf{E}_{\mathcal{F}} \subseteq \mathbf{E}_2(G)$, while each member of $\mathbf{E}_2(G)$ is G-conjugate to a member of $\mathbf{E}_{\mathcal{F}}$. If $W \in \widehat{\mathcal{Z}}(\mathcal{F})$ and $P = C_S(W)$, then by the following lemma, restriction defines a surjection from $\mathrm{Out}_G(P)$ onto $\mathrm{Aut}_G(W)$ with kernel of odd order. Hence $\mathrm{Out}_G(P)$ also has a strongly embedded subgroup, and $P \in \mathbf{E}_{\mathcal{F}}$.

Lemma 4.2. *Fix a finite group G and $S \in \mathrm{Syl}_2(G)$, and set $\mathcal{F} = \mathcal{F}_S(G)$.*

(a) *If $W \le P \le G$ are 2-subgroups such that $W = \Omega_1(Z(P))$ and $P \in \mathrm{Syl}_2(C_G(W))$, then restriction induces a surjection $\mathrm{Out}_G(P) \longrightarrow \mathrm{Aut}_G(W)$ with kernel of odd order.*

(b) *If $W \in \widehat{\mathcal{Z}}(\mathcal{F})$ and $P = C_S(W)$, then $P \in \mathbf{E}_{\mathcal{F}}$.*

Proof. **(a)** By the Frattini argument, $N_G(W) \le N_G(P)C_G(W)$, with equality since W is characteristic in P. So the natural homomorphism

$$\mathrm{Out}_G(P) \cong N_G(P)/C_G(P)P \longrightarrow N_G(W)/C_G(W) \cong \mathrm{Aut}_G(W),$$

induced by restriction of automorphisms or by the inclusion $N_G(P) \le N_G(W)$ is surjective with kernel $(N_G(P) \cap C_G(W))/C_G(P)P$ of odd order.

(b) If $W \in \widehat{\mathscr{Z}}(\mathcal{F})$ and $P = C_S(W)$, then $P \in \mathrm{Syl}_2(C_G(W))$ and $W = \Omega_1(Z(P))$ by definition. So we are in the situation of (a), and $\mathrm{Out}_G(P)$ has a strongly embedded subgroup since $\mathrm{Aut}_G(W)$ does. Also, $N_G(P) \le N_G(W)$, while $N_S(P) = N_S(W) \in \mathrm{Syl}_2(N_G(W))$ since W is fully normalized in $\mathcal{F}$. Hence $N_S(P) \in \mathrm{Syl}_2(N_G(P))$, so P is also fully normalized and $P \in \mathbf{E}_{\mathcal{F}}$. □

Our proof that $\mathrm{Ker}(\mu_G) = 1$ in all cases is based on the following proposition, which is a modified version of similar results in [AOV1] and [BMO]. In most cases handled in the next section, point (e) suffices to prove that $\mathrm{Ker}(\mu_G) = 1$.

When $\alpha \in \mathrm{Aut}(\mathcal{L})$ and P is an object in $\mathcal{L}$, we let $\alpha_P\colon \mathrm{Aut}_{\mathcal{L}}(P) \longrightarrow \mathrm{Aut}_{\mathcal{L}}(\alpha(P))$ denote the restriction of α to $\mathrm{Aut}_{\mathcal{L}}(P)$.

Proposition 4.3. *Fix a finite group G, choose $S \in \mathrm{Syl}_2(G)$, and set $\mathcal{F} = \mathcal{F}_S(G)$ and $\mathcal{L} = \mathcal{L}_S^c(G)$. Each element in $\mathrm{Ker}(\mu_G)$ is represented by some $\alpha \in \mathrm{Aut}(\mathcal{L})$ such that $\alpha_S = \mathrm{Id}_{\mathrm{Aut}_{\mathcal{L}}(S)}$. For each such α, there are elements $g_P \in C_{Z(P)}(\mathrm{Aut}_S(P)) = Z(N_S(P))$, defined for each fully normalized subgroup $P \in \mathrm{Ob}(\mathcal{L})$, for which the following hold:*

(a) *The automorphism $\alpha_P \in \mathrm{Aut}(\mathrm{Aut}_{\mathcal{L}}(P))$ is conjugation by $[g_P] \in \mathrm{Aut}_{\mathcal{L}}(P)$, and g_P is uniquely determined by α modulo $C_{Z(P)}(\mathrm{Aut}_{\mathcal{F}}(P))$. In particular, $\alpha_P = \mathrm{Id}_{\mathrm{Aut}_{\mathcal{L}}(P)}$ if and only if $g_P \in C_{Z(P)}(\mathrm{Aut}_{\mathcal{F}}(P))$.*

(b) *Assume $P, Q \in \mathrm{Ob}(\mathcal{L})$ are both fully normalized in $\mathcal{F}$. If $Q = {}^aP$ for some $a \in S$, then we can choose $g_Q = {}^ag_P$.*

(c) *If $Q \le P$ are both fully normalized and are objects in $\mathcal{L}$, then $g_P \equiv g_Q$ modulo $C_{Z(Q)}(N_G(P) \cap N_G(Q))$.*

(d) *Assume, for each $W \in \widehat{\mathscr{Z}}(\mathcal{F})$ and $P = C_S(W)$, that $g_P \in C_{Z(P)}(\mathrm{Aut}_{\mathcal{F}}(P))$ (equivalently, that $\alpha_P = \mathrm{Id}_{\mathrm{Aut}_{\mathcal{L}}(P)}$). Then $\alpha = \mathrm{Id}$.*

(e) *If $\widehat{\mathscr{Z}}(\mathcal{F}) = \varnothing$, then $\mathrm{Ker}(\mu_G) = 1$. If $|\widehat{\mathscr{Z}}(\mathcal{F})| = 1$, and $|Z(S)| = 2$ or (more generally) $\mathrm{Aut}_{\mathcal{F}}(\Omega_1(Z(S))) = 1$, then $\mathrm{Ker}(\mu_G) = 1$.*

Proof. Points (a)–(c) are part of [AOV1, Proposition 4.2], (d) follows from [BMO, Proposition A.2(d)], and (e) combines parts (a) and (b) in [BMO, Proposition A.2]. □

The following notation will be useful in the next lemma, and in the next section.

Definition 4.4. *For each finite group G and each $k \ge 0$, let $\mathscr{I}_k(G)$ be the set of subgroups $H \le G$ such that $[G{:}H] = 2^k{\cdot}m$ for some odd m. Let $\mathscr{I}_{\le k}(G)$ be the union of the sets $\mathscr{I}_\ell(G)$ for $0 \le \ell \le k$.*

Lemma 4.5. *Let H be a finite group, fix $T \in \mathrm{Syl}_2(H)$, and set $\mathcal{F} = \mathcal{F}_T(H)$. Set $Q = O_2(H)$, and assume $C_H(Q) \le Q$. Assume $W \in \widehat{\mathscr{Z}}(\mathcal{F})$, and set $P = C_T(W)$. Set $V = \Omega_1(Z(Q))$, and set $H^* = \mathrm{Aut}_H(V)$, $P^* = \mathrm{Aut}_P(V)$, $T^* = \mathrm{Aut}_T(V)$, and $\mathcal{F}^* = \mathcal{F}_{T^*}(H^*)$.*

(a) *We have $W \le V$, $\mathrm{Aut}_H(W) = \mathrm{Aut}_{H^*}(W)$ has a strongly embedded subgroup, P^* is a radical 2-subgroup of H^*, and $N_{H^*}(P^*)/P^*$ has a strongly embedded subgroup.*

(b) *If H^* is a Chevalley group (i.e., untwisted) over the field $\mathbb{F}_2$, then $P^* \in \mathbf{E}_{\mathcal{F}^*} \subseteq \mathscr{I}_1(H^*)$. If $H^* \cong SU_{2n}(2)$ or $\Omega_{2n}^-(2)$ for $n \geq 2$, then $P^* \in \mathbf{E}_{\mathcal{F}^*} \subseteq \mathscr{I}_{\leq 2}(H^*)$.*

(c) *If $H^* \cong A_6$, A_7, or M_{24}, then $P^* \in \mathbf{E}_{\mathcal{F}^*} \subseteq \mathscr{I}_1(H^*)$. If $H^* \cong M_{22}$ or M_{23}, then $P^* \in \mathbf{E}_{\mathcal{F}^*} \subseteq \mathscr{I}_{\leq 2}(H^*)$. If $H^* \cong S_5$, then $P^* \in \mathscr{I}_{\leq 2}(H^*)$.*

(d) *If $H^* \cong \mathrm{Aut}(M_{22})$, then $P^* \in \mathbf{E}_{\mathcal{F}^*} \subseteq \mathscr{I}_{\leq 2}(H^*)$, and $P^* \cap O^2(H^*) \in \mathbf{E}_2(O^2(H^*))$.*

Proof. Fix $W \in \widehat{\mathscr{Z}}(\mathcal{F})$, and set $P = C_T(W)$ as above. Then $P \in \mathbf{E}_{\mathcal{F}}$ by Lemma 4.2(b). Also, $W = \Omega_1(Z(P))$ and $P \geq O_2(C_H(V)) = Q$, and hence $W \leq \Omega_1(Z(Q)) = V$.

