p-STUBBORN SUBGROUPS OF CLASSICAL COMPACT LIE GROUPS

by Bob Oliver

For any compact Lie group G and any prime p, a p-stubborn subgroup $P \subseteq G$ is a subgroup which satisfies the following conditions:

(a) P is p-toral (i.e., an extension of a torus by a finite p-group)

(b) $N(P)/P$ is finite

(c) $\mathcal{O}_p(N(P)/P) = 1$: the Sylow p-subgroups of $N(P)/P$ intersect trivially (or equivalently, there is no nontrivial normal p-subgroup $1 \neq Q \lhd N(P)/P$).

These are the subgroups which were used in [JMO] to approximate the space BG at the prime p. More precisely, let $\mathcal{O}(G)$ denote the "orbit category" of G: the category whose objects are the orbits G/H for closed subgroups $H \subseteq G$, and whose morphisms are all G-maps between orbits. Let $\mathcal{R}_p(G)$ denote the full subcategory of $\mathcal{O}(G)$ consisting of those orbits G/P for p-stubborn $P \subseteq G$. Then for any G and p, the natural projection map

$$\underset{G/P \in \mathcal{R}_p(G)}{\underset{\longrightarrow}{\text{hocolim}}} \ (EG/P) \longrightarrow BG \qquad\qquad (EG/P \simeq BP)$$

induces an equivalence in $\mathbb{F}_p$-homology [JMO, §§1–2]. This decomposition of BG, and the category $\mathcal{R}_p(G)$, play a central role in [JMO] and [JMO2] as a tool for describing sets of homotopy classes of maps from BG to BH for any (other) compact connected Lie group H.

The main result of this paper — a list of the p-stubborn subgroups of each classical compact Lie group — was originally obtained while doing preliminary work on [JMO]; but in fact such an explicit result was not needed in that paper. However, recent results of Notbohm [N], proving in many cases the uniqueness of the completed classifying spaces $BG^\wedge_p$ (uniqueness among spaces with the same mod p cohomology), do require a more precise description of the p-stubborn subgroups of the classical compact Lie groups. And that provides the motivation for publishing this paper now.

In Theorems 6 and 8 below, the p-stubborn subgroups of the matrix groups $\mathrm{U}(n)$, $\mathrm{O}(n)$, and $\mathrm{Sp}(n)$ are described explicitly for each n and p — in terms of subgroups defined in Definitions 1 and 2. These results are then extended, in Theorems 10 and 12, to describe the p-stubborn subgroups of $\mathrm{SU}(n)$ and $\mathrm{SO}(n)$. In all cases, the p-stubborn subgroups show a surprisingly simple pattern: being generated from a small collection of "basic" p-stubborn subgroups by products and wreath products.

By [JMO, Proposition 1.6(i)], $\mathcal{R}_p(\widetilde{G}) \cong \mathcal{R}_p(G)$ for any compact connected Lie group G and any finite covering group $\widetilde{G}$ of G. So the results here also describe the p-stubborn subgroups of the simple groups $\mathrm{Spin}(n)$, $\mathrm{PSU}(n)$, etc.

For a finite group G, the "p-stubborn" subgroups of G are precisely the same as the "radical" p-subgroups which play a role in the representation theory of finite groups (cf. [Al]). The description given here of p-stubborn subgroups of the classical matrix groups is very similar in nature to the description by Alperin & Fong of radical subgroups of symmetric groups and finite general linear groups [AF, Theorems 2A & 4A].

We first define certain p-stubborn subgroups of Σ_n, $\mathrm{O}(n)$, $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$: subgroups which will be seen to generate all other p-stubborn subgroups of the classical groups.

Let $\sigma_0, \ldots, \sigma_{k-1} \in \Sigma_{p^k}$ denote the permutations

$$\sigma_r(i) = \begin{cases} i + p^r & \text{if } i \equiv 1, \ldots, (p-1)p^r \pmod{p^{r+1}} \\ i - (p-1)p^r & \text{if } i \equiv (p-1)p^r + 1, \ldots, p^{r+1} \pmod{p^{r+1}}. \end{cases}$$

These generate an elementary abelian p-subgroup $\langle \sigma_0, \ldots, \sigma_{k-1} \rangle \cong (C_p)^k$ of rank k, which can be identified with the translation action of $(C_p)^k$ on itself.

Set $\zeta = e^{2\pi i/p}$, a primitive p-th root of unity. Define matrices

$$A_0, \ldots, A_{k-1}, B_0, \ldots, B_{k-1} \in \mathrm{U}(p^k)$$

by setting

$$(A_r)_{ij} = \begin{cases} \zeta^{[(i-1)/p^r]} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad \text{and} \qquad (B_r)_{ij} = \begin{cases} 1 & \text{if } \sigma_r(i) = j \\ 0 & \text{if } \sigma_r(i) \neq j \end{cases}$$

(where $[-]$ denotes greatest integer). The A_r are thus all diagonal matrices, and the B_r are the permutation matrices for the σ_r. These matrices satisfy the commutator relations

$$[A_r, A_s] = I = [B_r, B_s] = [B_r, A_s] \quad (r \neq s), \qquad \text{and} \qquad [B_r, A_r] = \zeta \cdot I.$$

Finally, let $Q(8) \subseteq S^1(j) \subseteq \mathbb{H}^*$ denote the subgroups

$$Q(8) = \{\pm 1, \pm i, \pm j, \pm k\} \qquad \text{and} \qquad S^1(j) = \{a + bi, aj + bk \mid a^2 + b^2 = 1\}.$$

Definition 1. *For each prime p and each $k \geq 0$, subgroups*

$$E_{p^k} \subseteq \Sigma_{p^k} \qquad \textit{and} \qquad \Gamma^{\mathrm{U}}_{p^k} \subseteq \mathrm{U}(p^k) \quad \Big(\subseteq \mathrm{O}(2p^k),\ \mathrm{Sp}(p^k)\Big)$$

are defined by setting

$$E_{p^k} = \langle \sigma_0, \ldots, \sigma_{k-1} \rangle \cong (C_p)^k$$

and

$$\Gamma^{\mathrm{U}}_{p^k} = \langle u{\cdot}I, A_r, B_r \mid u \in S^1,\ 0 \leq r \leq k{-}1 \rangle \subseteq \mathrm{U}(p^k).$$

If $p=2$, then $A_r, B_r \in \mathrm{O}(2^k)$, and we define

$$\Gamma^{\mathrm{O}}_{2^k} \subseteq \overline{\Gamma}^{\mathrm{O}}_{2^k} \subseteq \mathrm{O}(2^k) \quad \text{and} \quad \Gamma^{\mathrm{Sp}}_{2^k} \subseteq \overline{\Gamma}^{\mathrm{Sp}}_{2^k} \subseteq \mathrm{Sp}(2^k)$$

by setting

$$\begin{aligned} \Gamma^{\mathrm{O}}_{2^k} &= \langle -I, A_r, B_r \mid 0 \le r \le k-1 \rangle, \\ \overline{\Gamma}^{\mathrm{O}}_{2^k} &= \langle \alpha^{\oplus 2^{k-1}}, A_r, B_r \mid \alpha \in \mathrm{SO}(2),\ 0 \le r \le k-1 \rangle, \\ \Gamma^{\mathrm{Sp}}_{2^k} &= \langle u \cdot I, A_r, B_r \mid u \in Q(8),\ 0 \le r \le k-1 \rangle, \\ \overline{\Gamma}^{\mathrm{Sp}}_{2^k} &= \langle u \cdot I, A_r, B_r \mid u \in S^1(j),\ 0 \le r \le k-1 \rangle. \end{aligned}$$

The groups $\Gamma^X_{p^k}$ sit in central extensions

$$1 \to S^1 \to \Gamma^{\mathrm{U}}_{p^k} \to (C_p)^{2k} \to 1$$

$$1 \to \{\pm 1\} \to \Gamma^{\mathrm{O}}_{2^k} \to (C_2)^{2k} \to 1, \qquad 1 \to \{\pm 1\} \to \Gamma^{\mathrm{Sp}}_{2^k} \to (C_2)^{2k+2} \to 1;$$

and the groups $\overline{\Gamma}^X_{p^k}$ in (non-central) extensions

$$1 \to S^1 \to \overline{\Gamma}^{\mathrm{O}}_{2^k} \to (C_2)^{2k-1} \to 1, \qquad 1 \to S^1 \to \overline{\Gamma}^{\mathrm{Sp}}_{2^k} \to (C_2)^{2k+1} \to 1$$

or

$$1 \to \mathrm{O}(2) \to \overline{\Gamma}^{\mathrm{O}}_{2^k} \to (C_2)^{2k-2} \to 1, \qquad 1 \to S^1(j) \to \overline{\Gamma}^{\mathrm{Sp}}_{2^k} \to (C_2)^{2k} \to 1.$$

When describing p-stubborn subgroups of the classical Lie groups, it will be convenient to let $\mathbb{G}$ denote one of the classes O, U, or Sp. We make here the usual identifications $\mathbb{G}(k) \times \mathbb{G}(m) \subseteq \mathbb{G}(k+m)$ and $\mathbb{G}(k) \wr \Sigma_m \subseteq \mathbb{G}(km)$ for $k, m \ge 1$. A subgroup $P \subseteq \mathbb{G}(n)$ will be called irreducible if the corresponding P-representation on $\mathbb{R}^n$, $\mathbb{C}^n$, or $\mathbb{H}^n$ is irreducible.

Definition 2. *For fixed $G = \mathbb{G}(n)$ and fixed p, let $\mathcal{T}_\Gamma(p, G) \subseteq \mathcal{T}_{\mathrm{irr}}(p, G) \subseteq \mathcal{T}_{\mathrm{prod}}(p, G)$ be the sets of p-toral subgroups of G defined as follows:*

(i) $\mathcal{T}_\Gamma(p, \mathrm{U}(p^k)) = \{\Gamma^{\mathrm{U}}_{p^k}\}$ for any p and any $k \ge 0$,

$\mathcal{T}_\Gamma(p, \mathrm{O}(2p^k)) = \mathcal{T}_\Gamma(p, \mathrm{Sp}(p^k)) = \{\Gamma^{\mathrm{U}}_{p^k}\}$ *for any odd p and any $k \ge 0$,*

$\mathcal{T}_\Gamma(2, \mathrm{O}(1)) = \{\Gamma^{\mathrm{O}}_1 = \mathrm{O}(1)\}$, $\quad \mathcal{T}_\Gamma(2, \mathrm{O}(2)) = \{\overline{\Gamma}^{\mathrm{O}}_2 = \mathrm{O}(2)\}$,

$\mathcal{T}_\Gamma(2, \mathrm{O}(2^k)) = \{\Gamma^{\mathrm{O}}_{2^k}, \overline{\Gamma}^{\mathrm{O}}_{2^k}\}$ *for any $k \ge 2$,*

$\mathcal{T}_\Gamma(2, \mathrm{Sp}(2^k)) = \{\Gamma^{\mathrm{Sp}}_{2^k}, \overline{\Gamma}^{\mathrm{Sp}}_{2^k}\}$ *for any $k \ge 0$;*

and $\mathcal{T}_\Gamma(p, \mathbb{G}(n)) = \emptyset$ in all other cases.

(ii) $\mathcal{T}_{\mathrm{irr}}(p, G)$ is the set of those wreath products in G of the form

$$P = \Gamma \wr E_{q_1} \wr \cdots \wr E_{q_r}$$

where $\Gamma \in \mathcal{T}_\Gamma(p, \mathbb{G}(m))$, $q_i = p^{t_i} > 1$, *and* $n = m \cdot q_1 \cdots q_r$; *and where* $q_1 \geq 4$ ($t_1 \geq 2$) *if* $\Gamma = \Gamma_1^{\mathrm{O}} = \mathrm{O}(1)$.