(a) Since $V \trianglelefteq H$, each $\alpha \in \mathrm{Aut}_H(W)$ extends to $\bar{\alpha} \in \mathrm{Aut}_H(V) = H^*$, and thus $\mathrm{Aut}_{H^*}(W) = \mathrm{Aut}_H(W)$. Hence

$$N_{H^*}(P^*)/P^* \cong N_{H/Q}(P/Q)\big/(P/Q) \cong N_H(P)/P \cong \mathrm{Out}_H(P),$$

so this group has a strongly embedded subgroup. In particular, $P^* = O_2(N_{H^*}(P^*))$ (see [AKO, Proposition A.7(c)]), so P^* is a radical 2-subgroup of H^*.

(b) Since $W \leq V$, $W = \Omega_1(Z(P)) = C_V(P^*)$. By (a), $N_{H^*}(P^*)/P^*$ has a strongly embedded subgroup, and $O_2(N_{H^*}(P^*)) = P^*$.

If H^* is a group of Lie type over the field $\mathbb{F}_2$, then by the Borel-Tits theorem (see [GLS, Corollary 3.1.5]), $N_{H^*}(P^*)$ is a parabolic subgroup and $P^* = O_2(N_{H^*}(P^*))$. Hence $P^* \in \mathbf{E}_{\mathcal{F}^*}$ in this case. Also, $O^{2'}(\mathrm{Out}_{H^*}(P)) \cong O^{2'}(H/P)$ is a central product of groups of Lie type in characteristic 2 (cf. [GLS, Proposition 2.6.5(f,g)]). Since it has a strongly embedded subgroup, it must be isomorphic to $SL_2(2) \cong S_3$ (hence $P \in \mathscr{I}_1(H^*)$), or possibly to $A_5 \cong SL_2(4) \cong \Omega_4^-(2)$ if $H^* \cong SU_{2n}(2)$ or $\Omega_{2n}^-(2)$ for $n \geq 2$ (in which case $P \in \mathscr{I}_2(H^*)$). Note that we cannot get $SU_3(2)$ since we only consider even dimensional unitary groups.

(c) If $H^* \cong M_n$ for $n = 22, 23, 24$, then by [GL, pp. 42–44], it is of characteristic 2 type, in the sense that all 2-local subgroups are strictly 2-constrained. So $N_{H^*}(P^*)$ is strictly 2-constrained, P^* is centric in this group, and hence $P^* \in \mathbf{E}_{\mathcal{F}^*}$. Also, $\mathbf{E}_{\mathcal{F}^*} \subseteq \mathscr{I}_1(H^*)$ if $H^* \cong M_{24}$ [OV, Proposition 6.5], while $\mathbf{E}_{\mathcal{F}^*} \subseteq \mathscr{I}_{\leq 2}(H^*)$ if $H^* \cong M_{22}$ or M_{23} [OV, Table 5.2].

The remaining cases ($H^* \cong A_6$, A_7, or S_5) are elementary.

(d) The radical 2-subgroups of $H^* \cong \mathrm{Aut}(M_{22})$ are listed in [Y, Table VIII]. There are just three classes of such subgroups Q for which $N(Q)/Q$ has a strongly embedded subgroup, of which the members of two have index 2 in a Sylow 2-subgroup and those of the third have index 4. Each of them is essential in $\mathrm{Aut}(M_{22})$, and contains with index 2 an essential 2-subgroup of M_{22}. □

We will need to identify the elements of $\widehat{\mathscr{Z}}(\mathcal{F})$, when $\mathcal{F} = \mathcal{F}_S(G)$ for a sporadic group G and $S \in \mathrm{Syl}_2(G)$. In most cases, it will turn out that $\widehat{\mathscr{Z}}(\mathcal{F}) = \{Z_2(S)\}$, which is why we need some tools for identifying this subgroup.

Lemma 4.6. *Let S be a 2-group, and assume $W \leq S$ is elementary abelian. If $[S{:}C_S(W)] = 2$, then $W \leq Z_2(S)$ and $\mathrm{rk}(W) \leq 2{\cdot}\mathrm{rk}(Z(S))$.*

Proof. Set $Q = C_S(W)$ for short; $Q \trianglelefteq S$ since it has index 2. Then $W \leq \Omega_1(Z(Q))$, and upon replacing W by $\Omega_1(Z(Q))$, we can arrange that $W \trianglelefteq S$.

Fix $x \in S \smallsetminus Q$. Since $x^2 \in Q = C_S(W)$, we have $[W, S] = [W, x] \leq C_W(x) \leq Z(S)$. So $W \leq Z_2(S)$, and $\mathrm{rk}(W) \leq 2{\cdot}\mathrm{rk}(Z(S))$. □

Lemma 4.7. *Fix a finite group G and a Sylow 2-subgroup $S \in \mathrm{Syl}_2(G)$.*

(a) *If G is one of the sporadic groups J_4, Co_2, Co_1, Suz, Ru, Fi'_{24}, F_5, F_3, F_2, or F_1, then $|Z(S)| = 2$ and $Z_2(S) \cong E_4$. If $G \cong Co_2$, then $Z_2(S)$ has type* **2ABB**, *while in all other cases, the three involutions in $Z_2(S)$ lie in the same G-conjugacy class.*

(b) *If $G \cong Fi_{22}$, then $Z_2(S) \cong E_8$ is of type $\mathbf{2A}_2\mathbf{B}_3\mathbf{C}_2$ and contains a subgroup of type $\mathbf{2B}^2$. If $G \cong Fi_{23}$, then $Z_2(S) \cong E_{16}$.*

(c) *If $G \cong HS$, $O'N$, or Co_3, then $|Z(S)| = 2$ and $Z_2(S) \cong C_4 \times C_2$.*

Proof. **(a)** In each of these cases, we choose $Q \trianglelefteq S$ and $H = N_G(Q)$ as follows, where $H^* = H/Q \cong \mathrm{Aut}_H(Q)$:

G	Co_1	Suz	Ru	F_5	F_3	F_2	F_1	J_4	Co_2	Fi'_{24}
Q	2^{1+8}_+	2^{1+6}_-	2.2^{4+6}	2^{1+8}_+	2^{1+8}_+	2^{1+22}_+	2^{1+24}_+	$E_{2^{11}}$	$E_{2^{10}}$	$E_{2^{11}}$
H^*	$\Omega_8^+(2)$	$\Omega_6^-(2)$	S_5	$\Omega_4^+(4){:}2$	A_9	Co_2	Co_1	M_{24}	$M_{22}{:}2$	M_{24}

References for all of these subgroups are given in the next section.

Assume that $|Z(Q)| = 2$; i.e., that we are in one of the first seven cases. Then $|Z(S)| = 2$, and $Z_2(S) \le Q$ since H^* acts faithfully on $Q/Z(Q)$. Set $\overline{Q} = Q/Z(Q)$, so that $Z_2(S)/Z(S) = C_{Z(\overline{Q})}(S/Q)$. If Q is extraspecial, then $\mathrm{rk}(Z_2(S)) \ge 2$: since $\overline{Q}$ has an odd number of isotropic points (cf. [Ta, Theorem 11.5]), at least one is fixed by S.

When $G \cong Co_1$ or Suz, $\overline{Q}$ is the natural (orthogonal) module for H^*, so $|C_{\overline{Q}}(S)| = 2$ (see [Cu, Theorem 6.15] or [GLS, Theorem 2.8.9]), and hence $Z_2(S) \cong E_4$.

When $G \cong Ru$, $\overline{Q}$ is special of type 2^{4+6}, $Z_2(Q) \cong E_{32}$, and H/Q acts on $Z_2(Q)/Z(Q)$ via the natural action of $\Sigma L_2(4)$ [W4, § 1.4]. So $|C_{Z(\overline{Q})}(S/Q)| = 2$ in this case, and $Z_2(S) \cong E_4$.

When $G \cong F_5$, a Sylow 2-subgroup of $O^2(H/Q) \cong \Omega_4^+(4)$ acts on $\overline{Q} \cong (\mathbb{F}_4)^4$ with 1-dimensional fixed subgroup. This subgroup lifts to $V_3 < Q$, where $V_3 \cong E_8$ and $\mathrm{Aut}_G(V_3) \cong GL_3(2)$ (see [NW, p. 365]). Thus $[V_3, S] > Z(S)$, so $Z_2(S) < V_3$, and $Z_2(S) \cong E_4$.

When $G \cong F_3$, $\overline{Q}$ as an $\mathbb{F}_2A_9$-module satisfies the hypotheses of Lemma 1.6 by [Pa, 3.7], and hence $|C_{\overline{Q}}(S/Q)| = 2$ by that lemma.

Assume $G \cong F_1$ or F_2. Thus $H^* \cong Co_1$ or Co_2, respectively. Set $T = S/Q \in \mathrm{Syl}_2(H^*)$, and let $V \trianglelefteq T$ and $K = N_{H^*}(V)$ be such that $K \cong 2^{11}.M_{24}$ or $2^{10}.M_{22}{:}2$ and $V = O_2(K)$. By [MStr, Lemmas 3.7.b & 3.8.b], $|C_{\overline{Q}}(V)| = 2$, and hence $|C_{\overline{Q}}(S/Q)| = 2$. So $Z_2(S) \cong E_4$ in both cases.