(iii) If $p = 2$, *or if* $\mathbb{G} = \mathrm{U}$ *or* Sp, *or if* $\mathbb{G} = \mathrm{O}$ *and* n *is even, then* $\mathcal{T}_{\mathrm{prod}}(p, G)$ *is the set of all products of the form*

$$P = P_1 \times P_2 \times \cdots \times P_s,$$

where $P_i \in \mathcal{T}_{\mathrm{irr}}(p, \mathbb{G}(n_i))$ *for each* i *(and* $n = n_1 + \cdots + n_s$*). If* p *is odd and* $G = \mathrm{O}(2m{+}1)$ $(m \geq 0)$, *then* $\mathcal{T}_{\mathrm{prod}}(p, G)$ *is the set of all products of the form* $P \times 1$ *for* $P \in \mathcal{T}_{\mathrm{prod}}(p, \mathrm{O}(2m))$.

Note that $\mathcal{T}_\Gamma(p, G)$ and $\mathcal{T}_{\mathrm{irr}}(p, G)$ are both empty if $G = \mathbb{G}(n)$ and n is not a power of p (or not twice a power of p if $\mathbb{G} = \mathrm{O}$ and p is odd).

The main result of this paper is that for G as in Definition 2, every subgroup in $\mathcal{T}_{\mathrm{irr}}(p, G)$ is p-stubborn (Theorem 6), and every p-stubborn subgroup of G is conjugate to a subgroup in $\mathcal{T}_{\mathrm{prod}}(p, G)$ (Theorem 8). It will also be specified in Theorem 6 exactly which elements of $\mathcal{T}_{\mathrm{prod}}(p, G)$ are p-stubborn. The elements of $\mathcal{T}_{\mathrm{irr}}(p, G)$ are conjugacy class representatives of those p-stubborn subgroups which are irreducible as representations.

The proofs of Theorems 6 and 8 below involve computing normalizers of the p-stubborn subgroups, and in particular for the products and wreath products which occur in Definition 2. The following lemma lists some of the relations which will be useful when doing this.

Lemma 3. *Let* $\mathbb{G} = \mathrm{O}$, U, *or* Sp, *and set* $G = \mathbb{G}(n)$.

(i) Assume $H \subseteq G$ *has the form*

$$H = (H_1)^{m_1} \times \cdots \times (H_k)^{m_k},$$

where $1 \neq H_i \subseteq \mathbb{G}(n_i)$ *is irreducible for each* i *(and* $n = \sum m_i n_i$*), and where for* $i \neq j$, *either* $n_i \neq n_j$, *or* H_i *and* H_j *are not conjugate in* $\mathbb{G}(n_i)$. *Then*

$$N_G(H) = (N_{\mathbb{G}(n_1)}(H_1)) \wr \Sigma_{m_1} \times \cdots \times (N_{\mathbb{G}(n_k)}(H_k)) \wr \Sigma_{m_k}.$$

(ii) Assume that $H \subseteq G$ *has the form* $H = H_0 \wr L$, *where* $1 \neq H_0 \subseteq \mathbb{G}(m)$ *is irreducible and* $L \subseteq \Sigma_k$ *acts transitively on* $\{1, \ldots, k\}$ *(and* $n = mk$*). Then* H *is irreducible (as a subgroup of* $\mathbb{G}(n)$*). If in addition,* $(H_0)^k \lhd N_G(H)$, *then*

$$N_{\mathbb{G}(n)}(H)/H = N_{\mathbb{G}(m)}(H_0)/H_0 \times N_{\Sigma_k}(L)/L.$$

(iii) Let $L \subseteq \Sigma_k$ *be any subgroup which acts freely and transitively on* $\{1, \ldots, k\}$. *Then* $N_{\Sigma_k}(L)/L \cong \mathrm{Aut}(L)$.

Proof. (i) Let V denote the n-dimensional representation of $H \subseteq \mathbb{G}(n)$. Then $V \cong (V_1)^{m_1} \times \cdots \times (V_k)^{m_k}$, where each V_i is an irreducible H_i-representation. The m_i

factors V_i are pairwise nonisomorphic as H-representations, since they are acted upon by distinct factors H_i in H. Hence, any element in $N(H)$ leaves each $(V_i)^{m_i}$ invariant, and permutes the individual factors V_i. It follows that

$$N(H) \subseteq (N_{\mathbb{G}(n_1)}(H_1)) \wr \Sigma_{m_1} \times \cdots \times (N_{\mathbb{G}(n_r)}(H_r)) \wr \Sigma_{m_r},$$

and the opposite inclusion is clear.

(ii) If V is the n-dimensional representation of H, then $V|(H_0)^k$ splits as a sum of k pairwise nonisomorphic irreducible representations, which are permuted transitively by L. So V is irreducible as an H-representation. If $(H_0)^k \lhd N_G(H)$, then $N_G(H) \subseteq N_{\mathbb{G}(n)}((H_0)^k) = N_{\mathbb{G}(m)}(H_0) \wr \Sigma_k$ by part (i).

Fix any element $\xi = \sigma\cdot(g_1, \ldots, g_k) \in N_G((H_0)^k)$, where $g_i \in N_{\mathbb{G}(m)}(H_0)$ and $\sigma \in \Sigma_k$. Then $\xi \in N_G(H)$ if and only if for all $\lambda \in L \subseteq \Sigma_k$,

$$\begin{aligned}\xi\lambda\xi^{-1} &= \sigma\cdot(g_1, \ldots, g_k)\cdot\lambda\cdot(g_1, \ldots, g_k)^{-1}\cdot\sigma^{-1} \\ &= \big(\sigma(g_1 g_{\lambda(1)}^{-1}, \ldots, g_k g_{\lambda(k)}^{-1})\sigma^{-1}\big)\cdot\sigma\lambda\sigma^{-1} \in H_0 \wr L.\end{aligned}$$

And since L acts transitively, $\xi \in N_G(H)$ if and only if $g_1 \equiv \cdots \equiv g_k \pmod{H_0}$, and $\sigma \in N_{\Sigma_k}(L)$.

(iii) Identify Σ_k with the group $\mathrm{Bij}(L)$ of bijections from the set L to itself, where $L \subseteq \mathrm{Bij}(L)$ is the subgroup given by left translation. Then

$$\begin{aligned}N_{\mathrm{Bij}(L)}(L)/L &\cong \{\alpha \in N_{\mathrm{Bij}(L)}(L) \,|\, \alpha(1) = 1\} \\ &= \big\{\alpha \in \mathrm{Bij}(L) \,|\, \alpha(1) = 1,\ \alpha(g\cdot\alpha^{-1}(x)) = \alpha(g)\cdot x \ (\text{all } x, g \in L)\big\} \\ &= \mathrm{Aut}(L). \qquad \square\end{aligned}$$

Centralizers of subgroups will also play an important role in identifying p-stubborn subgroups. The following description of the centralizers of subgroups of the classical groups is well known, but is included here since it will be referred to frequently.

Proposition 4. *Let G be one of the classical groups $\mathrm{O}(n)$, $\mathrm{U}(n)$, or $\mathrm{Sp}(n)$, and let $H \subseteq G$ be any (closed) subgroup. Let V denote the corresponding H-representation: an n-dimensional vector space over $K = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, respectively. Write $V = (V_1)^{m_1} \times \cdots \times (V_k)^{m_k}$, where $V_1, \ldots, V_k$ are distinct (pairwise nonisomorphic) irreducible H-representations. Then*

$$C_G(H) \cong G_1 \times \cdots \times G_k,$$

where the G_i are described as follows:

(i) If $G = \mathrm{U}(n)$ $(K = \mathbb{C})$, then $G_i \cong \mathrm{U}(m_i)$ for each i

(ii) If $G = \mathrm{O}(n)$ $(K = \mathbb{R})$, then $G_i \cong \mathrm{O}(m_i)$ if V_i has real type, $G_i \cong \mathrm{U}(m_i)$ if V_i has complex type, and $G_i \cong \mathrm{Sp}(m_i)$ if V_i has quaternion type. Here, V_i has quaternion type if it can be given the structure of an $\mathbb{H}$-vector space (upon which H acts $\mathbb{H}$-linearly);

otherwise V_i has complex type if it can be given the structure of a complex vector space; otherwise V_i has real type.

(iii) If $G = \mathrm{Sp}(n)$ ($K = \mathbb{H}$), then $G_i \cong \mathrm{Sp}(m_i)$ if V_i has real type, $G_i \cong \mathrm{U}(m_i)$ if V_i has complex type, and $G_i \cong \mathrm{O}(m_i)$ if V_i has quaternion type. Here, V_i has real type if $V_i = \mathbb{H} \otimes_{\mathbb{R}} W$ for some H-representation W over $\mathbb{R}$; otherwise V_i has complex type if it is induced up from a $\mathbb{C}$-representation of H; otherwise V_i has quaternion type.

Proof. By definition, $C_G(H)$ is the group of invertible matrices in

$$\mathrm{End}_G(V) \cong \prod_{i=1}^{k} M_{m_i}(\mathrm{End}_G(V_i))$$

which preserve the inner product. And by Schur's lemma, each $\mathrm{End}_G(V_i)$ is one of the division algebras $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ as described above. (See [Ad, §3] or [Se, §13.2] for more details about the distinction between irreducible representations of real, complex, or quaternion type.) $\square$

When working with subgroups of $G = \mathbb{G}(n)$, regarded as representations on the appropriate n-dimensional vector space V, it is useful to consider the character (i.e., trace) $\chi_V(g)$ of elements $g \in G$. When $\mathbb{G} = \mathrm{Sp}$, $\chi_V(g)$ means the real part of the sum of the diagonal elements in the matrix g (this clearly depends only on the conjugacy class of $g \in G$).

One tool used here for analyzing p-stubborn subgroups is to consider the subgroup of a given $P \subseteq G$ generated by all elements with nonzero character. The next lemma describes how this works for subgroups in $\mathcal{T}_{\mathrm{prod}}(p, G)$, and helps to motivate some of the later proofs.

Lemma 5. *Fix a prime p, and let $G = \mathbb{G}(n)$ be as in Definition 2. Fix any subgroup $P \in \mathcal{T}_{\mathrm{prod}}(p, G)$, and define*

$$\delta(P) = \langle g{\in}P \mid \mathrm{Tr}(g) \neq 0 \rangle.$$

Then (up to conjugacy) one of the following holds:

(a) If $P \notin \mathcal{T}_{\mathrm{irr}}(p, G)$ — if $P = P_1 \times P_2$, where $P_i \subseteq \mathbb{G}(n_i)$, $n_i > 0$, and $n = n_1 + n_2$ — then either $\delta(P) = P$, or $(G, P) = (\mathrm{O}(2), \mathrm{O}(1) {\times} \mathrm{O}(1))$.

(b) If $P = \Gamma \wr E_{q_1} \wr \cdots \wr E_{q_r}$ (with $r \geq 1$), then $\delta(P) = \big(\Gamma \wr E_{q_1} \wr \cdots \wr E_{q_{r-1}}\big)^{q_r}$.

(c1) If $P = \Gamma^{\mathrm{U}}_{p^k}$ or $\overline{\Gamma}^{\mathrm{O}}_{2^k}$ or $\overline{\Gamma}^{\mathrm{Sp}}_{2^k}$, then $\delta(P) \cong S^1$

(c2) If $P = \Gamma^{\mathrm{O}}_{2^k}$ or $\Gamma^{\mathrm{Sp}}_{2^k}$, then $\delta(P) = \{\pm I\}$.

Proof. Let V denote the n-dimensional representation of $P \subseteq \mathbb{G}(n)$.

(a) Assume first that P splits as a product only in a way such that one of the factors is trivial. By definition of $\mathcal{T}_{\mathrm{prod}}(p, G)$, this can only occur if p is odd, $G = \mathrm{O}(2m+1)$, $P = P_1 \times 1$, $P_1 \in \mathcal{T}_{\mathrm{irr}}(p, \mathrm{O}(2m))$, and $1 \subseteq \mathrm{O}(1)$. By inspection (see Definition 2), there exists a proper normal subgroup $H \lhd P_1$ such that all elements of $P_1 \smallsetminus H$ have zero trace (in $\mathrm{O}(2m)$). Hence all elements of $(P_1 \smallsetminus H) \times 1$ have trace 1 in $\mathrm{O}(2m+1)$; these elements generate P, and so $\delta(P) = P$ in this case.

Assume now that $P_1 \neq 1 \neq P_2$. Set

$$P_1' = \langle g \in P_1 \,|\, \chi_V(g,1) \neq 0 \rangle \quad \text{and} \quad P_2' = \langle g \in P_2 \,|\, \chi_V(1,g) \neq 0 \rangle.$$

If $P_1' \neq 1$, fix elements $1 \neq g_1 \in P_1$ and $1 \neq g_2 \in P_2$ such that $\chi_V(g_1,1) \neq 0$. Then $(g_1,1) \in \delta(P)$, $(1,x)$ or (g_1,x) is in $\delta(P)$ for all $x \in P_2$, and $(x,1)$ or (x,g_2) is in $\delta(P)$ for all $x \in P_1$; and together this implies that $\delta(P) = P$. Similarly, $\delta(P) = P$ if $P_2' \neq 1$.

If $P_1' = 1 = P_2'$, then $V|(P_1 \times 1)$ and $V|(1 \times P_2)$ must both be multiples of the regular representation (all non-identity elements have zero character). Then $n \geq |P_1|{\cdot}n_2$ and $n \geq |P_2|{\cdot}n_1$, so $n_1 = n_2$ and $|P_1| = |P_2| = 2$, and hence $(G,P) = (\mathrm{O}(2), \mathrm{O}(1) \times \mathrm{O}(1))$ in this case.

(b) By part (a), $\delta(P) \supseteq \bigl(\Gamma \wr E_{q_1} \wr \cdots \wr E_{q_{r-1}}\bigr)^{q_r}$. Note that the exceptional case in (a) does not occur, since by definition, $\Gamma_1^{\mathrm{O}} \wr E_2 \notin \mathcal{T}_{\mathrm{irr}}(2, \mathrm{O}(2))$. The opposite inclusion is clear, since all other matrices in P have zeroes on the diagonal.

(c1,2) These formulas follow immediately from the definitions of the groups. $\square$

We are now ready to describe $N(P)/P$ for $P \in \mathcal{T}_{\mathrm{prod}}(p,G)$, and to determine which of these subgroups are p-stubborn. Recall that a p-toral subgroup $P \subseteq G$ is p-stubborn if and only if $N(P)/P$ is finite and $\mathcal{O}_p(N(P)/P) = 1$, where $\mathcal{O}_p(N(P)/P)$ is the intersection of the Sylow p-subgroups in $N(P)/P$.

Theorem 6. *Fix a prime p, set $\mathbb{G} = \mathrm{O}$, U, or Sp, and let $G = \mathbb{G}(n)$ for some $n \geq 1$.*

(i) Any subgroup $P \in \mathcal{T}_{\mathrm{irr}}(p,G)$ is both irreducible and p-stubborn in G. If $P \in \mathcal{T}_{\mathrm{irr}}(p,G)$ is an iterated wreath product of the form

$$P = \Gamma \wr E_{q_1} \wr \cdots \wr E_{q_r} \quad (\Gamma \in \mathcal{T}_\Gamma(p, \mathbb{G}(m)),\ q_i = p^{t_i} > 1,\ n = m{\cdot}q_1 \cdots q_r) \tag{1}$$

as in Definition 2, then

$$N_{\mathbb{G}(n)}(P)/P \cong N_{\mathbb{G}(m)}(\Gamma)/\Gamma \times \mathrm{GL}_{t_1}(\mathbb{F}_p) \times \cdots \times \mathrm{GL}_{t_r}(\mathbb{F}_p),$$

and $N_{\mathbb{G}(m)}(\Gamma)/(\Gamma)$ is as given in the following table:

Γ	$\mathbb{G}(m)$	$N(\Gamma)/\Gamma$
$\Gamma_{p^k}^{\mathrm{U}}$	$\mathrm{U}(p^k)$	$\mathrm{Sp}_{2k}(\mathbb{F}_p)$
$\Gamma_{p^k}^{\mathrm{U}}$ (p odd)	$\mathrm{O}(2p^k)$, $\mathrm{Sp}(p^k)$	$C_2 \times \mathrm{Sp}_{2k}(\mathbb{F}_p)$
$\Gamma_{2^k}^{\mathrm{O}}$ ($k \neq 1$)	$\mathrm{O}(2^k)$	$\mathrm{O}_{2k}^{+}(\mathbb{F}_2)$
$\overline{\Gamma}_{2^k}^{\mathrm{O}}$ ($k \neq 0$)	$\mathrm{O}(2^k)$	$\mathrm{Sp}_{2k-2}(\mathbb{F}_2)$
$\Gamma_{2^k}^{\mathrm{Sp}}$	$\mathrm{Sp}(2^k)$	$\mathrm{O}_{2k+2}^{-}(\mathbb{F}_2)$
$\overline{\Gamma}_{2^k}^{\mathrm{Sp}}$	$\mathrm{Sp}(2^k)$	$\mathrm{Sp}_{2k}(\mathbb{F}_2)$

(2)

(ii) A subgroup $P \in \mathcal{T}_{\mathrm{prod}}(p,G)$ is p-stubborn in G if and only if when written as a product

$$P = P_1 \times \cdots \times P_s\ [\times 1] \qquad\qquad (P_i \in \mathcal{T}_{\mathrm{irr}}(p,\mathbb{G}(n_i)),\ n = n_1 + \cdots + n_s\ [+1]),$$

there is no factor P_i with $N_{\mathbb{G}(n_i)}(P_i)/P_i = 1$ which occurs (up to conjugacy) with multiplicity exactly 2 or 4 (if $p=2$) or 3 (if $p=3$).

Proof. This will be split into three cases. Part (i) is shown for $P \in \mathcal{T}_\Gamma(p,G)$ in Case 1, and for general $P \in \mathcal{T}_{\mathrm{irr}}(p,G)$ in Case 2. Part (ii) is shown in Case 3.

Case 1 Consider first $P = \Gamma^{\mathrm{U}}_{p^k} \subseteq \mathrm{U}(p^k)$. As noted earlier, P sits in a central extension

$$1 \to S^1 \to P \to (C_p)^{2k} \to 1,$$

where $S^1 = Z(P) = Z(\mathrm{U}(p^k))$ is the group of multiples of the identity in $\mathrm{U}(p^k)$. Also, P/S^1 has basis $\{A_0,\dots,A_{k-1},B_0,\dots,B_{k-1}\}$, where

$$[A_i,A_j] = I = [B_i,B_j] = [A_i,B_j]\ (i \neq j) \quad \text{and} \quad [B_i,A_i] = \zeta{\cdot}I, \tag{3}$$

and where $\zeta = e^{2\pi i/p}$ denotes a primitive p-th root of unity.

Let V denote the corresponding representation of P on $\mathbb{C}^{p^k}$. Consider the subgroup $P_0 = \langle S^1, A_0,\dots,A_{k-1}\rangle$. This is a group of diagonal matrices; and $V|P_0$ splits a sum of pairwise nonisomorphic 1-dimensional representations which are permuted transitively by $P/P_0 = \langle B_0,\dots,B_{k-1}\rangle$. Thus, V is irreducible (as a P-representation), and so $C_{\mathrm{U}(p^k)}(P) = S^1 \subseteq P$ by Proposition 4(i).

In particular, this shows that the homomorphism $N(P)/P \xrightarrow{\mathrm{conj}} \mathrm{Out}(P)$ is injective. To determine its image, for any $\alpha \in \mathrm{Aut}(P)$, let V_α denote the representation of P obtained by composing with α. Then α is induced by an element of $N(P)$ if and only if $V \cong V_\alpha$ as P-representations, if and only if $\chi_V(g) = \chi_V(\alpha(g))$ for all $g \in P$. And since $\chi_V(g) = 0$ for all $g \in P \smallsetminus S^1$ (Lemma 5), this is the case if and only if $\alpha|S^1 = \mathrm{Id}$.

We have now shown that $N_{\mathrm{U}(p^k)}(P)/P$ can be identified with the group $\mathrm{Out}_{S^1}(P)$ of outer automorphisms of P which are the identity on S^1. Consider the homomorphism

$$\sigma : \mathrm{Out}_{S^1}(P) \longrightarrow \mathrm{Aut}(P/S^1) \cong \mathrm{GL}_{2k}(\mathbb{F}_p)$$

induced by taking quotients. If $\alpha \in \mathrm{Aut}(P)$ induces the identity automorphism on S^1 and on the quotient, then it must have the form

$$\alpha(A_i) = \zeta^{r_i}{\cdot}A_i \qquad \text{and} \qquad \alpha(B_i) = \zeta^{s_i}{\cdot}B_i$$

for some choice of $r_i, s_i \in \mathbb{F}_p$, and α is seen using the relations in (3) to be an inner automorphism. Thus, σ is a monomorphism. Its image is the group $\mathrm{Sp}_{2k}(\mathbb{F}_p)$ of all matrices which preserve the nonsingular form $\begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}$ (where $I_k \in \mathrm{GL}_k(\mathbb{F}_p)$ is the identity). So we have now shown that $N(P)/P \cong \mathrm{Sp}_{2k}(\mathbb{F}_p)$.

Let $\mathrm{UT}_k(\mathbb{F}_p)$ and $\mathrm{Sym}_k(\mathbb{F}_p)$ denote the sets of $k\times k$ upper triangular and symmetric matrices, respectively, over $\mathbb{F}_p$. Here, upper triangular means with 1's along the diagonal. The sets

$$S_1 = \left\{ \begin{pmatrix} A & AX \\ 0 & (A^t)^{-1} \end{pmatrix} \,\Big|\, A\in \mathrm{UT}_k(\mathbb{F}_p),\ X\in \mathrm{Sym}_k(\mathbb{F}_p) \right\}$$

and

$$S_2 = \left\{ \begin{pmatrix} (A^t)^{-1} & 0 \\ AX & A \end{pmatrix} \,\Big|\, A\in \mathrm{UT}_k(\mathbb{F}_p),\ X\in \mathrm{Sym}_k(\mathbb{F}_p) \right\}$$

are subgroups of $\mathrm{Sp}_{2k}(\mathbb{F}_p)$, and are seen (by counting) to be Sylow p-subgroups. Since $S_1\cap S_2=1$, this shows that $\mathcal{O}_p(\mathrm{Sp}_{2k}(\mathbb{F}_p))=1$, and finishes the proof that $P=\Gamma^{\mathrm{U}}_{p^k}$ is p-stubborn in $\mathrm{U}(p^k)$.

Now assume that $P=\Gamma^{\mathrm{U}}_{p^k}$, and that $G=\mathrm{O}(2p^k)$ or $\mathrm{Sp}(p^k)$ (where $\mathrm{U}(p^k)$ is regarded as a subgroup of G in the usual way). The same argument as before shows that P is irreducible, and that

$$N_G(P)/P \cong \mathrm{Out}(P) \cong \mathrm{Out}_{S^1}(P)\times \mathrm{Aut}(S^1) \cong \mathrm{Sp}_{2k}(\mathbb{F}_p)\times C_2. \tag{4}$$

In particular, if p is odd, then $\mathcal{O}_p(N(P)/P)=1$, and P is again p-stubborn in G.

Now assume $p=2$, and $G=\mathrm{O}(2^{k+1})$ or $\mathrm{Sp}(2^k)$. Set $P=\overline{\Gamma}^{\mathrm{O}}_{2^{k+1}}$ or $\overline{\Gamma}^{\mathrm{Sp}}_{2^k}$, respectively. Let $P'=\Gamma^{\mathrm{U}}_{2^k}$: a subgroup of index 2 in P. Then $P'=C_P(S^1)$, where $S^1=Z(\mathrm{U}(2^k))$ denotes the identity component of P. In particular, $N_G(P)\subseteq N_G(P')$. And since P/P' is the C_2-factor in $N_G(P')/P'\cong \mathrm{Sp}_{2k}(\mathbb{F}_2)\times C_2$ (see (4)), we see that $N_G(P)=N_G(P')$, and that $N(P)/P\cong \mathrm{Sp}_{2k}(\mathbb{F}_2)$. Since $\mathcal{O}_2(\mathrm{Sp}_{2k}(\mathbb{F}_2))=1$, P is 2-stubborn in G.