In the remaining three cases, Q is elementary abelian. When $G \cong Co_2$, $Q \cong E_{2^{10}}$ is the Golay module (dual Todd module) for $H^* \cong M_{22}{:}2$. Let $K < H^*$ be the hexad subgroup $K \cong 2^4{:}S_6$, chosen so that $K > S^* = S/Q$, and set $R = O_2(K) \cong E_{16}$. Set $Q_1 = C_Q(R)$ and $Q_5 = [R, Q]$. By [MStr, Lemma 3.3.b], $\mathrm{rk}(Q_1) = 1$, $\mathrm{rk}(Q_5) = 5$, and Q_5/Q_1 is the natural module for $S_6 \cong Sp_4(2)$. Hence $Z(S) = C_Q(S^*) = Q_1$, and $Z_2(S)/Z(S) = C_{Q/Q_1}(S^*) = C_{Q_5/Q_1}(S^*)$ also has rank 1. So $Z_2(S) \cong E_4$. The two elements in $Z_2(S) \smallsetminus Z(S)$ are S-conjugate, and do not lie in **2C** since $C_G(x) \in \mathscr{I}_3(G)$ for $x \in$ **2C** (see [W1, Table II]). By [W1, Table II] again, each **2A**-element acts on the Leech lattice with character -8, so a subgroup of type $\mathbf{2A}^2$ would act fixing only the zero vector, hence cannot be in Co_2. Thus $Z_2(S)$ has type **2ABB**.

Assume $G \cong Fi'_{24}$ or J_4. In both cases, $Q \cong E_{2^{11}}$ is the Todd module for $H^* \cong M_{24}$ (see [A3, 34.9] and [J, Theorem A.4]). Let $K < H/Q$ be the sextet subgroup $K \cong 2^6{:}3S_6$, chosen so that $K > S^* = S/Q \in \mathrm{Syl}_2(H^*)$, and set $R = O_2(K) \cong E_{64}$. By [MStr, Lemma 3.5.b], there are $\mathbb{F}_2K$-submodules $Q_1 < Q_7 < Q$ of rank 1 and 7, respectively, where $Q_1 = C_Q(R)$

and $Q_2 = [R, Q]$, and where $K/R \cong 3S_6$ acts on Q_7/Q_1 as the dual module to R. Thus $Z(S) = Q_1$ and $Z_2(S)/Z(S) = C_{Q_7/Q_1}(S^*) \cong R/[S^*, R]$. Since $S^* \cong UT_5(2)$ contains only two subgroups of rank 6, one easily sees that $|R/[S^*, R]| = 2$, and hence $Z_2(S) \cong E_4$.

In all of the above cases except Co_2, S contains a normal elementary abelian subgroup V of rank at least 2 all of whose involutions lie in the same G-conjugacy class. We refer to the lists of maximal 2-local subgroups in the next section, where we can take $V = V_i = Z(O_2(H_i))$, for $i = 2$ (when $G \cong Co_1$, Suz, F_2, or F_1), $i = 3$ (when $G \cong J_4$, Ru, or F_5), or $i = 5$ (for $G \cong Fi_{24}'$ or F_3). Since each normal subgroup of order at least 4 contains $Z_2(S)$, the involutions in $Z_2(S)$ also lie in the same class.

(b) When $G \cong Fi_{22}$ and $S \in \mathrm{Syl}_2(G)$, $Z(S) = \langle z \rangle$ has order 2, and $H = C_G(z) \cong (2 \times 2_+^{1+8}){:}U_4(2){:}2$. Set $Q = O_2(H)$. Then $O^2(H/Q)$ acts faithfully on $\bar{Q} = Q/Z(Q)$ as a 4-dimensional unitary space over $\mathbb{F}_4$, so $\dim_{\mathbb{F}_4}(C_{\bar{Q}}(S \cap O^2(H))) = 1$ [Cu, Theorem 6.15]. An involution hQ with $h \in H \smallsetminus O^2(H)$ acts as a field automorphism on the unitary space $\bar{Q}$, so $\dim_{\mathbb{F}_2}(C_{\bar{Q}}(S)) = 1$. Since $|Z(Q)| = 4$, this proves that $|Z_2(S)| \leq 8$.

To see that $Z_2(S)$ does contain a subgroup of rank 3, consider a hexad group $V \cong E_{32}$ normal in S, generated by six transpositions $\{a_1, \dots, a_6\}$ (where $a_1 \cdots a_6 = 1$), ordered so that $\mathrm{Aut}_S(V) = \langle (1\,2)(3\,4), (1\,2)(5\,6), (1\,3)(2\,4) \rangle$. Then

$$Z(S) = C_V(S) = \langle a_5a_6 \rangle \qquad \text{and} \qquad Z_2(S) = \langle a_1a_2, a_3a_4, a_5, a_6 \rangle \text{ is of type } \mathbf{2A}_2\mathbf{B}_3\mathbf{C}_2,$$

and $\langle a_1a_2, a_3a_4 \rangle < Z_2(S)$ has type $\mathbf{2B}^2$.

When $G \cong Fi_{23}$ and $S \in \mathrm{Syl}_2(G)$, $Z(S) \cong E_4$ contains involutions x, y, z in each of the three classes **2A**, **2B**, and **2C**, respectively. Also, $C_G(x) \cong 2Fi_{22}$, so we can identify $S/\langle x \rangle$ as a Sylow 2-subgroup of Fi_{22}, whose center lifts to a pair of elements of class **2B** and **2C** in G. Thus $S/Z(S) \cong T/Z(T)$ when $T \in \mathrm{Syl}_2(Fi_{22})$, we already saw that $|Z(T/Z(T))| = 4$, and so $|Z_2(S)| = 16$. All involutions in Fi_{22} lift to involutions in $2{\cdot}Fi_{22} < G$, so $Z_2(S)$ is elementary abelian.

(c) When $G \cong HS$ or $O'N$, this follows from the descriptions by Alperin [Alp, Corollary 1] and O'Nan [O'N, § 1] of S as being contained in an extension of the form $4^3.L_3(2)$. (In terms of their presentations, $Z(S) = \langle v_1^2v_3^2 \rangle$, while $Z_2(S) = \langle v_1v_3, v_1^2v_2^2 \rangle$.) When $G \cong Co_3$, it follows from a similar presentation of $S \leq 4^3.(2 \times L_3(2))$ (see, e.g., [OV, § 7]). □

5. Injectivity of μ_G

We are now ready to prove, when $p = 2$, that $\mathrm{Ker}(\mu_G) = 1$ for each of the sporadic groups G not handled in Proposition 2.1. This will be done in each case by determining the set $\widehat{\mathcal{Z}}(\mathcal{F})$ and then applying Proposition 4.3. One can determine $\widehat{\mathcal{Z}}(\mathcal{F})$ using the lists of radical 2-subgroups found in [Y] and other papers. However, we decided to do this instead using lists of maximal 2-local subgroups, to emphasize that the details needed to prove this result are only a small part of what is needed to determine the radical subgroups.

Proposition 5.1. *Assume $p = 2$, and let G be a sporadic simple group whose Sylow 2-subgroups have order at least 2^{10}. Then* $\mathrm{Ker}(\mu_G) = 1$.

Proof. There are fifteen groups to consider, and we go through the list one or two at a time. In each case, we fix $S \in \mathrm{Syl}_2(G)$ and set $\mathcal{F} = \mathcal{F}_S(G)$, $\mathcal{L} = \mathcal{L}_S^c(G)$, and $\widehat{\mathcal{Z}} = \widehat{\mathcal{Z}}(\mathcal{F})$. When we list representatives for the conjugacy classes of maximal 2-local subgroups of G, we always choose them so that each such H satisfies $S \cap H \in \mathrm{Syl}_2(H)$. In particular, if H has odd index in G, then $H \geq S$ and hence $O_2(H) \trianglelefteq S$ and $Z(O_2(H)) \trianglelefteq S$ (making the choice of H unique in most cases).

In four of the cases, when $G \cong M_{24}$, He, Co_2, or Fi_{23}, $\widehat{\mathscr{Z}}$ has two members, and we use Proposition 4.3(b,c,d) to prove that μ_G is injective. In all of the other cases, $|\widehat{\mathscr{Z}}| = 1$ and $|Z(S)| = 2$, and we can apply Proposition 4.3(e). Recall that by Proposition 4.3, each class in $\mathrm{Ker}(\mu_G)$ contains an element $\alpha \in \mathrm{Aut}(\mathcal{L})$ which acts as the identity on $\mathrm{Aut}_{\mathcal{L}}(S)$.

Note that whenever $|Z(S)| = 2$ and $W \cong E_4$ is normal in S, $[S{:}C_S(W)] = 2$, and hence $W \leq Z_2(S)$ by Lemma 4.6.

For convenience, we sometimes write $A \sim_H B$ to mean that A is H-conjugate to B, and $A \leq_H B$ to mean that A is H-conjugate to a subgroup of B.

$\boldsymbol{G \cong M_{24}, He}$ **:** We identify S with $UT_5(2)$, the group of (5×5) upper triangular matrices over F_2. Let $e_{ij} \in S$ (for $i < j$) be the matrix with 1's on the diagonal, and with unique nonzero off-diagonal entry 1 in position (i, j). Set $W_1 = \langle e_{15}, e_{25}\rangle$ and $W_4 = \langle e_{14}, e_{15}\rangle$, $Q_i = C_S(W_i)$ for $i = 1, 4$, and $Q_{14} = Q_1 \cap Q_4$. By [OV, Propositions 6.2 & 6.9], Q_1 and Q_4 are essential in G, and are the only essential subgroups with noncyclic center. Hence by Lemma 4.2, $\widehat{\mathscr{Z}} = \{W_1, W_4\}$. Also, $Q_{14} = A_1A_2$, where A_1 and A_2 are the unique subgroups of S of type E_{64}, and hence $Q_{14} = J(S)$ is characteristic in S, Q_1, and Q_4.