Finally, the same arguments applied to the central extensions

$$1\to\{\pm1\}\to\Gamma^{\mathrm{O}}_{2^k}\to(C_2)^{2k}\to 1, \qquad 1\to\{\pm1\}\to\Gamma^{\mathrm{Sp}}_{2^{k-1}}\to(C_2)^{2k}\to 1,$$

show that they are irreducible (as representations on $\mathbb{R}^{2^k}$ or $\mathbb{H}^{2^{k-1}}$), and that

$$N_{\mathrm{O}(2^k)}(\Gamma^{\mathrm{O}}_{2^k})/\Gamma^{\mathrm{O}}_{2^k} \cong \mathrm{Out}(\Gamma^{\mathrm{O}}_{2^k}) \cong \mathrm{O}^+_{2k}(\mathbb{F}_2)$$

and

$$N_{\mathrm{Sp}(2^{k-1})}(\Gamma^{\mathrm{Sp}}_{2^{k-1}})/\Gamma^{\mathrm{Sp}}_{2^{k-1}} \cong \mathrm{Out}(\Gamma^{\mathrm{Sp}}_{2^{k-1}}) \cong \mathrm{O}^-_{2k}(\mathbb{F}_2).$$

Here, $\mathrm{O}^+_{2k}(\mathbb{F}_2)$ is the group of automorphisms of $(\mathbb{F}_2)^{2k}$ which leave invariant the quadratic form

$$q(x_0,\dots,x_{k-1},y_0,\dots,y_{k-1}) = x_0y_0+\cdots+x_{k-1}y_{k-1};$$

and $\mathrm{O}^-_{2k}(\mathbb{F}_2)$ is the group of automorphisms which leave invariant the form

$$q(x_0,\dots,x_{k-1},y_0,\dots,y_{k-1}) = (x_0^2+x_0y_0+y_0^2)+x_1y_1+\cdots+x_{k-1}y_{k-1}.$$

These are the forms induced by the sqaring maps in $\Gamma^{\mathrm{O}}_{2^k}$ and $\Gamma^{\mathrm{Sp}}_{2^{k-1}}$, respectively. By a theorem of Dieudonné [D, pp. 47–51], for $k\geq 3$, any nontrivial normal subgroup of $\mathrm{O}^{\pm}_{2k}(\mathbb{F}_2)$ contains its commutator subgroup, which is simple and nonabelian. So $\mathcal{O}_2(\mathrm{O}^{\pm}_{2k}(\mathbb{F}_2))=1$ when $k\ge 3$; and the following list shows that $\mathcal{O}_2(\mathrm{O}^{\pm}_{2k}(\mathbb{F}_2))=1$ in all other cases except for $\mathrm{O}^{+}_{2}(\mathbb{F}_2)$:

$$\mathrm{O}^{+}_{2}(\mathbb{F}_2)\cong C_2,\ \mathrm{O}^{-}_{2}(\mathbb{F}_2)\cong \Sigma_3,\ \mathrm{O}^{+}_{4}(\mathbb{F}_2)\cong \Sigma_3\wr\Sigma_2,\ \text{and}\ \mathrm{O}^{-}_{4}(\mathbb{F}_2)\cong \Sigma_5.$$

Thus $\Gamma^{\mathrm{O}}_{2^k}$ and $\Gamma^{\mathrm{Sp}}_{2^k}$ are 2-stubborn for all $k\ge 0$, except for the case $\Gamma^{\mathrm{O}}_{2}\subseteq \mathrm{O}(2)$ (dihedral of order 8).

Case 2 Now fix a subgroup of the form $P=\Gamma\wr E_{q_1}\wr\cdots\wr E_{q_r}\subseteq\mathbb{G}(n)$, where $r\ge 1$, $q_i=p^{t_i}$, $\Gamma\in\mathcal{T}_\Gamma(p,\mathbb{G}(m))$, $n=m{\cdot}q_1{\cdots}q_r$; and $q_1\ge 4$ if $\Gamma=\Gamma^{\mathrm{O}}_1=\mathrm{O}(1)$. Set $n'=n/q_r=m{\cdot}q_1{\cdots}q_{r-1}$, and write $P'=\Gamma\wr E_{q_1}\wr\cdots\wr E_{q_{r-1}}\subseteq\mathbb{G}(n')$. We may assume inductively that P' is irreducible. By Lemma 5, $(P')^{q_r}$ is the subgroup generated by all elements of P of nonzero trace, and is therefore normal in $N(P)$. So by Lemma 3(ii,iii), $P=P'\wr E_{q_r}$ is also irreducible, and

$$N_{\mathbb{G}(n)}(P)/P\cong N_{\mathbb{G}(n')}(P')/P'\times N_{\Sigma_{q_r}}(E_{q_r})/E_{q_r}\cong N_{\mathbb{G}(n')}(P')/P'\times \mathrm{GL}_{t_r}(\mathbb{F}_p).$$

It now follows by induction on r that

$$N(P)/P\cong N_{\mathbb{G}(m)}(\Gamma)/\Gamma\times\mathrm{GL}_{t_1}(\mathbb{F}_p)\times\cdots\times\mathrm{GL}_{t_r}(\mathbb{F}_p).$$

Also, $\mathcal{O}_p(\mathrm{GL}_t(\mathbb{F}_p))=1$ for all t (the subgroups of upper and lower triangular matrices are two Sylow p-subgroups with trivial intersection); and $\mathcal{O}_p(N(\Gamma)/\Gamma)=1$ by Case 1. So $N(P)/P$ is finite, $\mathcal{O}_p(N(P)/P)=1$, and P is p-stubborn.

Case 3 Now assume that P is reducible: write $P=(P_1)^{m_1}\times\cdots\times(P_r)^{m_r}$ $[\times 1]$, where $P_i\in\mathcal{T}_{\mathrm{irr}}(p,\mathbb{G}(n_i))$, $n=m_1n_1+\cdots+m_rn_r$ (or $n=\sum m_in_i+1$ if p is odd, $\mathbb{G}=\mathrm{O}$, and n is odd); and where the P_i are pairwise nonisomorphic as representations (of the P_i). Then by Lemma 3(i),

$$N(P)/P\cong\big(N_{\mathbb{G}(n_1)}(P_1)/P_1\big)\wr\Sigma_{m_1}\times\cdots\times\big(N_{\mathbb{G}(n_r)}(P_r)/P_r\big)\wr\Sigma_{m_r}\ [\times\{\pm1\}].$$

Also, $\mathcal{O}_p(N(P_i)/P_i)=1$ for all i by Cases 1 and 2. If $N(P_i)/P_i\ne 1$, then any nontrivial normal subgroup of $(N(P_i)/P_i)\wr\Sigma_{m_i}$ intersects nontrivially with $(N(P_i)/P_i)^{m_i}$, and so $\mathcal{O}_p((N(P_i)/P_i)\wr\Sigma_{m_i})=1$. And if $N(P_i)/P_i=1$, then $\mathcal{O}_p((N(P_i)/P_i)\wr\Sigma_{m_i})=\mathcal{O}_p(\Sigma_{m_i})$; and $\mathcal{O}_p(\Sigma_m)\ne 1$ only when (p,m) is one of the pairs $(2,2)$, $(2,4)$, or $(3,3)$. This finishes the proof of point (ii). $\square$

We now know which of the subgroups in the $\mathcal{T}_{\mathrm{prod}}(p,G)$ are p-stubborn, and it remains to show that they are the only p-stubborn subgroups of the classical groups. The following general properties of p-stubborn subgroups will be needed.

Lemma 7. *Let P be a p-stubborn subgroup of a compact Lie group G. Then the following hold.*

(i) Any p-toral subgroup $H\subseteq G$ which is normalized by $N(P)$ (i. e., $N(P)\subseteq N(H)$) is contained in P.

(ii) $C_{G_0}(P) \subseteq Z(P)$*, and* $C_G(P) = Z(P)$ *if* G/G_0 *is a* p*-group.*

Proof. Part (ii) is shown in [JMO, Lemma 1.5(ii)]. Part (i) is essentially shown in the same lemma, but with a slightly different formulation. For that reason, we repeat the proof here.

Assume that $H \nsubseteq P$, and that $N(P)$ normalizes H. Set $H' = \langle H, P\rangle \supsetneqq P$. Since P normalizes H, $H \lhd H'$, and H' is p-toral since H'/H is a quotient group of P. Also, $N(P) \subseteq N(H')$; and

$$\mathrm{Ker}[N(P)/P \to N(H')/H'] = (H' \cap N(P))/P = N_{H'}(P)/P$$

is a nontrivial normal p-subgroup of $N(P)/P$ (cf. [JMO, Lemmas A.2 & A.3]). Which contradicts the assumption that P is p-stubborn. $\square$

We are now ready to show that all p-stubborn subgroups of the classical groups lie (up to conjugacy) in the $\mathcal{T}_{\mathrm{prod}}(p, -)$.

Theorem 8. *Fix a prime* p*, set* $\mathbb{G} = \mathrm{O}$*,* U*, or* Sp*, and let* $G = \mathbb{G}(n)$ *for some* $n \geq 1$*. Then every* p*-stubborn subgroup of* G *is conjugate to a subgroup in* $\mathcal{T}_{\mathrm{prod}}(p, G)$*. And if two subgroups in* $\mathcal{T}_{\mathrm{prod}}(p, G)$ *are conjugate in* G*, then they are equal, after permuting irreducible factors if necessary.*

Proof. The first statement (all p-stubborn subgroups are conjugate to subgroups in $\mathcal{T}_{\mathrm{prod}}(p, G)$) will be shown in Steps 1–3: corresponding to the cases of subgroups in $\mathcal{T}_{\Gamma}(p, G)$, $\mathcal{T}_{\mathrm{irr}}(p, G)$, or $\mathcal{T}_{\mathrm{prod}}(p, G)$, respectively. The last statement will be shown in Step 4.

Fix $P \subseteq \mathbb{G}(n)$, and let V be the corresponding n-dimensional P-representation. We write $\chi_V(g)$ for the character (i.e., trace) of an element $g \in G$. As before, when $\mathbb{G} = \mathrm{Sp}$, $\chi_V(g)$ means the real part of the sum of the diagonal entries in the matrix g.

Throughout the first two steps, we assume that $P \neq 1$ is irreducible and p-stubborn. Consider again the subgroup

$$\delta(P) = \langle g \in P \,|\, \chi_V(g) \neq 0\rangle \subseteq P.$$

We first claim that

$$[P : \delta(P)] < \infty. \qquad \text{and} \qquad 1 \neq \delta(P) \lhd N(P) \tag{1}$$

The first statement follows since $\delta(P)$ contains a neighborhood of the identity in P (the trace is continuous). If G is connected or $p = 2$, then $\delta(P) \neq 1$ since $-I \in C_G(P) \subseteq P$ by Lemma 7(ii) (so $-I \in \delta(P)$). And if p is odd and $G = \mathrm{O}(n)$, then $C_{G_0}(P) \subseteq P$ by Lemma 7(ii) again; and so

$$\dim(\delta(P)) = \dim(P) \geq \dim(C_G(P)) > 0$$

by Proposition 4 (using [Ad, Lemma 3.62], one checks that any nonfixed irreducible representation of P has complex type, and hence has centralizer S^1).

Step 1 Assume first that for any subgroup $K\lhd P$ such that $\delta(P)\subseteq K\lhd N(P)$, either $V|K$ is irreducible, or it splits as a sum of isomorphic irreducible K-representations. We will show that P is conjugate to an element of $\mathcal{T}_\Gamma(p,G)$ in this case.

By assumption, $V|\delta(P)\cong W^s$ for some irreducible $\delta(P)$-representation W; and $\delta(P)$ is generated by elements g for which $\chi_W(g)\neq 0$. In particular, the representation W cannot be induced up from any proper subgroup of $\delta(P)$: since if $W=\mathrm{Ind}_H^{\delta(P)}(W')$ for some $H\lhd\delta(P)$ of index p then $\chi_W|(\delta(P)\smallsetminus H)=0$. Hence by [Se, §8.5, Th. 16] (or the fact that P is contained in the normalizer of a maximal torus in G), either $\dim(W)=1$, or $W\cong\mathbb{R}^2$. Then $\delta(P)$ is contained in O(2), U(1), or Sp(1); and by using again the fact that it is generated by elements with nonvanishing trace we see that $\delta(P)$ is cyclic or S^1.