Fix $\alpha \in \mathrm{Aut}(\mathcal{L})$ which is the identity on $\mathrm{Aut}_{\mathcal{L}}(S)$. By Proposition 4.3(a), there are elements $g_P \in C_{Z(P)}(\mathrm{Aut}_S(P))$, chosen for each $P \leq S$ which is fully normalized in $\mathcal{F}$ and 2-centric in G, such that $\alpha|_{\mathrm{Aut}_{\mathcal{L}}(P)}$ is conjugation by $[g_P]$. Then $g_{Q_1} = g_{Q_{14}} = g_{Q_4} \in Z(S)$ by point (c) in the proposition, since for $i = 1, 4$, $C_{Z(Q_{14})}(N_G(Q_i)) = 1$. Set $g = g_{Q_1}$; upon replacing α by $c_g^{-1} \circ \alpha$, we can arrange that $\alpha|_{\mathrm{Aut}_{\mathcal{L}}(Q_i)} = \mathrm{Id}$ for $i = 1, 4$ without changing $\alpha|_{\mathrm{Aut}_{\mathcal{L}}(S)}$. Hence $\mathrm{Ker}(\mu_G) = 1$ by Proposition 4.3(d).

$\boldsymbol{G \cong J_4}$ **:** By [KW, § 2], there are four conjugacy classes of maximal 2-local subgroups, represented by:

$$H_1 \cong 2_+^{1+12}.3M_{22}{:}2, \quad H_3 \cong 2^{3+12}.(\Sigma_5 \times L_3(2)), \quad H_{10} \cong 2^{10}{:}L_5(2), \quad H_{11} \cong 2^{11}{:}M_{24}.$$

Set $Q_i = O_2(H_i)$ and $V_i = Z(Q_i) \cong E_{2^i}$. Note that $H_{10} \in \mathscr{I}_1(G)$, while $H_i \geq S$ for $i \neq 10$.

Fix $W \in \widehat{\mathscr{Z}}$ and set $P = C_S(W)$. Then $N_G(W) \leq_G H_i$ for some i, in which case $P \geq_G Q_i$ and $W \leq_G V_i$ by Lemma 4.5(a). Thus $i > 1$ since $\mathrm{rk}(W) \geq 2$. By Lemma 4.5(b,c), $\mathrm{Aut}_P(V_i) \in \mathbf{E}_2(\mathrm{Aut}_{H_i}(V_i)) \subseteq \mathscr{I}_1(\mathrm{Aut}_{H_i}(V_i))$, and hence $P \in \mathscr{I}_1(H_i)$.

Thus either $[S{:}P] = 2$, in which case $W = Z_2(S) \cong E_4$ by Lemmas 4.6 and 4.7(a); or $i = 10$ and $[S{:}P] = 4$. In the latter case, since $H_{10}/V_{10} \cong L_5(2)$ acts on V_{10} as $\Lambda^2(\mathbb{F}_2^5)$, we have $\mathrm{rk}(W) = \mathrm{rk}(C_{V_{10}}(P/V_{10})) \leq \mathrm{rk}\big(C_{V_{10}}([S^*, S^*])\big) = 2$ for $S^* \in \mathrm{Syl}_2(H_{10}/V_{10})$. So $W \cong E_4$ in all cases.

By [KW, Table 1], there are two classes of four-groups in G whose centralizer has order a multiple of 2^{19}, denoted $AAA^{(1)}$ and $ABB^{(1)}$, with centralizers of order $2^{20}{\cdot}3{\cdot}5$ and $2^{19}{\cdot}3{\cdot}5$, respectively. Thus $AAA^{(1)} \sim_G Z_2(S)$ (Lemma 4.6), and W lies in one of the two classes. Since $\mathrm{Aut}_G(ABB^{(1)})$ is a 2-group, $W \not\sim_G ABB^{(1)}$. Hence $\widehat{\mathscr{Z}} = \{Z_2(S)\}$, and μ_G is injective by Proposition 4.3(e).

$\boldsymbol{G \cong Co_3}$ **:** By [OV, Proposition 7.3], there is at most one essential subgroup with noncyclic center (denoted R_1); and $R_1 \in \mathbf{E}_{\mathcal{F}}$ since otherwise $N_G(Z(S))$ would control fusion in G. Also, $\mathrm{Out}_G(R_1) \cong S_3$ and $Z(R_1) \in \widehat{\mathscr{Z}}$ by [OV, Propsition 7.5]. So $|\widehat{\mathscr{Z}}| = 1$, and $\mathrm{Ker}(\mu_G) = 1$ by Proposition 4.3(e). (In fact, it is not hard to see that $\widehat{\mathscr{Z}} = \{\Omega_1(Z_2(S))\}$.)

$G \cong Co_2$: By [W1, pp. 113–114], each 2-local subgroup of G is contained up to conjugacy in one of the following subgroups:

$$H_1 \cong 2_+^{1+8}.Sp_6(2), \quad H_4 \cong 2^{4+10}.(S_3 \times S_5), \quad H_5 \cong (2^4 \times 2_+^{1+6}).A_8, \quad H_{10} \cong 2^{10}{:}M_{22}{:}2$$

$$K_1 \cong U_6(2){:}2, \quad K_2 \cong McL, \quad K_3 \cong M_{23}.$$

For $i = 1, 4, 5, 10$, set $Q_i = O_2(H_i)$ and $V_i = Z(Q_i) \cong E_{2^i}$.

Recall (Lemma 4.7(a)) that $Z_2(S)$ has type $\mathbf{2ABB}$. Set $Z_2(S) = \{1, x, y_1, y_2\}$, where $x \in \mathbf{2A}$ and $y_1, y_2 \in \mathbf{2B}$. Thus $Z(S) = \langle x \rangle$, $H_1 = C_G(x)$, and we can assume $H_5 = C_G(y_1)$.

Fix $W \in \widehat{\mathscr{Z}}$, and set $P = C_S(W)$. Then $W \geq Z(S)$, so $W \cap \mathbf{2A} \neq \varnothing$. If $\text{rk}(W) = 2$, then W must have type $\mathbf{2A}^2$. Since each $\mathbf{2A}$-element acts on the Leech lattice with character -8 [W1, Table II], W would fix only the zero element, and hence cannot be contained in Co_2. Thus $\text{rk}(W) \geq 3$. If $[S{:}P] = 2$, then $W \leq Z_2(S)$ by Lemma 4.6, which is impossible since $\text{rk}(Z_2(S)) = 2$. So $[S{:}P] \geq 4$.

If $N_G(W) \leq_G K_2 \cong McL$ or $N_G(W) \leq_G K_3 \cong M_{23}$, then by the list of essential subgroups in these groups in [OV, Table 5.2], $\text{rk}(W) = \text{rk}(Z(P)) \leq 2$. So these cases are impossible.

The subgroup $K_1 \cong U_6(2){:}2$ in Co_2 is the stabilizer of a triple of 2-vectors in the Leech lattice [Cu1, pp. 561–2], which we can choose to have the form $(4, 4, 0, \dots)$, $(0, -4, 4, \dots)$, and $(-4, 0, -4, \dots)$. Using this, we see that the maximal parabolic subgroups $2_+^{1+8}{:}U_4(2){:}2$, $2^9{:}L_3(4){:}2$, and $2^{4+8}{:}(S_3 \times S_5)$ in K_1 can be chosen to be contained in H_1, H_{10}, and H_4, respectively. If $N_G(W) \leq_G K_1$, then it is contained in one of the maximal parabolics by the Borel-Tits theorem, and so $N_G(W)$ is also conjugate to a subgroup of one of the H_i.

Thus in all cases, we can assume that $N_G(W) \leq H_i$ for some $i = 1, 4, 5, 10$. Then $P \geq Q_i$ and $W = Z(P) \leq V_i$, so $i \neq 1$.

Assume $i = 5$, and recall that $\text{Fr}(Q_5) = \langle y_1 \rangle$. The image of W in $V_5/\langle y_1 \rangle \cong E_{16}$ has rank at least 2 since $\text{rk}(W) \geq 3$, so $\text{Aut}_{N_G(W)}(V_5/\langle y_1 \rangle)$ is the stabilizer subgroup of a projective line and plane in $A_8 \cong SL_4(2)$ (a line and plane determined by S). So there is at most one member of $\widehat{\mathscr{Z}}$ whose normalizer is in $H_5 = C_G(y_1)$, and it has rank 3 if it exists.

Now, $V_4 \geq Z_2(S)$ since it is normal in S. Since $Z_2(S)$ has type $\mathbf{2ABB}$, S_5 must act on $V_4^\#$ with orbits of order 5 and 10, and has type $\mathbf{2A}_5\mathbf{B}_{10}$. So if $i = 4$, then W is a rank 3 subgroup of the form $\mathbf{2A}_3\mathbf{B}_4$ (the centralizer of a 2-cycle in S_5). There is exactly one $\mathbf{2B}$-element in W whose product with each of the other $\mathbf{2B}$-elements is in class $\mathbf{2A}$, so $N_G(W) \leq_G H_5 = N_G(\mathbf{2B})$: a case which we have already handled.