Set

$$P'=\bigl\{g\in C_P(\delta(P))\,|\,g^p\in\delta(P),[g,P]\subseteq\delta(P)\bigr\}\lhd N(P).$$

Since $\delta(P)$ is abelian, we have $\delta(P)\subseteq Z(P')\subseteq P'$. Also, $Z(P')\lhd N(P)$ by construction, and hence $V|Z(P')$ also splits as a sum of isomorphic irreducible representations (by the assumption at the start of Step 1). Hence $Z(P')$ must also be cyclic or S^1, since it is abelian and has an effective irreducible representation.

Assume now that $|Z(P')|>2$ or $\mathbb{G}=\mathrm{U}$. Recall that $V|Z(P')$ is a sum of isomorphic (effective) irreducible representations: representations of complex type if $|Z(P')|>2$. Hence $C_G(Z(P'))$ is a unitary group by Proposition 4; and in particular $Z(C_G(Z(P')))\cong S^1$. This S^1 is normalized by $N_G(P)$; and so $S^1\subseteq P$ by Lemma 7(i). And since $[P{:}\delta(P)]<\infty$ by (1), this implies that $\delta(P)\cong S^1$. Also, since the center of a p-toral group is p-toral (cf. [JMO, Lemma A.3]), $|Z(P')|=2$ only if $p=2$. So we have now shown that

$$\delta(P)=Z(P')\cong\begin{cases}S^1 & \text{if } \mathbb{G}=\mathrm{U} \text{ or } p \text{ is odd}\\ \{\pm1\} \text{ or } S^1 & \text{if } p=2, \text{ and } \mathbb{G}=\mathrm{O} \text{ or Sp}.\end{cases}\tag{2}$$

We next claim that

$$[P,P']\subseteq\delta(P)\qquad\text{and}\qquad C_P(P')=\delta(P).\tag{3}$$

The first statement follows from the definition of P'. To see the second, assume that $C_P(P')\supsetneqq Z(P')=\delta(P)$. Since $C_P(P')/\delta(P)\lhd P/\delta(P)$, and since any nontrivial normal subgroup of a finite p-group intersects nontrivially with its center, there exists

$$g{\cdot}\delta(P)\in(C_P(P')/\delta(P))\cap Z(P/\delta(P))$$

of order p. Then $g\in P'$ by construction, and so $g\in P'\cap C_P(P')=Z(P')=\delta(P)$. And this contradicts the assumption that $g{\cdot}\delta(P)$ has order p.

By construction, $P'/\delta(P)$ is elementary abelian. So there is a central extension

$$1\to\delta(P)\to P'\to(C_p)^m\to 1$$

for some m. The map

$$\mu : (C_p)^m \times (C_p)^m \longrightarrow \delta(P),$$

defined by setting $\mu(g{\cdot}\delta(P), h{\cdot}\delta(P)) = [g,h]$, is bilinear and antisymmetric, and is nonsingular since $\delta(P) = Z(P')$. Hence $m = 2k$ for some k, and there is a basis $a_1,\ldots,a_k,b_1,\ldots,b_k$ for $P'/\delta(P)$ such that the a_i and b_i satisfy the commutator relations

$$[a_i,a_j] = [b_i,b_j] = [b_i,a_j] = 1 \ \ (i \neq j) \quad \text{and} \quad [b_i,a_i] = \zeta := e^{2\pi i/p}. \tag{4}$$

Case A Assume here that $\delta(P){\cong}S^1$. Since S^1 is infinitely divisible, we can choose the a_i and b_i such that $(a_i)^p = 1 = (b_i)^p$ for all i. Hence $P' \cong \Gamma^{\mathrm{U}}_{p^k}$.

Set $P'' = C_P(\delta(P)) \supseteq P'$. By (3), $[P,P']{\subseteq}\delta(P){\cong}S^1$. So for any $x{\in}P''$, conjugation by x is an automorphism of P' which induces the identity on $P'/\delta(P)$ and on $\delta(P)$. Any such automorphism is inner (as was seen in Case 1 of the proof of Theorem 6); and so $x \in \langle P', C_P(P')\rangle$. But $C_P(P') \subseteq P'$ by (3) again, and so we have shown that $C_P(\delta(P)) = P' \cong \Gamma^{\mathrm{U}}_{p^k}$. Also, since $S^1 \cong \delta(P) \lhd P$, $[P : P'] \leq |\operatorname{Aut}(S^1)| = 2$.

Now, V is an irreducible P-representation, and $[P : P'] \leq 2$. If $P = P'$, then by comparing characters ($\chi_V|(P{\smallsetminus}S^1) = 0$ by assumption), we see that V must be the standard representation of $\Gamma^{\mathrm{U}}_{p^k}$ on $\mathbb{C}^{p^k}$, $\mathbb{H}^{p^k}$, or $\mathbb{R}^{2p^k}$ (i.e, the representation given by the inclusion $\Gamma^{\mathrm{U}}_{p^k} \subseteq \mathrm{U}(p^k)$). Thus, $G = \mathbb{G}(n) = \mathrm{U}(p^k)$, $\mathrm{Sp}(p^k)$, or $\mathrm{O}(2p^k)$, and P is conjugate in G to $\Gamma^{\mathrm{U}}_{p^k}$. As was seen in the proof of Theorem 6,

$$N(P')/P' \cong \begin{cases} \mathrm{Sp}_{2k}(\mathbb{F}_p) & \text{if } \mathbb{G} = \mathrm{U} \\ C_2 \times \mathrm{Sp}_{2k}(\mathbb{F}_p) & \text{if } \mathbb{G} = \mathrm{O}, \mathrm{Sp}\,. \end{cases} \tag{5}$$

Now assume that $[P : P'] = 2$. Then $p = 2$ (P is p-toral), and $C_G(P) = Z(P) = Z(P')^{P/P'} \cong C_2$ by Lemma 7(ii). In particular, G is an orthogonal or symplectic group (Proposition 4(i)). Upon comparing characters, we see that $V|P' \cong \overline{V}^s$ for some s, where $\overline{V}$ is the standard representation of $\Gamma^{\mathrm{U}}_{2^k}$ on $\mathbb{R}^{2^{k+1}}$, or $\mathbb{H}^{2^k}$. Hence $C_G(P') \cong \mathrm{U}(s)$. Also, $C_G(P')$ has an involution with finite fixed point set (conjugation by any element of $P{\smallsetminus}P'$); this must be $(x \mapsto -x)$ on the Lie algebra, and hence $(g \mapsto g^{-1})$ on $C_G(P') \cong \mathrm{U}(s)$. So $s = 1$, and $V|P'$ is irreducible. Also, $N(P){\subseteq}N(P')$ ($P' \lhd N(P)$ by construction). It follows that P/P' must be the C_2-factor in $N(P')/P'$ (see (5)), since otherwise $N(P)/P \cong N_{C_2\times\mathrm{Sp}_{2k}(\mathbb{F}_2)}(P/P')/(P/P')$ has a normal subgroup of order 2. In other words, P is conjugate to $\overline{\Gamma}^{\mathrm{O}}_{2^{k+1}}{\subseteq}\,\mathrm{O}(2^{k+1})$ or $\overline{\Gamma}^{\mathrm{Sp}}_{2^k}{\subseteq}\,\mathrm{Sp}(2^k)$.

Case B Now assume that $\delta(P) \cong \{\pm 1\}$. In particular, by (2), $p = 2$ and $\mathbb{G} = \mathrm{O}$ or Sp. Also, $\operatorname{Aut}(\delta(P)) = 1$, so $\delta(P) = Z(P') = Z(P)$.

Using the commutator relations (4), one checks that all automorphisms of P' which fix $\delta(P)$ and $P'/\delta(P)$ are inner. Relations (3) now apply to show that $P = P'$. Also, $C_G(P) = Z(P) \cong C_2$ (Lemma 7(ii)). So by Proposition 4, P is irreducible of real type (if $\mathbb{G} = \mathrm{O}$) or of quaternion type (if $\mathbb{G} = \mathrm{Sp}$).

Since $Z(P) = [P,P] \cong C_2$, P is an extra special 2-group (cf. [G, p.183]). So by [G, Theorem 5.5.2], P is isomorphic, either to a central product of copies of $D(8)$, or to a central product of one copy of $Q(8)$ and copies of $D(8)$. Here, $D(8)$ and $Q(8)$ denote the dihedral and quaternion groups of order 8. In the first case, $P \cong \Gamma^{\mathrm{O}}_{2^k}$, and in the second case $P \cong \Gamma^{\mathrm{Sp}}_{2^{k-1}}$. And by comparing characters (use Lemma 5(c2) and recall that $\chi_V(P{\smallsetminus}Z(P)) = 0$), we see that either $G = \mathrm{O}(2^k)$ and P is conjugate to $\Gamma^{\mathrm{O}}_{2^k}$, or $G = \mathrm{Sp}(2^{k-1})$ and P is conjugate to $\Gamma^{\mathrm{Sp}}_{2^{k-1}}$.

Step 2 Now assume that there exists a subgroup $K \lhd P$ such that $\delta(P) \subseteq K \lhd N(P)$, and such that $V|K$ splits as a sum of irreducible K-representations not all isomorphic to each other. Write $V|K \cong V_1 \times \cdots \times V_r$ $(r>1)$, where each V_i is a sum of isomorphic K-representations, and where for $i \neq j$ the irreducible summands of V_i and V_j are nonisomorphic.

Since V is irreducible as a P-representation, P/K permutes the V_i transitively. In particular, they all have the same dimension: set $m = n/r = \dim(V_i)$. So (after conjugating) we may assume that $K \subseteq \mathbb{G}(m)^r$.

For each i, let $P_i \subseteq P$ be the subgroup of elements which leave V_i invariant, and let $K_i \subseteq \mathbb{G}(m)$ be the image of P_i in the i-th factor. Then

$$K \subseteq \overline{K} := K_1 \times \cdots \times K_q \subseteq \mathbb{G}(n).$$

The P_i are p-toral ($P_i \supseteq \delta(P)$ has finite index in P by (1)), so K_i is p-toral, and the product $\overline{K}$ is also p-toral. Since $K \lhd N(P)$, the conjugation action of $N(P)$ permutes the V_i, and so $\overline{K}$ is also normalized by $N(P)$. Lemma 7(i) now applies to show that $\overline{K} \subseteq P$.

Now set $L = P/\overline{K}$. Then L permutes the V_i effectively and transitively. For any $g \in P{\smallsetminus}\overline{K}$, there exists by construction an element $g' \in g{\cdot}\overline{K}$ which acts via the identity on each V_i which is invariant under the action of g; and thus $\chi_V(g') \neq 0$ if it leaves any V_i invariant. But $\chi_V(P{\smallsetminus}\overline{K}) = 0$ by assumption, and hence no summands can be left invariant under the action of any $g \in P{\smallsetminus}\overline{K}$. We can thus regard L as a free transitive subgroup of Σ_q. In particular, each V_i is irreducible as a K_i-representation, since otherwise a splitting of V_i would extend to a splitting of V (and V is irreducible by assumption).

Now, since L permutes the factors transitively, $V_i \cong V_j$ for all i and j, and the K_i are all conjugate to each other in $\mathbb{G}(m)$. So after conjugating, we may assume that $K_1 = K_2 = \cdots = K_q$; and that for each i some $g_i \in P$ sends P_1 to P_i via the identity. Then, since L permutes the factors freely, we can identify it with the subgroup $L' \subseteq P$ of those elements which permute the K_i identically. In other words, $P = K_1 \wr L$; and

$$N_{\mathbb{G}(n)}(P)/P \cong N_{\mathbb{G}(m)}(K_1)/K_1 \times \mathrm{Aut}(L)$$

by Lemma 3(ii,iii). In particular, $\mathcal{O}_p(N_{\mathbb{G}(m)}(K_1)/K_1) = 1$, and K_1 must be p-stubborn in $\mathbb{G}(m)$. Also, $\mathcal{O}_p(\mathrm{Aut}(L)) = 1$, and L must be abelian since otherwise $\mathrm{Inn}(L)$ is a nontrivial normal p-subgroup of $\mathrm{Aut}(L)$. And L must be elementary abelian, since otherwise the group of automorphisms fixing ${}_pL$ (the p-torsion subgroup) and $L/{}_pL$ is a nontrivial normal p-subgroup. Thus, $L \cong (C_p)^k$ $(q = p^k)$, L acts freely and transitively on $\{1, \ldots, p^k\}$; and it follows that L is conjugate to $E_{p^k} \subseteq \Sigma_{p^k}$.