Assume $i = 10$, and set $H^* = H_{10}/V_{10} \cong \text{Aut}_{H_{10}}(V_{10}) \cong \text{Aut}(M_{22})$ and $P^* = P/V_{10}$. By Lemma 4.5(d), $P^* \cap O^2(H^*)$ is an essential 2-subgroup of $O^2(H^*) \cong M_{22}$. Since $P \notin \mathscr{I}_1(H_{10})$, $P^* \cap O^2(H^*)$ has the form $2^4{:}2 < 2^4{:}S_5$ (the duad subgroup) by [OV, Table 5.2], and this extends to $P^* \cong 2^5{:}2 < 2^5{:}S_5 < \text{Aut}(M_{22})$. But $V_{10}.2^5$ has center V_4 (see [MStr, Lemma 3.3]), and so we are back in the case $i = 4$.

Thus $\widehat{\mathscr{Z}} = \{W_1, W_2\}$, where $\text{rk}(W_i) = 3$ and $N_G(W_i) \leq C_G(y_i) \sim_G H_5$ for $i = 1, 2$. (These also correspond to the two 2-cycles in $\text{Aut}_S(V_4) < S_5$.) Set $P_i = C_S(W_i)$. Fix $\alpha \in \text{Aut}(\mathcal{L})$ which is the identity on $\text{Aut}_{\mathcal{L}}(S)$, and let $g_i = g_{P_i} \in C_{W_i}(\text{Aut}_S(P_i)) = Z_2(S)$ $(i = 1, 2)$ be as in Proposition 4.3. Thus $\alpha|_{\text{Aut}_{\mathcal{L}}(P_i)}$ is conjugation by g_i. Since $y_i \in Z(N_G(P_i))$, we can replace g_i by $g_i y_i$ if necessary and arrange that $g_i \in Z(S)$. Then $g_1 = g_2$ by Proposition 4.3(b) and since P_1 and P_2 are S-conjugate. Upon replacing α by $c_{g_1}^{-1} \circ \alpha$, we can arrange that $\alpha|_{\text{Aut}_{\mathcal{L}}(Q_i)} = \text{Id}$ for $i = 1, 4$ without changing $\alpha|_{\text{Aut}_{\mathcal{L}}(S)}$. Hence $\text{Ker}(\mu_G) = 1$ by Proposition 4.3(d).

$G \cong Co_1$: There are three conjugacy classes of involutions in G, of which those in $\mathbf{2A}$ are 2-central. By [Cu2, Theorem 2.1], each 2-local subgroup of G is contained up to conjugacy in one of the subgroups

$$H_1 \cong 2_+^{1+8}.\Omega_8^+(2), \quad H_2 \cong 2^{2+12}.(A_8 \times S_3), \quad H_4 \cong 2^{4+12}.(S_3 \times 3S_6), \quad H_{11} \cong 2^{11}M_{24};$$
$$K_1 \cong (A_4 \times G_2(4)){:}2, \quad K_2 \cong (A_6 \times U_3(3)){:}2.$$

Curtis also included Co_2 in his list, but it is not needed, as explained in [W1, p. 112]. Set $Q_i = O_2(H_i)$ and $V_i = Z(Q_i) \cong E_{2^i}$.

Assume $W \in \widehat{\mathscr{Z}}$. Then $W \geq Z(S)$, so $W \cap \mathbf{2A} \neq \varnothing$. If $W \cap \mathbf{2C} \neq \varnothing$, then $N_G(W) \leq H_i$ for some $i = 1, 2, 4, 11$ by [Cu2, Lemma 2.2] (where the involution centralizer in the statement is for an involution of type **2A** or **2C**). If W contains no **2C**-elements, then by the argument given in [Cu2, p. 417], based on the action of the elements on the Leech lattice, a product of distinct **2A**-elements in W must be of type **2A**. So in this case, $\langle W \cap \mathbf{2A} \rangle$ is **2A**-pure, and its normalizer is contained in some H_i by [Cu2, Lemma 2.5] (together with Wilson's remark [W1, p. 112]).

Set $P = C_S(W)$; then $P \geq Q_i$ and hence $W \leq V_i$. Also, $i \neq 1$ since $\operatorname{rk}(W) > 1$. By Lemmas 4.5(b,c), $\operatorname{Aut}_P(V_i) \in \mathbf{E}_2(\operatorname{Aut}_{H_i}(V_i)) \subseteq \mathscr{I}_1(\operatorname{Aut}_{H_i}(V_i))$. Since $H_i \geq S$, we have $[S{:}P] = 2$, and $W = \Omega_1(Z(P)) \leq Z_2(S)$ by Lemma 4.6, with equality since $|Z_2(S)| = 4$ by Lemma 4.7. It follows that $\widehat{\mathscr{Z}} = \{Z_2(S)\}$, and $\operatorname{Ker}(\mu_G) = 1$ by Proposition 4.3(e).

***G* ≅ *Suz* :** By [W2, § 2.4], there are three classes of maximal 2-local subgroups which are normalizers of **2A**-pure subgroups, represented by

$$H_1 \cong 2_-^{1+6}.\Omega_6^-(2), \qquad H_2 \cong 2^{2+8}.(A_5 \times S_3), \qquad H_4 \cong 2^{4+6}.3A_6.$$

Fix $W \in \widehat{\mathscr{Z}}$, and set $P = C_S(W)$. Since $W \geq Z(S)$, it contains **2A**-elements, and since $\langle W \cap \mathbf{2A} \rangle$ is **2A**-pure by [W2, p. 165], $N_G(W) \leq H_i$ for some $i \in \{1, 2, 4\}$. Then $P \geq O_2(H_i)$ and $W \leq V_i \overset{\text{def}}{=} Z(O_2(H_i))$ by Lemma 4.5(a), so $i \neq 1$ since $\operatorname{rk}(W) \geq 2$. Hence $i = 2$ or 4, so $\operatorname{Aut}_G(V_i) \cong S_3$ or A_6, and $\operatorname{Aut}_P(V_i) \in \mathbf{E}_2(\operatorname{Aut}_G(V_i)) \subseteq \mathscr{I}_1(\operatorname{Aut}_G(V_i))$ by Lemma 4.5(b,c). So $[S{:}P] = 2$, and $W \leq Z_2(S)$ by Lemma 4.6, with equality since $|Z_2(S)| = 4$ by Lemma 4.7. Thus $\widehat{\mathscr{Z}} = \{Z_2(S)\}$, and $\operatorname{Ker}(\mu_G) = 1$ by Proposition 4.3(e).

***G* ≅ *Ru* :** There are two conjugacy classes of involutions, of which the **2A**-elements are 2-central. By [W4, § 2.5], the normalizer of each **2A**-pure subgroup is contained up to conjugacy in one of the following subgroups:

$$H_1 \cong 2.2^{4+6}.S_5 \qquad H_3 \cong 2^{3+8}.L_3(2) \qquad H_6 \cong 2^6.G_2(2).$$

Set $Q_i = O_2(H_i)$ and $V_i = Z(Q_i)$. For each $i = 1, 3, 6$, V_i is elementary abelian of rank i and **2A**-pure.

Fix $W \in \widehat{\mathscr{Z}}$, and set $P = C_S(W) \in \mathbf{E}_{\mathcal{F}}$. Then $W \geq Z(S)$, so W contains **2A**-elements. Since the subgroup $W_0 = \langle W \cap \mathbf{2A} \rangle$ is **2A**-pure [W4, p. 550], $N_G(W) \leq N_G(W_0) \leq H_i$ for $i \in \{1, 3, 6\}$. Since H_i is 2-constrained, $P \geq Q_i = O_2(H_i)$ and $W \leq V_i$ by Lemma 4.5(a). Hence $i \neq 1$, since $\operatorname{rk}(W) \geq 2$.

For $i = 3, 6$, $\operatorname{Aut}_G(V_i)$ is a Chevalley group over $\mathbb{F}_2$, so by Lemma 4.5(b), $\operatorname{Aut}_P(V_i) \in \mathscr{I}_1(\operatorname{Aut}_G(V_i))$, and hence $P \in \mathscr{I}_1(H_i)$. So $|P| = 2^{13}$ (if $i = 3$) or 2^{11} (if $i = 6$). Also, W is **2A**-pure since V_i is. By [W4, § 2.4], there are four classes of subgroups of type $\mathbf{2A}^2$, of which only one has centralizer of order a multiple of 2^{11}, and that one must be the class of $Z_2(S)$ (Lemma 4.7). So $W = Z_2(S)$ if $i = 3$, or if $i = 6$ and $\operatorname{rk}(W) = 2$.

As explained in [W4, § 2.5], if $W \leq V_6$ and $\operatorname{rk}(W) \geq 3$, then either $N_G(W) \leq_G H_1$, or $N_G(W)$ is in the normalizer of a group of the form $\mathbf{2A}^2$ which must be conjugate to $Z_2(S)$ by the above remarks, or $C_G(W) = V_6$. The first case was already handled. If $N_G(W) \leq_G N_G(Z_2(S))$, then $N_G(W) \leq_G H_3$ by [W4, p. 550], and this case was already handled. If $C_G(W)=V_6$, then $W=P=V_6$, which is impossible since $G_2(2)$ does not have a strongly embedded subgroup. Thus $\widehat{\mathscr{Z}}=\{Z_2(S)\}$, and μ_G is injective by Proposition 4.3(e).

$G \cong Fi_{22}$, Fi_{23}, or Fi_{24}' : It will be simplest to handle these three groups together. Their maximal 2-local subgroups were determined in [W5, Proposition 4.4], [Fl], and [W8, Theorem D], and are listed in Table 5.1. To make it clearer how 2-local subgroups of one Fischer group lift to larger ones, we include the maximal 2-local subgroups in $Fi_{21}\cong PSU_6(2)$ (the maximal parabolic subgroups by the Borel-Tits theorem), and give the normalizers in Fi_{24} of the maximal 2-local subgroups of Fi_{24}'. Also, we include one subgroup which is not

	$PSU_6(2)=Fi_{21}$	Fi_{22}	Fi_{23}	Fi_{24}
K_1			$2{\cdot}Fi_{22}$	$(2\times 2{\cdot}Fi_{22}).2$
K_2		$2{\cdot}Fi_{21}$	$2^2{\cdot}Fi_{21}.2$	$(2\times 2^2{\cdot}Fi_{21}).S_3$
K_3			$S_4\times Sp_6(2)$	$S_4\times\Omega_8^+(2){:}S_3$
H_1	$2_+^{1+8}{:}U_4(2)$	$(2\times 2_+^{1+8}{:}U_4(2)).2$	$(2^2\times 2_+^{1+8}).(3\times U_4(2)).2$	$(2_+^{1+12}).3U_4(3).2^2$
H_2	$2^{4+8}{:}(A_5\times S_3)$	$2^{5+8}{:}(A_6\times S_3)$	$2^{6+8}{:}(A_7\times S_3)$	$2^{7+8}{:}(A_8\times S_3)$
H_3	$2^9{:}M_{21}$	$2^{10}.M_{22}$	$2^{11}.M_{23}$	$2^{12}.M_{24}$
H_4		$2^6.Sp_6(2)$	$[2^7.Sp_6(2)]$	$2^8{:}SO_8^-(2)$
H_5				$2^{3+12}(SL_3(2)\times S_6)$

TABLE 5.1

maximal: $H_4\le Fi_{23}$ is contained in K_1.

As usual, set $Q_i=O_2(H_i)$ and $V_i=Z(Q_i)$. For each of the four groups Fi_n, $H_i\ge S$ for $i=1,2,3,5$. We write $K_i^{(n)}$, $H_i^{(n)}$, $Q_i^{(n)}$, or $V_i^{(n)}$ when we need to distinguish K_i, H_i, Q_i, or V_i as a subgroup of Fi_n.

Each of the groups Fi_n for $21\le n\le 24$ is generated by a conjugacy class of 3-transpositions. By [A3, 37.4], for $22\le n\le 24$, Fi_n has classes of involutions $\mathscr{J}_m$, for $m=1,2,3$ when $n=22,23$ and for $1\le m\le 4$ when $n=24$. Each member of $\mathscr{J}_m$ is a product of m commuting transpositions (its *factors*): a unique such product except when $n=22$ and $m=3$ (in which case each $x\in\mathscr{J}_3$ has exactly two sets of factors) and when $n=24$ and $m=4$. Note that $\mathscr{J}_1=\mathbf{2A}$, $\mathscr{J}_2=\mathbf{2B}$, and $\mathscr{J}_3=\mathbf{2C}$ in Fi_{22} and Fi_{23}, while $\mathscr{J}_2=\mathbf{2A}$ and $\mathscr{J}_4=\mathbf{2B}$ in Fi_{24}' (and the other two classes are outer automorphisms).

In all cases, K_1, K_2, and H_1 are normalizers of sets of $(n{-}22)$, $(n{-}21)$, and $(n{-}20)$ pairwise commuting transpositions. Also, H_3 is the normalizer of the set of all n transpositions in S; these generate $Q_3=V_3$ of rank $n-12$, and form a Steiner system of type $(n-19,n-16,n)$. Then H_2 is the normalizer of a pentad, hexad, heptad, or octad of transpositions: one of the members in that Steiner system. From these descriptions, one sees, for example, that a subgroup of type K_i $(i=1,2)$ or H_i $(i=1,2,3)$ in Fi_{22} lifts to a subgroup of type K_i or H_i, respectively, in $2{\cdot}Fi_{22}<Fi_{23}$ and in $2Fi_{22}.2<Fi_{24}'$.

By [W5, Lemma 4.2], each $\mathbf{2B}$-pure elementary abelian subgroup of Fi_{22} ($\mathbf{2B}=\mathscr{J}_2$) supports a symplectic form for which $(x,y)=1$ exactly when conjugation by y exchanges the two factors of x. Then $V_4^{(22)}$ is characterized as a subgroup of type $\mathbf{2B}^6$ with nonsingular symplectic form. Since each $\mathbf{2B}$-element in Fi_{22} lifts to a $\mathbf{2B}$- and a $\mathbf{2C}$-element in $2{\cdot}Fi_{22}<Fi_{23}$, $H_4^{(22)}$ lifts to $H_4^{(23)}$ of the form $2^7.Sp_6(2)$.

By [W8, Corollary 3.2.3], each elementary abelian subgroup of $G\cong Fi_{24}'$ supports a symplectic form where $(x,y)=1$ if and only if y is in the "outer half" of $C_G(x)\cong 2{\cdot}Fi_{22}.2$

or $2_+^{1+12}.3U_4(3){:}2$. By [W8, Proposition 3.3.3], the form on $V_4^{(24)} \cong E_{2^8}$ is nonsingular, and $V_4^{(24)}$ contains elements in both classes $\mathbf{2A} = \mathscr{J}_2$ and $\mathbf{2B} = \mathscr{J}_4$. If $x \in V_4 \cap \mathbf{2A}$, then $V_4 \cap O^2(C_G(x))/\langle x\rangle$ has rank 6 with nonsingular symplectic form in Fi_{22}, and hence $\bigl(C_{H_4}(x) \cap O^2(C_G(x))\bigr)/\langle x\rangle$ is conjugate to $H_4^{(22)}$. Thus $H_4^{(24)}$ contains a lifting of $H_4^{(22)}$ via the inclusion $2{\cdot}Fi_{22} < Fi_{24}'$.

Fix $W \in \widehat{\mathscr{Z}}$, and set $P = C_S(W)$. If $N_G(W) \le K_i$ for $i = 1$ or 2, then since $W \ge Z(S)$, and $O_2(K_i)$ does not contain involutions of all classes represented in $Z(S)$ (note that $O_2(K_i^{(24)}) \cap Fi_{24}'$ is $\mathbf{2A}$-pure for $i = 1, 2$), we have $\overline{W} = (W \cap F^*(K_i))/O_2(K_i) \ne 1$. Thus $N_G(\overline{W})$ is a 2-local subgroup of $F^*(K_i)/O_2(K_i) \cong Fi_{22}$ or Fi_{21}, and hence is contained up to conjugacy in one of its maximal 2-local subgroups. So (after applying this reduction twice if $i = 1$), $N_G(W) \le H_i$ for some $1 \le i \le 4$. We will see below that we can also avoid the case $N_G(W) \le K_3$ (when $G \cong Fi_{23}$ or Fi_{24}'), and hence that in all cases, $N_G(W) \le_G H_i$ for some $1 \le i \le 5$.

When $G \cong Fi_{22}$, we just showed that (up to conjugacy) we can assume $N_G(W) \le H_i$ for some $i = 1, 2, 3, 4$. If $i = 4$, then by Lemma 4.5(b), $W = C_{V_4}(P/V_4)$ where $P/V_4 \in \mathbf{E}_2(H_4/V_4)$ and $H_4/V_4 \cong Sp_6(2)$, so W must be totally isotropic with respect to the symplectic form on V_4 described above. But in that case, by [W5, Lemma 3.1], the subgroup $W^* > W$ generated by all factors of involutions in W is again elementary abelian, and $N_G(W) \le N_G(W^*) \le H_j$ for some $j = 1, 2, 3$.

Thus $N_G(W) \le H_i$ where $i \in \{1, 2, 3\}$, H_i is 2-constrained, and so $P = C_S(W) \ge O_2(H_i)$ and $W = \Omega_1(Z(P)) \le V_i$. Also, $i \ne 1$ since V_1 has type $\mathbf{2AAB}$ (so $\mathrm{Aut}_G(V_1)$ is a 2-group). Hence $i = 2, 3$, and $H_i \in \mathscr{I}_0(G)$. By Lemma 4.5(c), $\mathrm{Aut}_P(V_i) \in \mathbf{E}_2(\mathrm{Aut}_{H_i}(V_i))$, and either $[S{:}P] = 2$, or $i = 3$ and $[S{:}P] = 4$. In this last case, $P/V_3 \cong 2^4{:}2$ is contained in a duad subgroup $D \cong 2^4{:}S_5$ in M_{22}. Also, $O_2(D) \cong E_{16}$ permutes $V_3 \cap \mathbf{2A}$ in five orbits of length 4, each of which forms a hexad together with the remaining two transpositions. Hence $C_{V_3}(O_2(D))$ has type $\mathbf{2AAB}$, and cannot contain W.

Thus $[S{:}P] = 2$, and hence $\mathrm{rk}(W) = 2$ and $W \le Z_2(S)$ by Lemma 4.6. By Lemma 4.7(b), $Z_2(S)$ has rank 3 and type $\mathbf{2A}_2\mathbf{B}_3\mathbf{C}_2$. Since $\mathrm{Aut}_G(W)$ is not a 2-group, W must be the $\mathbf{2B}$-pure subgroup of rank 2 in $Z_2(S)$. (Note that the factors of the involutions in W form a hexad.) Thus $|\widehat{\mathscr{Z}}| = 1$, and $\mathrm{Ker}(\mu_G) = 1$ by Proposition 4.3(e).