By induction, P is conjugate to an iterated wreath product $\Gamma \wr E_{q_1} \wr \cdots \wr E_{q_r}$, where $q_i = p^{t_i}$ and (by Step 1) $\Gamma \in \mathcal{T}_\Gamma(p, \mathbb{G}(m))$. Note in particular that $q_1 \geq 4$ if $\Gamma = \mathrm{O}(1)$ (and $p = 2$): since $\mathrm{O}(1) \wr \Sigma_2$ is not 2-stubborn in $\mathrm{O}(2)$. This finishes the proof that $P \in \mathcal{T}_{\mathrm{irr}}(p, G)$ (up to conjugation) when P is irreducible.

Step 3 Now let $P \subseteq \mathbb{G}(n)$ be an arbitrary p-stubborn subgroup. Assume that the corresponding P-representation factors as a product $V_1 \times \ldots \times V_s$ of irreducible representations. In other words, after conjugating, we may assume that $P \subseteq \mathbb{G}(n_1) \times \cdots \times \mathbb{G}(n_s)$, where $n_i = \dim(V_i)$, and where the image $P_i \subseteq \mathbb{G}(n_i)$ of P in the i-th factor is irreducible for each i.

By Lemma 7(ii), $C_{\mathbb{G}(n)}(P) = Z(P)$ — unless possibly p is odd and $\mathbb{G} = \mathrm{O}$ in which case $[C_{\mathbb{G}(n)}(P) : Z(P)] \leq 2$. In this latter case, since all nonfixed irreducible P-representations have complex type, $C_{\mathbb{G}(n)}(P)$ is a product of unitary groups and one copy of $\mathrm{O}(m)$ (where $m = \dim(V^P)$). Also, $m \leq 1$, since $\mathrm{O}(m) \subseteq C_{\mathbb{G}(n)}(P)/Z(P)$; and m is the number of trivial summands in V.

In either case, $C_{\mathbb{G}(n)}(P)$ is abelian, so the V_i are distinct as P-representations (Proposition 4), and are permuted by $N(P)$. Hence $N(P) \subseteq N(P_1 \times \cdots \times P_s)$, and Lemma 7(i) now applies to show that $P = P_1 \times \cdots \times P_s$.

After reindexing (and conjugating), we can write $P = (P_1)^{m_1} \times \cdots \times (P_r)^{m_r}$, where the $P_i \subseteq \mathbb{G}(n_i)$ are irreducible and pairwise nonisomorphic as representations. Then

$$N(P)/P \cong \left(N_{\mathbb{G}(n_1)}(P_1)/P_1\right) \wr \Sigma_{m_1} \times \cdots \times \left(N_{\mathbb{G}(n_r)}(P_r)/P_r\right) \wr \Sigma_{m_r}$$

by Lemma 3(i). Since $\mathcal{O}_p(N(P)/P) = 1$ by assumption, $\mathcal{O}_p(N(P_i)/P_i) = 1$ for each i, and so each P_i is p-stubborn in $\mathbb{G}(n_i)$. By Steps 1 and 2, P_i is conjugate to an element of $\mathcal{T}_{\mathrm{irr}}(p, \mathbb{G}(n_i))$ for each i, and hence P is conjugate to an element of $\mathcal{T}_{\mathrm{prod}}(p, G)$.

Step 4 Now assume that $P, P' \in \mathcal{T}_{\mathrm{prod}}(p, G)$ are conjugate in G. Then $P \in \mathcal{T}_\Gamma(p, G)$ if and only if $V|\delta(P)$ splits as a sum of pairwise isomorphic irreducible representations. So $P \in \mathcal{T}_\Gamma(p, G)$ if and only if $P' \in \mathcal{T}_\Gamma(p, G)$; in which case one easily sees that $P = P'$. If $P, P' \in \mathcal{T}_{\mathrm{irr}}(p, G) \setminus \mathcal{T}_\Gamma(p, G)$, then $P = P_0 \wr E_q$ and $P' = P'_0 \wr E_q$, where $P_0, P'_0 \in \mathcal{T}_{\mathrm{irr}}(p, \mathbb{G}(n/q))$ and $q = [P : \delta(P)] = [P' : \delta(P')]$. So by induction on n, $P = P'$ in this case, and the general case follows immediately. $\square$

When working with concrete problems involving the orbit categories $\mathcal{R}_p(G)$, it is useful to know not only the p-stubborn subgroups themselves, but also how they are included in each other. We next note the conditions for inclusion (up to conjugacy) between elements of $\mathcal{T}_{\mathrm{prod}}(p, \mathbb{G}(n))$.

Proposition 9. *Let p be a prime, let $G = \mathbb{G}(n)$ be as in Theorem 8, and fix subgroups $P', P \in \mathcal{T}_{\mathrm{prod}}(p, G)$ such that P' is conjugate to a subgroup of P. Then $xP'x^{-1} \subseteq P$ for some permutation matrix $x \in G = \mathbb{G}(n)$ which permutes the irreducible factors of P'. In particular, $xP'x^{-1} \in \mathcal{T}_{\mathrm{prod}}(p, G)$, and $P' \subseteq P$ if P' is irreducible. Finally, if $P' \subseteq P$, the inclusion is a composite of products of inclusions of the following types:*

(a) $\Gamma^X_{qq'} \wr E_Q \subseteq \Gamma^X_q \wr E_{q'} \wr E_Q$ *or* $\overline{\Gamma}^X_{qq'} \wr E_Q \subseteq \overline{\Gamma}^X_q \wr E_{q'} \wr E_Q$

(b) $\Gamma \wr E_Q \wr E_{qq'} \wr E_{Q'} \subseteq \Gamma \wr E_Q \wr E_q \wr E_{q'} \wr E_{Q'}$

(c) $\Gamma^X_{2^k} \wr E_Q \subseteq \overline{\Gamma}^X_{2^k} \wr E_Q$

(d) $(\Gamma \wr E_Q)^q \subseteq \Gamma \wr E_Q \wr E_q$

(e) $\mathrm{O}(1) \times \mathrm{O}(1) \subseteq \mathrm{O}(2)$

(f) $\mathrm{O}(1) \wr E_{2q} \wr E_Q \subseteq \mathrm{O}(2) \wr E_q \wr E_Q$

Here, E_Q and $E_{Q'}$ denote arbitrary (possibly trivial) wreath products of the E_q, and Γ denotes any of the groups Γ^X_q or $\overline{\Gamma}^X_q$.

Proof. Fix $g \in \mathbb{G}(n)$ such that $gP'g^{-1} \subseteq P$. Then $g\delta(P')g^{-1} \subseteq \delta(P)$, where $\delta(-)$ is as defined in Lemma 5.

Case 1 Assume first that P is reducible. Write $P = P_1 \times \cdots \times P_s$, where $P_i \in \mathcal{T}_{\mathrm{irr}}(p, \mathbb{G}(n_i))$ (or possibly $P_s = 1$ and $\mathbb{G}(n_s) = \mathrm{O}(1)$), and $n = \sum n_i$. Then P' is also reducible. And after permuting its irreducible factors (or without permuting them if $P' \subseteq P$), we get that $P' = P'_1 \times \cdots \times P'_s$ for some subgroups $P'_i \in \mathcal{T}_{\mathrm{prod}}(p, \mathbb{G}(n_i))$ (or $P_s = 1$) such that P'_i is (conjugate in $\mathbb{G}(n_i)$ to) a subgroup of P_i. So it remains to prove the proposition in the case where $P \in \mathcal{T}_{\mathrm{irr}}(p, G)$.

Case 2 If P' is reducible and P is irreducible, then by Lemma 5(a), either $P' \cong \mathrm{O}(1) \times \mathrm{O}(1)$ (case (e)), or $gP'g^{-1} = g\delta(P')g^{-1} \subseteq \delta(P)$. And in the latter case, P is a wreath product of the form

$$P = \Gamma \wr E_{q_1} \wr \cdots \wr E_{q_r},$$

(otherwise $gP'g^{-1} \subseteq \delta(P) \cong S^1$ or $\{\pm 1\}$); and

$$gP'g^{-1} \subseteq \delta(P) = \left(\Gamma \wr E_{q_1} \wr \cdots \wr E_{q_{r-1}}\right)^{q_r}$$

(Lemma 5(b)). So ($gP'g^{-1} \subseteq P$) is the composite of an inclusion of type (d) with an inclusion of reducible subgroups. And by induction on n, we are done in this case.

Case 3 Assume here that $P' \in \mathcal{T}_\Gamma(p, G)$ and $P \in \mathcal{T}_{\mathrm{irr}}(p, G)$. From Definitions 1 and 2, one sees that $P' \subseteq P$ for any such pair for which $\dim(P') \leq \dim(P)$ (without any prior assumption of containment up to conjugacy). Also, the inclusion is a composite of inclusions of type (a) and (c) above. So we are done in this case.

Case 4 Finally, assume that $P', P \in \mathcal{T}_{\mathrm{irr}}(p, G)$, and that $P' \notin \mathcal{T}_\Gamma(p, G)$. Write

$$P' = \Gamma' \wr E_{q'_1} \wr \cdots \wr E_{q'_s} \qquad \text{and} \qquad P = \Gamma \wr E_{q_1} \wr \cdots \wr E_{q_r}.$$

Here $s > 0$ by assumption; and $r > 0$ since otherwise $\delta(P) \cong C_2$ or S^1 (in which case $g\delta(P')g^{-1} \nsubseteq \delta(P)$). Also,

$$g\delta(P')g^{-1} = g\left(\Gamma' \wr E_{q'_1} \wr \cdots \wr E_{q'_{s-1}}\right)^{q'_s} g^{-1} \subseteq \left(\Gamma \wr E_{q_1} \wr \cdots \wr E_{q_{r-1}}\right)^{q_r} = \delta(P),$$

where the individual factors are irreducible. Hence $q_r | q'_s$. By induction on n (and since permuting the irreducible factors leaves the groups unchanged)

$$\left(\Gamma' \wr E_{q'_1} \wr \cdots \wr E_{q'_{s-1}}\right)^{q'_s/q_r} \subseteq \Gamma \wr E_{q_1} \wr \cdots \wr E_{q_{r-1}} \tag{1}$$

for some $g' \in \mathbb{G}(n/q_r)$. If $q_r = q'_s$, then we are reduced to the case of a smaller inclusion of irreducible p-stubborn subgroups. If $q_r < q'_s$, then the inclusion (1) is of the type handled in Case 2. So either it is the inclusion $\mathrm{O}(1) \times \mathrm{O}(1) \subseteq \mathrm{O}(2)$ (and $gP'g^{-1} \subseteq P$ is an inclusion of type (f)); or $r > 1$, $q_{r-1}|(q'_s/q_r)$ and

$$gP'g^{-1} \subseteq \Gamma \wr E_{q_1} \wr \cdots \wr E_{q_{r-1}q_r} \subseteq \Gamma \wr E_{q_1} \wr \cdots \wr E_{q_{r-1}} \wr E_{q_r} = P.$$

In other words, we have factored through an inclusion of type (b) in this case; and this finishes the proof. $\square$

Theorems 6 and 8 describe the p-stubborn subgroups of the matrix groups $\mathrm{O}(n)$, $\mathrm{U}(n)$, and $\mathrm{Sp}(n)$; but we are mostly interested in the connected simple groups. The rest of the paper deals with the connection between the p-stubborn subgroups of $\mathrm{SU}(n)$ and of $\mathrm{U}(n)$; and between the 2-stubborn subgroups of $\mathrm{SO}(n)$ and $\mathrm{O}(n)$. In fact, the categories $\mathcal{R}_p(\mathrm{SU}(n))$ and $\mathcal{R}_p(\mathrm{U}(n))$ are always isomorphic.