When $G \cong Fi_{23}$, $W = \Omega_1(Z(P))$ strictly contains $Z(S)$. Hence $\mathrm{rk}(W) \ge 3$, and W contains involutions of each type $\mathbf{2A}$, $\mathbf{2B}$, and $\mathbf{2C}$. If $|W \cap \mathbf{2A}| = 1$ 2, or 3, then $N_G(W) \le K_1$, K_2, or H_1, respectively, while if $|W \cap \mathbf{2A}| \ge 4$, then $N_G(W) \le H_2$ or H_3, depending on whether or not the transpositions in W are contained in a heptad. So by the above remarks, we can assume in all cases that $N_G(W) \le H_i$ for some $i = 1, 2, 3, 4$. Since H_i is strictly 2-constrained, $P \ge Q_i$ and $W \le V_i$. If $i = 1$, then $W = V_1$ since it has rank at least 3, and thus W has type $\mathbf{2A}_3\mathbf{B}_3\mathbf{C}$. The case $i = 4$ can be eliminated in the same way as it was when $G \cong Fi_{22}$.

Assume $N_G(W) \le H_2$ and $W \le V_2$, where $\mathrm{Aut}_G(V_2) \cong A_7$. Write $V_2 \cap \mathbf{2A} = \{a_1, \ldots, a_7\}$, permuted by $\mathrm{Aut}_G(V_2) \cong A_7$ in the canonical way. Then (up to choice of indexing), $\mathrm{Aut}_P(V_2)$ is one of the two essential subgroups $P_1^* = \langle (1\,2)(3\,4), (1\,2)(5\,6)\rangle$ and $P_2^* = \langle (1\,2)(3\,4), (1\,3)(2\,4)\rangle$. Set $W_j = C_{V_2}(P_j^*)$ and $P_j = C_S(W_j)$; thus $P_j^* = \mathrm{Aut}_{P_j}(V_2)$ and hence $[S{:}P_j] = 2$. Also, $W_1 = \langle a_1a_2, a_3a_4, a_5a_6\rangle$ has type $\mathbf{2AB}_3\mathbf{C}_3$, and $W_2 = \langle a_5, a_6, a_7\rangle$ has type $\mathbf{2A}_3\mathbf{B}_3\mathbf{C}$ (thus $W_2 \sim_G V_1$).

If $N_G(W) \le H_3$ and $W \le V_3$, then $\mathrm{Aut}_G(V_3) \cong M_{23}$ has three essential subgroups, of which two are contained in the heptad group $2^4{:}A_7$ and one in the triad group $2^4{:}(3 \times A_5){:}2$. In the first case, the subgroup 2^4 acts on $V_3 \cap \mathbf{2A}$ fixing a heptad, and we are back in the case

$N_G(W) \le H_2$. In the second case, the subgroup 2^4 fixes a rank 3 subgroup in V_3 generated by three tranpositions, and so the essential subgroup $2^4{:}2$ fixes only $Z(S)$.

Thus $\widehat{\mathscr{Z}} = \{W_1, W_2\}$, where $W_1, W_2 \le Z_2(S)$ by Lemma 4.6, and $W_1, W_2 < V_2$. Also, $\sigma_2 = (5\,6\,7)$ normalizes P_2 and Q_2 and permutes the three **2B**-elements in W_1 cyclically, while $\sigma_1 = (1\,3\,5)(2\,4\,6)$ normalizes P_1 and Q_2 and permutes the three **2A**-elements in W_2 cyclically.

Fix $\alpha \in \mathrm{Aut}(\mathcal{L})$ which is the identity on $\mathrm{Aut}_{\mathcal{L}}(S)$. Let $g_P \in C_{Z(P)}(\mathrm{Aut}_S(P))$, for all $P \in \mathrm{Ob}(\mathcal{L})$ fully normalized in $\mathcal{F}$, be as in Proposition 4.3. Thus $\alpha|_{\mathrm{Aut}_{\mathcal{L}}(P)}$ is conjugation by g_P. Set $g = g_{Q_2} \in C_{Z(Q_2)}(\mathrm{Aut}_S(Q_2)) = Z(S)$. Upon replacing α by $c_g^{-1} \circ \alpha$, we can arrange that $g_{Q_2} = 1$, and hence that α is the identity on $\mathrm{Aut}_{\mathcal{L}}(Q_2)$. Since $Z(S) = Z(N_G(S))$ (recall $Z(S)$ has type **2ABC**), α is still the identity on $\mathrm{Aut}_{\mathcal{L}}(S)$.

Set $P_j = C_S(W_j)$ $(j = 1, 2)$. By Proposition 4.3(c) and since σ_j normalizes P_j and Q_2, $g_{P_1} \equiv g_{Q_2} = 1$ modulo $\langle W_1 \cap \mathbf{2A}\rangle$, and $g_{P_2} \equiv g_{Q_2} = 1$ modulo $\langle W_2 \cap \mathbf{2C}\rangle$. Also, $\langle W_1 \cap \mathbf{2A}\rangle \le Z(N_G(P_1))$ and $\langle W_2 \cap \mathbf{2C}\rangle \le Z(N_G(P_2))$ (since $N_G(P_i) \le N_G(W_i)$). Thus $\alpha|_{\mathrm{Aut}_{\mathcal{L}}(P_j)} = \mathrm{Id}$ for $j = 1, 2$, so $\alpha = \mathrm{Id}$ by Proposition 4.3(d). This proves that $\mathrm{Ker}(\mu_G) = 1$.

When $G \cong Fi_{24}'$, since $W \ge Z(S)$, it contains at least one **2B**-element (recall $\mathbf{2A} = \mathcal{J}_2$ and $\mathbf{2B} = \mathcal{J}_4$). By Propositions 3.3.1, 3.3.3, 3.4.1, and 3.4.2 in [W8] (corrected in [LW, § 2]), the normalizer of every elementary abelian 2-subgroup of G is contained up to conjugacy in K_1, K_2, or one of the H_i for $i \le 5$, except when it is **2A**-pure and the symplectic form described above is nonsingular. So we can assume that $N_G(W)$ is contained in one of these groups. Together with earlier remarks, this means that we can eliminate all of the K_i, and assume that $N_G(W) \le H_i$ for some $1 \le i \le 5$. So $P \ge Q_i$ and $W \le V_i$, and $i \ne 1$ since $\mathrm{rk}(V_1) = 1$.

By Lemma 4.5(b,c), $W = C_{V_i}(P^*)$, where $P^* = \mathrm{Aut}_P(V_i)$ is an essential 2-subgroup of $H_i^* = \mathrm{Aut}_{H_i}(V_i)$. If $i = 2, 3, 5$, then $P^* \in \mathscr{I}_1(H_i^*)$ by Lemma 4.5(b,c), and hence $[S{:}P] = 2$ since $H_i \ge S$. So $W = Z_2(S)$ in these cases by Lemmas 4.7(a) and 4.6.

If $i = 4$, then $H_i^* \cong \Omega_8^-(2)$, and the conditions $P^* \in \mathbf{E}_2(H_4^*)$ and $\mathrm{rk}(C_V(P^*)) \ge 2$ imply that $N_G(W) \cong 2^8.(2^{3+6}.(S_4 \times 3))$ (the stabilizer of an isotropic line and plane in the projective space of V_4). Hence $\mathrm{rk}(W) = 2$ and $|P| = 2^{19}$. By [W8, Table 15], there are only two classes of four-groups in G with centralizer large enough, one of type **2AAB** (impossible since $\mathrm{Aut}(W)$ is not a 2-group), and the other $Z_2(S)$ of type $\mathbf{2B}^2$. Thus $\widehat{\mathscr{Z}} = \{Z_2(S)\}$, and $\mathrm{Ker}(\mu_G) = 1$ by Proposition 4.3(e).

$G \cong F_5$: By [NW, § 3.1], each 2-local subgroup of G is contained up to conjugacy in one of the subgroups

$$H_1 \cong 2_+^{1+8}.(A_5 \times A_5){:}2, \qquad H_3 \cong 2^3.2^2.2^6.(3 \times L_3(2)), \qquad H_6 \cong 2^6{\cdot}U_4(2),$$

$$K_1 \cong 2{\cdot}HS{:}2, \qquad K_2 \cong (A_4 \times A_8){:}2 < A_{12}.$$

As usual, set $Q_i = O_2(H_i)$ and $V_i = Z(Q_i)$ for $i = 1, 3, 6$. Then V_1 and V_3 are **2B**-pure, and $O_2(K_1)$ and $O_2(K_2)$ are **2A**-pure. By [NW, § 3.1], for each elementary abelian 2-subgroup $V \le G$, there is a quadratic form $\mathfrak{q}\colon V \to \mathbb{F}_2$ defined by sending **2A**-elements to 1 and **2B**-elements to 0.

Fix $W \in \widehat{\mathscr{Z}}$, and set $P = C_S(W)$. Then $W \ge Z(S)$, so $W \cap \mathbf{2B} \ne \varnothing$. So either the quadratic form $\mathfrak{q}$ on W is nondegenerate and $\mathrm{rk}(W) \ge 3$, or there is a **2B**-pure subgroup $W_0 \le W$ such that $N_G(W) \le N_G(W_0)$. By [NW, § 3.1], in this last case, $N_G(W_0) \le H_i$ for $i = 1$ or 3.

If $N_G(W) \le N_G(W_0) \le H_i$ for $i = 1, 3$, then $P \ge O_2(H_i)$, so $W \le V_i$. In particular, $i \ne 1$. If $N_G(W) \le H_3$, then P has index 2 in S since $\mathrm{Aut}_G(V_i) \cong L_3(2)$, so $W = Z_2(S)$ by Lemmas 4.6 and 4.7(a).

Now assume $\mathfrak{q}$ is nondegenerate as a quadratic form (and $\mathrm{rk}(W) \ge 3$). Let $W^* < W$ be a $\mathbf{2A}$-pure subgroup of rank 2, and identify $C_G(W^*)$ with $(2^2 \times A_8) < A_{12} < G$. If $\mathrm{rk}(W) = 3$, then we can identify W with $\langle (1\,2)(3\,4), (1\,3)(2\,4), (5\,6)(7\,8) \rangle$, so $C_G(W) \cong 2^2 \times (2^2 \times A_4){:}2$, $P = C_S(W) \cong 2^2 \times (2^4{:}2)$, $Z(P) \cong 2^4$, which contradicts the assumption that $W = \Omega_1(Z(P))$. If $\mathrm{rk}(W) \ge 4$, then it must be conjugate to one of the subgroups (1), (2), or (3) defined in [NW, p. 364] (or contains (2) or (3) if $\mathrm{rk}(W) = 5$). Then $C_G(W) \cong E_{2^6}$ or $E_{16} \times A_4$, so $P = W \sim_G V_6$, which is impossible since $\mathrm{Aut}_G(V_6) \cong U_4(2)$ does not contain a strongly embedded subgroup.

Thus $\widehat{\mathscr{Z}} = \{Z_2(S)\}$, and μ_G is injective by Proposition 4.3(e).

$\boldsymbol{G \cong F_3}$ **:** By [W11, Theorem 2.2], there are two classes of maximal 2-local subgroups of G, represented by $H_1 \cong 2_+^{1+8}.A_9$ and $H_5 \cong 2^5.SL_5(2)$. Set $Q_i = O_2(H_i)$ and $V_i = Z(Q_i) \cong E_{2^i}$ $(i = 1, 5)$.

Fix $W \in \widehat{\mathscr{Z}}$, set $P = C_S(W)$, and let $i = 1, 5$ be such that $N_G(W) \le H_i$. Then $P \ge O_2(H_i)$ and $W \le V_i$, so $i = 5$. By Lemma 4.5(b), $P/V_5 \in \mathbf{E}_2(H_5/V_5)$ (where $H_5/V_5 \cong L_5(2)$) and $[S{:}P] = 2$. Hence $W \le Z_2(S)$ by Lemma 4.6. Since $|Z_2(S)| = 4$ by Lemma 4.7, this proves that $\widehat{\mathscr{Z}} = \{Z_2(S)\}$, and hence that $\mathrm{Ker}(\mu_G) = 1$ by Proposition 4.3(e).

$\boldsymbol{G \cong F_2,\ F_1}$ **:** If $G \cong F_1$, then by [MS, Theorem 1], there are maximal 2-local subgroups of the form

$$H_1 \cong 2^{1+24}.Co_1, \quad H_2 \cong 2^2.[2^{33}].(M_{24} \times S_3), \quad H_3 \cong 2^3.[2^{36}].(L_3(2) \times 3{\cdot}S_6),$$
$$H_5 \cong 2^5.[2^{30}].(S_3 \times L_5(2)), \quad H_{10} \cong 2^{10+16}{\cdot}\Omega_{10}^+(2),$$

If $G \cong F_2$, then by [MS, Theorem 2], there are maximal 2-local subgroups of the form

$$H_1 \cong 2^{1+22}.Co_2, \quad H_2 \cong 2^2.[2^{30}].(M_{22}{:}2 \times S_3), \quad H_3 \cong 2^3.[2^{32}].(L_3(2) \times S_5),$$
$$H_5 \cong 2^5.[2^{25}].L_5(2), \quad H_9 \cong 2^{9+16}{\cdot}Sp_8(2),$$

As usual, we set $Q_i = O_2(H_i)$, and $V_i = Z(Q_i) \cong E_{2^i}$. In both cases ($G \cong F_1$ or F_2), $H_1 = C_G(x) \ge S$ for $x \in \mathbf{2B}$, and $H_i > S$ ($V_i \trianglelefteq S$) for each i.

Fix $W \in \widehat{\mathscr{Z}}$, and set $P = C_S(W)$. Then $W \ge Z(S)$, and hence W contains $\mathbf{2B}$-elements. By [Mei, Lemma 2.2], W is "of 2-type", in the sense that $C_G(O_2(C_G(W)))$ is a 2-group, since the subgroup generated by a $\mathbf{2B}$-element is of 2-type. In particular, $C_G(P)$ is a 2-group and hence $C_G(P) = Z(P)$.

A $\mathbf{2B}$-pure elementary abelian 2-subgroup $V \le G$ is called *singular* if $V \le O_2(C_G(x))$ for each $x \in V^\#$. If $G \cong F_1$, then by [MS, Proposition 9.1], applied with P in the role of Q and $t = 1$, there is a subgroup $W_0 \le W$ such that $N_G(W) \le N_G(W_0)$, and either W_0 is $\mathbf{2B}$-pure and singular or $W = W_0 \sim_G V_{10}$. Since $\mathrm{Aut}_G(V_{10}) \cong \Omega_{10}^+(2)$ has no strongly embedded subgroup, W_0 must be $\mathbf{2B}$-pure and singular, and hence $N_G(W) \le H_i$ for some $i = 1, 2, 3, 5$ by [MS, Theorem 1]. Thus $P \ge Q_i$ and $W \le V_i$, so W is also $\mathbf{2B}$-pure and singular.

If $G \cong F_2$, identify $G = C_M(x)/x$, where $M \cong F_1$ and x is a $\mathbf{2A}$-element in M. Let $\widetilde{P} \le C_M(x)$ be such that $x \in \widetilde{P}$ and $\widetilde{P}/\langle x \rangle = P$, and set $\widetilde{W} = \Omega_1(Z(\widetilde{P}))$. Then $x \in \widetilde{W}$ and $\widetilde{W}/\langle x \rangle \le W$, and $(W \cap \mathbf{2B}) \subseteq \widetilde{W}/\langle x \rangle$ since $\mathbf{2B}$-elements in G lift to pairs of involutions of classes $\mathbf{2A}$ and $\mathbf{2B}$ in M (coming from a subgroup of type $\mathbf{2BAA}$ in $Q_1 < M$). By [MS, Proposition 9.1] again, applied with $\widetilde{P}$ in the role of Q and $t = x$, there is a subgroup $\widetilde{W}_0 \le \widetilde{W}$ such that $N_M(\widetilde{W}) \le N_M(\widetilde{W}_0)$, and either $\widetilde{W}_0$ is $\mathbf{2B}$-pure and singular or $\widetilde{W} = \widetilde{W}_0 \sim_M V_{10}^{(M)}$. In the latter case, $W \sim_G V_9$, which is impossible since $\mathrm{Aut}_G(V_9) \cong Sp_8(2)$ has no strongly embedded subgroup. Again, we conclude that $N_G(W) \leq H_i$ for some $i = 1, 2, 3, 5$ [MS, Theorem 2], and that $W \leq V_i$ by Lemma 4.5(a) and hence is **2B**-pure and singular.

By [MS, Lemma 4.2.2], applied with $W = 1$ (if $G \cong F_1$) or $W = \langle x \rangle$ ($G \cong F_2$), the automizer of a singular subgroup is its full automorphism group. Since $GL_n(2)$ has no strongly embedded subgroup for $n \neq 2$, this implies that $\mathrm{rk}(W) = 2$. By [MS, Lemma 4.4], if we identify $Q_1/V_1 \cong E_{2^{24}}$ with the mod 2 Leech lattice, then **2A**-elements correspond to the 2-vectors and **2B**-elements to the classes of 4-vectors, and hence $H_1/Q_1 \cong Co_1$ acts transitively on each. So F_1 contains a unique class of singular subgroups of rank 2. A similar argument, using [MS, Corollary 4.6], now shows that F_2 also contains a unique class of singular subgroup of rank 2. Since $H_2 > S$, each of these classes has a representative normal in S, so $W \trianglelefteq S$, and $W = Z_2(S)$ by Lemmas 4.7(a) and 4.6.

To conclude, we have now shown that $\widehat{\mathscr{Z}} = \{Z_2(S)\}$ in both cases. Hence $\mathrm{Ker}(\mu_G) = 1$ by Proposition 4.3(e).

This finishes the proof of Proposition 5.1. □

By inspection in the above proof, in all cases where $Z_2(S) \cong E_4$ and its involutions are G-conjugate, we have $\widehat{\mathscr{Z}}(\mathcal{F}) = \{Z_2(S)\}$. A general result of this type could greatly shorten the proof of Proposition 5.1, but we have been unable to find one. The following example shows that this is not true without at least some additional conditions.

Set $G = 2^4{:}15{:}4 \cong \mathbb{F}_{16} \rtimes \mathit{\Gamma} L_1(16)$. Set $E = O_2(G) \cong E_{16}$, fix $S \in \mathrm{Syl}_2(G)$, and let $P \trianglelefteq S$ be the subgroup of index 2 containing E. Then $Z_2(S) = Z(P) \cong E_4$ and $\mathrm{Aut}_G(Z_2(S)) \cong S_3$, while $\widehat{\mathscr{Z}}(\mathcal{F}_S(G)) = \{Z_2(S), E\}$.

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