Theorem 10. *For and p and any $n > 0$, the correspondences*

$$P \mapsto \langle P, Z(\mathrm{U}(n))\rangle \qquad \text{(for } P \subseteq \mathrm{SU}(n)\text{)}$$

and

$$\overline{P} \mapsto (\overline{P} \cap \mathrm{SU}(n)) \qquad \text{(for } \overline{P} \subseteq \mathrm{U}(n)\text{)}$$

define a one-to-one correspondence between the p-stubborn subgroups of $\mathrm{SU}(n)$ and those of $\mathrm{U}(n)$; and induce an isomorphism of categories

$$\mathcal{R}_p(\mathrm{SU}(n)) \xrightarrow{\ \cong\ } \mathcal{R}_p(\mathrm{U}(n)).$$

Proof. By [JMO, Proposition 1.6(i)], for any connected G, a subgroup $P \subseteq G$ is p-stubborn if and only if $P \supseteq Z(G)$ and $P/Z(G)$ is p-stubborn in $G/Z(G)$. We thus get bijections between the p-stubborn subgroups of $\mathrm{SU}(n)$, $\mathrm{SU}(n)/Z(\mathrm{SU}(n)) \cong \mathrm{U}(n)/Z(\mathrm{U}(n))$, and $\mathrm{U}(n)$. And since $\mathrm{SU}(n)/P \cong \mathrm{U}(n)/\langle P, Z(\mathrm{U}(n))\rangle$ as orbits, this correspondence induces an isomorphism between the orbit categories $\mathcal{R}_p(-)$. $\square$

The relation between 2-stubborn subgroups of $\mathrm{SO}(n)$ and $\mathrm{O}(n)$ is more complicated.

Proposition 11. *For any 2-stubborn subgroup $P \subseteq \mathrm{SO}(n)$, there is a unique 2-stubborn subgroup $\epsilon P \subseteq \mathrm{O}(n)$ such that $P = \epsilon P \cap \mathrm{SO}(n)$ and $N_{\mathrm{O}(n)}(\epsilon P) = N_{\mathrm{O}(n)}(P)$. If $P_1 \subseteq P_2$ is a pair of 2-stubborn subgroups of $\mathrm{SO}(n)$, then $\epsilon P_1 \subseteq \epsilon P_2$.*

Proof. If $P \subseteq \mathrm{SO}(n)$ is 2-stubborn, then define $\epsilon P \subseteq \mathrm{O}(n)$ to be the subgroup such that $\epsilon P/P = \mathcal{O}_2(N_{\mathrm{O}(n)}(P)/P)$. Then $\epsilon P \cap \mathrm{SO}(n) = P$, since $\mathcal{O}_2(N_{\mathrm{SO}(n)}(P)/P) = 1$. It follows that $N_{\mathrm{O}(n)}(\epsilon P) \subseteq N_{\mathrm{O}(n)}(P)$, and $N_{\mathrm{O}(n)}(P) \subseteq N_{\mathrm{O}(n)}(\epsilon P)$ by construction. Hence $\mathcal{O}_2(N_{\mathrm{O}(n)}(\epsilon P)/\epsilon P) = \mathcal{O}_2(N_{\mathrm{O}(n)}(P)/\epsilon P) = 1$, and so ϵP is 2-stubborn. The uniqueness of ϵP is clear.

It remains to show that $\epsilon P_1 \subseteq \epsilon P_2$ whenever $P_1 \subseteq P_2$. This will be done by induction on n. In order to carry out the induction step, we consider the following slightly more general situation. Let $\overline{P_1}, \overline{P_2} \subseteq \mathrm{O}(n)$ be a pair of subgroups, such that $x_i\overline{P_i}x_i^{-1} \in \mathcal{T}_{\mathrm{prod}}(2, \mathrm{O}(n))$ for some $x_1, x_2 \in \mathrm{O}(n)$. Set $P_i = \overline{P_i} \cap \mathrm{SO}(n)$, and assume that $P_1 \subseteq P_2$. Assume in addition that (a) $N_{\mathrm{O}(n)}(P_i) = N_{\mathrm{O}(n)}(\overline{P_i})$, and (b) $\mathrm{O}(1)$ does not have multiplicity exactly 2 or 4 in the irreducible decomposition of $x_1\overline{P_1}x_1^{-1}$ (compare with Theorem 6(ii)). We claim that under these assumptions, $\overline{P_1} \subseteq \overline{P_2}$. We may clearly assume that $\overline{P_1} \nsubseteq \mathrm{SO}(n)$ ($\overline{P_1} \neq P_1$).

Throughout the following arguments, it is useful to note that (by Definition 2) the subgroups in $\mathcal{T}_{\mathrm{irr}}(2, \mathrm{O}(n))$ which are not contained in $\mathrm{SO}(n)$ are precisely those of the form

$$\mathrm{O}(m) \wr E_{q_1} \wr \cdots \wr E_{q_r}, \tag{1}$$

where $m = 1$ or 2, $r \geq 0$, and $n = m{\cdot}q_1 \cdots q_r$ (and $q_1 \geq 4$ if $m = 1$ and $r \geq 1$). In particular, each such group contains all diagonal matrices $\mathrm{diag}(\pm 1, \ldots, \pm 1)$.

We first check that for each i, P_i has the same decomposition into irreducible representations as $\overline{P_i}$. To see this, it suffices to show that for any (irreducible) $\overline{P} \in \mathcal{T}_{\mathrm{irr}}(2, \mathrm{O}(m))$, the subgroup $P := \overline{P} \cap \mathrm{SO}(m)$ is also irreducible. If V denotes the $\overline{P}$-representation on $\mathbb{R}^m$, and $V|P = V_1 \oplus V_2$ is reducible, then $V \cong \mathrm{Ind}_P^{\overline{P}}(V_1)$, and $\chi_V(g) = 0$ for all $g \in \overline{P} \smallsetminus P$. Also, $\overline{P}$ has the form in (1) above (since it is not contained in $\mathrm{SO}(m)$). And the only subgroup of that form in which all elements of determinant -1 have zero trace is the case $\overline{P} = \mathrm{O}(2)$ (and $\mathrm{SO}(2)$ is irreducible).

Case 1 Assume first that $\overline{P_2}$ is reducible. We can assume that $\overline{P_2} \in \mathcal{T}_{\mathrm{prod}}(2, \mathrm{O}(n))$ (i.e., $x_2 = 1$); and write $\overline{P_2} = Q_{21} \times Q_{22}$ where $Q_{2j} \subseteq \mathrm{O}(n_j)$ ($j = 1, 2$) and $n = n_1 + n_2$ ($0 < n_j < n$). Then, since P_1 and $\overline{P_1}$ decompose in the same way, and since $\overline{P_1}$ splits as a product of irreducible subgroups, we have $\overline{P_1} = Q_{11} \times Q_{12}$, where $Q_{1j} \subseteq \mathrm{O}(n_j)$. If hypothesis (b) holds for both pairs (Q_{2j}, Q_{1j}), then $Q_{1j} \subseteq Q_{2j}$ by induction on n, and hence $\overline{P_1} \subseteq \overline{P_2}$. If hypothesis (b) does not hold, then Q_{11} and Q_{12} both contain factors $\mathrm{O}(1)$, and hence neither lies in $\mathrm{SO}(n_j)$. This implies that Q_{1j} is the image of P_1 under the projection to $\mathrm{O}(n_j)$ ($j = 1, 2$), hence that $Q_{1j} \subseteq Q_{2j}$, and hence that $\overline{P_1} \subseteq \overline{P_2}$.

The remaining cases will be handled using the subgroups

$$\delta(P_i) = \langle g{\in}P_i \mid \mathrm{Tr}(g) \neq 0 \rangle \qquad \text{and} \qquad \delta(\overline{P_i}) = \langle g \in \overline{P_i} \mid \mathrm{Tr}(g) \neq 0 \rangle$$

of Lemma 5. Clearly, $\delta(P_1) \subseteq \delta(P_2)$. If $\overline{P_i} \nsubseteq \mathrm{SO}(n)$, then $x_i\overline{P_i}x_i^{-1}$ contains at least one factor of the form in (1) above, and hence some diagonal matrix D with a single entry -1. In particular, since $\mathrm{Tr}(D) \neq 0$ when $n \neq 2$,

$$\overline{P_i} = \langle \delta(\overline{P_i}), P_i \rangle \qquad \text{whenever } n \neq 2. \tag{2}$$

It thus suffices in the remaining cases to show that $\delta(\overline{P_1}) \subseteq \overline{P_2}$.

Case 2 Assume now that $\overline{P_2}$ is irreducible, but that $\overline{P_1}$ is reducible. In this case, it will be convenient to assume that $\overline{P_1} \in \mathcal{T}_{\mathrm{prod}}(2, \mathrm{O}(n))$ (i.e., $x_1 = 1$). Note that n is a power of 2, since $\overline{P_2}$ is irreducible. And since $\overline{P_1}$ is reducible but cannot be a product involving exactly 2 or 4 copies of $\mathrm{O}(1)$ (and since the case $n = 2$ is trivial), we may assume that either $\overline{P_1} = \mathrm{O}(2) \times \mathrm{O}(2)$, or $n \geq 8$. And if $\overline{P_1} = \mathrm{O}(2) \times \mathrm{O}(2)$, then the identity component of P_1 is a maximal torus in $O(4)$, so $\overline{P_2} = \mathrm{O}(2) \wr E_2$ (the normalizer of that maximal torus); and $\overline{P_1} \subseteq \overline{P_2}$.

Assume now that $n \geq 8$. We claim that $\delta(P_1) = P_1$. This will follow from Lemma 5(a) if we can show that $\delta(P_1) = P_1 \cap \delta(\overline{P_1})$. This means showing, for any elements $A_1, A_2 \in \overline{P_1}$ such that $\det(A_i) = -1$ and $\mathrm{Tr}(A_i) \neq 0$ $(i = 1, 2)$, that $A_1A_2 \in \delta(P_1)$. Let D_i (for $i = 1, \ldots, n$) be the diagonal matrix with entry -1 in the i-th position and 1's elsewhere; and set $V = \{i \,|\, D_i \in \overline{P_1}\}$. Since $\overline{P_1} \nsubseteq \mathrm{SO}(n)$, $\overline{P_1}$ contains at least one summand of the form in (1); and all irreducible summands not of that form have dimension a multiple of 4. It follows that $|V| > 0$, and that $|V| \equiv n \equiv 0 \pmod 4$. For each $i = 1, 2$, either there exists $j_i \in V$ such that $\mathrm{Tr}(A_iD_{j_i}) \neq 0$; or all of the diagonal components $(A_i)_{jj}$ for $j \in V$ are the same, and $\mathrm{Tr}(A_iD) \neq 0$ whenever D is the product of three of the D_j's for $j \in V$. In either case, we can find elements $A'_i \in P_1$ such that $\mathrm{Tr}(A'_i) \neq 0$, and such that $A_1A_2 = A'_1A'_2D$ for some diagonal matrix D with at most two entries -1. Since $n \geq 8$, $\mathrm{Tr}(D) \neq 0$, and so $A_1A_2 \in \delta(P_1)$.

This shows that

$$P_1 = \delta(P_1) \subseteq \delta(P_2) \subseteq \delta(\overline{P_2}).$$

By Lemma 5, $\overline{P_2}$ must be a nontrivial wreath product:

$$x_2\overline{P_2}x_2^{-1} = \Gamma \wr E_{q_1} \wr \cdots \wr E_{q_r} \quad (r \geq 1) \qquad \text{and} \qquad x_2\delta(\overline{P_2})x_2^{-1} = \left(\Gamma \wr E_{q_1} \wr \cdots \wr E_{q_{r-1}}\right)^{q_r}.$$

Thus, $\delta(\overline{P_2})$ is reducible and conjugate to an element of $\mathcal{T}_{\mathrm{prod}}(2, \mathrm{O}(n))$; so $\overline{P_1} \subseteq \delta(\overline{P_2}) \subseteq \overline{P_2}$ by Case 1, and $\overline{P_1} \subseteq \overline{P_2}$ by (2).

Case 3 Finally, assume that $\overline{P_1}$ and $\overline{P_2}$ are both irreducible. If $\overline{P_1} = \mathrm{O}(1)$ or $\mathrm{O}(2)$, then there is nothing to prove. Otherwise (if we take $x_1 = 1$), $\overline{P_1} \in \mathcal{T}_{\mathrm{irr}}(2, \mathrm{O}(n))$ is a nontrivial wreath product of the form in (1):

$$\overline{P_1} = \mathrm{O}(m) \wr E_{q_1} \wr \cdots \wr E_{q_r} \quad (r \geq 1) \qquad \text{and} \qquad \delta(\overline{P_1}) = \left(\mathrm{O}(m) \wr E_{q_1} \wr \cdots \wr E_{q_{r-1}}\right)^{q_r}$$

($m = 1$ or 2). Then $\delta(\overline{P_1}) \cap \mathrm{SO}(n) \subseteq P_2$, and $\delta(\overline{P_1}) \in \mathcal{T}_{\mathrm{prod}}(2, \mathrm{O}(n))$. So $\delta(\overline{P_1}) \subseteq \overline{P_2}$ (and hence $\overline{P_1} \subseteq \overline{P_2}$), unless $\delta(\overline{P_1}) = (\mathrm{O}(1))^4$ and $\overline{P_1} = \mathrm{O}(1) \wr E_4$ (i.e., condition (b) fails). And this case cannot occur, since $(\mathrm{O}(1) \wr E_4) \cap \mathrm{SO}(4) = \Gamma_4^{\mathrm{O}}$ and $N_{\mathrm{O}(4)}(\mathrm{O}(1) \wr E_4) \neq N_{\mathrm{O}(4)}(\Gamma_4^{\mathrm{O}})$.
□

By Proposition 11, for each n, there is a functor

$$\mathcal{E}_n : \mathcal{R}_2(\mathrm{SO}(n)) \to \mathcal{R}_2(\mathrm{O}(n)),$$

defined by setting $\mathcal{E}_n(\mathrm{SO}(n)/P) = \mathrm{O}(n)/\epsilon P$ whenever $P \subseteq \mathrm{SO}(n)$ and $\epsilon P \subseteq \mathrm{O}(n)$ are 2-stubborn subgroups such that $P = \epsilon P \cap \mathrm{SO}(n)$ and $N_{\mathrm{O}(n)}(\epsilon P) = N_{\mathrm{O}(n)}(P)$.

Theorem 12. *If $4 \nmid n$, then $\mathcal{E}_n$ is an isomorphism of categories. Otherwise, its failure to be an isomorphism is described (in part) as follows:*

(i) For any 2-stubborn subgroup $\overline{P}\subseteq \mathrm{O}(n)$, $P := \overline{P}\cap \mathrm{SO}(n)$ is 2-stubborn; and $\overline{P} = \epsilon P$ unless $\overline{P}$ is conjugate to a subgroup of the form $(\mathrm{O}(1)\wr E_4)\times P'$ for $P'\subseteq \mathrm{SO}(n-4)$ (in which case $\overline{P}\neq \epsilon P$ and $\mathrm{O}(n)/\overline{P}\notin \mathrm{Im}(\mathcal{E}_n)$).

(ii) Let $P\subseteq \mathrm{SO}(n)$ be any 2-stubborn subgroup. Then

$$\{gPg^{-1}\,|\,g\in \mathrm{O}(n)\} = \{g\epsilon Pg^{-1}\cap \mathrm{SO}(n)\,|\,g\in \mathrm{O}(n)\}$$

consists of one $\mathrm{SO}(n)$-conjugacy class if $N(\epsilon P)\nsubseteq \mathrm{SO}(n)$, and consists of two conjugacy classes otherwise.

(iii) For any pair P_1, P_2 of p-stubborn subgroups of $\mathrm{SO}(n)$, the map

$$\mathcal{E}_n : \mathrm{Mor}(\mathrm{SO}(n)/P_1, \mathrm{SO}(n)/P_2) \longrightarrow \mathrm{Mor}(\mathrm{O}(n)/\epsilon P_1, \mathrm{O}(n)/\epsilon P_2)$$

is injective, and is bijective if $\epsilon P_2\nsubseteq \mathrm{SO}(n)$. Also (when $P_1 = P_2 = P$),

$$\mathrm{Aut}(\mathrm{SO}(n)/P)\cong \mathrm{Aut}(\mathrm{O}(n)/\epsilon P)$$

unless $\epsilon P\subseteq \mathrm{SO}(n)$ and $N_{\mathrm{O}(n)}(\epsilon P)\nsubseteq \mathrm{SO}(n)$.

(iv) Fix an irreducible 2-stubborn subgroup $\overline{P}\subseteq \mathrm{O}(2^k)$ (any $k\geq 0$). Then $\overline{P}\subseteq \mathrm{SO}(2^k)$ if and only if $\overline{P}$ is conjugate to one of the groups

$$\Gamma^{\mathrm{O}}_{q_0}\wr E_{q_1}\wr\cdots\wr E_{q_r}\ (q_0\geq 4) \quad \text{or} \quad \overline{\Gamma}^{\mathrm{O}}_{q_0}\wr E_{q_1}\wr\cdots\wr E_{q_r}\ (q_0\geq 4). \tag{1}$$

And for such $\overline{P}$, $N(\overline{P})\subseteq \mathrm{SO}(2^k)$ unless $\overline{P}$ is conjugate to Γ^{O}_4.

Proof. If n is odd, then $\mathrm{O}(n)\cong \mathrm{SO}(n)\times\{\pm I\}$, and $\mathcal{E}_n$ is clearly an isomorphism of categories.

If $n = 4k+2$, then any 2-stubborn subgroup in $\mathrm{O}(n)$ has, up to conjugacy, the form $\overline{P} = \overline{P'}\times \mathrm{O}(2)$ or $\overline{P} = \overline{P'}\times \mathrm{O}(1)\times \mathrm{O}(1)$ for some $\overline{P'}\subseteq \mathrm{O}(4k)$. In particular, for any 2-stubborn $\overline{P}\subseteq \mathrm{SO}(n)$, $\overline{P} = (\epsilon P\cap \mathrm{SO}(n))$ by (i). So $\mathcal{E}_n$ is surjective on objects; and is injective since $P = \epsilon P\cap \mathrm{SO}(n)$ for any P. And finally, since no 2-stubborn subgroup of $\mathrm{O}(n)$ is contained in $\mathrm{SO}(n)$, $\mathcal{E}_n$ induces bijections on all morphism sets by (iii).

It remains to prove points (i) to (iv).

(i) Let $\overline{P}\subseteq \mathrm{O}(n)$ be any 2-stubborn subgroup, and set $P = \overline{P}\cap \mathrm{SO}(n)$. By Theorem 8, we may assume that $\epsilon P\in \mathcal{T}_{\mathrm{prod}}(2, \mathrm{O}(n))$; and in particular that it splits as a product of irreducible p-stubborn subgroups $P_i\in \mathcal{T}_{\mathrm{irr}}(2, \mathrm{O}(n_i))$ (where $n = \sum n_i$). Let r be the number of factors for which $P_i\nsubseteq \mathrm{SO}(n_i)$. If $r = 0$, then $P = \overline{P}$ is 2-stubborn in $\mathrm{SO}(n)$ — since every Sylow 2-subgroup of $N_{\mathrm{O}(n)}(P)/P$ contains a Sylow 2-subgroup of $N_{\mathrm{SO}(n)}(P)/P$ — and $\epsilon P = \overline{P}$. If $r\geq 2$, then $N_{\mathrm{O}(n)}(P) = N_{\mathrm{O}(n)}(\overline{P})$ ($\overline{P}$ is the product of the projections of P into the irreducible factors); and so again P is 2-stubborn and $\epsilon P = \overline{P}$. We are left with the case $r = 1$; and we can just as easily assume here that $\overline{P}\nsubseteq \mathrm{SO}(n)$ is irreducible. And a quick check of the list of normalizers in Theorem 6(i,ii) shows that $\overline{P} = \mathrm{O}(1)\wr E_4\subseteq \mathrm{O}(4)$ is the only case where $N_{\mathrm{O}(n)}(\overline{P})\neq N_{\mathrm{O}(n)}(P)$ (and $(\mathrm{O}(1)\wr E_4)\cap \mathrm{SO}(4) = \Gamma^{\mathrm{O}}_4$ is 2-stubborn in $\mathrm{SO}(4)$).

(ii) The set $\{gPg^{-1}\,|\,g\in\mathrm{O}(n)\}$ contains at most $[\mathrm{O}(n){:}\,\mathrm{SO}(n)]{=}2$ $\mathrm{SO}(n)$-conjugacy classes. It contains just one conjugacy class if and only if $gPg^{-1}=xPx^{-1}$ for some $g\in\mathrm{O}(n)\smallsetminus\mathrm{SO}(n)$ and $x\in\mathrm{SO}(n)$, if and only if $N_{\mathrm{O}(n)}(P)=N_{\mathrm{O}(n)}(\epsilon P)\not\subseteq\mathrm{SO}(n)$.

(iii) Set

$$X=\bigl\{g\in\mathrm{O}(n)\,|\,P_1\subseteq gP_2g^{-1}\bigr\}=\bigl\{g\in\mathrm{O}(n)\,|\,\epsilon P_1\subseteq g(\epsilon P_2)g^{-1}\bigr\}.$$

Then $\mathrm{Mor}(\mathrm{SO}(n)/P_1,\mathrm{SO}(n)/P_2)\cong(X\cap\mathrm{SO}(n))/P_2$, and $\mathrm{Mor}(\mathrm{O}(n)/\epsilon P_1,\mathrm{O}(n)/\epsilon P_2)\cong X/\epsilon P_2$. So $\mathcal{E}_n$ induces an injection between these morphism sets, and a bijection if $\epsilon P_2\not\subseteq\mathrm{SO}(n)$. And if $P_1=P_2=P$, then $X=N_{\mathrm{O}(n)}(P)=N_{\mathrm{O}(n)}(\epsilon P)$, and $N_{\mathrm{SO}(n)}(P)/P\cong N_{\mathrm{O}(n)}(\epsilon P)/\epsilon P$ if and only if $\epsilon P\not\subseteq\mathrm{SO}(n)$ or $N_{\mathrm{O}(n)}(\epsilon P)\subseteq\mathrm{SO}(n)$.

(iv) This follows easily from Theorem 8 (and Definition 2), except for showing that $N_{\mathrm{O}(2^k)}(\overline{P})\subseteq\mathrm{SO}(2^k)$ when $\overline{P}=\Gamma^{\mathrm{O}}_{2^k}$ $(k\geq 3)$ or $\overline{\Gamma}^{\mathrm{O}}_{2^k}$ $(k\geq 2)$. When $\overline{P}=\overline{\Gamma}^{\mathrm{O}}_{2^k}$, this follows since $N(\overline{P})/\overline{P}\cong\mathrm{Sp}_{2k-2}(\mathbb{F}_2)$ is generated by elements in $\mathrm{U}(2^{k-1})\subseteq\mathrm{SO}(2^k)$.

Now assume $P=\overline{P}=\Gamma^{\mathrm{O}}_{2^k}=\langle A_0,B_0,\ldots,A_{k-1},B_{k-1}\rangle$ (see Definition 1); and $k\geq 3$. Then for $1\leq i\leq k{-}1$, A_i and B_i lie in the simply connected subgroup $\mathrm{SU}(2^{k-1})\subseteq\mathrm{SO}(2^k)$. So the commutators $[A_i,B_i]$ $(1\leq i\leq k{-}1)$ all lift to the same commutator in $\widetilde{P}\subseteq\mathrm{Spin}(2^k)$. By symmetry, $[A_0,B_0]$ lifts to the same commutator $(k\geq 3)$; and so only one lifting of $-I$ lies in $[\widetilde{P},\widetilde{P}]$. On the other hand, for any $x\in\mathrm{O}(2^k)\smallsetminus\mathrm{SO}(2^k)$, $\mathrm{conj}(x)$ lifts to a unique automorphism of $\mathrm{Spin}(2^k)$ which switches the two liftings of $-I$; and so $x\notin N(P)$. Thus $N(P)=N(\Gamma^{\mathrm{O}}_{2^k})\subseteq\mathrm{SO}(2^k)$. $\square$

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